



On a Dynamical Origin for Fermion Generations

Jim Bashford

Ph.D. Thesis

Department of Physics and Mathematical Physics

University of Adelaide

July 2003

Contents

0.1	Statement	3
0.2	Acknowledgements	4
1	Introduction	5
2	Preliminary	13
2.1	Chiral symmetry	13
2.2	Chiral anomalies	15
2.3	Effective action	18
2.4	Schwinger-Dyson equations	20
2.5	Renormalisability and triviality	23
2.5.1	Wegner-Houghton RGE	27
3	Models	29
3.1	Electroweak model	29
3.1.1	Generations and anomaly cancellation	30
3.1.2	Spontaneous symmetry breaking and mass	31
3.1.3	Flavour mixing and CP violation	33
3.1.4	Non-perturbative aspects	34
3.2	The NJL model	35
3.3	QED_4 : dynamical chiral symmetry breaking	37
3.3.1	Quenched rainbow approximation	38
3.3.2	Improved quenched approximations	40
3.3.3	Unquenched approximations	44
3.4	Gauged model	45
4	Four-fermion model of generations	48
4.1	Digression	50
4.2	Mass generation	52
4.2.1	Conventional chiral symmetric phase	53

4.2.2	Explicitly broken phase	54
4.2.3	Anomalous phase	55
4.3	Fermion generations	55
4.4	CP-violation	57
4.4.1	Triviality	57
4.4.2	Summary	61
4.5	Four-fermion model	63
4.5.1	Mass generation	65
5	Quenched hypercharge	68
5.1	$D\chi$ SB in hypercharge $U(1)$	69
5.1.1	Rainbow approximation	69
5.1.2	Renormalisable vertex	72
5.2	Role of composite scalars	80
5.2.1	Anomalous vertex	82
6	Conclusion and outlook	86
6.0.2	Chiral-breaking model	86
6.0.3	Quenched hypercharge	88
6.1	Outlook	90
A	Fierz transformations	95
B	1-loop running couplings	97
C	Transverse boson propagator	99

Abstract

We investigate a proposal to address several outstanding shortcomings of the perturbative Standard Model (SM) of particle physics, specifically a common, dynamical origin for the number of fermion generations, the spectrum of fermion masses and for Charge-Parity (CP) violating processes. The appeal of this proposal is that these features are a manifestation of the non-perturbative sector of the SM, requiring no assumptions about new physics beyond presently attainable experimental limits.

In this thesis we apply non-perturbative techniques, which have been used to investigate dynamical symmetry breakdown in other quantum field theories, to the $U(1)$ hypercharge sector of the SM, peculiar among chiral gauge theories in that the same gauge field mediates both interacting sectors.

We consider two models, a toy 4-fermion model containing explicit chiral symmetry-breaking terms (of an anomalous origin) and the quenched hypercharge gauge interaction, which complement each other. The key difference between this theory and studies of “conventional” breaking (such as in the gauged Nambu Jona Lasinio model) is the realisation that here fermion pairing terms associated with dynamical chiral symmetry breakdown are a necessary but not sufficient requirement for dynamical mass generation, analogous to the pseudogap phenomenon observed in systems of strongly-correlated electrons.

Understanding of how the mass, generations and CP-violation might arise are first investigated in the toy 4-fermion model. It is shown that different scale-invariant 4-fermion operators are present for the three subspaces of the full theory, enabling self-consistent introduction of three fermion generations.

The second part of the thesis is concerned with dynamical fermion mass generation in the quenched hypercharge interaction. In particular we follow the successful procedure developed for QED, developing a 1-loop renormalisable vertex ansatz for solution of the fermion self-energy Dyson-Schwinger equation. In the absence of dynamical fermion-antifermion bound states it is found there exist two mass “gaps” corresponding to two types of condensate. These “gaps” cannot be interpreted as physical fermion mass. It is suggested that only after the incorporation of the composite scalars does the self-energy equation admit multiple (physical) solutions. An alternative

possibility, that of a rearrangement of fermionic degrees of freedom analogous to spin-charge separation (SCS) in condensed matter physics, is also briefly outlined.

0.1 Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

Jim Bashford

0.2 Acknowledgements

- First and foremost, I wish thank to my supervisor Tony Thomas, for his unfailing support and enormous patience.
- To my wonderful partner Virginia and also to Hayden and Emily for their continuing insights into the equation-free world at large. And to Lydia for providing the necessary impetus to finally finish.
- To my parents and sisters for their moral support.
- To Stuart Corney for friendship and bravely trusting me with the run of his machine at the University of Tasmania when facilities in Adelaide were incapacitated.
- To Peter Jarvis, for encouragement, friendship and several helpful discussions.
- To my friends Natalie, Michael and Sam, who made life in Adelaide enjoyable.
- I wish to acknowledge financial support from the University of Adelaide through an APRA scholarship, and a generous extension from the Centre for the Subatomic Structure of Matter.

Chapter 1

Introduction

The Standard Model (SM) of particle physics has had remarkable success describing the behaviour of subatomic matter (for a review see [1]). The scalar sector of the SM, however is constructed to accommodate, rather than explain, certain phenomenological features which, therefore, must be regarded as poorly understood. Chief among these are the Charge-Parity (CP)-violating electroweak processes, the origin and magnitude of mass and the number of fundamental fermion species.

In this thesis we investigate a proposal [2] which suggested a common, dynamical origin for the mass and fermionic generations without the requirement of additional gauge unification. The nature of electroweak flavour mixing - and more specifically, CP-violating processes - is known to be sensitive to both the ratios of quark mass eigenstates and the number of quark flavours. Therefore the origin of CP violation could also reasonably be expected to lie within the hypothesis [2].

The known fundamental fermions are naturally classified into 3 families or “generations”, each consisting of two quarks (colour degrees of freedom are largely irrelevant to this thesis and shall be omitted here) q_\uparrow , q_\downarrow , a charged lepton ℓ and its associated neutrino ν_ℓ . The weak isospin τ_3 , electromagnetic charge Q and mass m_i of each generation are summarised in the columns of Table (1.1) below. Within current experimental limits (for example for lepton universality [3],[4]), the fermions of one generation are identical to those of another in terms of gauge interactions, being distinguished only by their orders-of-magnitude mass difference. Essentially the two more massive generations behave as unstable and largely redundant copies of the lightest, of which the vast majority of stable matter in the observed universe is composed.

Table 1.1: Fundamental fermion properties. Note (estimated) quark masses are in GeV while lepton masses are in MeV. All masses are obtained from [5].

	τ_3	Q	m_1	m_2	m_3
q_\uparrow	$+\frac{1}{2}$	$+\frac{2}{3}$	$0.001 - 0.005$	$1.15 - 1.35$	174.3 ± 5.1
q_\downarrow	$-\frac{1}{2}$	$-\frac{1}{3}$	$0.003 - 0.009$	$0.075 - 0.170$	$4.0 - 4.4$
ℓ	$-\frac{1}{2}$	-1	0.511	105.7	1777
ν_ℓ	$+\frac{1}{2}$	0	≤ 0.003	≤ 0.19	≤ 18.2

An important experimental limit on fermion numbers comes from the LEP measurements of partial invisible Z -decay width [5], where the number of “light” neutrinos (with mass $\leq M_Z/2$) is determined as

$$n_\nu = 2.984 \pm 0.008, \quad (1.1)$$

from which it is inferred - if the cancellation of anomalous electroweak processes is to occur - that there are only three fermion generations. The existence of extra “half generations” of quarks has been considered (e.g. [6]), however a new confining gauge interaction is needed to form composites which mimic the absent leptons.

A number of theories explaining the number of generations exist, for example new “horizontal” flavour gauge interactions. Variations within this theme include gauge interactions underlying composite quarks and leptons [7],[8] or linking the number of generations to the mass hierarchy problem via spontaneously-broken $SU(3)$ (see [9] and references contained within). The simplest horizontal SM extension - an Abelian interaction - taken to be non-anomalous [10] or anomalous [11],[12], [13] at the level of the supersymmetric standard model readily reproduces the required mass hierarchy. Here it should be noted that for the latter case [11],[12] anomaly cancellation is restored at the string-unification scale via the Green-Schwartz mechanism [14].

While dynamical chiral symmetry breaking ($D\chi SB$) has been used for decades to study the origin of mass in relativistic quantum field theories such as the Nambu Jona Lasinio (NJL) model [15] and strongly-coupled quantum electrodynamics (QED) [16] (for reviews see [17] or [18]), the possibility of a dynamical origin for fermion generations has a smaller and relatively recent literature. Some recent theories requiring interesting gauge interactions include broken discrete chiral symmetry [19] and the $SU(3)$ colour analogue of electromagnetic duality [20].

The possibility that fermion generations could arise dynamically from interplay of scalar fields and elementary fermions without new gauge interactions has also recently been explored in several contexts [21], [22]. In the former [21], the multiple generations are a manifestation of a specially-constrained Higgs potential possessing a discrete symmetry of degenerate vacuum minima, while in the latter [22] higher generations effectively appear as a bound state of some elementary fermion with excitations of the Higgs scalar. However both mechanisms make use of at least one elementary Higgs field and in both cases the number of generations is still a parameter to be put in by hand. A motivation for the former has been outlined in [23].

In contrast, an earlier paper [2] based upon the non-perturbative $U(1)$ hypercharge interaction suggests the existence of 3 families and the absence of a fundamental Higgs. The hypothesis, which is the main theme of this thesis, is based upon two observations:

1) Analysis of strong-coupling QED in a quenched approximation (e.g. see [16]) suggests vector Abelian theories have 2 phases separated by a second-order transition associated with breakdown of chiral symmetry. The possibility of a non-trivial ultraviolet (UV)-fixed point is interesting for several reasons, firstly because the total charge-screening of perturbative QED at large scales fails to eventuate, facilitating a consistent definition of non-perturbative QED. Moreover a self-consistent treatment of this non-perturbative QED may dynamically generate fermion mass, abolishing the requirement for undetected elementary Higgs scalars.

2) Assuming no further unification physics, the large momentum limit of the Standard Model is dominated by the chiral $U(1)$ (hypercharge) sector, with interaction Lagrangian term

$$\mathcal{L}_I = -\frac{g_2}{\cos \theta_W} (q \sin^2 \theta_W \bar{\psi}_R \gamma^\mu \psi_R + (-\tau_3 + q \sin^2 \theta_W) \bar{\psi}_L \gamma^\mu \psi_L) Z_\mu. \quad (1.2)$$

Here g_2 is the weak coupling constant, and θ_W is the Weinberg angle. The key feature is that, unlike QED, the coupling is dependent on fermion chirality. The hypercharge theory is special amongst chiral gauge theories in that the *same* gauge field couples to both chiralities. In particular integration of the gauge-boson field from the lagrangian Eq.(1.2) leads to effective 4-fermi interactions of the form (with the perturbative boson propagator)

$$\mathcal{L}_{eff} = (c_L \psi_L \gamma^\mu \psi_L + c_R \psi_R \gamma^\mu \psi_R)(c_L \psi_L \gamma_\mu \psi_L + c_R \psi_R \gamma_\mu \psi_R). \quad (1.3)$$

Upon application of Fierz identities (see Appendix A) the terms $\sim c_L^2$, $\sim c_R^2$ are seen to provide the conventional chiral result, i.e. upon integration over

separate left- and right- gauge fields, the vector, axial and anomalous 4-fermi terms are obtained. The "cross" terms $\sim c_L c_R$, peculiar to the hypercharge theory provide scalar 4-fermi interactions

$$(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2,$$

familiar to D χ SB studies of QED or the gauged NJL (GNJL) [24] theory. In principle there are *three* running couplings (right-right, right-left and left-left) and the theory could be anticipated to have a more complicated phase structure than QED.

The stability of non-asymptotically free (NAF) $U(1)$ vector theories has been studied extensively (e.g. [25], [26]). While the question of whether the two-phase structure of strongly-coupled QED carries over into the unquenched sector has yet to be fully resolved, even less is known about the nature of the corresponding chiral theory, although a preliminary lattice study [27] suggests it suffers from the same charge-screening problem. In the absence of any consensus, if one assumes the above phase structure, the features of fermion mass and generations arise in the qualitative picture of [2] as follows:

Fermion masses arise in analogy to the QED case. In quenched rainbow QED [24], it was suggested that at large momentum scales attractive scalar fermion couplings became strong and offset the total virtual-pair-screening of a point-like fermion charge [28]. In the renormalisation group picture non-renormalisable (at "low" momentum) 4-fermi operators acquire a large anomalous dimension, mixing with the gauge interaction. At a critical coupling value the vacuum undergoes a transition and the fields must be re-quantised with respect to the strong-coupling ground state. The difference in zero-point energies of the two phases is manifested as a scalar potential in the chirally-symmetric phase and thus a fermion mass is dynamically generated [2].

For the more complicated chiral case, which has three couplings, three such phase transitions are suggested [2] to occur, each contributing to the scalar potential.

If the SM couplings evolved to high momentum scales are used ($\sin^2 \theta_W \rightarrow 1$) the right-right coupling is first to become critical. In the neighbourhood of the critical point attractive scalar pairing between right-handed fermions comes to dominate over the gauge interaction. If there is a vacuum decay, it will be associated with a (parity-violating) scalar condensate of right-handed fermions. The (dynamical) left-handed fermions are highly excited with respect to the condensed right sector. The Adler-Bell-Jackiw (ABJ) anomaly

[29], [30] causes a nett production of right-handed states (which, being supercritical condense) at the expense of left ones, leading to the vacuum decay of the left sector. It is in this way that both chiralities of the lightest generation "freeze out" at the right-right critical coupling, leading to the requisite scalar potential at lower scales. At this stage the right-handed components of the heavier generations exist as resonances; right-handed bound states of a heavy fermion and light antifermion. These generations are anticipated to condense at the right-left and left-left critical couplings respectively. The ABJ anomaly, once again ensuring that the condensation of left- and right-fermions occurs on an equal footing. In this picture a single dynamical fermion behaves as three "quasi-particles" of different masses at distinct scales, leading to the appearance of generation structure.

Gauge-boson mass is generated by the same dynamical chiral symmetry breaking via the mechanism outlined in [31], [32]: Massive fermions automatically generate a mass term for axial fields through the vacuum polarisation [32], which to be unitary, requires the existence of non-transverse degrees of freedom [33]. The nature of these (composite) pseudoscalars has been discussed in [31].

The structure of this thesis is as follows. In chapter 2 we review the relevant techniques required to investigate the phase structure of such a theory, specifically the effective action and Schwinger-Dyson formalism. The Schwinger-Dyson equations (SDEs) for the hypercharge interaction are derived before the renormalisation group (RG) ideas of Wilson [34], [35] are introduced and a short discussion on triviality of the vector and chiral NAF theories is presented.

Chapter 3 consists of two sections, the first dealing with the incorporation of a scalar sector in the electroweak interactions illustrates the conventional mechanisms for including mass and CP violation. The second reviews the progress made in understanding $D\chi$ SB of the gauged NJL model, the motivation being to draw upon these techniques for an analysis of our model.

The following two chapters present different, complementary approaches taken towards understanding the problem. In chapter 4 we present a simple model of fermion pairing interactions, the motivation being that the 4-fermi Wilson potential [35] of the hypercharge theory could contain explicit chiral-breaking terms due to the anomaly. Specifically fermion pairings of the form

$$2\tilde{x}\bar{\psi}_L\psi_R\bar{\psi}_R\psi_L + \tilde{r}(\bar{\psi}_L\psi_R)^2 - \tilde{r}^*(\bar{\psi}_R\psi_L)^2,$$

where \tilde{x} , \tilde{r} and \tilde{r}^* are dimensionful couplings. Denoting the renormalisation scale by Λ_χ and writing the dimensionless couplings $x = \tilde{x}/\Lambda_\chi$, $r = \tilde{r}/\Lambda_\chi$,

their running is investigated at 1-loop level by renormalisation group equations. It is found that there are three renormalised null-trajectories through different subspaces of the $\{x, r, r^*\}$ coupling-constant space, depending on whether one or both of the x and r terms are present, corresponding to three scalar four-fermi operators

$$O_1 = (\bar{\psi}_L \psi_R)^2 + (\bar{\psi}_R \psi_L)^2; \quad x = 0, r - r^* = 0, \quad (1.4)$$

$$O_2 = \bar{\psi}_L \psi_R \bar{\psi}_R \psi_L; \quad r = 0 = r^*, x \neq 0, \quad (1.5)$$

$$O_3 = (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)^2; \quad x + r + r^* = 0. \quad (1.6)$$

The potential emergence of three scalar condensates associated with these operators provides further [23] motivation to consider a dynamical mechanism like that of the special 3-Higgs potential of [21]. Here the Higgs would be composites bound by the underlying attractive fermion couplings which are required to prevent the total charge-screening scenario, in direct analogy to the GNJL augmentation of QED. That is, in the language of the renormalisation group, the above 4-fermi interactions are non-renormalisable at low momenta, however close to a UV fixed point they are expected to obtain a large anomalous dimension, becoming strictly renormalisable (marginal) and thus mixing with the gauge interaction.

In contrast to what is found for chirally-symmetric matter-only theories, the auxiliary method requires our composite fields to be, in general complex, chiral objects. With the self-consistent introduction of a “fundamental” fermion with a component living in each of the three coupling subspaces and an extended vacuum

$$|0, 0, 0\rangle = |0\rangle_x \otimes |0\rangle_r \otimes |0\rangle_\ell,$$

a range of CP-violating terms are possible. It is noted that with the imposition of a discrete symmetry ($Z_3 \otimes Z_2$, in contrast to [21], where Z_3 is used; here Z_n is the permutation group of order n) this number is reduced to one and the calculation [21] of the CKM matrix elements to tree-level accuracy could be reproduced.

Chiral symmetry breakdown in the three potentials defined by operators O_1 - O_3 , presumably corresponding to different critical scales is then studied. All dimensionful couplings were found to have the critical value $4\pi^2/\Lambda$ with the respective couplings \tilde{r} , \tilde{x} and the combination

$$\tilde{\ell} \equiv \frac{\tilde{x}}{1 + \frac{\tilde{r}\tilde{r}^*}{\tilde{x}^2}} \frac{1}{1 - \text{Re}(\tilde{r})(1 + \tilde{r}\tilde{r}^*/\tilde{x}^2)} \quad (1.7)$$

for O_3 . Here Λ is the ultraviolet momentum cutoff for the fermion loop. Dimensionless couplings, with the critical value $4\pi^2\Lambda_\chi/\Lambda$ are finite at large scales if $\Lambda_\chi \rightarrow \infty$ and $\Lambda \rightarrow \infty$ in such a way that their ratio is finite. It turns out moreover, that the null trajectory for the operator O_3 in Eq.(1.6), flows to a finite (nonzero) IR limit, making this a suitable candidate for the effective theory at intermediate or higher scales.

In summary the model is seen to have all the desirable features; the dynamical mass generation at different scales, the phase structure and the existence of three scalar condensates, amenable to a Kiselev-type [21] model of generations and CP violation. It is not clear, however, how these parity- and chiral-breaking scalar terms arise in the context of hypercharge: in the Fierz-reordering above they are cancelled separately in the right-right and left-left terms, and also in the combination of left-right with right-left terms (see Appendix A). The only possibility, therefore, seems to be some dynamical effect leading to a mismatch between the latter, cross terms encountered when unquenching the boson propagator. Thus in the second part of the chapter we consider the Dyson-Schwinger equation for the fermion self-energy in the chirally symmetric, P-violating Wilson potential associated with the anomaly-free hypercharge theory.

Chapter 5 discusses the question of criticality in the hypercharge theory. In [2] it was suggested that at the lowest (right) critical scale the lightest generation “froze out” of the theory; fermions with right chirality formed a condensate $\bar{\psi}_L\psi_R$. The left-handed vacuum was highly excited relative to its right counterpart and via gauge transformations the ABJ anomaly caused a flux of (dynamical) left fermions over into the condensed phase. It is for this reason that both chiralities are expected to feel the same scalar potential far away from the critical point, the manifestation of which is the appearance of a mass term.

In 5.1.1 in the (non-anomalous) quenched rainbow approximation we find some indication of the effects of different dynamics of the left and right fermion chiralities in that they acquire different form factors A_\pm , B_\pm . As in the case for quenched rainbow QED, we find above a critical coupling strength that chiral symmetry is dynamically-broken, $B_\pm \neq 0$, but here separate left- and right- gaps are found, corresponding to the distinct condensation of chiral fermions. A physical fermion mass, in the context of Lorentz symmetry and Hermiticity, is unable to be constructed. The effect is found to disappear in the Landau gauge whereupon we require more a more sophisticated approximation for the vertex appearing in the fermion SDE. We repeat the analysis with an improved, naively 1-loop renormalisable vertex and find the effect, although considerably reduced, persists in

arbitrary gauge.

The anticipated role of composite scalars in enabling physical fermion mass generation is then investigated. In order to have a low-energy effective theory with a unitary perturbative limit it is necessary to include the dynamically-generated Goldstone boson loop. The effect upon the fermion self energy SDE is found to be that scalar form factors B_{\pm} are now found to have terms depending upon three couplings, suggesting distinct behaviour at three separate scales, paving the way for self-consistent introduction of three fermion generations as outlined in [2]. For completeness we also attempt to include the anomalous corrections to the Goldstone vertex, however these are found to dominate the equations, causing all scalar form factors to vanish. Possible reasons for failure are discussed, in particular it is pointed out that correct implementation would involve moving beyond the quenched approximation.

In chapter 6 we present our conclusions, and noting superficial similarities between the problem at hand and strongly correlated electrons in 1+1 dimensions, outline an alternative mechanism for an origin of the fermion generations consistent with existing theories (for example [81]).

Chapter 2

Preliminary

In this chapter we introduce the formalism for non-perturbative analyses of a Quantum Field Theory which we shall need in later chapters. The material is largely standard and in particular we follow [37] for the Schwinger-Dyson formalism, [17] for the discussions of effective action technique and gauge anomalies and [38] for renormalisation group.

2.1 Chiral symmetry

Consider the Lagrangian for N species of free fermions in 4 dimensions

$$\mathcal{L}_F = \bar{\psi} \gamma^\mu \partial_\mu \psi, \quad (2.1)$$

where $\psi = (\psi_1, \dots, \psi_N)$. Defining the matrix γ^5 via [37]

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad \{\gamma^5, \gamma^\mu\} = 0 \quad (\gamma^5)^2 = 1, \quad (2.2)$$

the fermion Lagrangian is invariant under the global $U(N)$ chiral transformation

$$\psi \rightarrow \exp(i\omega^a \lambda_a \gamma^5) \psi \quad \bar{\psi} \rightarrow \bar{\psi} \exp(i\omega^a \lambda_a \gamma^5), \quad (2.3)$$

where $\omega = (\omega_1, \dots, \omega_N)$ is an N -dimensional constant vector and $\lambda = (\lambda_1, \dots, \lambda_N)$ are the $U(N)$ group generators. Now \mathcal{L}_F is also invariant under the global $U(N)$ transformation

$$\psi \rightarrow \exp(i\beta^a \lambda_a) \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp(-i\beta^a \lambda_a), \quad (2.4)$$

and has the corresponding conserved Noether currents

$$J_5^{\alpha\mu} = \bar{\psi} \gamma^\mu \gamma^5 \lambda^\alpha \psi, \quad (2.5)$$

$$J^{\alpha\mu} = \bar{\psi} \gamma^\mu \lambda^\alpha \psi. \quad (2.6)$$

Equivalently \mathcal{L}_F can be rewritten in terms of the chiral spinors

$$\mathcal{L}_F = \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R \quad (2.7)$$

$$\psi_a = \chi_a \psi \quad (2.8)$$

where χ_a are the projection operators

$$\chi_{L,R} = (1 \mp \gamma^5)/2. \quad (2.9)$$

That is, left-handed and right-handed components commute and the theory has chiral symmetry $U(1)_L \otimes U(1)_R$, with corresponding Noether currents

$$J_{L,R}^{\alpha\mu} = J^{\alpha\mu} \mp J_5^{\alpha\mu}. \quad (2.10)$$

However a mass term $m\bar{\psi}\psi$ destroys the invariance under the transformations Eq.(2.3), mixing left and right currents:

$$m\bar{\psi}\psi \rightarrow m\bar{\psi} \exp(2i\omega^a \lambda_a \gamma^5) \psi \quad (2.11)$$

and the chiral symmetry group is broken to its diagonal subgroup.

$$U(N)_L \otimes U(N)_R \supset U(N)_{L+R} \equiv U(N)_V. \quad (2.12)$$

Here only the vector current is conserved

$$\partial_\mu J^{\alpha\mu} = 0, \quad (2.13)$$

$$\partial_\mu J_5^{\alpha\mu} = 2im\bar{\psi}\gamma^5\lambda^\alpha\psi, \quad (2.14)$$

while due to Goldstone's Theorem (see chapter 3 of [17]), the global symmetry breaking Eq.(2.12) is accompanied by the production of $N^2 - 1$ massless scalar particles - one for each broken group generator - or Goldstone bosons. For the more useful instances of interacting theories, models of scalar interactions (such as the NJL model in section 3.2) respecting global chiral symmetry may be constructed. Moreover vector and axial gauge-interactions

$$\bar{\psi}\gamma^\mu(V_\mu + A_\mu\gamma^5)\psi,$$

necessarily preserve global chiral symmetry as they anticommute with γ^5 . The case of local chiral symmetry is more subtle, due to a non-vanishing surface term in the computation of the associated Noether current. Chiral gauge anomalies will be dealt with below in 2.2.

2.2 Chiral anomalies

Gauge anomalies arise in a quantum field theory from the violation of a classical symmetry by the second-quantisation procedure. The chiral gauge anomaly, which is violation of local chiral symmetry in gauge theories, was originally discovered [39] as a lack of gauge-invariance in the computation of the $\pi_0 \rightarrow 2\gamma$ decay “triangle” diagram Δ^{abc} with one axial and two vector vertices, as shown in Figure 2.1. Since the definitive analyses of Adler [29],

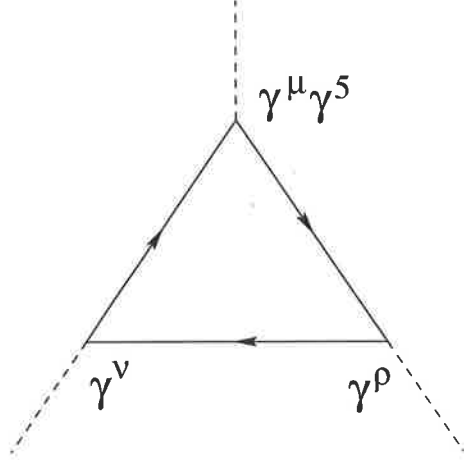


Figure 2.1: Anomalous triangle diagram Δ^{abc} .

Bell and Jackiw [30] (ABJ) in 1969 illustrating the lack of a chiral-invariant regularisation procedure, the anomaly has been interpreted and derived in a number of ways. Perhaps the most intuitive picture is provided by the Dirac sea picture ([40],[41],[42]). Here second-quantisation is achieved by filling the (infinite) negative-energy states. On introducing a regularisation, that is giving the sea a finite “depth”, anomalies appear as an exchange of degrees of freedom (such as charge, chirality or spin) across the cut-off momentum scale. In this thesis the path-integral approach of Fujikawa [43] will be the most convenient method of computation. As outlined below, here the anomaly arises from the failure of the path-integral measure to be invariant under local chiral symmetry transformations.

Consider the $U(1)_V \otimes U(1)_A$ gauge theory of massless fermions with Lagrangian

$$\mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi + F_{\mu\nu}^V F^{V\mu\nu} + F_{\mu\nu}^A F^{A\mu\nu}, \quad (2.15)$$

$$D_\mu = \partial_\mu - (c_L + c_R) V_\mu - (c_R - c_L) i \gamma^5 A_\mu, \quad (2.16)$$

$$F_{\mu\nu}^X = \partial_\mu X_\nu - \partial_\nu X_\mu; \quad X = V, A. \quad (2.17)$$

It has the classical conserved currents Eqs.(2.1, 2.5) (with $\lambda^0 = 1$)

$$\frac{i}{c_L + c_R} \partial_\mu \langle J^\mu \rangle = 0, \quad (2.18)$$

$$\frac{i}{c_R - c_L} \partial_\mu \langle J_5^\mu \rangle = 0; \quad c_R \neq c_L. \quad (2.19)$$

The generating functional of the theory is

$$\mathcal{Z} = \int DV DAD\psi D\bar{\psi} \exp(i \int d^4x \mathcal{L}). \quad (2.20)$$

To rigorously define the path-integral measure, the fermion fields are expanded in terms of complete eigenfunctions ϕ_n of the Dirac operator $\gamma^\mu D_\mu$, which are defined by

$$\psi = \sum_n a_n \phi_n, \quad \bar{\psi} = \sum_n \phi_n^\dagger b_n, \quad (2.21)$$

$$\int d^4x \phi_n^\dagger \phi_m = \delta_{nm}, \quad (2.22)$$

where a_n, b_n are Grassmann numbers. The fermion integral measure is invariant under the transformation Eq.(2.21) and the generating functional Z becomes

$$Z = \int DZ \prod_n Db \prod_n Da \exp(i \int d^4x \mathcal{L}), \quad (2.23)$$

where here the Grassmann measures are defined as the left-derivative.

Under the local chiral transformations

$$\psi \rightarrow \exp i\alpha(x) \gamma^5 \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp i\alpha(x) \gamma^5 \quad (2.24)$$

the Lagrangian and path integral measure transform as

$$\mathcal{L} \rightarrow \mathcal{L} - \bar{\psi} \gamma^\mu \partial_\mu \alpha(x) \gamma^5 \psi, \quad (2.25)$$

$$\prod_n Da_n \rightarrow (\det A_{nm})^{-1} \prod_n Da_n, \quad (2.26)$$

$$A_{mn} = \sum_m \int d^4x \phi_m^\dagger \exp(i\alpha(x) \gamma^5) \phi_n, \quad (2.27)$$

$$\prod_n Db_n \rightarrow \prod_n Db_n (\det A_{nm})^{-1}. \quad (2.28)$$

That is, neither the action nor the path-integral measure are invariant under chiral transformations. If the path integral is to remain invariant Eq.(2.25) must compensate for Eqs. (2.26,2.28). For small $\alpha(x)$ the determinant is computed as

$$\begin{aligned} (\det A_{mn})^{-1} &= \exp(i \sum_n \int d^4x \alpha(x) \phi_n^\dagger \gamma^5 \phi_n) + O(\alpha^2) \\ &\equiv \exp(i \int d^4x \alpha(x) \mathcal{A}(x)). \end{aligned} \quad (2.29)$$

The compensation condition is then

$$2\text{Tr}(\mathcal{A}) = \langle -i \int d^4x \partial_\mu \alpha(x) \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle, \quad (2.30)$$

where the trace Tr is understood to contain path integrals. Performing the x integration by parts on the right-hand side and neglecting an inessential surface term, we see the classical axial current Eq.(2.5). is no longer conserved (cf Eq.(2.19))

$$\frac{1}{c_R - c_L} \partial_\mu \langle J_5^\mu \rangle = -2i\mathcal{A}. \quad (2.31)$$

After regularisation of $\text{Tr}(\alpha(x)\gamma^5)$ (see for example [44]) a lengthy expansion in plane waves yields the expression [45]

$$\begin{aligned} \mathcal{A} &= \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \left(F_{\mu\nu}^V F_{\rho\sigma}^V + \frac{1}{3} F_{\mu\nu}^A F_{\rho\sigma}^A + \frac{32}{3} A_\mu A_\nu A_\rho A_\sigma \right. \\ &\quad \left. - \frac{8}{3} (F_{\mu\nu}^V A_\rho A_\sigma + A_\nu F_{\nu\rho}^V A_\sigma + A_\mu A_\nu F_{\rho\sigma}^V) \right). \end{aligned} \quad (2.32)$$

While, conventionally, gauge theories (such as the Standard Model) require anomaly cancellation in order to define renormalisable perturbations, recent study of “anomalous” gauge theories in 2 dimensions [46], [47], [48], [49] should be briefly mentioned. It has been shown for the chiral Schwinger model

$$\mathcal{L} = -F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu (i\partial_\mu + eA_\mu \chi_+) \psi, \quad (2.33)$$

either the introduction of Wess-Zumino [50], [51] fields, [48], [49] or non-gauge-invariant quantisation [46], [48], enables construction of a unitary, perturbatively renormalisable theory. Little work, aside from [47], has been done in higher dimensions.

2.3 Effective action

The effective action is a standard method by which to study the behaviour of vacuum degrees of freedom in quantum field theories. Specifically, from the point of view of dynamical chiral symmetry breaking it enables calculation of permissible non-vanishing vacuum-expectation values of scalar condensates. Historically concepts such as the classical “order parameter” were borrowed from superfluidity and since Coleman and Weinberg [52] have been successfully applied to relativistic theories.

Given a scalar theory with fields ϕ_i and Lagrangian density $\mathcal{L}(\phi_1 \dots \phi_n)$ (we shall discuss the generalisation to fermions presently), the generating functional Z is defined as

$$Z[J, I] = \langle 0 | \int \prod_{i=1}^n D\phi_i T \exp \{ i \int d^4x (\mathcal{L}(\phi_1(x), \dots \phi_n(x)) + \Phi(x) \cdot J(x)) + \int d^4y \Phi(x)_i \Phi(y)_i I^{ii}(x, y) \} | 0 \rangle. \quad (2.34)$$

where $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ and $J(x) = (J_1(x), \dots, J_n(x))$, with $J_i(x)$ a classical external source term coupling to field $\phi_i(x)$. $I_{ii}(x, y)$ is a bilocal source. It is clear that these sources require the same statistics, thus the sources coupling to fermion fields are Grassmann numbers. Eq.(2.34) has an series expansion in terms of Greens functions

$$Z[J, I] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G^{(n)}(x_1, \dots, x_n) J_1(x_1) \dots J_n(x_n), \quad (2.35)$$

where the n-point Greens function $G^{(n)}$ in the presence of source J is given by

$$G^{(n)}(x_1, \dots, x_n; J) = (-i)^n \frac{\delta^n Z[J, I]}{\delta J(x_1) \dots \delta J(x_n)} = \langle 0 | T \phi_1(x_1) \dots \phi_n(x_n) | 0 \rangle_J, \quad (2.36)$$

and care needs to be taken with minus signs arising from the anticommutativity of Grassmann sources. The notation δ denotes a functional derivative. For a discussion of Grassmann differentiation and integration, see for example [44].

The connected generating functional, so named because it is expanded in terms of Greens functions $G_c^{(n)}$ which have connected Feynman diagrams in the corresponding perturbation theory, is defined by

$$W[J, I] = -i \text{Ln} Z[J, I], \quad (2.37)$$

or in terms of connected Greens functions

$$W[J, I] = \sum_{n=1}^{\infty} \int d^4x_1 \dots d^4x_n d^4y_1 \dots d^4y_n G_c^{(n)}(x_1, \dots, x_n; J) I_1(x_1, y_1) \dots I_n(x_n, y_n)$$

where $G_c^{(n)}$ is readily determined by comparison of Eqs.(2.34,2.37)

With the restriction to one-particle-irreducible amplitudes, the vacuum expectation value of operator $\phi(x)$ in the presence of current J is

$$\begin{aligned} \phi(x; J) &\equiv \frac{\langle 0 | \phi(x) | 0 \rangle_J}{\langle 0 | 0 \rangle_{J, I}}, \\ &= \frac{i}{Z[J, I]} \frac{\delta}{\delta J(x)} Z[J, I], \\ &= \frac{\delta}{J(x)} W[J, I]. \end{aligned} \quad (2.38)$$

If $\phi(x; J)$ has the value $\phi_c(x)$ for the particular currents $J(x) = J_c(x)$, $I(x, y) = I_c(x, y)$ and

$$\frac{\delta W[J, I]}{\delta I(x, y)} = \phi_c(x) \phi_c(y) - iG(x, y),$$

where G is the 2-point Greens function in Eq.(2.36), then the effective action is defined as the double Legendre transform

$$\begin{aligned} \Gamma[\phi, G] &= W[J_c, I_c] - \int d^4x \phi_c(x) J_c(x) \\ &\quad + \int d^4x d^4y (iG(x, y) I_c(x, y) - \phi_c(x) \phi_c(y) I_c(x, y)) \end{aligned} \quad (2.39)$$

The physical significance of this is that, analogous to the action of a classical theory, in the absence of fictitious external currents the permissible values of $\phi_c(x)$ are given by the stationary points of Γ . Taking a functional derivative of Eq. (2.39) with respect to ϕ_c

$$\frac{\delta \Gamma[\phi, G]}{\delta \phi_c(x)} = -J_c(x) - 2 \int d^4y \phi_c(y) I(x, y) \quad (2.40)$$

the right-hand side vanishes readily when $J_c(x) = 0 = I(x, y)$. Note that the other stationarity equation

$$\frac{\delta \Gamma}{\delta G(x, y)} = 0 \quad (2.41)$$

corresponds to the Schwinger-Dyson equation for propagator G , discussed in section 2.4 below.

2.4 Schwinger-Dyson equations

The Schwinger-Dyson equations (SDEs) of a quantum field theory determine relations between its Greens functions. They are derived from the fact that the total functional derivative of a path integral, given appropriate boundary conditions, vanishes. For example, consider the generating functional for the $U(1)_V \otimes U(1)_A$ theory with Lagrangian Eq. (2.15)

$$Z[J_V, J_A, \eta, \bar{\eta}] = \mathcal{N} \int DV DAD\psi D\bar{\psi} \exp\{iS[V, A, \psi, \bar{\psi}] + \int d^4x (J_V^\mu V_\mu + J_A^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta)\} \quad (2.42)$$

$$S[V, A, \psi, \bar{\psi}] = \int d^4x \{ \bar{\psi}(\partial_\mu - g_V V_\mu - ig_A \gamma^5 A_\mu) \gamma^\mu \psi + F_{\mu\nu}^V F^{V\mu\nu} + F_{\mu\nu}^A F^{A\mu\nu} - (\frac{1}{2v} \partial^\mu V_\mu)^2 - (\frac{1}{2a} \partial^\mu A_\mu)^2 \} \quad (2.43)$$

where J_V , J_A and the Grassmann-valued functions η , $\bar{\eta}$ are external sources coupling to the gauge bosons and fermions respectively. The conventional covariant gauge-fixing prescription is used, with v and a the gauge parameters for vector and axial fields respectively. In the renormalisation group (RG) picture outlined below, S is a regulated, low energy effective action and Z has a cutoff dependence via the running couplings, however this shall be taken as implicit. While the ABJ anomaly is present the theory is unregulated, and several of the results below are defined at a formal level, further refinement is needed for well-defined quantities.

The SDEs for the two-point Greens functions follow from the four formal equations

$$\left(\frac{\delta}{\delta\phi} S[V, A, \psi, \bar{\psi}] + J_\phi \right) Z[J_V, J_A, \eta, \bar{\eta}] = 0 \quad (2.44)$$

where ϕ with corresponding source J_ϕ ranges over the fields $\{V, A, \psi, \bar{\psi}\}$. Following the previous section, the connected generating functional is defined, formally at least, by

$$W[J_V, J_A, \eta, \bar{\eta}] = -i \text{Ln} Z[J_V, J_A, \eta, \bar{\eta}] \quad (2.45)$$

It is customary to decompose W into parity-even and -odd parts

$$W[J_V, J_A, \eta, \bar{\eta}] = W_+[J_V, J_A, \eta, \bar{\eta}] + W_-[J_V, J_A, \eta, \bar{\eta}], \quad (2.46)$$

where the even part W_+ is understood to be regularisable and its computation follows the standard procedure used for vector theories. Calculation

of the W_- term, which may also be rendered finite [53] contains the ABJ anomaly and a $2\pi i$ multivaluation, will be considered in a later chapter. To illustrate the derivation of SDEs simply we shall consider only the parity-even part below.

The effective action corresponding to W_+ is obtained from the Legendre-transform (c.f. Eq.(2.39))

$$i\Gamma[V, A, \psi, \bar{\psi}] = W[J_V, J_A, \eta, \bar{\eta}] - \int d^4x (J_V^\mu V_\mu + J_A^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta) \quad (2.47)$$

where, for $Z = V, A$

$$\begin{aligned} Z_\mu(x) &= \frac{\delta W}{i\delta J_Z^\mu(x)}, \quad \psi(x) = \frac{\delta W}{i\delta \bar{\eta}(x)}, \quad \bar{\psi}(x) = -\frac{\delta W}{i\delta \eta(x)}, \\ J_Z^\mu(x) &= -\frac{\delta \Gamma}{\delta Z_\mu(x)}, \quad \bar{\eta}(x) = \frac{\delta \Gamma}{\delta \psi(x)}, \quad \eta(x) = -\frac{\delta \Gamma}{\delta \bar{\psi}(x)}. \end{aligned}$$

With vanishing external currents Eq.(2.44) with $\phi = V$ leads to

$$\frac{\delta \Gamma(V, A, \psi, \bar{\psi})}{\delta V_\mu(x)} = (\partial^2 g_{\mu\nu} - (1-v)\partial_\mu\partial_\nu)V^\nu(x) - ig_V \text{tr} \left(\gamma^\mu \frac{\delta^2 \Gamma}{\delta \psi \delta \bar{\psi}}(x, x) \right)^{-1} \quad (2.48)$$

where the identity

$$\int d^4x \frac{\delta^2 G}{\delta \eta(y) \delta \bar{\eta}(x)} \frac{\delta^2 \Gamma}{\delta \psi(x) \delta \bar{\psi}(z)} = i\delta^4(y-z); \quad \begin{aligned} \eta &= 0 = \bar{\eta} \\ \psi &= 0 = \bar{\psi} \end{aligned} \quad (2.49)$$

has been used. Taking the functional derivative, again with respect to V , and setting all fields to zero, the SDE for the inverse two-point vector Greens function is obtained:

$$\begin{aligned} (D_V^{-1})^{\mu\nu}(x, y) &\equiv \left(\frac{\delta^2 \Gamma}{\delta V^\mu(x) \delta V^\nu(y)} \right)_{V=A=0=\psi=\bar{\psi}} \\ &= (D_{V0}^{-1})^{\mu\nu}(x, y) + \Pi_V^{\mu\nu}(x, y), \end{aligned} \quad (2.50)$$

where $D_{V0}^{\mu\nu}(x, y)$ denotes the bare vector boson propagator and $\Pi_V^{\mu\nu}(x, y)$ is the vacuum polarisation:

$$(D_{V0}^{-1})^{\mu\nu}(x, y) = (\partial^2 g^{\mu\nu} - (1-v)\partial^\mu\partial^\nu)\delta^4(x-y), \quad (2.51)$$

$$\Pi_V^{\mu\nu}(x, y) = ig_V \text{Tr}(\gamma^\mu S \Gamma^{V\nu} S). \quad (2.52)$$

Here the trace includes integration over fermion loops, while S and $\Gamma^{V\mu}$ are the inverse 2-point fermion and irreducible vector-fermion vertex Greens

functions respectively. The latter is distinguished from the effective action Γ by the presence of a Lorentz index. These functions are defined by

$$S(x, y) = \left(\frac{\delta^2 \Gamma}{\delta \bar{\eta}(x) \delta \eta(y)} \right)_{V=A=0=\psi=\bar{\psi}}^{-1} \quad (2.53)$$

$$\Gamma^{V\mu}(x, y, z) = \left(\frac{\delta^3 \Gamma}{\delta V_\mu(x) \delta \bar{\eta}(y) \delta \eta(z)} \right)_{V=A=0=\psi=\bar{\psi}} \quad (2.54)$$

and may be computed from Eq.(2.47) accordingly. The propagator for the axial boson, given by

$$D_{\mu\nu}^A(x, y) \equiv \left(\frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\nu(y)} \right)_{V=A=0=\psi=\bar{\psi}}^{-1} \quad (2.55)$$

is computed analogous to the vector propagator above. With the irreducible axial-fermion vertex defined by

$$\Gamma_\mu^A(x, y, z) = \left(\frac{\delta^3 \Gamma}{\delta A^\mu(x) \delta \bar{\eta}(y) \delta \eta(z)} \right)_{V=A=0=\psi=\bar{\psi}} \quad (2.56)$$

the full set of (unrenormalised) 2-point SDEs for the theory Eq.(2.15) is now given, in momentum space by

$$D_V^{\mu\nu}(k) = (k^2 g^{\mu\nu} + \frac{v-1}{v} k^\mu k^\nu) - \Pi_V^{\mu\nu}(k) \quad (2.57)$$

$$\Pi_V^{\mu\nu}(k) = \int \frac{d^4 q}{(2\pi)^4} \text{tr}((-ig_V \gamma^\mu)(iS(q))(-i\Gamma_V^\nu(q, k-q))iS(k-q)) \quad (2.58)$$

$$D_A^{\mu\nu}(k) = (k^2 g^{\mu\nu} + \frac{a-1}{a} k^\mu k^\nu) - \Pi_A^{\mu\nu}(k) \quad (2.59)$$

$$\Pi_A^{\mu\nu}(k) = \int \frac{d^4 q}{(2\pi)^4} \text{tr}((-ig_A \gamma^5 \gamma^\mu)(iS(q))(-i\Gamma_A^\nu(q, k-q))iS(k-q))$$

$$S^{-1}(p) = \gamma^\mu p_\mu - \Sigma(p) \quad (2.60)$$

$$\begin{aligned} \Sigma(p) = & \int \frac{d^4 q}{(2\pi)^4} ((-ig_V \gamma^\mu)(iS(q))(-i\Gamma_V^\nu(q, q-p)))D_{\mu\nu}^V(q-p) \\ & + (-ig_A \gamma^5 \gamma^\mu)(iS(q))(-i\Gamma_A^\nu(q, q-p))D_{\mu\nu}^A(q-p) \end{aligned} \quad (2.61)$$

Note, that due to the presence of the 3-point functions $\Gamma_{V,A}^\mu$ the system does not close and constitutes a countably infinite tower of coupled equations. The standard computational procedure, which truncates the set (typically at the 3-point level) with a set of approximations, will be followed in chapters below.

2.5 Renormalisability and triviality

While QED is often cited as the most successful theory in terms of experimental data, this is entirely within a perturbative framework. The “zero charge problem” encountered at very short distances from a point charge was first reported in 1955 [54], and has remained mathematically unresolved since. The effect is common to all NAF gauge theories, where the bare coupling is allowed to run unchecked at short distances leading to a total screening of the bare charge by the vacuum. Such a screening renders the theory “trivial” or non-interacting. In this section we briefly review the salient features of Wilson’s [34], [35] renormalisation group (RG) as presented in [38].

Consider an action on D -dimensional spacetime

$$S = \int d^4x \sum_i g_i O_i, \quad (2.62)$$

where the g_i are coupling constants and the operators O_i are local monomials of elementary fields, ϕ , with canonical dimension d_i . The standard definition of a perturbatively renormalisable theory is one for which S is composed of operators with $d_i \leq D$.

Upon the introduction of a cutoff, the resulting action S_Λ may be considered as that of a low-energy (with respect to Λ) approximation to a “fundamental” theory S_0 . This requires the (non-trivial) assumption that the fields appearing in the path integral can be divided into high- and low energy parts

$$\phi = \phi_- + \phi_+, \quad (2.63)$$

so that

$$\int D\phi_- D\phi_+ e^{iS_0[\phi_-, \phi_+]} = \int D\phi_- e^{iS_\Lambda[\phi_-]}. \quad (2.64)$$

that is, S_Λ is obtained by “integrating out” the heavy degrees of freedom:

$$e^{iS_\Lambda[\phi_-]} = \int D\phi_+ e^{iS_0[\phi_-, \phi_+]}. \quad (2.65)$$

Above the cutoff S_Λ is nonlocal, but in the low energy region it may be approximated by a local expansion of the form Eq.(2.62). The operators O^i are classified in terms of their dimension; those with $d < D$ correspond to relevant or superrenormalisable interactions, while those with $d = D$ are called marginal. A perturbatively renormalisable theory, composed of these types of interaction monomial contains only a finite number of divergent

diagrams and can be formulated independently of a given cutoff value Λ . Non-renormalisable interactions, those for which $d_i > 4$ lead to new divergences at each order of perturbation are accommodated by the assumption [34] that their (dimensionless) couplings λ_i are of the order of unity. Such a term,

$$\frac{\lambda_i}{\Lambda^{d_i-D}} O_i, \quad (2.66)$$

is then suppressed by positive powers of Λ . Its divergent diagrams are now removed by relevant or marginal counterterms, or equivalently absorbed into “running” bare quantities.

For large (i.e. non-perturbative) couplings this running may change the dimension d_i significantly, so that e.g. perturbatively irrelevant interactions become marginal (such as in the GNJL below)

The transformation of shifting renormalisation scale, readily demonstrated to have semigroup properties, hence dubbed the “renormalisation group” is the most suitable method for analysing such running behaviour. The differential equations describing the response of the Greens functions and couplings of a theory to a small shift in renormalisation scale are known as renormalisation group equations (RGEs). Consider a perturbatively renormalisable theory, such as four dimensional QED with, for simplicity, a single fermion flavour:

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu(\partial_\mu + eA_\mu) - m_0)\psi - F_{\mu\nu}F^{\mu\nu} - \frac{1}{2z}(\partial^\mu A_\mu)^2 \quad (2.67)$$

where z is the covariant gauge-fixing parameter and m, e denote the bare mass and coupling respectively. The renormalisability property means that the divergences may be removed order-by order upon the field redefinitions

$$\psi = \sqrt{Z_2}\psi \quad A_\mu = \sqrt{Z_3}A_\mu$$

and vertex renormalisation

$$\gamma^\mu = Z_1\gamma^\mu$$

Consider now the 2-point fermion Greens function renormalised at an arbitrary scale μ .

$$S(p, g_R, m_R; \mu) = Z_2(\mu)S(p, g_0, m_0) \quad (2.68)$$

The statement that the unrenormalised function S is independent of the renormalisation scale for fixed g_0, m_0

$$\frac{d}{d\mu} S(p, g_0, m_0)|_{g_0, m_0} = 0 \quad (2.69)$$

can thus be rearranged, on inverting Eq.(2.68) as

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g_R} - \gamma_m m_R \frac{\partial}{\partial m_R} - 2\gamma \right) S|_{g_0, m_0} = 0 \quad (2.70)$$

where

$$\beta = \mu \frac{\partial g_R}{\partial \mu} |_{g_0, m_0} \quad (2.71)$$

$$\gamma_m = -\frac{\mu}{m_R} \frac{\partial m_R}{\partial \mu} |_{g_0, m_0} \quad (2.72)$$

$$\gamma = \frac{\mu}{2Z_2} \frac{\partial Z_2}{\partial \mu} |_{g_0, m_0} \quad (2.73)$$

The actual parameter-dependence of these functions varies with the RG method used. For example in the Gell-Mann Low scheme m_R is fixed as the physical fermion mass, $\gamma_m = 0$, while g_R , Z_2 are defined in terms of the off-shell subtraction $p^2 = -\mu^2$. In this case $\beta = \beta(g_R, m_R/\mu)$, $\gamma = \gamma(g_R, m_R/\mu)$ and the two point function can be written in terms of dimensionless variables as

$$S(p, g_R, m_R) = \mu^{4-2 \times 3/2} \tilde{S}\left(\frac{p}{\mu}, g_R, \frac{m_R}{\mu}\right) \quad (2.74)$$

For a shift in scale $p \rightarrow \lambda p$ the RG equation Eq.(2.74) can be re-expressed as

$$\left(-\lambda \frac{\partial}{\partial \lambda} + \beta(g_R, \frac{m_R}{\mu}) \frac{\partial}{\partial g} + 4 - 2\left(\frac{3}{2} + \gamma\right) \right) \tilde{S}\left(\frac{\lambda p}{\mu}, g_R, \frac{m_R}{\mu}\right) \quad (2.75)$$

In the massless case, setting $t = \ln \lambda$ this may be solved (the result coincides for a number of RG schemes) to give [38]

$$\tilde{S}\left(\frac{\lambda p}{\mu}, g_R, \frac{m_R}{\mu}\right) = \tilde{S}(p, \bar{g}(t), \mu) \exp\left\{t - 2 \int_0^t dt' \gamma(\bar{g}(t'))\right\} \quad (2.76)$$

$$t = \int_g^{\bar{g}(t)} \frac{dg'}{\beta(g')} \quad (2.77)$$

The second expression, Eq.(2.77), gives the behaviour of the running coupling in the following way. Any root of β at the corresponds to a fixed point of Eq.(2.77). If $\beta(g_c) = 0$ for some coupling g_c then the behaviour at asymptotic scales is given by

$$\frac{d\beta}{d\bar{g}}|_{g=g_c} > 0, \quad \bar{g}(t) \rightarrow g_c; \quad t \rightarrow \infty, \quad (2.78)$$

$$\frac{d\beta}{d\bar{g}}|_{g=g_c} < 0, \quad \bar{g}(t) \rightarrow g_c; \quad t \rightarrow -\infty. \quad (2.79)$$

The point g_c is then known as a UV or IR (Gaussian) fixed point respectively. At such a point, scale invariance is recovered and the solution of Eq.(2.76) becomes

$$\tilde{S}(p, g_c) \lambda^{4-2(3/2+\gamma)}, \quad (2.80)$$

that is, the fermion fields have dynamically acquired an “anomalous” dimension γ , given by Eq.(2.73), in the neighborhood of the fixed point. It is apparent that at this scale operators in Eq.(2.62) which are monomials in these fields have a different dimension than expected from naive power-counting. It is in this way that (for example) irrelevant 4-fermi operators become marginal and could mix with the QED interaction close to such a point (see section 3.4 below).

If, however, there is no nontrivial root the coupling grows without limit and the theory is described as being non-asymptotically free (NAF). Finite answers are only obtained for the noninteracting limit, $g(t) = 0$, in which case the theory is said to be trivial. The running behaviour of the coupling β is readily computed via loop corrections

$$\beta(g_R) \simeq \sum \beta_i g_R^{2i+1} \quad (2.81)$$

and it is precisely the perturbative approximation which leads to triviality of NAF theories. For example, substituting the one-loop QED result $\beta = \beta_1 \bar{g}^3$ into Eq.(2.77) gives

$$\bar{g}^2(t) = g_R^2 \frac{1}{1 - 2\beta_1 g_R^2 t}, \quad (2.82)$$

which has a singularity at the scale

$$\Lambda_L = e^{1/2\beta_0 g^2}. \quad (2.83)$$

The singularity persists for higher loop calculations, moreover being driven to lower scales. For example, in two loop perturbation theory the “Landau pole” occurs [26] at

$$\Lambda_L = m_R \exp\left\{ \frac{e_R^2}{\beta_1} \left(\frac{\beta_2 e_R^2}{\beta_1 + \beta_2 e_R^2} \right)^{\beta_2/\beta_1^2} \right\}, \quad (2.84)$$

which, with the particle spectrum of the standard model gives $\Lambda_L \simeq 10^{34} \text{GeV}$. For “realistic” SUSY theories [26] the singularity may be pushed as low as 10^{17}GeV , adding to the necessity for resolution of the Landau ghost problem in NAF gauge theories. The problem is of a perturbative nature, therefore, in the absence of higher physics, the solution must be non-perturbative.

Göckeler *et al.* in a lattice study [26] suggested that the singularity is avoided due to spontaneous breakdown of chiral symmetry. Their results however were consistent with triviality. Moreover [27] it was subsequently suggested that the chiral $U(1)$ suffers a similar fate.

As will be discussed in sec 3.4 there is growing evidence that the best hope for nontrivial, singularity-free NAF theories such as the vector and chiral $U(1)$ gauge theories is the existence of a UV fixed point, whereby marginal 4-fermi operators mix with the gauge interaction. In this picture triviality is argued [28] to be averted due to the strong fermi self-interactions, suppressing the charge-screening effect.

2.5.1 Wegner-Houghton RGE

We conclude the discussion of RG methods with a brief summary of the Wegner-Houghton [55] scheme, used to determine the variation of the effective action S_Λ appearing in Eq.(2.65). Let us define the dimensionless scale parameter as

$$t = \ln \Lambda_0 / \Lambda \quad (2.85)$$

where now Λ_0 denotes the cutoff of Eq.(2.65) and explicitly write the field-dependence $S_\Lambda = S_\Lambda[\phi_i, \dots, \phi_n]$. The scaling response of the effective action to the infinitesimal shift in cutoff $\Lambda_0 \rightarrow \Lambda_0 - \delta\Lambda$ is given by

$$\begin{aligned} \frac{\partial S_\Lambda}{\partial t} = & 4S_\Lambda - \int \frac{d^4 p}{(2\pi)^4} \phi_i(p) \left(\frac{2d_i - 4 - \gamma_i}{2} - p^\mu \frac{\partial}{\partial p^\mu} \right) \frac{\delta S_\Lambda}{\delta \phi_i(p)} \\ & - \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \delta(|p| - 1) \left\{ \frac{\delta S_\Lambda}{\delta \phi_i(p)} \left(\frac{\delta^2 S_\Lambda}{\delta \phi_i(p) \delta \phi_j(-p)} \right)^{-1} \frac{\delta S_\Lambda}{\delta \phi_j(-p)} \right. \\ & \left. - \text{tr} \ln \left(\left(\frac{\delta^2 S_\Lambda}{\delta \phi_i(p) \delta \phi_j(-p)} \right) \right) \right\}. \end{aligned} \quad (2.86)$$

Here d_i and γ_i denote the canonical and anomalous dimensions of field ϕ_i respectively, while the trace appearing in the second line is graded (for fermion fields the sign changes). The most common simplification, the local potential approximation (LPA) corresponds to keeping only the evolution of the potential part. That is, radiative corrections to operators containing derivatives are ignored, as are field renormalisations and anomalous dimensions. If the potential term of the action is denoted V_Λ , then in the LPA approximation the Wegner-Houghton RGE Eq.(2.86) reduces to

$$\frac{\partial V_\Lambda}{\partial t} = 4V_\Lambda - \frac{4 - d_i}{2} \phi_i \frac{\partial V_\Lambda}{\partial \phi_i} + \frac{1}{4\pi^2} \ln \left(1 + \frac{\partial^2 V_\Lambda}{\partial \phi_i \partial \phi_j} \right). \quad (2.87)$$

This result shall be used to compute the β functions of the 4-fermi couplings in section 4.4.1, for direct comparison with the GNJL result [56].

Chapter 3

Models

Before introducing the model in the next chapter it is necessary to describe the phenomena we seek to explain. The other purpose of this chapter is to provide motivation for the model by discussing analogues from other physical systems. Section 3.1 introduces the electroweak sector of the Standard Model and briefly illustrates how CP violation, mass and fermion generations are accommodated with a lack of predictive power. The material presented is readily found in textbooks and lecture notes, here we follow [1] and [57]. The second half of the chapter, 3.4 is a review of the progress made in understanding dynamical chiral symmetry breakdown, specifically within QED and the GNJL.

3.1 Electroweak model

The Glashow-Weinberg-Salam (GSW) electroweak model [58], [59], [60] is a chiral gauge theory which, at low energies, has its full symmetry group spontaneously broken to an electromagnetic subgroup:

$$SU(2)_L \otimes U(1)_Y \supset U(1)_{QED}. \quad (3.1)$$

The gauge sector of the electroweak Lagrangian contains 4 boson fields, three $SU(2)$ bosons $\vec{W}^\mu = (W_1^\mu, W_2^\mu, W_3^\mu)$ plus the Abelian field B^μ . The kinetic terms (without gauge-fixing) are

$$\mathcal{L}_G = -\frac{1}{4}(\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} + B_{\mu\nu}B^{\mu\nu}), \quad (3.2)$$

$$\vec{W}^{\mu\nu} = \partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu + g_2 \vec{W}^\mu \times \vec{W}^\nu, \quad (3.3)$$

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu. \quad (3.4)$$

The weak boson fields couple only to left-handed fermions, which transform as $SU(2)_L$ isodoublets, one doublet each for quarks and leptons, while right-handed fermions transform as isoscalars. In terms of generic fermion isodoublets

$$\Psi_i = \begin{pmatrix} \psi_{\uparrow i} \\ \psi_{\downarrow i} \end{pmatrix}, \quad (3.5)$$

with $U(1)$ hypercharge y_i , the fermion-gauge interaction term is:

$$\mathcal{L}_I = \sum_{j=q,\ell} \bar{\Psi}_j i\gamma^\mu D_{j\mu} \Psi_j, \quad (3.6)$$

$$D_{j\mu} = \partial_\mu - ig_2 \frac{\vec{\tau}}{2} \cdot W_\mu \chi_L - ig_1 y_j B_\mu. \quad (3.7)$$

Here τ are the $SU(2)$ isospin matrices, g_2 and g_1 are respectively the $SU(2)$ and $U(1)$ couplings. Here it is understood that isospin indices are suppressed and that right-handed neutrinos ν_R have $y = 0$. Note the Lagrangian \mathcal{L}_I contains a sum over isodoublets and can in principle accommodate any number of fermions.

3.1.1 Generations and anomaly cancellation

The importance of the generation structure becomes apparent when considering anomalous processes [1], which arise from contributions of the triangle loop diagrams Δ^{abc} of Fig.2.1. The total sum of anomalous contributions Δ^{abc} , \mathcal{A}^{abc} is proportional to [44]

$$\mathcal{A}^{abc} \sim \text{tr}(\{\lambda^a, \lambda^b\} \lambda^c)_L - \text{tr}(\{\lambda^a, \lambda^b\} \lambda^c)_R, \quad (3.8)$$

where the group generators are $\lambda = \tau/2$, Y . The possible combinations of λ 's are

$$\begin{aligned} \text{tr}(\{\tau^a, \tau^b\} \tau^c) &\sim 0 & \text{tr}(Y^2 \tau^c) &\sim 0, \\ \text{tr}(\{\tau^a, \tau^b\} Y) &\sim Q & \text{tr}(Y^3) &\sim Q, \end{aligned}$$

where Q is the sum of electromagnetic charges of the contributing internal fermions. If Q is summed over a generation:

$$Q = \sum Q_i = Q_\ell + Q_\nu + N_C(Q_{q\downarrow} + Q_{q\uparrow}) = \frac{N_C}{3} - 1, \quad (3.9)$$

and thus the electroweak anomalies cancel precisely when quarks come in 3 colours. However while the requirement of anomaly cancellation suggests

a classification of fermions into families, the number of such families is still a free parameter. The remarkable conspiracy of fermions within a generation to have a vanishing sum of charges Q is often touted as evidence for higher physics (for a recent review see [61]) although the problem of anomaly cancellation shall not concern us here.

3.1.2 Spontaneous symmetry breaking and mass

As seen in section 2.1, bare fermion masses mix left- and right-handed states, violating the chiral gauge-invariance and hence necessarily remain absent from a renormalisable theory. Moreover the addition of mass terms $m^2 W^2$ to the gauge-boson Lagrangian Eq.(3.2) in order to accommodate the massive W^\pm and Z bosons also violates $SU(2)_L$ gauge-invariance. In the pure SM gauge-boson and fermion masses are accommodated via the Higgs-Kibble mechanism [62], [63] of spontaneous symmetry breaking (SSB). That is, an extra $SU(2)_L$ isodoublet ϕ of scalar fields is postulated

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad (3.10)$$

with the corresponding gauged scalar Lagrangian

$$\mathcal{L}_H = (D_\mu \phi_i)^\dagger D^\mu \phi_i - \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2, \quad (3.11)$$

$$D_\mu = \partial_\mu - i g_2 \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu - i \frac{g_1}{2} B_\mu. \quad (3.12)$$

For $\lambda > 0$, $\mu^2 < 0$ the scalar potential terms have the famous “Mexican hat” shape with a continuum of vacuum minima satisfying

$$| \langle 0 | \phi | 0 \rangle | = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{\nu}{\sqrt{2}}. \quad (3.13)$$

Upon choosing a single ground state, the electroweak symmetry is broken to its QED subgroup Eq.(3.1) and from Goldstone’s theorem (see section 2.1) three massless states $\vec{\theta}$ appear. Rewriting ϕ in terms of the real fields $\vec{\theta}$ and the Higgs field H

$$\phi = \exp \left(\frac{i}{2} \vec{\tau} \cdot \vec{\theta} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu + H(x) \end{pmatrix}, \quad (3.14)$$

$SU(2)_L$ -invariance allows the unphysical (due to gauge invariance) $\vec{\theta}$ states to be rotated away, in which case \mathcal{L}_H becomes

$$\mathcal{L}_H = \frac{1}{4} \lambda \nu^4 + \frac{1}{2} (\partial H)^2 - \frac{1}{2} M_H^2 (H^2 - \frac{1}{\nu} H - \frac{1}{4\nu^2} H^4)$$

$$+ (1 + \frac{H}{\nu})^2 (M_W^2 W^2 + \frac{1}{2} M_Z^2 Z^2), \quad (3.15)$$

$$M_H = \sqrt{2\lambda\nu}, \quad (3.16)$$

$$M_W = \frac{\nu g}{2} = M_Z \cos \theta_W. \quad (3.17)$$

where now, due to the symmetry breaking, it is more convenient physically to rewrite the Lagrangian in terms of the intermediate gauge fields A_μ , Z_μ , W_μ^\pm defined by

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \quad (3.18)$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \pm W_\mu^2). \quad (3.19)$$

where θ_W is the Weinberg mixing angle. The SSB has been constructed in such a way that these fields are identifiable with the photon of QED and the experimentally observed massive intermediate vector bosons. That is, the unphysical Goldstone particles have been transformed into longitudinal gauge degrees of freedom and give rise to mass terms M_W^2 and M_Z^2 . In terms of the intermediate fields defined in Eqs.(3.18, 3.19) the fermion Lagrangian \mathcal{L}_I (given by Eq.(3.6)) becomes

$$\begin{aligned} \mathcal{L}_I = & \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - \frac{ig_2}{2\sqrt{2}} (\bar{\psi}_\uparrow \gamma^\mu W_\mu^- \chi_L \psi_\downarrow + \bar{\psi}_\downarrow \gamma^\mu W_\mu^+ \chi_L \psi_\uparrow) \\ & - \bar{\Psi} \gamma \cdot \left(A_\mu \left\{ g_2 \frac{\tau_3}{2} \sin \theta_W + g_1 y \cos \theta_W \right\} + Z_\mu \left\{ g_2 \frac{\tau_3}{2} \cos \theta_W - g_1 y \sin \theta_W \right\} \right) \Psi \end{aligned} \quad (3.20)$$

and all that remains to extract QED from the A terms is to impose

$$g_2 \sin \theta_W = g_1 \cos \theta_W \equiv e \quad (3.21)$$

$$y = q - \tau_3 \quad (3.22)$$

By postulating the existence of a complex scalar isodoublet there is the possibility for introducing a fermion mass via a gauge-invariant scalar-fermion coupling

$$\mathcal{L}_Y = -\kappa \phi \bar{\Psi}_L \Psi_R - (\kappa \phi)^\dagger \bar{\Psi}_R \Psi_L. \quad (3.23)$$

which for a generic fermion isodoublet Ψ , after SSB becomes

$$\mathcal{L}_Y = -\frac{\nu}{\sqrt{2}} (\nu + H) (\kappa_1 \bar{\psi}_\uparrow \psi_\uparrow + \kappa_2 \bar{\psi}_\downarrow \psi_\downarrow). \quad (3.24)$$

for arbitrary complex parameters κ . In particular if there are n identical generations κ is an $n \times n$ matrix.

Thus, the electroweak model is seen to accommodate fermion and boson masses in a gauge-invariant way, however the price to be paid for multiple generations is the introduction of a proliferation of Yukawa couplings κ_{ij} . The main drawback of the Higgs-Kibble mechanism is, however, the lack of experimental evidence for a physical Higgs particle (for a recent review see [64]). A number of alternative mechanisms for generating gauge-invariant boson masses have been proposed (for example [65], [66]) which do not suffer the shortcomings of elementary scalar fields. Dynamical symmetry breaking mechanisms are also used to generate fermion masses (for example the technicolour [67],[68] or top-condensation [69],[70] scenarios), and in addition can generate gauge boson masses [31].

3.1.3 Flavour mixing and CP violation

A priori, the coupling matrix κ in Eq.(3.23) may be non-diagonal and complex, constrained by the requirement that it commute with the charge operator q of QED. To obtain the mass eigenstates (i.e. the physical fermion basis) one must diagonalise κ by unitary phase-redefinitions of the fermions

$$\psi_L \rightarrow \psi'_L = U_L \psi_L, \quad (3.25)$$

$$\psi_R \rightarrow \psi'_R = U_R \psi_R, \quad (3.26)$$

such that

$$\bar{\psi}_L U_L^\dagger \kappa U_R \psi_R \rightarrow \bar{\psi}_L M \psi_R, \quad (3.27)$$

however this phase-redefinition transforms fermion interaction terms as

$$\bar{\psi}_\uparrow \gamma^\mu \chi_L \psi_\downarrow \rightarrow \bar{\psi}_\uparrow \gamma^\mu \chi_L V \psi_\downarrow, \quad (3.28)$$

where V is an $n \times n$ unitary matrix. Essentially the mass and weak eigenstates are incompatible, and thus flavour-changing charged currents are predicted.

As pointed out by Kobayashi and Maskawa [71], for n generations the number of complex elements κ_{ij} which cannot be removed by unitary redefinitions of fermion fields n_ϵ is

$$n_\epsilon = (n-1)(n-2)/2, \quad (3.29)$$

and hence for $n > 2$, complex couplings must remain, which lead to Charge-conjugation- and Parity- (CP-)violating interactions. In fact the existence

of CP violating electroweak processes had been previously confirmed in the decays of neutral kaons [72]. Hence the SM with CP-violation provides a lower bound on the number of quark generations, $n > 2$. From Eq.(3.29), for $n = 3$ there exists a single complex phase in the quark sector. With $\psi_\uparrow = (u, c, t)$ and $\psi_\downarrow = (d, s, b)$ the isospin-changing (i.e. W^\pm -fermion) interactions in Eq.(3.20) become

$$\bar{\psi}_\uparrow \gamma^\mu W_\mu^- \chi_L V_{CKM} \psi_\downarrow + \bar{\psi}_\downarrow \gamma^\mu W_\mu^+ \chi_L V_{CKM}^\dagger \psi_\downarrow, \quad (3.30)$$

where the Cabbibo-Kobayashi-Maskawa (CKM) matrix is commonly parametrised in terms of the angles θ_{ij} , the mixing angle between the i^{th} and j^{th} generations [5]:

$$\begin{aligned} V_{CKM} &\equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \end{aligned}$$

where c_{ij} and s_{ij} denote $\cos \theta_{ij}$ and $\sin \theta_{ij}$ respectively. Note that with recent experimental evidence for neutrino masses [73],[74] a CKM matrix and CP-violation potentially exist in the leptonic sector also. As only one CP-violating parameter δ_{13} is predicted considerable experimental focus is currently being placed on tests of the CKM model (for a recent review see [75], for a survey of generalised CP violation in the SM see [76]). Recently it has been argued [77] that only three types of interactions in gauge theories lead to CP violation, fermion-scalar interactions (as above), scalar-scalar interactions (for example [78],[79]) and gauge terms $F\tilde{F}$ (e.g. the strong CP problem). However it has also been pointed out [81] that non-perturbative sources of flavour mixing also exist as the SM fermion mass matrices also undergo rotation as a result of scale changes.

3.1.4 Non-perturbative aspects

With historical emphasis in theoretical physics on grand unification the non-perturbative region of the SM has been largely ignored. As pointed out in [2], at large momenta the hypercharge sector dominates, due to the asymptotic freedom of the colour and flavour interactions. Not only is the $U(1)$ hypercharge NAF, but numerical simulations [25] suggest that for elementary Higgs the renormalised quartic scalar coupling also vanishes. Indeed the

fact that triviality can apparently be averted by embedding the SM group in a higher semisimple group adds to the appeal of higher physics. However, as demonstrated in the last chapter the GNJL offers an alternative, in the case of NAF or “pure” QED.

In another paper Kiselev [23] postulated a strongly self-interacting (SSIR) (composite) Higgs region intermediate to Λ and a higher (e.g. unification) scale M . Qualitatively, condensates of three bilocal fermion composites (corresponding to $\bar{\tau}\tau$, $\bar{t}t$ and $\bar{b}b$) were found to acquire non zero VEV’s below the scale Λ , breaking EW symmetry. The masses of these three fermions may be predicted accurately (e.g. $m_t = 165 \pm 1$ GeV [23]) from the SSIR IR fixed point conditions. The model is also consistent with [21], where three Higgs sharing a specially-constrained potential led to the assertion of an extended vacuum with Z_3 -symmetry. Here the existence of multiple local vacuum minima was suggested to be an origin for the fermion generations.

A realistic description of the fermion generations via this method requires new symmetry breaking for the lighter fermion generations to acquire masses, however the successes of [23] provide further motivation for a study of the qualitative features of a hypercharge equivalent of the GNJL.

In conclusion, this section has illustrated how the electroweak model accommodates as arbitrary parameters the number of fermions (with some constraint given by anomaly cancellation), the fermion masses, mixings and CP violation. For three fermion generations, one Higgs isodoublet and no lepton CKM matrix the poor understanding of the scalar sector implies 14 *ad hoc* parameters (9 Yukawa couplings, 3 mixing angles, 1 phase and the Higgs VEV) are required.

3.2 The NJL model

Historically the Nambu-Jona-Lasinio model was the first relativistic quantum field theory in which dynamical chiral symmetry breaking was studied [15]. It is an important sector of contemporary theories including top-condensation models (see, for example [69],[70]) and the GNJL [24]. In this section we illustrate a number of its qualitative features following Miransky [17]. The NJL Lagrangian is obtained from the GNJL Lagrangian Eq.(3.98) upon setting the gauge coupling $e = 0$:

$$\mathcal{L}_{NJL} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + G \sum_{a=0}^{N^2-1} ((\bar{\psi}\frac{\lambda^a}{2}\psi)^2 + (\bar{\psi}\frac{\lambda^a}{2}\gamma^5\psi)^2), \quad (3.31)$$

$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}_L M \psi_R - \bar{\psi}_R M^\dagger \psi_L - \frac{1}{2G} \text{tr} M^\dagger M, \quad (3.32)$$

where the auxiliary field M satisfies the Euler-Lagrange equation

$$M^{ab} = -2G\bar{\psi}_L^b\psi_R^a, \quad (3.33)$$

i.e. the coupling G has canonical dimension -2 , and in L_{NJL} we have suppressed all colour indices.

The corresponding generating functional is

$$Z_{NJL}(J) = \mathcal{N} \int DMDM^\dagger D\psi D\bar{\psi} \exp\{i \int d^4x (L_{NJL} + \sigma)\}, \quad (3.34)$$

$$\sigma = \bar{\eta}\psi + \bar{\psi}\eta + J_M M + J_{M^\dagger} M^\dagger \quad (3.35)$$

where \mathcal{N} is a normalisation factor. Now the fermion fields simply appear in Eq.(3.35) as a Gaussian integral and may be formally integrated out:

$$\int D\bar{\psi} D\psi \exp\{i \int d^4x \bar{\psi} \Delta \psi\} \sim (\text{Det} i\Delta)^{N_c}, \quad (3.36)$$

$$\Delta = (i\gamma^\mu \cdot \partial_\mu - M\chi_L - M^\dagger\chi_R). \quad (3.37)$$

Upon substituting in Eq.(3.35) the effective NJL action may then be read off as

$$S_{eff} = -N_c(i \ln \text{Det} i(i\gamma^\mu \partial_\mu - M\chi_L - M^\dagger\chi_R) + \frac{1}{2GN_c} \int d^4x \text{tr} M^\dagger M) \quad (3.38)$$

For $N_c \rightarrow \infty$, the path integral

$$Z_{NJL} \sim \int DMDM^\dagger \exp\{-iS_{eff}(M, M^\dagger) + \int d^4x \sigma(x)\}, \quad (3.39)$$

is dominated by the stationary points of S_{eff} and the dynamics is classical. Thus ignoring quantum fluctuations of M, M^\dagger , we approximate M by its mean-field value ν . Using chiral symmetry any vacuum solution may be transformed via chiral symmetry into

$$M = \nu = M^\dagger, \quad (3.40)$$

so in momentum space we consider

$$0 = \frac{\delta S_{eff}}{\delta M} = \frac{1}{GN_c} \nu - i \text{tr} \frac{\nu}{\gamma^\mu \cdot p_\mu - \nu + i\epsilon}. \quad (3.41)$$

Evaluating the trace

$$\nu = 4iGN_c \int \frac{d^4p}{(2\pi)^4} \frac{\nu}{p^2 - \nu^2 + i\epsilon}. \quad (3.42)$$

it is immediately clear that the solution to this equation is constant and the trivial solution $\nu = 0$ exists. Rotating into Euclidean space and computing the momentum integral yields the algebraic equation

$$\nu \left(\frac{GN_c \Lambda^2}{4\pi^2} - 1 \right) - \frac{GN_c}{4\pi^2} \nu^3 \ln \frac{\Lambda^2 + \nu^2}{\nu^2} = 0. \quad (3.43)$$

For $0 < G < 4\pi^2/N_c \Lambda^2$ both terms on the left-hand side of Eq.(3.43) are negative if $\nu > 0$ and in this case only the trivial solution $\nu = 0$ exists. It follows then that to obtain a non-trivial real value of ν , G must exceed the critical value $4\pi^2/N_c \Lambda^2$. Moreover this solution is unique. It is clear that in the subcritical case the single zero corresponds to a stable extremum and chiral symmetry is preserved. However for the supercritical case this zero corresponds to a maximum, while the nontrivial value is the stable vacuum solution. In this supercritical phase then a mass term is dynamically generated, spontaneously breaking the chiral symmetry of the bare Lagrangian.

3.3 QED₄: dynamical chiral symmetry breaking

A number of early studies [16],[82],[83],[84] in massless, quenched, rainbow QED indicated a 2-phase structure, with a critical coupling value $\alpha = \frac{\pi}{3}$ above which chiral symmetry was broken. Maskawa and Nakajima[16] found this to be possible only for a finite cutoff. In general [82] the supercritical solution is found to be proportional to the cutoff, requiring judicious fine-tuning of the coupling constant to render the solution finite.

The general strategy for studying dynamical fermion mass generation in the literature has been to truncate the system of Schwinger-Dyson equations at the stage of 2-point Greens functions. Typically only the fermion propagator is retained and a test form is used for the photon propagator. In QED₄ the 2-point equations to be solved are

$$\Pi(k) = -\frac{1}{3}g_{\mu\nu}N \int \frac{d^4r}{(2\pi)^4} \text{tr}\{\Gamma_0^\mu S(r)\Gamma^\nu(r, r+k)S(r+k)\} \quad (3.44)$$

$$\Sigma(p) = \int \frac{d^4q}{(2\pi)^4} D_{\mu\nu}(p-q)\Gamma^\nu(p, q)S(q)\Gamma_0^\mu, \quad (3.45)$$

where $D_{\mu\nu}$ is the gauge-boson propagator, while Γ_0^μ and $\Gamma^\nu(q, p)$ denote the bare and dressed vertex functions respectively.

From Lorentz invariance and parity conservation the most general decomposition of the inverse fermion propagator is, in Euclidean space

$$S^{-1}(p) = iA(p)\gamma \cdot p + B(p) \equiv i\gamma \cdot p + m_0 + \Sigma(p), \quad (3.46)$$

where m_0 is the (unrenormalised) bare mass, $A(p)$, $B(p)$ are scalar form factors and $\Sigma(p)$ is the fermion self-energy. At this point a number of simplifying assumptions must be made for the photon propagator $D_{\mu\nu}$ and the full vertex $\Gamma^\nu(q, p)$ in order to obtain approximate solutions.

3.3.1 Quenched rainbow approximation

In this, and the following subsection the running of the coupling is ignored (quenched approximation) and the photon propagator is approximated by its bare counterpart:

$$D_{\mu\nu}(k) = (\delta_{\mu\nu} + (z - 1) \frac{k_\mu k_\nu}{k^2}) \frac{1}{k^2}, \quad (3.47)$$

where z is the gauge parameter. Further, approximating the full vertex $\Gamma^\mu(q, p)$ by the bare γ^μ (rainbow approximation) and performing the trace over Eq.(3.46) yields:

$$A(x) = 1 + \frac{z\alpha}{4\pi} \int_0^{\Lambda^2} dy \frac{A(y)}{yA^2(y) + B^2(y)} \left(\frac{y^2}{x^2} \theta_+ + \theta_- \right), \quad (3.48)$$

$$B(x) = m_0 + \frac{(3+z)\alpha}{4\pi} \int_0^{\Lambda^2} dy \frac{B(y)}{yA^2(y) + B^2(y)} \left(\frac{y}{x} \theta_+ + \theta_- \right). \quad (3.49)$$

where x and y are momentum-squared while $\theta_\pm = \theta(\pm(x - y))$ is the Heaviside step function. In the Landau gauge, $z = 0$, this system reduces to a single integral

$$B(x) = m_0 + \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dy \frac{B(y)}{y + B^2(y)} \left\{ \frac{y}{x} \theta_+ + \theta_- \right\}, \quad (3.50)$$

which was extensively studied by Fukuda and Kugo [84] in its differential form:

$$y \frac{d^2 B}{dy^2} + 2 \frac{dB}{dy} + \frac{3\alpha}{4\pi} \frac{B}{y + B^2} = 0. \quad (3.51)$$

Here the integration limits become the boundary conditions

$$\lim_{y \rightarrow 0} \frac{d}{dy} (y^2 B) = 0 \quad (3.52)$$

$$\lim_{y \rightarrow \Lambda^2} \frac{d}{dy} (yB) = m_0 \quad (3.53)$$

The expression Eq.(3.51) has no analytic solution, however analyses via bifurcation techniques [85], [86] have been performed, while linearisation

$B^2(y) \sim B^2(0)$ has been shown to be a good approximation in both infra-red and ultra-violet regions [87]. Adopting the latter approach, the solution in the UV regime is [17]

$$B(y) \simeq B(0) \left\{ \frac{\Gamma(\omega)}{\Gamma(\frac{1+\omega}{2})\Gamma(\frac{3+\omega}{2})} \left(\frac{y}{B^2(0)} \right)^{(\omega-1)/2} + \frac{\Gamma(-\omega)}{\Gamma(\frac{1-\omega}{2})\Gamma(\frac{3-\omega}{2})} \left(\frac{y}{B^2(0)} \right)^{-(\omega+1)/2} \right\}; \alpha < \frac{\pi}{3}, \quad (3.54)$$

$$B(y) \simeq \frac{2}{\pi} \frac{B^2(0)}{\sqrt{y}} (\ln \frac{y}{B^2(0)} + \ln 16 - 2); \alpha = \frac{\pi}{3}, \quad (3.55)$$

$$B(y) \simeq B^2(0) \sqrt{\frac{8 \coth \frac{\pi \tilde{\omega}}{2}}{y \pi \tilde{\omega} (\tilde{\omega}^2 + 1)}} \sin(\frac{\tilde{\omega}}{2} \ln \frac{y}{B^2(0)} + \tilde{\omega} (\ln 4 - 1)); \alpha > \frac{\pi}{3}, \quad (3.56)$$

where $\omega = \sqrt{\frac{3\alpha}{\pi} - 1}$ and $\tilde{\omega} = \sqrt{1 - \frac{3\alpha}{\pi}}$. Substituting these into the UV boundary condition Eq.(3.53) yields

$$m_0 \simeq \begin{cases} B(0) \frac{\Gamma(\omega)}{\Gamma(\frac{1+\omega}{2})\Gamma(\frac{3+\omega}{2})} \left(\frac{\Lambda^2}{B(0)} \right)^{\omega-1} & ; \alpha < \frac{\pi}{3}, \\ \frac{4B(0)^2}{\pi \Lambda^2} (\ln \frac{\Lambda^2}{B(0)} + \ln 4 - 1) & ; \alpha = \frac{\pi}{3}, \\ \frac{B(0)^2}{\Lambda} \sqrt{\frac{2 \coth \frac{\pi \tilde{\omega}}{2}}{\pi \tilde{\omega}}} \sin(\tilde{\omega} \ln \frac{\Lambda^2}{B(0)} + \arg \frac{\Gamma(1+i\tilde{\omega})}{\Gamma^2(\frac{1+i\tilde{\omega}}{2})}) & ; \alpha > \frac{\pi}{3}. \end{cases} \quad (3.57)$$

Taking the chiral limit $m_0 \rightarrow 0$ when $\alpha \leq \pi/3$ in Eq.(3.57) for finite Λ^2 it is clear that $B(0)$ is also trivial. For $\Lambda^2 \rightarrow \infty$, m cannot naively be set to zero in a consistent way. The argument [17] follows from the conservation of the (renormalised) axial current j_5^μ , which is required for chiral symmetry-breaking

$$\lim_{\Lambda^2 \rightarrow \infty} \partial_\mu j_5^\mu = \lim_{\Lambda^2 \rightarrow \infty} i Z_m m_0 (\Lambda^2) \bar{\psi} \gamma^5 \psi = 0 \quad (3.58)$$

Upon substituting Eq.(3.58) into the subcritical solution of Eq.(3.57) it follows that only the trivial solution $B(0) = 0$ satisfies the ultra-violet boundary condition Eq.(3.53).

However when $\alpha > \pi/3$, infinitely many solutions of Eq.(3.57) are possible. For $\alpha - \pi/3 << 1$

$$\sin(\tilde{\omega} \ln \frac{\Lambda^2}{B(0)}) = 0 \quad (3.59)$$

or re-arranging for $B(0)$

$$B(0) \sim \Lambda^2 \exp(-\frac{\pi n}{\tilde{\omega}} + \ln 4); n = 1, 2, \dots \quad (3.60)$$

Miransky [88] showed that only the case $n = 1$ corresponds to a stable vacuum; solutions with higher values are interpreted as radial Goldstone excitations.

3.3.2 Improved quenched approximations

There are a number of general criteria that an ansatz for the vector vertex function $\Gamma^\mu(q, p)$ should address as outlined by Burden and Roberts [89]. Despite being a useful first study for dynamical chiral symmetry breaking, quenched rainbow QED₄ fails to satisfy several of these.

- $\Gamma^\mu(q, p)$ must transform in the same way as the bare vertex γ^μ under Lorentz transformations and satisfy charge conservation.
- It must satisfy the vector Ward Identity

$$k \cdot \Gamma_V(q, p) = S^{-1}(q) - S^{-1}(p) \quad (3.61)$$

In the rainbow approximation, substituting Eq.(3.46) and $\Gamma_V^\mu(q, p) = \gamma^\mu$ this gives

$$(q - p) \cdot \gamma = A(q)q \cdot \gamma + B(q) - (A(p)p \cdot \gamma + B(p)) \quad (3.62)$$

which is satisfied only when $A(q) = 1$, that is, in Landau gauge. However the Ward Identity only constrains a portion of the vertex Γ^μ ; a “transverse” piece transverse remains unconstrained. The full gauge covariance of QED is guaranteed by the Landau-Khalatnikov transformations [90]. The strategy of making vertex ansatze satisfying individual Ward-Takahashi identities provides only approximate gauge-covariance. However the LK transforms are difficult to implement for many choices of $\Gamma^\mu(q, p)$ and consequently much of the literature for QED [91],[92],[93] is aimed at pinning down the behaviour of the transverse piece by the other vertex requirements. An alternative approach is to also consider the so-called “Transverse Ward-Takahashi identity” [94],[95]. Here, in addition to the divergence $\partial_\mu \Gamma^\mu$, one considers the curl of the vertex $\partial_\mu \Gamma^\nu - \partial_\nu \Gamma^\mu$. In 2-dimensional QED the latter simply reduces to the axial Ward-Takahashi Identity [96] and moreover the model is exactly solvable [95].

- The vertex must reduce to γ^μ when dressed fermion and boson propagators are replaced by bare ones so that perturbation theory is recovered in the free-field limit. The absence of kinematic singularities guarantees a unique $q \rightarrow p$ limit for the vertex. Such a vertex is given in Feynman gauge to two loops [36] perturbation theory while a 1 loop, singularity-free form in arbitrary covariant gauge was given in [92].

- It must allow multiplicative renormalisation of the SDE it appears in. The restrictions that this places on the transverse vertex piece have been

considered in quenched QED₄ [91]. The most general, non-perturbative, multiplicatively-renormalisable form for the vertex has also been considered [93].

The methods for applying these constraints to the case of QED will be discussed in what follows. The most general Lorentz-invariant, spin- $\frac{1}{2}$ vertex may be written as [97]

$$\Gamma^\mu(p, q) = \sum_{i=1}^{12} (f_i(p^2, q^2) + g_i(p^2, q^2) \gamma^5) v_i^\mu \quad (3.63)$$

where $q_\pm = (q \pm p)$ and v_i are given by Eq.(3.64)

$$\begin{aligned} v_1^\mu &= q_+^\mu & v_2^\mu &= q_-^\mu & v_3^\mu &= \gamma^\mu \\ v_4^\mu &= \sigma^{\mu\nu} q_{+\nu} & v_5^\mu &= \sigma^{\mu\nu} q_{-\nu} & v_6^\mu &= \gamma \cdot q_+ q_+^\mu \\ v_7^\mu &= \gamma \cdot q_+ q_-^\mu & v_8^\mu &= \gamma \cdot q_- q_+^\mu & v_9^\mu &= \gamma \cdot q_- q_-^\mu \\ v_{10}^\mu &= \gamma^\mu \sigma^{\nu\lambda} q_{+\nu} q_{-\lambda} & v_{11}^\mu &= \gamma \cdot q_+ \gamma \cdot q_- q_+^\mu & v_{12}^\mu &= \gamma \cdot q_+ \gamma \cdot q_- q_-^\mu \end{aligned} \quad (3.64)$$

For a parity-conserving theory the form factors g_i vanish, and the further constraint of charge-conjugation invariance

$$C \Gamma_V^\mu(q, p) C^{-1} = -(\Gamma^\mu)^T(-p, -q) \quad (3.65)$$

demands that each of the 12 terms be symmetric under $q \iff p$.

The requirement that the vertex satisfy the vector-current Ward identity Eq.(3.61), and in particular as $p \rightarrow q$ to avoid the appearance of kinematic singularities

$$\frac{\partial S^{-1}(p)}{\partial p^\mu} = \Gamma^\mu(p, p) \quad (3.66)$$

was considered in [36]. Indeed a criterion for choosing the basis vectors is that each of the 12 terms be independently singularity-free. Substituting

$$S^{-1}(p) = A(p^2) \gamma \cdot p + B(p^2) \quad (3.67)$$

into Eq.(3.66) yields

$$\Gamma^\mu(p, p) = \gamma^\mu A(p^2) + 2p^\mu \left(\gamma \cdot p \frac{\partial A}{\partial p^2} + \frac{\partial B}{\partial p^2} \right) \quad (3.68)$$

which may be rewritten in a p, q -symmetric way as

$$\Gamma_{BC}^\mu(q, p) \equiv \frac{A(q^2) + A(p^2)}{2} v_3^\mu + \frac{A(q^2) - A(p^2)}{2(q^2 - p^2)} v_6^\mu + \frac{B(q^2) - B(p^2)}{q^2 - p^2} v_1^\mu \quad (3.69)$$

In addition to restricting 3 of the 12 form factors, the absence of $q^\mu p^\nu \sigma_{\mu\nu}$ terms from Eq. (3.61) causes another to vanish. The resulting (Ball-Chiu) vertex may now be written as

$$\Gamma^\mu(q, p) = \Gamma_{BC}^\mu + \sum_{i=1}^8 A_i T_i^\mu \quad (3.70)$$

where the Ball-Chiu basis vectors for the transverse part of the vector vertex are

$$T_1^\mu = p \cdot (q - p) q^\mu - q \cdot (q - p) p^\mu, \quad (3.71)$$

$$T_2^\mu = \gamma \cdot (q + p) T_1^\mu, \quad (3.72)$$

$$T_3^\mu = (q - p)^2 \gamma^\mu - (q - p)^\mu \gamma \cdot (q - p), \quad (3.73)$$

$$T_4^\mu = p^\nu q^\rho \sigma_{\nu\rho} T_1^\mu, \quad (3.74)$$

$$T_5^\mu = (q - p)_\nu \sigma^{\nu\mu}, \quad (3.75)$$

$$T_6^\mu = \gamma^\mu (q^2 - p^2) - (q + p)^\mu \gamma \cdot (q - p), \quad (3.76)$$

$$T_7^\mu = \frac{1}{2} (q^2 - p^2) (\gamma^\mu \gamma \cdot (q + p) - (q + p)^\mu) + (q + p)^\mu p^\nu q^\rho \sigma_{\nu\rho}, \quad (3.77)$$

$$T_8^\mu = -\gamma^\mu p^\nu q^\rho \sigma_{\nu\rho} + p^\mu \gamma \cdot q - q^\mu \gamma \cdot p. \quad (3.78)$$

To two loops in Feynman [36] gauge $A_4 = 0 = A_5 = A_7$ while if m_0 , $A_1 = 0$ also. The analysis has been repeated to 1 loop in arbitrary covariant gauge [92] and misprints in the coefficients A_2, A_3 of [36] corrected

Constraining the transverse vertex by requiring the self-energy Eq.(3.45) be multiplicatively-renormalisable led to the well-known Curtis-Pennington vertex ansatz [91]:

$$\Gamma_{CP}^\mu(q, p) = \Gamma_{BC}^\mu(q, p) + \frac{A(q) + A(p)}{2} \frac{T_6^\mu}{d(q, p)} \quad (3.79)$$

where

$$d(q, p) = \frac{(q^2 - p^2)^2 + ((B(q)/A(q))^2 + (B(q)/A(q))^2)^2}{q^2 + p^2}. \quad (3.80)$$

Substituting this into Eq.(3.45) and tracing out the vector and scalar form-factors as above one arrives at

$$\begin{aligned} A(x) = & 1 - \frac{\alpha}{4\pi} \int_0^{\Lambda^2} \frac{dy}{yA^2(y) + B^2(y)} \left\{ \frac{y^2}{x^2} \theta_+(-zA^2(y) - \frac{zB(y)}{y}(B(y) - B(x))) \right. \\ & \left. - \theta_- zA(x)A(y) + \left(\frac{3B(y)}{2(y-x)}(B(y) - B(x)) \right) \right\} \end{aligned}$$

$$+ \frac{3(y+x)}{4(y-x)} \left(1 - \frac{(y-x)^2}{d(y,x)}\right) A(y)(A(y) - A(x)) \left(\frac{y^2}{x^2} \theta_+ + \theta_-\right\}. \quad (3.81)$$

$$\begin{aligned} B(x) = & m_0 + \frac{\alpha}{4\pi} \int_0^{\Lambda^2} \frac{dy}{yA^2(y) + B^2(y)} \{z(\frac{y}{x} \theta_+ A(x) B(y) + \theta_- A(y) B(x)) \\ & - \frac{3x}{2(y-x)} (A(x) B(y) - A(y) B(x)) (\frac{y^2}{x^2} \theta_+ + \theta_-) \\ & + \frac{3B(y)}{2} (A(y) + A(x) + (A(y) - A(x)) \frac{y-x}{d(y,x)}) (\frac{y}{x} \theta_+ + \theta_-)\}. \end{aligned} \quad (3.82)$$

Naturally the corresponding equations for the Ball-Chiu vertex are obtained by dropping the terms containing $d(q, p)$. Numerical solutions of this system of equations have yielded estimates of the critical coupling in the range of $0.91 < \alpha_C < 0.98$ in Landau gauge [98] and $\alpha_C \sim 0.92$ for Landau, Feynman and Yennie gauges [91]. The gauge-dependence of the critical coupling has been investigated both numerically [91] and analytically [99]. Numerically the “Euclidean mass” [91] is associated with the appearance of a fixed point of the equation

$$M(p^2 = -m_E^2) = m_E; \quad M(x) = B(x)/A(x). \quad (3.83)$$

In the latter study a bifurcation technique [85], namely functional differentiation of this self-mapping, was employed to locate the change between the oscillatory and non-oscillatory behaviours of the mass function. Differentiating Eqs.(3.81,3.82) with respect to B and evaluating the system at the trivial point $B(x) = 0$, the momentum integrals may be performed [99]. Scale invariance is recovered for the large cutoff limit, $\Lambda \rightarrow \infty$, with the solution

$$A(x) = (1 + \frac{\alpha z}{8\pi}) (\frac{x}{\Lambda^2})^\nu, \quad (3.84)$$

$$B(x) = A(x) x^{-s}, \quad (3.85)$$

where

$$\nu = \frac{2\alpha z}{8\pi + \alpha z} < 1, \quad (3.86)$$

and $0 < s < 2$ satisfies

$$\begin{aligned} z = & \frac{3\nu(\nu - s + 1)}{2(1 - s)} (3\pi \cot \pi(\nu - s) + 2\pi \cot \pi s - \pi \cot \pi \nu \\ & + \frac{1}{\nu} + \frac{1}{\nu + 1} + \frac{2}{1 - s} + \frac{3}{s - \nu} + \frac{1}{s - \nu - 1}). \end{aligned} \quad (3.87)$$

A bifurcation point occurs when 2 solutions of Eq.(3.87) become equal, and, in particular, criticality is reached when a nontrivial solution of Eq.(3.82) bifurcates away from the trivial one, $B(x) = 0$. In Landau gauge Eq.(3.87) has two solutions, and the critical coupling was found to be $\alpha_C = 0.933667$. More than 2 solutions exist in an arbitrary gauge and only the solutions continuously connected to the Landau solutions are of interest. Differentiating the right-hand side of Eq.(3.87) w.r.t. s gives

$$2\pi^2 \csc^2 \pi s - 3\pi^2 \csc^2 \pi(\nu - s) + \frac{3}{(v - s)^2} - \frac{2}{(1 - s)^2} + \frac{1 - \frac{2z}{3}}{(1 + \nu - s)^2} = 0. \quad (3.88)$$

Solving Eqs.(3.87),(3.88) simultaneously, Atkinson et al. [99] found only 11% deviation from the Landau gauge result for gauge parameter $-2 < z < 20$, while for $z < -3$ the solution B becomes infra-red-divergent and the gauge-dependence of α_C increases markedly.

Further improvements upon the Curtis-Pennington vertex must ultimately improve the gauge-dependence of the critical coupling, and include the effects of unquenching the photon propagator.

3.3.3 Unquenched approximations

The unquenched behaviour of the SDE equations in QED is of vital importance: the running behaviour of the coupling leads problems such as the Landau pole in perturbation theory, while removing the cutoff ($\Lambda^2 \rightarrow \infty$) leads to the requirement that QED (and non-asymptotically free theories in general) be trivial [25]. Including fermion loops introduces an extra dimensionless parameter into the theory, N , the number of flavours of contributing fermions, and is responsible for any possible dynamically-generated photon mass. The effect that quenching the photon propagator has upon the phase transition in the unquenched theory has been investigated in massless, rainbow approximation in a number of studies [100], [101],[102], [103]. Qualitatively the phase transition has been found to be pushed to larger coupling scales, for $N = 1$ estimates (to be compared with $\alpha_C = \frac{\pi}{3}$ in quenched, rainbow case) include $\alpha_C \sim 2.00$ [100] and $\alpha_C \sim 2.25$ [103].

The simplest way to improve upon the quenched approximation of section 2.1 is to replace the photon propagator Eq.(3.47) by

$$D_{\mu\nu}(k) = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2(1 + \Pi(k^2))}) + G \frac{k_\mu k_\nu}{k^2} \quad (3.89)$$

and re-derive the coupled equations for the fermion form factors A and B :

$$A(p) = 1 + \alpha \int \frac{d^4 q}{(2\pi)^4} \frac{A(q)}{q^2 A^2(q) + B^2(q)} \left(\frac{z}{(p-q)^2} + \frac{1}{(p-q)^2(1+\Pi(q))} + \frac{2p \cdot (p-q)}{(p-q)^2} \left(1 - \frac{p \cdot q}{q^2} \right) \left(\frac{z}{(p-q)^2} - \frac{1}{(p-q)^2(1+\Pi(q))} \right) \right) \quad (3.90)$$

$$B(p) = m_0 + \alpha \int \frac{d^4 q}{(2\pi)^4} \left(\frac{3}{1+\Pi(q)} + z \right) \frac{B(q)}{q^2 A^2(q) + B^2(q)}. \quad (3.91)$$

The simplest non-trivial ansatz for $\Pi(k^2)$ is to take the large momentum limit of the one-loop contribution of N massless fermion species:

$$\Pi(k^2) = \frac{N\alpha}{3\pi} \ln \frac{\Lambda^2}{k^2} \quad (3.92)$$

However the so-called “Landau ghost problem” arises at large momenta: setting $k^2 = \epsilon \Lambda^2$ an unremoveable singularity appears when

$$1 + \Pi(k^2) = 0 \iff \alpha = -\frac{3\pi}{N \ln \epsilon^{-1}} \quad (3.93)$$

i.e. when $\epsilon \geq 1$. In particular at the cutoff when ϵ can equal 4, this constrains N to a few permissible values $N < \frac{3\pi}{\alpha \ln 4}$. Setting $\alpha_C = 2.00, 2.25$ it is clear that to avoid the singularity $N \leq 3$. Indeed one study [102] found nontrivial solutions for only $N = 1, 2$. The solution to Eqs.(3.91) was found [101],[102] to possess mean field behaviour

$$B(0) \sim \Lambda^2 (\alpha - \alpha_C)^{\frac{1}{2}} \quad (3.94)$$

compared with the exponential behaviour Eq.(3.60) in the unquenched case.

3.4 Gauged model

The suggestion that 4-fermi operators of the type contained in the NJL could mix with the QED gauge interaction was first made by Bardeen *et al.* [24]. In the quenched rainbow QED approximation in the neighbourhood of the UV fixed point they showed that the naively irrelevant (6-dimensional) operators

$$(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2 \quad (3.95)$$

become marginal, i.e. acquire an anomalous dimension ≥ -2 . In terms of the Wilson RG, the GNJL can be thought of as a low-energy effective action arising in the following way

$$S_{eff}^\Lambda = \int d^4 x \mathcal{L}_{QED} + V(\bar{\psi}, \psi) \quad (3.96)$$

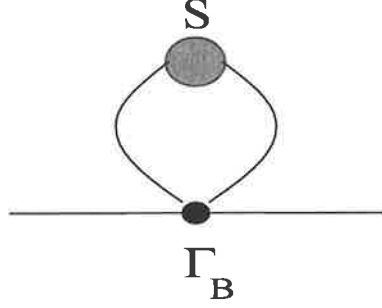


Figure 3.1: Contribution of 4-fermi interactions to fermion self energy

where V is a chiral- and parity-invariant potential. In the local potential approximation (LPA) (i.e. in the absence of derivative terms) the lowest order terms are given by

$$\frac{g_S}{\Lambda^2}((\bar{\psi}\lambda^\alpha\psi)^2 + (\bar{\psi}\lambda^\alpha i\gamma^5\psi)^2) + \frac{g_V}{\Lambda^2}((\bar{\psi}\lambda^\alpha\gamma^\mu\psi)^2 + (\bar{\psi}\lambda^\alpha\gamma^\mu\gamma^5\psi)^2), \quad (3.97)$$

chiral symmetry guaranteeing that the next lowest order terms are of order eight. Several analyses [24], [106] have shown that for quenched QED, the scalar term $\sim g_S$ becomes marginal while the vector couplings [56] remain irrelevant.

The GNJL (in covariant gauge) for N fermion flavours is therefore given by:

$$\mathcal{L}_{GNJL} = \mathcal{L}_{QED} + G \sum_{a=0}^{N^2-1} ((\bar{\psi}\frac{\lambda^a}{2}\psi)^2 + (\bar{\psi}\frac{\lambda^a}{2}\gamma^5\psi)^2), \quad (3.98)$$

where G is a dimensionful four-fermion coupling constant, identified with g_S/Λ^2 , while λ^a are global $U(N_f)$ flavour generators normalised by

$$\text{tr}\lambda^a\lambda^b = 2\delta_{ab}, \quad (3.99)$$

and the QED Lagrangian, Eq.2.67 is

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu(\partial_\mu + eA_\mu) - m_0)\psi - F_{\mu\nu}F^{\mu\nu} - \frac{1}{2z}(\partial^\mu A_\mu)^2. \quad (3.100)$$

The extra 4-fermi term contributing to the fermion self-energy is given by Figure 3.1. In the quenched rainbow approximation this leads to the modification (cf. Eq.(3.49))

$$\begin{aligned} B(x) = & m_0 + G \int_0^{\Lambda^2} dy \frac{B(y)}{yA^2(y) + B^2(y)} \\ & + \frac{(3+z)\alpha}{4\pi} \int_0^{\Lambda^2} dy \frac{B(y)}{yA^2(y) + B^2(y)} \left\{ \frac{y}{x}\theta_+ + \theta_- \right\}. \end{aligned} \quad (3.101)$$

In the Landau gauge, $z = 0$, this system is equivalent to the differential equation Eq.(3.51)

$$y \frac{d^2 B}{dy^2} + 2 \frac{dB}{dy} + \frac{3\alpha}{4\pi} \frac{B}{y + B^2} = 0 \quad (3.102)$$

where the 4-fermi coupling enters through the UV boundary condition Eq.(3.104)

$$\lim_{y \rightarrow 0} \frac{d}{dy} (y^2 B) = 0, \quad (3.103)$$

$$\lim_{y \rightarrow \Lambda^2} \left(1 + \frac{4\pi G}{3\alpha} \frac{dB}{dy} + B \right) = m_0. \quad (3.104)$$

In this case the large y , scale-invariant solution reveals a critical curve in coupling constant space given by [85]

$$G_c = \frac{1}{4} \left(1 + \sqrt{\frac{3\alpha}{4\pi}} \right) \quad (3.105)$$

Beyond this approximation, progress is made upon introducing auxiliary σ and π fields of the type used in the NJL. Recently it has been argued in detail [28] that interactions of these scalar composites with fermions suppress the charge screening fermion loops.

Chapter 4

Four-fermion model of generations

The crucial point is to make an appropriate choice of variables, able to capture the physics which is most important for the problem at hand.

Antonio Pich

You can use any variables at all to analyze a problem, but if you use the wrong variables you'll be sorry.

Steven Weinberg

In this chapter we construct a toy four-fermion model in order to develop the qualitative discussion of [2]). The key feature of the proposal is for the interactions between fermion chiralities to occur at separate momentum scales, specifically

$$\lambda_R^2 < \lambda_X^2 < \lambda_L^2, \quad (4.1)$$

where R , X and L are the scales for right-right, left-right and left-left couplings respectively. This is anticipated to lead to a rich set of phase transitions and fermion-antifermion condensates. Under the proposal of [2] a “fundamental” fermion consisting of three flavours corresponding to the observed generations may then be self-consistently introduced

We begin with the large momentum-limit of the standard model where, due to asymptotic freedom, the $SU(3)$ colour and $SU(2)_L$ isospin interactions have become negligible. The dynamics is governed by the non-asymptotically free hypercharge $U(1)$ interaction. To this end we consider the generating functional \mathcal{Z} for a chiral Abelian theory containing, for sim-

plicity, a single charged, massless fermion:

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int D\bar{\psi} D\psi DZ e^{iS(\bar{\psi}, \psi, Z)} \\ S(\bar{\psi}, \psi, Z) &= \int \bar{\psi} \gamma^\mu (i\partial_\mu + (c_L \chi_- + c_R \chi_+) Z_\mu) \psi - Z^\mu \Delta_{\mu\nu} Z^\nu \end{aligned} \quad (4.2)$$

Here \mathcal{N} is a normalisation factor and $\Delta_{\mu\nu}$ is the inverse boson propagator. In the absence of any anomaly-matching, strictly speaking the parity-odd part of the action S contains not only a $2\pi i$ -multivalued chiral invariant remainder, but also a Wess-Zumino-Witten functional which saturates the anomaly

$$S_-(\bar{\psi}, \psi, Z) \equiv \Gamma_{WZW} + 2\pi i \eta(\bar{\psi}, \psi, Z).$$

We shall postpone treatment of the anomaly until the next chapter, with the understanding that the following discussion holds for the parity-even part of the action at a formal level.

Completing the square in Z and formally integrating out the resulting Gaussian leads to the appearance of three effective interactions, proportional to c_L^2 , c_R^2 , and $c_L c_R$ respectively:

$$\begin{aligned} \mathcal{Z} &= \mathcal{N}' (\det \Delta^{-1})^{-1/2} \int D\bar{\psi} D\psi \exp\left(\int (\bar{\psi} i \gamma \cdot \partial \psi + \mathcal{L}_{eff})\right) \quad (4.3) \\ \mathcal{L}_{eff} &\sim (c_L \psi_L \gamma^\mu \psi_L + c_R \psi_R \gamma^\mu \psi_R) \Delta_{\mu\nu} (c_L \psi_L \gamma^\nu \psi_L + c_R \psi_R \gamma^\nu \psi_R) \end{aligned} \quad (4.4)$$

In order to proceed one must make an approximation to the quartic fermion terms. In the conventional GNJL, the bosons acquire an effective mass below the chiral symmetry breaking scale Λ_χ :

$$\Delta_{\mu\nu}(k) \sim g_{\mu\nu} / \Lambda_\chi.$$

Here the various terms could be critical at *separate* scales, however the simplest form for the massive boson propagator is

$$\tilde{g}^2 \Delta_{\mu\nu}(k) \sim -(\tilde{c}_R^2 + \tilde{c}_L^2 + \tilde{c}_X^2) g_{\mu\nu}. \quad (4.5)$$

where it is understood that the couplings denoted with a “ \sim ” have absorbed the scale factor Λ_χ and hence have canonical dimension -2 .

Some *ansatz* must, in general be made for the separately-running SM couplings $c_i(k)$, however we shall initially consider three independent coupling constants. The effective Lagrangian density is now:

$$\begin{aligned} \mathcal{L} &= \bar{\psi} i \gamma^\mu \partial_\mu \psi - \tilde{c}_R^2 \bar{\psi}_R \gamma^\mu \psi_R \bar{\psi}_R \gamma_\mu \psi_R - \tilde{c}_L^2 \bar{\psi}_L \gamma^\mu \psi_L \bar{\psi}_L \gamma_\mu \psi_L \\ &\quad - \tilde{c}_X^2 (\bar{\psi}_R \gamma^\mu \psi_R \bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_L \gamma^\mu \psi_L \bar{\psi}_R \gamma_\mu \psi_R). \end{aligned} \quad (4.6)$$

Using Fierz identities (see Appendix A) the general effective quartic interaction Eq.(4.4) is arrived at:

$$\begin{aligned}\mathcal{L}_{eff} = & -\tilde{g}_1\{(\bar{\psi}\gamma^\nu\psi)^2 + (\bar{\psi}\gamma^5\gamma^\nu\psi)^2\} + \tilde{g}_2\{(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2\} \\ & + \tilde{g}_3(\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\psi\bar{\psi}\gamma_\mu\psi)\end{aligned}\quad (4.7)$$

The first two terms are just the extended NJL if we identify

$$\begin{aligned}\tilde{g}_1 &= \tilde{c}_R^2 + \tilde{c}_L^2, \\ \tilde{g}_2 &= 2\tilde{c}_X^2.\end{aligned}$$

The third, anomalous term $\sim \tilde{g}_3$ violates parity, chiral symmetry and contributes to the odd part of the effective action. It is therefore clear that an analysis of D χ SB of the even part would proceed along the same lines as the extended NJL.

4.1 Digression

Due to the ABJ anomaly, chiral symmetry in the single-fermion hypercharge theory would be broken by gauge transformations. In a first attempt therefore to distinguish the current model from the extended NJL we could add chiral-breaking 4-fermi interactions. These interactions would be constrained by the requirement that the resulting Dirac operator D has non-negative eigenvalues in order for the theory to be regularisable in Euclidean space. In computing the quantity $\ln \det D$, and in the absence of external vector and axial fields, the most general expression for the mass matrix is (see e.g. appendix A, [104])

$$(s^2 + (ip)^2) + 4p^2 - i\gamma^5(sp + ps)$$

where s and p are, respectively, external scalar and pseudoscalar flavour matrices. Here the first term is recognisable as the chiral-invariant NJL terms, while the second and third are new. The above expression is Hermitian in the limit of vanishing p , however recently it has been demonstrated [105] that Hermiticity, while a sufficient condition for non-negativity is not necessary. Invariance under PT transformations was conjectured [105] to be the fundamental requirement.

Assuming *a priori* that there are no difficulties in regulating S (e.g. via the ζ -function approach [104]), our toy model will have 4-fermi interactions

$$\begin{aligned}\mathcal{L}_{4F} &= -\tilde{g}_2\{(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2\} + i\tilde{g}_3\{\bar{\psi}\psi\bar{\psi}i\gamma^5\psi + \bar{\psi}i\gamma^5\psi\bar{\psi}\psi\} \\ &\equiv -2\tilde{g}_2\bar{\psi}_R\psi_L\bar{\psi}_L\psi_R - \tilde{g}_3((\bar{\psi}_L\psi_R)^2 - (\bar{\psi}_R\psi_L)^2)\end{aligned}\quad (4.8)$$

where couplings $\tilde{g}_2 \equiv \tilde{x}$ and \tilde{g}_3 are taken to be real.

Since chiral symmetry is explicitly broken a chiral rotation

$$\psi \rightarrow \exp(i\gamma^5\omega)\psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp(i\gamma^5\omega),$$

has the effect of transforming the anomalous coupling into a complex quantity

$$\mathcal{L}'_{4F} = -2\tilde{x}\bar{\psi}_R\psi_L\bar{\psi}_L\psi_R - \tilde{r}(\omega)(\bar{\psi}_L\psi_R)^2 - \tilde{r}^*(\omega)(\bar{\psi}_R\psi_L)^2 \quad (4.9)$$

$$\equiv 2\frac{x}{\Lambda_\chi}\bar{\psi}_R\psi_L\bar{\psi}_L\psi_R - \frac{r(\omega)}{\Lambda_\chi}(\bar{\psi}_L\psi_R)^2 - \frac{r^*(\omega)}{\Lambda_\chi}(\bar{\psi}_R\psi_L)^2 \quad (4.10)$$

where Λ_χ is some renormalisation scale, which we set to the scale of chiral symmetry breaking. As in the case without explicit chiral symmetry breaking, progress is made via the mean field approximation, however here the choice of auxiliary field varies differs for both couplings \tilde{x}, \tilde{r} nonzero.

When $\tilde{r} = 0, \tilde{x} \neq 0$ we encounter the familiar NJL auxiliary field

$$M = -2\tilde{x}\bar{\psi}_L\psi_R, \quad (4.11)$$

For $\tilde{x} = 0, \tilde{r} \neq 0$ the choice will be

$$\mu = -\tilde{r}\bar{\psi}_R\psi_L, \quad (4.12)$$

while for $\tilde{x}, \tilde{r} \neq 0$ it is the complex, chiral matrix

$$\nu = -2\tilde{x}\bar{\psi}_L\psi_R - \frac{\tilde{r}}{2\tilde{x}}\bar{\psi}_R\psi_L. \quad (4.13)$$

For convenience of analysis, let us consider the general case by introducing fermions ψ_1, ψ_2, ψ_3 , with respective coupling values

$$(\tilde{x}, \tilde{r}) = \{(\tilde{x}, 0), (0, \tilde{r}), (\tilde{x}, \tilde{r})\}. \quad (4.14)$$

The general 4-fermi terms in Eq.(4.9) may then be rewritten as a (Hermitian) mass matrix as

$$\begin{aligned} \mathcal{L}_M &= \bar{\psi}_{1L}M\psi_{1R} + \bar{\psi}_{2L}\mu\psi_{2R} + \bar{\psi}_{3L}\nu\psi_{3R} + h.c. \\ &\quad - \frac{1}{\tilde{x}}MM^\dagger - \frac{1}{2}\left(\frac{\mu^2}{\tilde{r}} + \frac{(\mu^\dagger)^2}{\tilde{r}^*}\right) \\ &\quad - \frac{\epsilon}{2\tilde{x}}\left(\frac{1}{\epsilon}\nu\nu^\dagger - \frac{\tilde{r}}{2\tilde{x}}\nu^2 - \frac{\tilde{r}^*}{2\tilde{x}}(\nu^\dagger)^2\right) \end{aligned} \quad (4.15)$$

where

$$\epsilon = 1 + \frac{\tilde{r}\tilde{r}^*}{\tilde{x}^2} \quad (4.16)$$

4.2 Mass generation

Although chiral symmetry is explicitly broken in the model Eq.(4.8), we shall now investigate the case where the magnitude of this effect is small. An analysis of dynamically-generated mass associated with the breaking of the chiral-symmetric component can still be legitimately compared to that in other quantum field theories. Meanwhile the other two features of the toy model, fermionic generations and CP violation are entirely dependent on the possibility of mass generation.

We consider the theory with three introduced fermion species with the coupling constants specified in (4.14). With the mass terms of Eq.(4.15) the Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \sum_{j=1}^3 \bar{\psi}_j (i\gamma \cdot \partial - m_j \chi_R - m_j^\dagger \chi_L) \psi_j + \mathcal{L}_M \\ \mathcal{L}_M &= \sum_{j=1}^3 (\bar{\psi}_{jL} m_j \psi_{jR} + h.c.) - \frac{1}{\tilde{x}} m_1 m_1^\dagger - \frac{1}{2} \left(\frac{m_2^2}{\tilde{r}} + \frac{(m_2^\dagger)^2}{\tilde{r}^*} \right) \\ &\quad - \frac{\epsilon}{2\tilde{x}} \left(\frac{1}{\epsilon} m_3 m_3^\dagger - \frac{\tilde{r}}{2\tilde{x}} m_3^2 - \frac{\tilde{r}^*}{2\tilde{x}} (m_3^\dagger)^2 \right)\end{aligned}\tag{4.17}$$

with ϵ given by Eq.(4.16). From the corresponding generating functional

$$Z = \mathcal{N}' \int [D\bar{\psi}] [D\psi] [D\mu] [D\mu^\dagger] \exp(i \int d^4x \mathcal{L}(x)).\tag{4.18}$$

one formally integrates out the fermions to obtains the effective action

$$S_{4P}^{eff} = -i \sum_{j=1}^3 \text{Tr} \text{Ln} i G_j - \int d^4x \mathcal{L}_m\tag{4.19}$$

where the Dirac operators G_j are, as discussed previously, assumed non-negative in Euclidean space and therefore regularisable. If this action is dominated by the stationary points

$$\frac{\delta S_{4P}^{eff}}{\delta m_j} = 0 = \frac{\delta S_{4P}^{eff}}{\delta m_j^\dagger}\tag{4.20}$$

there are 3 pairs of coupled NJL-like equations, which in momentum space are found to be

$$\text{tr}[G_1^{-1} \chi_R] = \frac{1}{\tilde{x}} m_1^\dagger,\tag{4.21}$$

$$\text{tr}[G_1^{-1}\chi_L] = \frac{1}{\tilde{x}}m_1, \quad (4.22)$$

$$\text{tr}[G_2^{-1}\chi_R] = \frac{1}{\tilde{r}}m_2, \quad (4.23)$$

$$\text{tr}[G_2^{-1}\chi_L] = \frac{1}{\tilde{r}^*}m_2^\dagger, \quad (4.24)$$

$$\text{tr}[G_3^{-1}\chi_R] = \frac{\epsilon^2}{2\tilde{x}}\left(\frac{2}{\epsilon}m_3^\dagger - \frac{\tilde{r}}{\tilde{x}}m_3\right), \quad (4.25)$$

$$\text{tr}[G_3^{-1}\chi_L] = \frac{\epsilon^2}{2\tilde{x}}\left(\frac{2}{\epsilon}m_3 - \frac{\tilde{r}^*}{\tilde{x}}m_3^\dagger\right), \quad (4.26)$$

Neglecting the momentum dependence of the couplings, in the mean field approximation this system has a constant solution. The momentum integrals are evaluated in polar coordinates and it is convenient to define the integrand function

$$\mathcal{F}(m) = \int_0^\Lambda dz \frac{mz}{z+m^2} = m\Lambda - m^3 \ln \frac{\Lambda+m^2}{m^2} \quad (4.27)$$

where z is the Euclidean squared momentum. Given that m_j is in general a complex quantity, $m_j = a_j + ib_j$, we shall frequently express this function as

$$\begin{aligned} \mathcal{F}(m) &\equiv \mathcal{F}_1 + i\mathcal{F}_2 \\ \mathcal{F}_1 &= a\Lambda - (a^3 - 3ab^2)L + (3a^2b - b^3)A \\ \mathcal{F}_2 &= b\Lambda - (a^3 - 3ab^2)A - (3a^2b - b^3)L \\ L &\equiv \frac{1}{2} \log \left(1 + \frac{\Lambda^2 + 2\Lambda(a^2 - b^2)}{a^2 + b^2} \right) \\ A &\equiv \arctan \left(\frac{2ab}{\Lambda + a^2 - b^2} \right) - \arctan \left(\frac{2ab}{a^2 - b^2} \right) \end{aligned} \quad (4.28)$$

where for the complex logarithm the usual branch cut along the negative imaginary axis has been made and it has been assumed that $a > b$.

The mean-field expectation values thus obtained are complex, however in the results obtained below it is understood that a suitable unitary redefinition of the fermion field in question enables the phase to be eliminated. For the question of mass generation it therefore suffices to seek non-trivial real values a , which minimise the effective potential.

4.2.1 Conventional chiral symmetric phase

Let us first consider the chiral-invariant limit of the model, $r = 0$, where the behaviour should emulate that of the conventional NJL. That is, above a

critical coupling value the chiral-symmetric vacuum becomes unstable and undergoes a phase transition to a configuration where chiral symmetry is dynamically broken. Integrating Eqs.(4.21,4.22) up to the UV cutoff Λ yields

$$\frac{1}{\tilde{x}}m_1 = \frac{1}{4\pi^2}\mathcal{F}(\Lambda, 0)m_1^\dagger, \quad (4.29)$$

$$\frac{1}{\tilde{x}}m_1^\dagger = \frac{1}{4\pi^2}\mathcal{F}(\Lambda, 0)m_1. \quad (4.30)$$

In terms of real and imaginary parts $m_1 = a_1 + ib_1$ there are two “gap” equations

$$a_1 = \frac{\tilde{x}}{4\pi^2}\mathcal{F}_1 \quad (4.31)$$

$$b_1 = -\frac{\tilde{x}}{4\pi^2}\mathcal{F}_2 \quad (4.32)$$

In the case of real fermion mass, $b_1 = 0$, $\mathcal{F}_2 = 0$ and Eq.(3.43) is just the mean-field NJL equation (3.43). Hence nontrivial solutions to Eq.(4.31) exist when both sides of the equation have the same sign, i.e.

$$\tilde{x}_c \geq 4\pi^2/\Lambda, \quad (4.33)$$

or, in terms of the dimensionless couplings defined in Eq.(4.9),

$$x_c \geq 4\pi^2 \frac{\Lambda_\chi}{\Lambda}, \quad (4.34)$$

The trivial solution $a_1 = 0$ also exists for both phases, but it corresponds to a vacuum maximum when $x > 4\pi^2/\Lambda$ (see section 3.2).

4.2.2 Explicitly broken phase

Consider now the purely explicit chiral-breaking interaction. Integrating over the fermion loop, Eqs.(4.23, 4.24) give

$$\frac{1}{\tilde{r}}m_2 = \frac{1}{4\pi^2}\mathcal{F}(m_2^\dagger), \quad (4.35)$$

$$\frac{1}{\tilde{r}^*}m_2^\dagger = \frac{1}{4\pi^2}\mathcal{F}(m_2). \quad (4.36)$$

Writing the coupling as $\tilde{r} = \tilde{\rho} + i\tilde{\sigma}$, the real and imaginary “gap” equations then read

$$a_2 = \tilde{\rho}\mathcal{F}_1 - \tilde{\sigma}\mathcal{F}_2 \quad (4.37)$$

$$b_2 = \tilde{\rho}\mathcal{F}_2 + \tilde{\sigma}\mathcal{F}_1 \quad (4.38)$$

Real, nontrivial solutions are therefore obtained when the imaginary component of the coupling $\tilde{\sigma}$ vanishes. The NJL equation (3.43) is obtained again, with the explicit chiral-breaking coupling \tilde{r} having the critical value

$$\tilde{r}_c = \rho_c = 4\pi^2/\Lambda \quad (4.39)$$

4.2.3 Anomalous phase

Finally we integrate the equations for G_3 up to scale Λ yielding

$$\frac{\epsilon^2}{2\tilde{x}}\left(\frac{2}{\epsilon}m_3^\dagger - \frac{\tilde{r}}{\tilde{x}}m_3\right) = \frac{1}{4\pi^2}\mathcal{F}(m_3) \quad (4.40)$$

$$\frac{\epsilon^2}{2\tilde{x}}\left(\frac{2}{\epsilon}m_3 - \frac{\tilde{r}^*}{\tilde{x}}m_3^\dagger\right) = \frac{1}{4\pi^2}\mathcal{F}(m_3^\dagger) \quad (4.41)$$

In this case when $b_3=0$, $\tilde{\sigma} = 0$ the NJL gap equation is found where the critical coupling curve is given by

$$\tilde{\ell} = \frac{\tilde{x}}{\epsilon} \frac{1}{1 - \tilde{\rho}\epsilon/2\tilde{x}} = \frac{4\pi^2}{\Lambda} \quad (4.42)$$

and ϵ is given by Eq.(4.16).

4.3 Fermion generations

It was seen in the previous section that the three fermion species introduced in Eq.(4.15), each living in their own 4-fermi potential experience a two-phase vacuum structure analogous to the NJL. Moreover there are three phase transitions in the theory, corresponding to the scales where \tilde{x} , \tilde{r} and $\tilde{\ell}$ each attain the critical value $4\pi^2/\Lambda$. This is the behaviour envisaged in [2] and fermion generations can be introduced as the components of a self-consistently defined fundamental fermion $\Psi = (\psi_1, \psi_2, \psi_3)$ via the mechanism of Kiselev [21], which we now briefly outline.

In the original study [21] 3 Higgs scalars were introduced into a specially-constrained potential leading to the possibility of an extended Z_3 -degenerate vacuum. Fermionic fields with Z_3 components corresponding to fermion generations are then self-consistently introduced into the theory.

For the model under consideration we can rewrite the three “masses” as rewritten in the polar form

$$m_j = \tilde{g}_j \nu_j e^{i(\phi_j + \omega_j \gamma^5)}; \quad j = r, x, \ell \quad (4.43)$$

where \tilde{g}_j are understood to be (real) Yukawa constants. These quantities could replace the three complex VEVs in [21] if we anticipate the full gauged version of the toy model in the following chapter. Without loss of generality the quantities $\phi_1, \omega_1, \omega_2$ could be eliminated in the unitary gauge with the remaining phases $\phi_{2,3}, \omega_3$ potentially leading to observable mass effects. The fermion “mass” terms could be recast in terms of auxiliary fermions, for example $\bar{\Psi} M_{LR} \Psi_R$ (c.f. [21], where $\omega_3 = 0$) given by

$$\bar{\Psi}_L = \frac{1}{\sqrt{3}} \begin{pmatrix} \bar{\psi}_L \\ e^{2i\phi_2} \bar{\psi}_L \\ e^{2i(\phi_3 + \omega_3 \gamma^5)} \bar{\psi}_L \end{pmatrix} \quad (4.44)$$

$$\begin{aligned} \Psi_R &= \frac{1}{\sqrt{3}} \begin{pmatrix} \psi_R, & e^{-i\phi_2} \psi_R, & e^{-i(\phi_3 + \omega_3 \gamma^5)} \psi_R \end{pmatrix}, \\ M_{LR} &= \begin{pmatrix} \nu_1 & \nu_2 e^{3i\phi_r} & \nu_3 e^{3i(\phi_3 + \omega_3 \gamma^5)} \\ \nu_2 & \nu_3 e^{i(\phi_2 + \phi_3 + \omega_3) \gamma^5} & \nu_1 e^{i(2\phi_3 - \phi_2 + 2\omega_3 \gamma^5)} \\ \nu_3 & \nu_1 e^{i(2\phi_3 - \phi_2 + 2\omega_3 \gamma^5)} & \nu_2 e^{i(\phi_2 + \phi_3 + \omega_3 \gamma^5)} \end{pmatrix} \end{aligned} \quad (4.45)$$

and similarly for the conjugate term $\sim M_{RL}$. As Kiselev pointed out, all ϕ -dependence is removed from the mass matrices if $\phi_r = -\phi_\ell = 2k\pi/3$. More generally it is clear that if in addition $\omega_3 = k\pi$, the above mass matrix is also scalar.

In [21] the discrete ϕ phase values result from a special choice of Higgs potential, whereby the vacuum configuration has a Z_3 degeneracy. With the extended vacuum state

$$|0, 0, 0\rangle = |0_1\rangle \otimes |0_2\rangle \otimes |0_3\rangle$$

a Z_3 symmetric model of fermions $\Psi, \bar{\Psi}$ can be self-consistently defined. See [21] for full details of the scalar potential. Once the chiral phase is also allowed, the required symmetry here is $Z_3 \otimes Z_2$. Although we have not yet considered internal fermionic degrees of freedom the Z_2 factor could be relevant to the isospin structure of leptons or quarks, particularly in the context of left-right symmetric models (see section 6.1 for further discussion).

We have outlined in this section how the existence of condensates associated with the three critical scales provides a mechanism for fermion generations to be self-consistently introduced. We note however that this auxiliary field technique is quite general and may in principle be extended beyond the current 2- and 3-generation models [21], [23]. Here the motivation for considering three generations is given by the observation that, once the requirement of chiral symmetry is dropped, there are three possible types of 4-fermi potential.

4.4 CP-violation

The model contains several potential sources of CP-violation. Firstly, as noted, the masses in Eq.(4.43) are complex, chiral objects. As outlined above, upon introducing three generations the phases may be eliminated if $\phi_j = 2k\pi/3$, $\omega_j = 2k\pi$. That is, if one appeals to the existence of a chiral $Z_3 \otimes Z_2$ symmetry.

In this case flavour mixing also arises from the fact that the mass matrix in Eq.(4.45) is only determined up to an ambiguity; an internal rotation which cancels at the level of the Z_3 constituent fields, leaving the eigenvalues of M_{LR} invariant. These CKM/MNS mixing angles are naturally expressed in terms of the ratios of quark/lepton generation masses and Kiselev has shown [21] that when the augmented mass matrix is expressed in the form

$$M = \begin{pmatrix} \eta_1 & \zeta + \theta & \zeta \\ \zeta + \theta & \eta_2 & \zeta - \theta \\ \zeta & \zeta - \theta & \eta_3 \end{pmatrix},$$

the resulting estimates for the CP-violating phase and certain CKM matrix elements are statistically equivalent to best current experimental limits (see [21]).

In the absence of the discrete chiral symmetry, the complex chiral phases will lead to further sources of CP violation. In the general parametrisation there are such three CP parameters: ϕ_2, ϕ_3, ω_3 .

4.4.1 Triviality

In summarising this section to date, we have introduced a toy fermionic theory which appears to contain mass generation and, upon self-consistent introduction of three fermion generations, CP-violating behaviour. It was constructed in such a way that perturbative renormalisability was implicit. However the magnitude of the running of the coupling was ignored until now. If the model is to represent a high-energy matter-only sector, rescuing the $U(1)$ hypercharge from triviality analogous to the GNJL, the model must itself be well behaved in the UV and IR limits.

The running behaviour of the dimensionless couplings x, r, r^* in the LPA approximation of the Wegner-Houghton RG scheme are readily obtained from using Eq.(2.87) with the Wilson potential given by \mathcal{L}'_{4F} in Eq.(4.9). Equivalently the form of the one-loop β functions can be deduced from the diagrams given in Appendix B. For vanishing bare fermion mass the running

couplings are described by (c.f. [56])

$$\frac{dx}{dt} = -2x + (x^2 + 2rr^*)f \quad (4.46)$$

$$\frac{dr}{dt} = -2r + (2rx + r^2)f \quad (4.47)$$

$$\frac{dr^*}{dt} = -2r^* + (2r^*x + r^{*2})f \quad (4.48)$$

where f is a constant associated with the evaluation of the fermion loop in figure B.2, and the dimensionless scale parameter t is defined in terms of the fixed scale Λ_X as

$$t = \ln \Lambda_X / \Lambda \quad (4.49)$$

Equivalently the equations for $r(t)$, $r^*(t)$ may be split into real and imaginary parts, as above:

$$\frac{d\rho}{dt} = -2\rho(xf + 1) + f(\rho^2 - \sigma^2) \quad (4.50)$$

$$\frac{d\sigma}{dt} = -2\sigma(fx + 1) + 2f\sigma\rho \quad (4.51)$$

The system of equations (4.46,4.50,4.51) is found, for $\rho(t) = 0 = \sigma(t)$, to have the same qualitative properties as existing analyses of the conventional 4-Fermi theory [106], [107]. In this case Eq.(4.46) has the analytic solution

$$x(t) = \frac{2}{f} \frac{1}{1 + ce^{2t}} \quad (4.52)$$

where c is the constant of integration. This solution evidently flows to a vanishing IR point ($t \rightarrow \infty$) and the non-trivial UV fixed point ($t \rightarrow -\infty$) at $x = 2/f$.

There is also a well-behaved analytic solution for Eqs.(4.47, 4.48) when $x = 0$

$$r(t) = \frac{2}{f} \frac{1}{1 + de^{2t}}, \quad (4.53)$$

$$r^*(t) = \frac{2}{f} \frac{1}{1 + d^*e^{2t}}, \quad (4.54)$$

i.e. it has vanishing IR and finite UV behaviour, similar to Eq.(4.52) above.

Once the symmetry-breaking couplings are switched on, the full system of equations can only be solved numerically. Again the model flows to a UV

fixed point, however in the general complex case the IR behaviour has been altered. Given the IR boundary conditions

$$x(t_0) = x_0 \quad (4.55)$$

$$r(t_0) = r_0 \quad (4.56)$$

$$\sigma(t_0) = \sigma_0 \quad (4.57)$$

it is found that for $\sigma_0 = 0$, the values x_0, r_0 are scale-independent, however $x(t)$ and $r(t)$ rapidly become singular below $t < t_0$.

The reason for this behaviour becomes clear when these equations are combined:

$$\frac{d}{dt}(x + r + r^*) = (x + r + r^*)(-2 + f(x + r + r^*)). \quad (4.58)$$

That is, when $x(t) = -2\rho(t)$, the flow through coupling constant space remains null, defining another renormalised trajectory. Along this path the RG equations simplify to

$$\frac{dx}{dt} = -2x(t) + f\left(\frac{3}{2}x^2(t) + 2\sigma^2(t)\right), \quad (4.59)$$

$$\rho(t) = -\frac{1}{2}x(t), \quad (4.60)$$

$$\sigma(t) \sim \exp\left(-\int dt(2 - fx(t))\right). \quad (4.61)$$

From Eq.(4.61) it is clear that a possibility for finite nontrivial solutions exists if the exponent is positive, i.e. for “small” enough $x(t)$. With vanishing imaginary coupling σ the running couplings are (c.f Eq.(4.52))

$$x(t) = \frac{3}{2f} \frac{1}{ce^{2t} + 1} = -\frac{1}{2}r(t). \quad (4.62)$$

This phase of the theory could, therefore, with suitable matching conditions upon x_0, r_0 at some scale λ be interpreted as a high ($p^2 > \lambda$) energy limit of some lower effective theory.

At this stage it is helpful to classify the three interesting coupling-constant trajectories. We shall refer to the subspace defined by

$$x(t) = 0 \quad (4.63)$$

$$r(t) = \frac{2}{f} \frac{1}{1 + de^{2t}} = r^*(t) \quad (4.64)$$

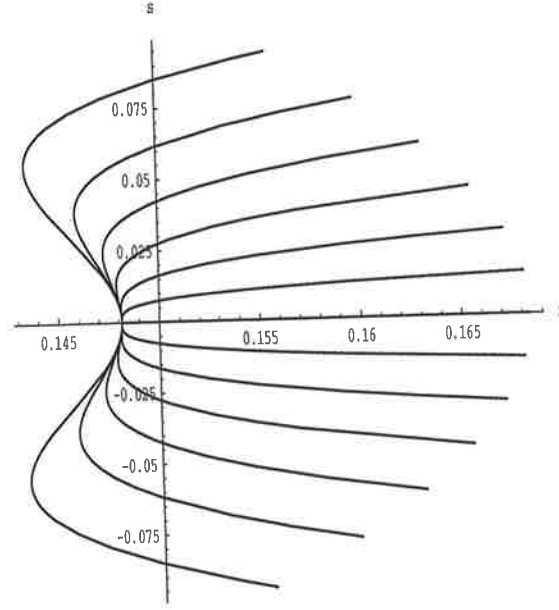


Figure 4.1: Behaviour of the solution of Eqs. 4.59-4.61. Here s denotes the imaginary coupling σ .

as region I. It is characterised by a real, positive coupling and the relevant low-energy 4-fermi operator is

$$O_1 = \tilde{r}(\bar{\psi}_{1L}\psi_{1R})^2 + \tilde{r}^*(\bar{\psi}_{1R}\psi_{1L})^2 \quad (4.65)$$

Region II is defined to be the trajectory

$$x(t) = \frac{2}{f} \frac{1}{1 + be^{2t}}, \quad (4.66)$$

$$r(t) = 0 = r^*(t) \quad (4.67)$$

which is just the conventional attractive scalar interaction of the GNJL

$$O_2 = \tilde{x}\bar{\psi}_{2L}\psi_{2R}\bar{\psi}_{2R}\psi_{2L}. \quad (4.68)$$

Finally there is region III, where both couplings are non-zero

$$x(t) = \frac{2}{f} \frac{1}{1 + ce^{2t}}, \quad (4.69)$$

$$\text{Re}(r(t)) = -x(t)/2, \quad (4.70)$$

where if x is now an attractive scalar interaction, r is necessarily repulsive, a similar situation to the 1+1-dimensional Luttinger liquid model of interactions between left- and right-moving charge densities. The relevant operator in this case is

$$O_3 = \tilde{x}(\bar{\psi}_{3L}\psi_{3R} - \bar{\psi}_{3R}\psi_{3L})^2 \quad (4.71)$$

Up until now our analysis has only considered one scale, however with the hierarchy $\lambda_1 < \lambda_2 < \lambda_3$ the set of phase transitions suggested in [2] could naturally be described by the following scenario.

Consider a chiral gauge theory defined at momentum scales $p^2 \ll \lambda_1$. Upon evolution to higher scales, an attractive, irrelevant scalar operator O_1 acquires a large anomalous dimension, and begins to mix with the gauge interaction, suppressing the non-asymptotically free coupling before becoming marginal in the neighborhood of λ_1 . This chiral symmetry- and parity-breaking interaction is analogous to the chiral condensates generated at the right-right critical scale in [2]. At progressively larger scales fluctuations of operator O_2 become apparent, before, at the scale λ_2 , also becoming relevant. As noted previously it is the left-right hypercharge interaction which gives rise to such a term, it is natural therefore to associate λ_2 with this critical scale. Finally, at λ_3 both couplings are switched on, however in this phase r has been driven negative. This phenomenon of competing attractive and repulsive scalar interactions also occurs in the Luttinger liquid models of strongly correlated electrons in 1+1 dimensions (see section 6.1 for further discussion).

4.4.2 Summary

In this section we proposed a toy 4-fermion model which, upon analysis was demonstrated to have behaviour matching that of the hypothesis [2] of dynamical mass, generations and CP violation.

To the best of our knowledge it is the first toy model exhibiting such dynamical features and can be considered essentially intermediate to two differing approaches to the generation problem, [81], [21], both of which are independent of the nature of the origin of mass and make good tree-level predictions for CKM matrix elements, heavy/light mass ratios etc.

The study of Kiselev [21] illustrates how, with the self-consistent introduction of Z_3 symmetric fermions and a special 3-Higgs potential, the generation structure emerges. Under our proposal the three Higgs scalars are composites, the underlying dynamics being the four-fermion operators needed to rescue the chiral $U(1)$ model from triviality in the Wilsonian RG

picture. This is in direct analogy to the GNJL as the extended version of QED.

The latter [81] notes that RG evolution of the SM fermion mass matrices offers a mechanism for flavour mixing and the fermion mass hierarchy. Given that these “masses” are coupling-dependent linear combinations of the conventional NJL auxiliary fields, the running of the couplings would generate flavour mixing in a similar manner.

The main drawback with this model is the origin and nature of the explicit chiral symmetry breaking. In the Fierz reordering of the (chiral-symmetric) hypercharge effective action above, such terms cancelled for the separate right-right and left-left interactions, also for the combination of left-right and right-left terms. The most probable cause is an anomalous mixing of pseudoscalar and vector/ axial fermion couplings, in which case the effect is small at low scales.

Secondly there is the question of how operators O_1 , O_3 , which seemingly break the chiral symmetry but, in the mean-field approximation at least, impart no fermion mass at subcritical scales. With two of the three fermion species (ψ_1 , ψ_3) behaving in a different manner to ψ_2 , it moreover appears logical to associate the former with the heavier, unstable generations.

A resolution to this may lie in the “pseudogap” phenomenon of strongly-correlated electrons [108]: It is well-known that mean-field, Bardeen-Cooper-Schrieffer (BCS) type theories of superconductivity are inadequate outside the regimes of weak coupling and high carrier density. A separation of the temperatures of pair formation and pair condensation occurs and, in the intermediate temperature range Cooper pairs exist, however due to large fluctuations no condensate is formed.

Recently it was shown [108] that the phenomenon exists in the Gross-Neveu model and it was suggested to apply to other relativistic matter-only models. In the theory under consideration the implication is that the Dirac vacuum is, in fact such a “pseudogap”-like phase. Although the pairing operators O_1 , O_3 should be written down, in the absence of a condensate the chiral symmetry is not broken.

While the origin of such an effect is speculative, the model contains all the desirable features needed for the hypothesis [2]. Thus, instead of rejecting the model in Eq.(4.8) outright, we shall now use the insights gained to attempt to construct a well-defined theory from an alternative approach.

4.5 Four-fermion model

We now proceed to investigate the effective 4-fermi theory containing chirally-invariant, parity-violating terms Eq. (4.7). In terms of left and right auxiliary fields we may write it as

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(i\gamma^\mu\partial_\mu + M\chi_R + M^\dagger\chi_L + \gamma^\mu\chi_R R_\mu + \gamma^\mu\chi_L L_\mu)\psi \\ & -\text{tr}\left(\frac{1}{x}MM^\dagger + \frac{1}{r}R^2 + \frac{1}{\ell}L^2\right) \end{aligned} \quad (4.72)$$

$$M = -2\tilde{x}\bar{\psi}_R\psi_L \quad (4.73)$$

$$M^\dagger = -2\tilde{x}\bar{\psi}_L\psi_R \quad (4.74)$$

$$R_\mu = -\tilde{r}\bar{\psi}_R\gamma_\mu\psi_R \quad (4.75)$$

$$L_\mu = -\tilde{\ell}\bar{\psi}_L\gamma_\mu\psi_L \quad (4.76)$$

In order to study chiral symmetry breakdown we shall improve on the mean-field approximation and consider dynamical fluctuations in the composite chiral bosons. The fermion self-energy SDE is readily obtained from the generating functional of Eq.(4.72) via a similar procedure to that in section 2.4. Alternatively it may be written diagrammatically as shown in Figure 4.2: The most general form of the inverse fermion propagator consistent

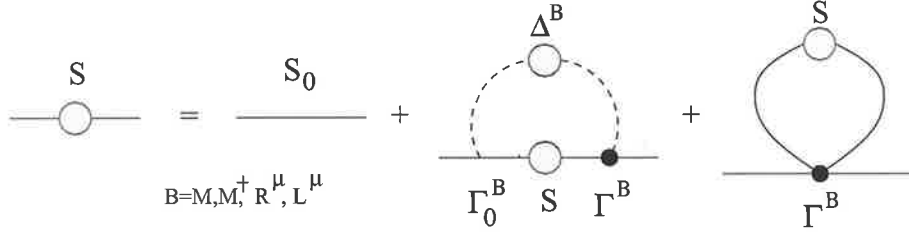


Figure 4.2: SDE for the fermion self-energy from the model Eq.(4.72).

with Lorentz invariance we consider is, in Euclidean space,

$$S(p) = \{(A(p) + a(p)\gamma^5)i\gamma \cdot p + B(p) + ib(p)\gamma^5\}^{-1}. \quad (4.77)$$

which may be rewritten in a number of ways, however we shall project out the chiral components of the propagator on the left simplifies to

$$S^{-1}(p) = \chi_+(iA_+(p)\gamma \cdot p + B_+(p))^{-1} + \chi_-(iA_-(p)\gamma \cdot p + B_-(p))^{-1}, \quad (4.78)$$

$$S(p) = \frac{(A_+(p)i\gamma \cdot p - B_+(p))}{D_+(p)}\chi_+ + \frac{(A_-(p)i\gamma \cdot p - B_-(p))}{D_-(p)}\chi_-, \quad (4.79)$$

$$D_\pm(p) = |A_\pm(p)|^2 p^2 + |B_\pm(p)|^2 \quad (4.80)$$

where the chiral form factors are defined by

$$\begin{aligned} B(p) &= (B_+(p) + B_-(p))/2 \\ b(p) &= (B_+(p) - B_-(p))/2 \\ A(p) &= (A_+(p) + A_-(p))/2 \\ a(p) &= (A_+(p) - A_-(p))/2 \end{aligned}$$

and the Euclidean mass "poles" and wavefunction normalisations are read off as

$$M_{\pm}(p) = \frac{|B_{\pm}(p)|}{|A_{\pm}(p)|}, \quad (4.81)$$

$$Z_{\pm}(p) = \frac{1}{|A_{\pm}(p)|}. \quad (4.82)$$

The presence of chirality-dependent form factors in the numerator of Eq.(4.79) can be interpreted as the fermion interacting with a non-trivial background. For example, in the presence of an external, spatially varying pseudoscalar field it has been shown [109] that a bias exists between particles and anti-particle distributions of the same chirality. We shall encounter this plus the case when Eq.(4.79) contains two distinct terms, $|A_+| \neq |A_-|$, $|B_+| \neq |B_-|$, in section 5.1.1.

The fermion self energy is now written down from Fig 3.45 as

$$\begin{aligned} \Sigma(p) &= \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left(\Gamma_0^M S(q) + \Gamma_0^{M\dagger} S(q) \right) \\ &+ \sum_{B=R^\mu, L^\mu} \int \frac{d^4 q}{(2\pi)^4} \Gamma_0^B S(q) \Gamma^B(q, p) \Delta^B(q - p) \end{aligned} \quad (4.83)$$

where the quantities Γ_0^B , $\Gamma^B(q, p)$ denote the bare and full boson vertices and Δ^B the inverse boson propagators. The bare vertices appear in Eq.(4.72) while the boson propagators satisfy the SDEs

$$\Delta_B^{-1}(k) = \Delta_{0B}^{-1}(k) + \Pi_B(k), \quad (4.84)$$

$$\Pi_B(k) = \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left((-i\Gamma_B^0)(iS(q))(-i\Gamma_B(q, k - q))(iS(k - q)) \right) \quad (4.85)$$

Here the bare boson propagators are

$$\Delta_{0M}^{-1}(k) = \frac{1}{\tilde{x}^2} = \Delta_{0M^\dagger}^{-1}(k), \quad (4.86)$$

$$(\Delta_{0R}^{\mu\nu})^{-1}(k) = \frac{\delta^{\mu\nu}}{\tilde{r}^2}, \quad (4.87)$$

$$(\Delta_{0L}^{\mu\nu})^{-1}(k) = \frac{\delta^{\mu\nu}}{\tilde{\ell}^2}. \quad (4.88)$$

The full vertices Γ_B also satisfy SDEs, however in addition to the two-point functions defined here they depend upon 4-point Greens functions. As mentioned in section 2.4 the full set of SDEs for a theory is a countably infinite set of coupled equations. In order to obtain an approximate solution it is necessary to truncate the system. Here we shall adopt the standard approach and truncate at the two-point level, making approximations to the vertex functions.

4.5.1 Mass generation

The simplest approximate solution of the fermion self-energy Eq.(4.83) is obtained by replacing all full vertices and boson propagators by their bare counterparts, for the latter this is achieved by neglecting the vacuum polarisation terms Π_B .

The (Euclidean space) self energy is also defined through the inverse fermion propagator Eq.(4.78)

$$\begin{aligned}\Sigma(p) &= S^{-1}(p) - i\gamma \cdot p - m \\ &= \chi_+ \{i\gamma \cdot p (A_+(p) - 1) + (B_+(p) - m_+)\} \\ &\quad + \chi_- \{i\gamma \cdot p (A_-(p) - 1) + (B_-(p) - m_-)\}\end{aligned}\quad (4.89)$$

Equating Eq.(4.89) and Eq.(4.83) projected into chiral components

$$\Sigma(p) = \chi_+ \Sigma_+(p) + \chi_- \Sigma_-(p),$$

expressions for the chiral form factors A_\pm , B_\pm may then be traced out. Computing the momentum integral in polar coordinates, these expressions are found to be:

$$\text{tr}(\gamma \Sigma_\pm)/4 = A_\pm(y) = 1 \quad (4.90)$$

$$\text{tr}(\Sigma_\pm)/4 = B_\pm(y) = B_\pm = \frac{\tilde{x}}{16\pi^2} \int_0^\Lambda dz \frac{B_\mp}{z + |B_\mp|^2} \quad (4.91)$$

which have an analytic solution given by (c.f. the NJL gap equation):

$$B_\pm = \frac{\tilde{x}^2}{16\pi^2} B_\mp \mathcal{F}(B_\mp) \quad (4.92)$$

As for the NJL model $B_+ = B_-$ are real solutions and one finds the critical coupling

$$\tilde{x}_c^2 = 16\pi^2/\Lambda^2. \quad (4.93)$$

To move beyond this approximation it is necessary to promote the bosons to dynamical particles. This is achieved by dropping the auxiliary “mass” terms and adding the kinetic piece

$$\mathcal{L}_{kin} = \frac{1}{x^2} \partial_\mu M \partial^\mu M^\dagger + \frac{1}{r^2} \mathcal{F}_R^{\mu\nu} \mathcal{F}_{R\mu\nu} + \frac{1}{\ell^2} \mathcal{F}_L^{\mu\nu} \mathcal{F}_{L\mu\nu}, \quad (4.94)$$

with

$$\mathcal{F}_R^{\mu\nu} = \partial^\mu X^\nu - \partial^\nu X^\mu,$$

to the Lagrangian Eq.(4.72). The bare boson propagators Δ_B are thus modified:

$$\Delta_M = \Delta_{M^\dagger} = \tilde{x}^2 \frac{1}{k^2}, \quad (4.95)$$

$$\Delta_R^{\mu\nu}(k) = \tilde{r}^2 \frac{1}{k^2} (\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}), \quad (4.96)$$

$$\Delta_L^{\mu\nu}(k) = \tilde{\ell}^2 \frac{1}{k^2} (\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}). \quad (4.97)$$

With these substitutions in Eq.(4.83), Eqs. (4.90, 4.91) become

$$A_\pm(y) = 1 \quad (4.98)$$

$$B_\pm(y) = \frac{\tilde{x}^2}{16\pi^2} \int_0^{\Lambda^2} dz \frac{x^2 B_\mp(z)}{D_\mp(z)} \{ \frac{z}{y} \theta_+ + \theta_- \} \quad (4.99)$$

$$D_\pm(z) = z + |B_\pm(z)|^2 \quad (4.100)$$

where, as before, $\theta_\pm = \theta(\pm(y - z))$ is the Heaviside step function and the squared momenta are $z = q^2$, $y = p^2$.

The solutions of the system Eqs.(4.99) are again real, $B_+ = B_- \equiv B$ and given by the equation of quenched rainbow QED in the Landau gauge with the 4-fermion interactions

$$\alpha \equiv \tilde{x}^2 \quad (4.101)$$

playing the role of the gauge coupling α . Analysis of the critical coupling may proceed in an almost identical fashion [18], for example converting Eq.(4.99) into a differential form

$$y \frac{d^2 B}{dy^2} + 2 \frac{dB}{dy} + \frac{\alpha}{16\pi^2} \frac{B}{y + |B|^2} = 0 \quad (4.102)$$

$$\lim_{y \rightarrow 0} \frac{d}{dy} (y^2 B) = 0, \quad (4.103)$$

$$\lim_{y \rightarrow \Lambda^2} \frac{d}{dy} (y B) = 0 \quad (4.104)$$

These equations are scale invariant for $y \gg |B|^2$ and have the solution

$$B_{<}(y) = y^{(-1+\sqrt{1-\frac{\alpha}{16\pi^3}})^2}; \alpha \leq 16\pi^2/3 \quad (4.105)$$

$$B_{>}(y) = y^{-0.5} e^{i(\sqrt{\frac{\alpha}{16\pi^3}-1} \ln y)/2}; \alpha \geq 16\pi^2/3 \quad (4.106)$$

Only the latter of these satisfies the ultraviolet boundary condition nontrivially:

$$B(0) = \Lambda e^{-\frac{\pi}{\alpha/16\pi^2-1}} \quad (4.107)$$

Therefore real positive solutions

$$B(y) = \text{Re}(B_{>\pm}(y)) \quad (4.108)$$

are obtained above the dimensionless critical coupling defined by (c.f. Eq.(4.93))

$$\alpha_c = x^2 = \frac{16\pi^2}{3}. \quad (4.109)$$

In the 4-fermi model therefore, one sees that in this approximation the couplings r, ℓ enter the expressions for the scalar form factors B_{\pm} on an equal footing and that the momentum integral of the kernel in Eq.(4.98) vanishes. This vanishing of the chiral dependency of form factors is a natural artifact of the choice of bare propagators and vertices. Thus while such an approximation is useful for study of mass generation there is no opportunity for CP-violation.

An improved vertex *ansatz*, leading to non-vanishing A_{\pm} contributions could be expected to remedy this problem. However study of the gauged 4-fermion model in the next chapter is also seen to lead to non-vanishing contributions of the required type.

Chapter 5

Quenched hypercharge

In this chapter we wish to investigate the question of criticality in the hypercharge theory. In contrast to QED there are three couplings and the behaviour of the two models could therefore reasonably be expected to differ.

In 5.1.1 for the (non-anomalous) quenched, rainbow approximation we find an indication of different dynamics for the left- and right- fermion chiralities in the DSE for the self-energy. Specifically the departure from the equalities $A_+ = A_-$, $B_+ = B_-$ in non-Landau gauges ($z \neq 0$) leads to the appearance of two poles in the fermion propagator, associated with two types of fermion condensate and the dynamical breaking of parity.

In addition to being badly gauge-dependent, these poles are degenerate in the Landau gauge. In section 5.1.2 therefore, following the successful Ball-Chiu [36] and Curtis-Pennington [91] approaches for QED we propose a (naively) multiplicatively-renormalisable hypercharge vertex. A bifurcation analysis of these equations suggests the existence of two critical curves, multiple intersections of which would signal a series of transitions between phases with different chiral symmetry breakings.

The second part of the chapter is concerned with incorporating composite scalars into the theory, preliminary to considering the full gauged 4-fermi model. In [2] it was suggested that at the lowest (right) critical scale the lightest generation “froze out” of the theory; fermions with right chirality formed a condensate $\bar{\psi}_L \psi_R$, consistent with the presence of two poles in the propagator. The left-handed vacuum was highly excited relative to its right counterpart and via gauge transformations the ABJ anomaly caused a flux of (dynamical) left fermions over into the condensed phase. It is for this reason that both chiralities are expected to feel the same scalar

potential. Moreover in the presence of global chiral symmetry breaking, inclusion of Goldstone bosons is necessary for construction of a unitary, low-energy effective theory. To this end we consider the contribution of the dynamically-generated composite Goldstones to the fermion self-energy and their mixing with the gauge boson via the anomalous correction to the vertex WTI are thus investigated in 5.2.1.

5.1 D χ SB in hypercharge $U(1)$

In the same way that approximate solutions to the QED SDEs led to early suggestions that non-perturbative QED was capable of dynamically generating a fermion mass, we commence with the analogous equations for the $U(1)$ hypercharge model. The relevant SDEs were derived in chapter 2. With a quenched boson propagator the main qualitative difference to QED is that the fermion gauge-boson coupling is a linear combination of vector and axial vector contributions; The bare (respectively right-and left-handed) fermion-boson vertices have the form

$$\Gamma_0^\mu = \gamma^\mu (c_+ \chi_+ + c_- \chi_-). \quad (5.1)$$

We shall adopt the same form of the inverse fermion propagator in Euclidean space, Eq.(4.78), as used in section 4.5

$$S^{-1}(p) = \chi_+ (iA_+(p)\gamma \cdot p + B_+(p))^{-1} + \chi_- (iA_-(p)\gamma \cdot p + B_-(p))^{-1}. \quad (5.2)$$

5.1.1 Rainbow approximation

From Eq.(4.78) we can write the fermionic self-energy as

$$\begin{aligned} \Sigma(p) &= \chi_+ \{i\gamma \cdot p (A_+(p) - 1) + (B_+(p) - m_+)\} \\ &\quad + \chi_- \{i\gamma \cdot p (A_-(p) - 1) + (B_-(p) - m_-)\} \\ &= \int \frac{d^4 q}{(2\pi)^4} D_{\mu\nu}(p-q) \Gamma^\nu(p, q) S(q) \Gamma_0^\mu. \end{aligned} \quad (5.3)$$

Approximating the full vertex $\Gamma^\nu(p, q)$ by Eq. (5.1) and the gauge boson propagator $D_{\mu\nu}$ by its quenched form

$$D_{\mu\nu}(k) = (\delta_{\mu\nu} + (G-1) \frac{k_\mu k_\nu}{k^2}) \frac{1}{k^2}, \quad (5.4)$$

one arrives at the analogue of the quenched rainbow approximation for the fermion self-energy. Tracing out the various projected components yields:

$$A_{\pm}(y) = 1 + \frac{Gc_{\mp}^2}{16\pi^2} \int_0^{\Lambda^2} dx \frac{A_{\pm}(x)}{D_{\pm}(x)} \left\{ \frac{x^2}{y^2} \theta_+ + \theta_- \right\}, \quad (5.5)$$

$$B_{\pm}(y) = m_{\pm} + \frac{(3+G)c_+c_-}{16\pi^2} \int_0^{\Lambda^2} dx \frac{B_{\mp}(x)}{D_{\mp}(x)} \left\{ \frac{x}{y} \theta_+ + \theta_- \right\},$$

$$D_{\pm}(x) = |A_{\pm}(x)|^2 x + |B_{\pm}(x)|^2, \quad (5.6)$$

where $x = q^2$, $y = p^2$ and $m_{\pm} = m_e \pm m_o$

In the Landau gauge $A_{\pm} \rightarrow 1$, $a \rightarrow 0$, the chiral phase vanishes and mass generation is obtained from the behaviour of the two coupled integral equations in B_{\pm} . In the limit $c_+ = c_-$ the system reduces to the integral equations of the bare-vertex 4 fermi approximation Eqs.(4.98,4.99). That is, the solutions are of the form $B_+(p) = B_-(p)$ and can be considered as equivalent to the QED scalar form factor. Converting Eq.(5.6) to differential form and repeating the latter analysis one then obtains the critical coupling

$$\frac{c_+c_-}{16\pi^2} = \frac{\pi}{3}. \quad (5.7)$$

That is, like quenched rainbow QED the hypercharge theory appears to have two phases separated by a transition associated with breakdown of chiral symmetry.

There are two important distinctions however: Eq.(5.7) now represents a curve (hyperbola) in 2-dimensional coupling constant space rather than a point on a line. Secondly the degeneracy $A_+ = A_-$ in Eq.(5.5) is lifted and consequently in Eq.(5.6), $B_+ \neq B_-$. Indeed solving Eqs.(5.5, 5.6) numerically by an iterative process confirms this, see for example Fig. 5.1. The appearance of two poles M_{\pm} in the fermion propagator Eq.(4.79) has a natural interpretation in terms of an extra mode of condensation [111]. Moreover the condensates were found to form above separate critical scales. Figure 5.2 shows the coupling-dependence of the fixed point equations,

$$M_{\pm}(p^2 = -M_{\pm}^2) = M_{\pm}, \quad (5.8)$$

analogous to the determination of the QED critical coupling [91]. In this example, for fixed c_+ , because $c_- > c_+$ the left-handed fermion acquires a gap first at $(c_-, c_+) \sim (4.44, 4)$ while the right chirality remains massless until the point $(c_-, c_+) \sim (6.605, 4)$. The behaviour of the chiralities is switched upon changing the values $c_+ \iff c_-$.

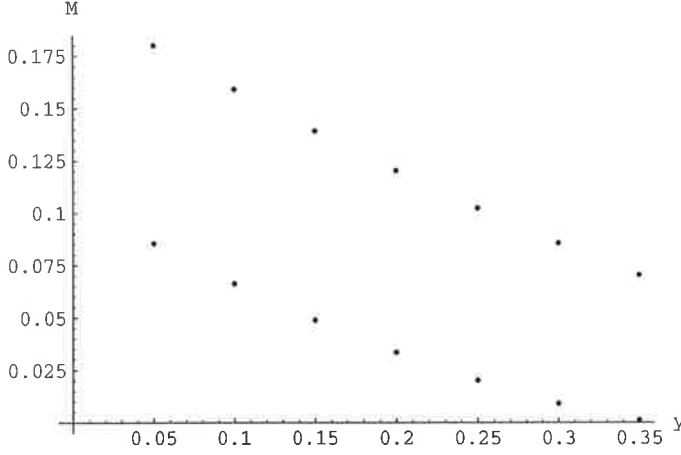


Figure 5.1: mass functions M_+ (upper) and M_- (lower) as a function of momentum-squared in the Feynman gauge. Coupling values used are $c_+ = 7$, $c_- = 5$.

This is precisely the behaviour anticipated [2] at the right-right and left-left critical coupling points, where right and left-handed condensates were expected to form.

It is clear that a physical (i.e. Hermitean, Lorentz invariant) fermion “mass” term requires

$$M_+^\dagger = M_- \quad (5.9)$$

and moreover, that unless $c_+ = c_-^*$, due to Eq. (5.5) this relation will not be satisfied. In [2] the mass term becomes possible due to the ABJ anomaly-driven collapse of the (dynamical) chiral sector with largest mass. In the case shown in Figure 5.2, above the left critical scale, there is a nett loss of gapped (left) fermions and gain of gapless right fermions, while the former remains as a resonant state. In this way both left and right fermions become coupled to the same condensates, allowing physical mass generation to occur.

Unlike QED, therefore, dynamical chiral symmetry breaking in hypercharge is a necessary but not sufficient condition for dynamical mass generation. The extra condition, namely the (anomalous) effective coupling of composite scalars with fermions will be considered in 5.2.

We conclude this subsection with the observation that that, as is the case for QED, only the choice of Landau gauge in the quenched, rainbow approximation satisfies the vertex WTI, required for gauge-invariant renor-

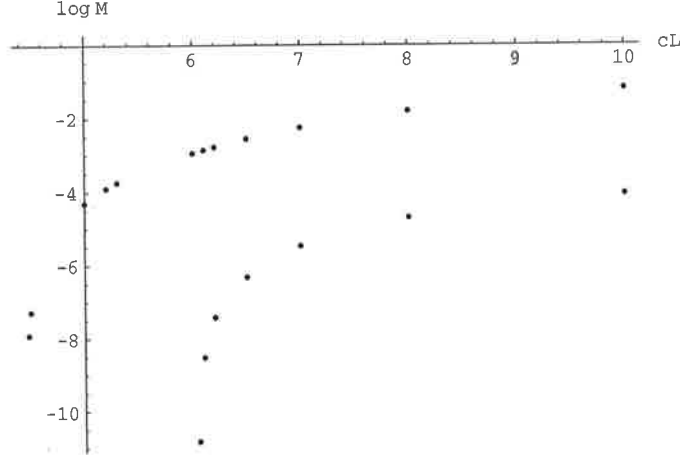


Figure 5.2: Fixed point of the mass functions M_- (upper) and M_+ (lower) as a function of the coupling c_- in the Feynman gauge. The right coupling is fixed to the value $c_+ = 4$.

malisability. Clearly any physically realistic truncation of the 3-point vertex Greens function DSE must also have this property. However as evidenced in Eqs.(5.6,5.7) the choice $z = 0$ is precisely where the degeneracy of the chiral condensates occurs, leading to qualitatively QED-like behaviour. It is vital therefore to move beyond rainbow approximation.

5.1.2 Renormalisable vertex

We now attempt to reduce the gauge-dependence of the dynamical mass by building a vertex respecting the appropriate WTI. Given the success of the method of Ball and Chiu [36] and improvement by Curtis and Pennington [91] for the dressed, multiplicatively-renormalisable QED vertex ansatz, it is logical to emulate this approach here.

We begin by considering the analogue of the Ball-Chiu ansatz [36], i.e. a vertex form which satisfies the 3-point Ward-Takahashi identity and reduces to the bare vertex in the perturbative limit. The WTI follows from the general vector and axial current identities

$$(q - p) \cdot \Gamma^V(q, p) = S^{-1}(q) - S^{-1}(p), \quad (5.10)$$

$$(q - p) \cdot \Gamma^A(q, p) = S^{-1}(q)\gamma^5 + \gamma^5 S^{-1}(p) - \bar{F}(q, p), \quad (5.11)$$

by forming the relevant linear combinations:

$$ik \cdot \Gamma^\pm(q, p) = c_\pm((S(q)\chi_\pm - \chi_\mp S(p)) \mp \bar{F}(q, p)) \quad (5.12)$$

$$= c_{\pm}(S(q) - S(p))\chi_{\pm} \pm c_{\pm}(\eta\chi_{\pm} - \bar{F}(q, p)) \quad (5.13)$$

where $k = q - p$. There are a number of immediate qualitative differences to QED. The first two terms of the right-hand side can be made to be singularity-free in the limit $k \rightarrow 0$, analogous to the QED identity Eq.(3.61). However, while the currents satisfying Eq.(5.13) are gauge invariant, they do not represent a true functional variation. As a result the two extra terms, the dynamical “Goldstone” term (c.f.[110])

$$\eta(p, q) = (B_+(q) - B_-(q)) \quad (5.14)$$

and a hard component $\bar{F}(q, p)$, related to the ABJ anomaly can give rise to kinematic singularities in the vertex. Initially we shall proceed by naively neglecting these corrections.

The remainder of the vertex function $\hat{\Gamma}^{\mu}(q, p)$ itself may be expanded in the same manner as that of QED:

$$\hat{\Gamma}^{\mu}(q, p) = \Gamma_0^{\mu}(q, p) + \sum_{r=5}^{12} \zeta_r(q, p) T_r^{\mu} \quad (5.15)$$

where T_r^{μ} are the Ball-Chiu [36] basis vectors. The longitudinal part of the vertex constrained by eq. Eq.(5.13) which is free of kinematic singularities is given by

$$\Gamma_L^{\mu}(q, p) = (\zeta_1 \gamma \cdot (q + p)(q + p)^{\mu} + \zeta_2 \gamma^{\mu} + i\zeta_3 (q + p)^{\mu}) (c_+ \chi_+ + c_- \chi_-) \quad (5.16)$$

with

$$\zeta_1(q, p) = \frac{1}{2} \frac{A_-(q) - A_-(p)}{q^2 - p^2} \chi_+ + \frac{1}{2} \frac{A_+(q) - A_+(p)}{q^2 - p^2} \chi_-, \quad (5.17)$$

$$\zeta_2(q, p) = \frac{1}{2} (A^-(q) + A^-(p)) \chi_+ + \frac{1}{2} (A^+(q) + A^+(p)) \chi_-, \quad (5.18)$$

$$\zeta_3(q, p) = \frac{B^+(q) - B^+(p)}{q^2 - p^2} \chi_+ + \frac{B^-(q) - B^-(p)}{q^2 - p^2} \chi_-. \quad (5.19)$$

Setting $\Gamma^{\mu}(q, p) = \Gamma_L^{\mu}(q, p)$ in Eq.(3.45) gives the longitudinal contribution to the self energy $\Sigma_L(p)$. Tracing out the various components yields

$$A_{\pm}^L(y) = 1 - \frac{c_{\pm}^2}{16\pi^2} \int_0^{\Lambda^2} dx \left\{ \left(\frac{x^2}{y^2} \theta_+ + \theta_- \right) \right. \\ \left. + \frac{3}{2(x-y)} \left(\frac{B_{\mp}(x)}{D_{\mp}(x)} (B_{\mp}(x) - B_{\mp}(y)) + \frac{x+y}{2} \frac{A_{\pm}}{D_{\pm}(x)} (x) (A_{\pm}(x) - A_{\pm}(y)) \right) \right\}$$

$$\begin{aligned}
& -G \frac{x^2}{y^2} \theta_+ \left(\frac{A_{\pm}(x)A_{\pm}(x)}{D_{\pm}(x)} + \frac{1}{x} \frac{B_{\mp}(x)}{D_{\mp}(x)} (B_{\mp}(x) - B_{\mp}(y)) \right. \\
& \left. - G \theta_- \frac{A_{\pm}(x)A_{\pm}(y)}{D_{\pm}(x)} \right), \tag{5.20}
\end{aligned}$$

$$\begin{aligned}
B_{\pm}^L(y) = & m_{\pm} + \frac{c_+ c_-}{16\pi^2} \int_0^{\Lambda^2} \frac{dx}{D_{\mp}(x)} \left\{ \left(\frac{x^2}{y^2} \theta_+ + \theta_- \right) \right. \\
& \frac{3x}{2(x-y)} (A_{\mp}(x)B_{\mp}(y) - B_{\mp}(y)A_{\mp}(x)) \\
& + \frac{3}{2} B_{\mp}(x) (A_{\mp}(x) + A_{\mp}(y)) \left(\frac{x}{y} \theta_+ + \theta_- \right) \\
& + \frac{x}{y} \theta_+ G B_{\mp}(x) A_{\mp}(y) \\
& \left. + \theta_- G B_{\mp}(x) A_{\mp}(y) \right\}, \tag{5.21}
\end{aligned}$$

where $x = q^2$ and $y = p^2$.

The form of the transverse part of the vertex, that is the term associated with the eight Ball-Chiu basis vectors, is constrained by a number of considerations. It is possible to derive a "transverse" Ward [95] identity of the form $\partial_{\mu}\Gamma_{\nu} - \partial_{\nu}\Gamma_{\mu}$ or to enforce physical requirements such as that of multiplicative renormalisability (MR) [91]. Finding closed forms for the transverse WTI is difficult for more than two dimensions, where it may be written exactly, while the MR requirement has been highly successful in quenched QED. We therefore choose to consider restrictions of the latter kind.

For the (massless) hypercharge theory, the renormalisation process involves rescaling the fields by

$$\psi_{\pm}^R = 1/\sqrt{Z_2^{\pm}} \psi_{\pm}, \quad Z_{\mu}^R = 1/\sqrt{Z_3} Z_{\mu},$$

and the couplings by

$$c_{\pm} = Z_2^{\pm} \sqrt{Z_3}/Z_1^{\pm} c_{\pm}^R.$$

where the superscript R denotes renormalised quantities. For the quenched approximation $Z_3 = 1$ and the (non-anomalous) WTI imposes $Z_1^{\pm} = Z_2^{\pm}$. The renormalised Lagrangian then reads

$$\begin{aligned}
\mathcal{L} = & Z_2^+ \bar{\psi}_+ \gamma \cdot (i\partial + c_+ Z) \psi_+ + Z_2^- \bar{\psi}_- \gamma \cdot (i\partial + c_- Z) \psi_- \\
& - F^R \cdot F^R - \frac{1}{2z} (\partial \cdot Z^R)^2, \tag{5.22}
\end{aligned}$$

from which the renormalised DSEs may be obtained, as in section 2.4 above. The renormalised chiral vector form functions must satisfy [91]

$$A_{\pm}^R(y; \mu_{\pm}) = Z_2^{\pm}(\Lambda, \mu_{\pm}) A_{\pm}(y; \Lambda) \quad (5.23)$$

where μ_{\pm} are the arbitrary renormalisation scales for the left- and right-sectors and for simplicity we assume they have the same UV cutoff.

Following [91], Eq.(5.23) must have the perturbative, leading logarithm expansion

$$A_{\pm}(y; \Lambda) = 1 + \sum_{i=1}^{\infty} \frac{1}{i!} \left(\frac{c_{\pm}^2}{4\pi} f \ln\left(\frac{y}{\Lambda^2}\right) \right)^i \quad (5.24)$$

where f is the coefficient associated with the 1-loop correction. The 1-loop vertex correction for $p \gg q$ is readily obtained as

$$\begin{aligned} \Gamma_{1\pm}^{\mu}(q+p, p) &= \frac{zc_{\pm}^2}{16\pi} (-\gamma^{\mu} - 2p^{\mu}\gamma \cdot q/p^2 + \frac{(q+p)^{\mu}}{q^2-p^2} \gamma \cdot (q+p)) \ln q^2/p^2 \\ &\simeq -\frac{zc_{\pm}^2}{16\pi^2} (\gamma^{\mu}(q^2-p^2) - (q+p)^{\mu}\gamma \cdot (q-p)). \end{aligned} \quad (5.25)$$

Substituting this vertex form into Eq.(4.83) yields the coefficient

$$f = \frac{z}{8\pi} \quad (5.26)$$

and thus the transverse vertex choice (c.f. [91])

$$\Gamma_{T\pm}^{\mu}(q, p) = \frac{A_{\pm}(q) - A_{\pm}(p)}{2} T_6^{\mu} \frac{1}{p^2} \quad (5.27)$$

guarantees renormalisability to 1 loop, where T_6^{μ} is the Ball-Chiu basis vector Eq.(3.76). This expression is obtained under the assumption $p \gg q$, thus removal of the apparent singularity in Eq.(5.27) can be justified by assuming the denominator is valid only to $O(q^2/p^2)$.

A form for this modified factor may be obtained upon consideration of the massive 1-loop fermion correction. In this case there are extra renormalised parameters

$$m_R^{\pm} = m_{\pm}/Z_m^{\pm} \quad (5.28)$$

appearing in the Lagrangian. Of course for the case of physical fermions, Eq.(5.9) must hold, in which case it is well known that the WTI Eq.(C.5) fails unless $Z_2^+ = Z_2^-$ and $c_+ = c_-$. However, as pointed out above, it is

the anomaly which catalyses mass generation, in which case gauge-invariant renormalisation is spoiled anyway. We shall thus focus here on reducing the gauge-dependence of the calculated quantities, with the understanding that the “mass” terms M_{\pm} obtained need not correspond to physical fermion masses.

The term associated with the “mass”, computed from the vertex diagram is for $q^2 \gg p^2 \gg m_{\pm}^2$ readily verified to be

$$\Gamma_{T2\pm}^{\mu}(q, p) = (3 + z) \frac{c_+ c_-}{4\pi} \frac{q^{\mu}}{q^2} m_{\mp} \ln \frac{q^2}{p^2}. \quad (5.29)$$

while perturbatively, the fermion self-energy to first order is

$$\Sigma_{\pm}(p) = m_{\pm} - \frac{3c_+ c_-}{16\pi^2} m_{\mp} \ln \frac{p^2}{\Lambda^2}. \quad (5.30)$$

It is apparent that substitution of Eq.(5.29) in Eq.(4.83) leads to the correct 1-loop self-energy result. The sum of vertex terms Eqs.(5.19, 5.27) are thus seen to reproduce the (naively) renormalised 1-loop self-energy in the leading log approximation. While, due to decoupling of the chiral sectors, the vertex for massless fermion corrections could be anticipated to be MR to all orders of the leading logarithm expansion as for QED [91] we have only verified it explicitly here at the one-loop level. This is for simplicity and because the interesting (i.e. massive fermion) case and, in section 2.2 the ABJ anomaly, in general spoils chiral-invariant renormalisability, therefore negating the motivation to proceed to higher orders analogous to [91].

All that remains to complete the vertex is to find an $O(q^2/p^2)$ modification for the denominator of Eq.(5.27) with the correct charge-conjugation property (i.e. symmetric under $q \longleftrightarrow p$).

We propose a generalised version of the function used in [91]

$$d_{\pm}(x, y) = \frac{(x - y)^2}{x + y} + \frac{((B_{\mp}(x)/A_{\mp}(x))^2 + (B_{\mp}(y)/A_{\mp}(y))^2)^2}{x + y}. \quad (5.31)$$

The kernel of the 1-loop renormalisable transverse vertex contribution is thus given by the contribution

$$\Gamma_0^{\mu} S(q) \left(\frac{A_+(q) - A_+(p)}{2d_+(q, p)} c_+ \chi_+ + \frac{A_-(q) - A_-(p)}{2d_-(q, p)} c_- \chi_- \right) T_6^{\nu} D_{\mu\nu}(q - p). \quad (5.32)$$

Computation of the angular integrals and tracing over the vector and scalar parts yields

$$A_{\pm}^T(y) = -\frac{3c_{\mp}^2}{16\pi^2} \int_0^{\Lambda} dx \frac{A_{\pm}(x)}{D_{\pm}(x)} (A_{\pm}(x) - A_{\pm}(y)) \frac{x^2 - y^2}{d_{\pm}(y, x)} \left(\frac{x^2}{y^2} \theta_- + \theta_+ \right),$$

(5.33)

$$B_{\pm}^T(y) = -\frac{3c_+c_-}{16\pi^2} \int_0^{\Lambda} dx \frac{B_{\mp}(x)}{D_{\mp}(x)} (A_{\mp}(x) - A_{\mp}(y)) \frac{x-y}{d_{\mp}(y,x)} \left(\frac{x}{y}\theta_- + \theta_+\right). \quad (5.34)$$

The full, 1-loop DSE equations are thus given by the combination of Eqs.(5.20, 5.33) and Eq.(5.21) with Eq.(5.34):

$$A_{\pm}(y) = A_{\pm}^L(y) + A_{\pm}^T(y), \quad (5.35)$$

$$B_{\pm}(y) = B_{\pm}^L(y) + B_{\pm}^T(y). \quad (5.36)$$

Following [85] we can undertake a bifurcation analysis of Eqs.(5.35, 5.36), the idea being that at the critical coupling their solution bifurcates into nontrivial terms. This is achieved by taking the functional derivative of the equations and enumerating the result at the trivial value $B_{\pm}(x) = 0$. It is clear that only the terms linear in B_{\pm} will survive this procedure. The behaviour of the pure gauge-interaction part of the theory is then obtained by proceeding along similar lines to bifurcation analyses of quenched QED [85], [86], [99]. Performing the functional derivative of Eqs.(5.35, 5.36) and evaluating them at the points $B_{\pm}(y) = 0$, the simplified equations then read, to $O(B_{\pm}^2)$,

$$A_{\pm}(y) = 1 - \frac{Gc_{\mp}^2}{16\pi^2} \int_0^{\Lambda^2} dx \left(\frac{x}{y^2} \theta_+ + A_{\pm}(y) \frac{1}{A_{\pm}(x)x} \theta_- \right), \quad (5.37)$$

$$\begin{aligned} B_{\pm}(y) = & \frac{c_+c_-}{16\pi^2} \int_0^{\Lambda^2} \frac{dx}{x} \left(\frac{3B_{\mp}(x)}{2A_{\mp}(x)} \left(1 + \frac{A_{\mp}(y)}{A_{\mp}(x)} + \frac{1+x}{1-x} \left(1 - \frac{A_{\mp}(y)}{A_{\mp}(x)}\right) \left(\frac{x}{y}\theta_+ + \theta_-\right) \right. \right. \\ & - \frac{3y}{2(x-y)} \left(\frac{A_{\mp}(y)}{A_{\mp}(x)} \left(\frac{B_{\mp}(x)}{A_{\mp}(x)} - \frac{B_{\mp}(x)}{A_{\mp}(x)} \right) \left(\frac{x^2}{y^2} \theta_+ + \theta_- \right) \right. \\ & \left. \left. + G \frac{B_{\mp}(x)}{A_{\mp}(x)} \frac{A_{\mp}(y)}{A_{\mp}(x)} \theta_+ + G \frac{B_{\mp}(y)}{A_{\mp}(y)} \frac{A_{\mp}(y)}{A_{\mp}(x)} \theta_- \right) \right). \end{aligned} \quad (5.38)$$

The first integral in Eq.(5.37) is readily computed leaving

$$A_{\pm}(y) = 1 - \frac{Gc_{\mp}^2}{16\pi^2} \left(\frac{1}{2} + A_{\pm}(y) \int_0^{\Lambda^2} dx \frac{1}{xA_{\pm}(x)} \right). \quad (5.39)$$

These are of the same form as the QED equation [99], and upon converting to differential form

$$\left(1 - \frac{Gc_{\mp}^2}{32\pi^2}\right) \frac{d}{dy} \left(\frac{1}{A_{\pm}(y)} \right) - \frac{1}{yA_{\pm}(y)} = 0,$$

are readily seen to have the unique solutions

$$A_{\pm}(y) = \left(1 + \frac{c_{\mp}^2 z}{32\pi^2}\right) \left(\frac{y}{\Lambda^2}\right)^{-\mu_{\pm}} \quad (5.40)$$

with the exponent

$$\mu_{\pm} = 2z / \left(\frac{32\pi^2}{c_{\mp}^2} + z\right). \quad (5.41)$$

Substituting Eq.(5.40) now into Eq.(5.38) one obtains the equation (c.f. [99])

$$\begin{aligned} \frac{B_{\pm}(y)}{A_{\pm}(y)} &= \frac{c_+ c_-}{\mu_{\pm} c_{\mp}^2} \left(\frac{y}{\Lambda^2}\right)^{\mu_{\mp}} \int_0^{\Lambda^2} dx \left(\frac{3B_{\mp}(x)}{2A_{\mp}(x)} \left(1 + \left(\frac{x}{y}\right)^{\mu_{\mp}}\right) \right. \\ &\quad + \frac{y+x}{y-x} \left(1 - \left(\frac{x}{y}\right)^{\mu_{\mp}}\right) \left(\frac{x}{y}\theta_+ + \theta_-\right) \\ &\quad + \frac{3y}{2(x-y)} \left(\frac{x}{y}\right)^{\mu_{\mp}} \left(\frac{B_{\mp}(x)}{A_{\mp}(x)} - \frac{B_{\mp}(y)}{A_{\mp}(y)}\right) \left(\frac{x^2}{y^2}\theta_+ + \theta_-\right) \\ &\quad \left. + \left(\frac{x}{y}\right)^{\mu_{\mp}+1} \frac{B_{\mp}(x)}{A_{\mp}(x)} \left(\frac{x}{y}\theta_+ + \theta_-\right) \right). \end{aligned} \quad (5.42)$$

Upon taking the limit $\Lambda^2 \rightarrow \infty$ we look for scale-invariant solutions to the equations (5.42), signalling the presence of a UV fixed point. The obvious candidate is

$$M_{\pm}(x) \equiv \frac{B_{\pm}(x)}{A_{\pm}(x)} = x^{s_{\pm}}, \quad (5.43)$$

where, due to the coupling of chirality in Eqs.(5.36)

$$x^{s_{\pm}} = \kappa_{\pm} x^{s_{\mp}} = \kappa_+ \kappa_- x^{s_{\pm}}. \quad (5.44)$$

That is, the exponents s_{\pm} must now be roots of (c.f. the QED case Eq.(3.87))

$$1 - \kappa_+ \kappa_- = 0 \quad (5.45)$$

where

$$\begin{aligned} \kappa_{\pm} &= \frac{3c_{\pm}}{2c_{\mp}z} \frac{\mu_{\mp}(\mu_{\mp} - s_{\mp} + 1)}{1 - s_{\mp}} \left(3\pi \cot \pi(\mu_{\mp} - s_{\mp}) - \pi \cot \pi\mu_{\mp} + 2\pi \cot \pi s_{\mp} \right. \\ &\quad \left. + \frac{1}{\mu_{\mp}} + \frac{2}{1 - s_{\mp}} + \frac{3}{s_{\mp} - \mu_{\mp}} + \frac{1}{\mu_{\mp} + 1} + \frac{1}{s_{\mp} - \mu_{\mp} - 1} \right) \end{aligned} \quad (5.46)$$

and convergence of the integrals in Eq.(5.42) is conditional upon $0 \leq s_{\pm} \leq 2$.

Numerical solution of Eq.(5.45) reveals the number of roots in the interval $0 \leq s^\pm \leq 2$ to be gauge-dependent. In the Landau gauge there are two such roots and therefore in a general gauge we wish to consider only the pair of roots which is continuously connected to this pair via the change of gauge parameter. Criticality corresponds to a choice of coupling constants c_\pm for which the two roots become equal. The results of such a computation in the Feynman gauge are shown in Table 5.1.2. The first observation to be

c_\pm	c_\mp	ν_\pm	ν_\mp	s_\pm	s_\mp	γ_m^\pm	γ_m^\mp
2.000	5.802	0.025	0.192	0.462	0.466	1.076	1.068
3.000	3.867	0.055	0.090	0.467	0.466	1.066	1.068
3.407	3.407	0.071	0.071	0.466	0.466	1.068	1.068
4.000	2.901	0.096	0.052	0.456	0.468	1.088	1.065
5.000	2.324	0.147	0.034	0.466	0.462	1.068	1.076

Table 5.1: critical coupling values and scaling exponents obtained for Feynman gauge, $z = 1$.

made about these results is that the critical curve is again a parabola in the two-dimensional coupling constant space (c_-, c_+) as shown in Fig. 5.3. In the vector limit the critical coupling is $c_+ = c_- \simeq 3.4065$ corresponding to a value $\alpha_C \simeq 0.9234$, identical with the value obtained for Feynman gauge QED [99]. In fact from Table 5.1.2 the equation of the critical parabola is given by

$$c_+ c_- \simeq 11.61 \simeq 0.9234 \times 4\pi. \quad (5.47)$$

There is an approximate symmetry between couplings $c_+ \iff c_-$ and the solutions $A_+ \iff A_-$, $B_+ \iff B_-$ which may also be pertinent for models of mirror matter. It is seen that at the critical point, both mass terms M_\pm have a large anomalous dimension

$$\gamma_m^\pm = 2(1 - s^\pm) \quad (5.48)$$

and become marginal operators. Over the coupling range considered, the mismatch between the anomalous dimensions of these operators is typically less than 3%. Therefore in the neighbourhood of the fixed point, the gaps due to each chiral condensate are approximately (i.e to $O(M_\pm^2)$) degenerate.

It is also found for a large range of values for z , the scaling exponents and critical couplings are quite robust, see for example the values obtained in Table 5.1.2.

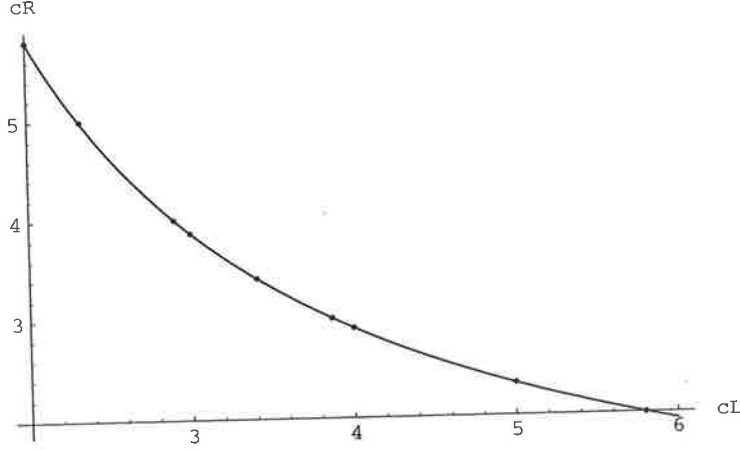


Figure 5.3: Critical curve Eq.(5.47) (solid) and numerical points obtained in Feynman gauge.

z	c_{\pm}	c_{\mp}	s_{\pm}	s_{\mp}
1	3.407	3.407	0.466	0.466
2	3.402	3.402	0.462	0.452
5	3.445	3.445	0.455	0.444
10	3.589	3.589	0.429	0.427

Table 5.2: gauge dependence of critical coupling values and scaling exponents

In this study, motivated by [36], [91], we have proposed a vertex with longitudinal part satisfying the non-anomalous WTI and the transverse part constrained to be (naively, the Goldstone-fermion coupling has been ignored) renormalisable to 1-loop. In addition to relatively gauge-insensitive results it was found that *both* left and right fermion pairings acquired large anomalous dimensions in the neighborhood of the phase transition with approximately equal exponents.

5.2 Role of composite scalars

We conclude this section with another correction which needs to be noted. The quenched, rainbow approximation is meant to use the perturbative boson propagator and vertex function, however as Gribov [31] and others have

pointed out, in order to construct a low-energy unitary perturbation theory in the presence of dynamically broken global chiral symmetry, the gauge boson-Goldstone mixing needs to be included. For the quenched rainbow hypercharge approximation this amounts to including diagrams where internal gauge boson lines are replaced by Goldstone propagators, with the bare vertex

$$\Gamma^\mu(q, p)_\eta = (c_+\chi_+ - c_-\chi_-)\eta(q, p)\frac{(q-p)^\mu}{(q-p)^2}.$$

The effective Goldstone-fermion coupling, required to balance the left- and right-hand sides of Eq.(5.13), is given by (c.f [110])

$$\eta(q, p) = B^+(q) - B^-(q), \quad (5.49)$$

where the inverted commas refer to the fact that, although Eq.(5.49) has odd parity and plays the role of a Goldstone, balancing the left and right-sides of the WTI, it is a chiral object. The fermion self-energy Eq.(5.3) has now two contributions, the first given by Eqs(5.35, 5.36) above, while the second corresponds to the "Goldstone" loop. The kernel of Eq.(4.83) contains the extra term

$$\frac{\delta_{\mu\nu}}{k^2}(c_+\chi_+ - c_-\chi_-)\eta(p, p+k)\frac{k^\mu}{k^2}S(k+p)(c_+\chi_+ - c_-\chi_-)\eta(k+p, p)\frac{k^\nu}{k^2}\gamma^5.$$

which gives the form factors

$$A'_\pm(y) = \frac{c_+c_-}{16\pi^2} \int_0^{\Lambda^2} \frac{dx}{D_\mp(x)} \left(A_\mp\eta(x, y)\eta(y, x)\frac{1}{x-y}\left(\frac{x^2}{y^2}\theta_+ - \theta_-\right) \right), \quad (5.50)$$

$$B'_\pm(y) = \frac{c_\mp^2}{16\pi^2} \int_0^{\Lambda^2} \frac{dx}{D_\pm(x)} \left(B_\pm(x)\eta(x, y)\eta(y, x)\frac{1}{x-y}\left(\frac{x}{y}\theta_+ - \theta_-\right) \right). \quad (5.51)$$

That is we may write the full expressions as

$$A_\pm(p) = A_\pm(p, c_\mp^2) + A'_\pm(p, c_+c_-), \quad (5.52)$$

$$B_\pm(p) = B_\pm(p, c_+c_-) + B'_\pm(p, c_\pm^2). \quad (5.53)$$

At this stage inspection of Eq.(5.53) reveals the possibility for generation of a fermion mass satisfying Eq.(5.9). While the fermion self-energy for B_\pm due to gauge terms (e.g. Eq.(5.36)) contained only terms with B_\mp in the kernel,

the Goldstone contributions are seen to be B_{\pm} . The equations for B_{\pm} no longer depend solely on B_{\mp} , condensation of one chirality (e.g. right if the SM couplings are used, as in [2]) is now partially offset by formation of a condensate of the other at the same scale, similar to the anomaly-catalysed collapse of the left-fermion sector [2].

An opportunity for obtaining three physical masses at three separate scales now exists: At small momentum scales with Eq.(5.53) in differential form there would be no nontrivial real solutions compatible with the UV boundary condition, analogous to quenched QED. When the critical right-right coupling is reached, the "Goldstone" term B'_+ admits a nonzero solution. Due to the coupling of the B_{\pm} equations both form factors acquire a non trivial real solution, hence a conventional "mass", that is one involving both left and right-handed condensates could be generated. The remaining two terms, (B_{\pm}^L and B'_-) would be "switched on" at the right-left and left-left critical points respectively, leading to three distinct mass terms. The two key steps, the appearance of three separate contributions to the equations B_{\pm} and the mixing of these equations, required to generate mass terms are made possible here by the Goldstone-gauge boson mixing. With "pure" gauge-boson contributions the equations for B_{\pm} only contain B_{\mp} terms in the kernel and always couple $\sim c_+c_-$.

5.2.1 Anomalous vertex

The last piece of the vertex WTI to be considered is the ABJ anomaly, the motivation being to see whether fermion mass generation can be dynamically generated in conjunction with the chiral symmetry breaking described previously in 5.1.2. In the proposal of [2] the mismatch between the chiral sectors (i.e right-condensed vs left-dynamical fermions) was suggested to be removed via the ABJ anomaly. In the model under consideration the discrepancy is manifested as the breakdown of Eq. (5.9), specifically $B_+(y) \neq B_-(y)$. The anomalous term \bar{F} omitted from the identity (Eq.5.13) is given by [37]

$$\bar{F}(q,p) = (c_+ - c_-) \frac{i(c_+^2 - c_-^2)}{4\pi^2} \int d^4x d^4y e^{i(q \cdot y - p \cdot x)} \langle \psi(x) \bar{\psi}(y) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}(0) \rangle. \quad (5.54)$$

We shall approximate this here by obtaining an effective anomalous vertex from integrating over the pseudoscalar-2 boson anomalous diagram shown in Fig. 5.4. Using a momentum-cutoff regulation and with soft external momenta a possible form for the 3-boson contact term is see [112] in Eq.(5.54)

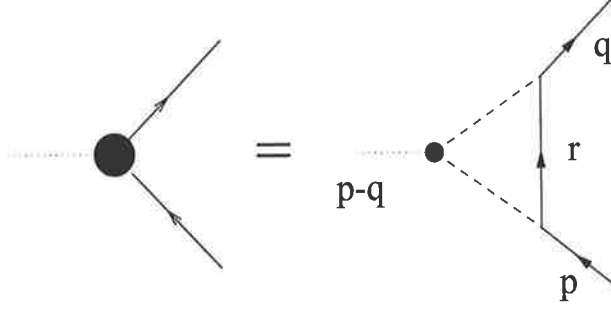


Figure 5.4: 1-loop anomalous vertex

is

$$\langle -(k+l)Z_\alpha(k)Z_\beta(l)\frac{g}{m}\phi \rangle \simeq \frac{i}{16\pi^2}\frac{g}{m}\epsilon_{\alpha\beta\mu\nu}k^\mu l^\nu \quad (5.55)$$

where ϕ is a massless composite pseudoscalar, m the fermion mass and g the Yukawa coupling. Inserting this expression into the loop diagram Fig. 5.4 one finds the effective vertex

$$\Gamma_\pm^F(q, p) = \mp i c_\pm (c_+^2 - c_-^2) \frac{c_+ + c_-}{64\pi^3} \epsilon_{\alpha\beta\mu\nu} p^\mu q^\nu \gamma^\alpha \gamma^\beta H(p, q; m) \chi_\pm \quad (5.56)$$

where H is a finite scalar function to be constrained by the physical properties of the anomaly: the anomalous vertex function must have the opposite CP properties to the gauge-boson vertex

$$\Gamma_\pm^F(p, q) = \Gamma_\pm^F(q, p) \quad (5.57)$$

i.e. H must be symmetric in p and q and it must not vanish for massless fermions. With the term

$$\frac{\delta^{\mu\nu}}{k^2} \Gamma^F(p, k+p) \frac{k^\mu}{k^2} S(q) \Gamma^F(k+p, p) \frac{k^\nu}{k^2} \quad (5.58)$$

added to the kernel of Eq.(3.45), computation of these contributions gives

$$A_\pm^F(y) = -\frac{\alpha_F^\pm}{8\pi} \int_0^{\Lambda^2} \frac{dx}{D(x)} A_\mp(x) H(x, y) H(y, x) \frac{xy}{x-y} (-\theta_- + \frac{x^2}{y^2} \theta_+), \quad (5.59)$$

$$B_\pm^F(y) = \frac{\alpha_F^\pm}{8\pi} \int_0^{\Lambda^2} \frac{dx}{D(x)} B_\mp(x) H(x, y) H(y, x) \frac{xy}{x-y} (-\theta_- + \frac{x}{y} \theta_+), \quad (5.60)$$

where we have defined the anomalous coupling

$$\alpha_F^\pm = \frac{1}{4\pi} (c^\pm (c_+^2 - c_-^2) \frac{c_+ c_-}{64\pi^3})^2. \quad (5.61)$$

Note that for the naive estimate H these contributions would diverge quadratically. In order to obtain the correct power counting behaviour it is clear that H must have mass dimension -1 . In fact in deriving the vertex function (5.56) from Eq.(5.55), such a factor, g/m has been effectively absorbed into H . A convenient form consistent with the above requirements and the momentum-squared rescaling $x \rightarrow \Lambda^2 x$ is then just

$$H(p, q) = \frac{1}{\Lambda} \left(\frac{A_+(p) + A_+(q)}{2} \chi_+ + \frac{A_-(p) + A_-(q)}{2} \chi_- \right). \quad (5.62)$$

The form factor equations to be solved are now given by Eqs.(5.52,5.53, 5.59, 5.60)

$$A_\pm(y) = A_\pm^L(y) + A_\pm^G(y) + A_\pm^F(y) \quad (5.63)$$

$$B_\pm(y) = B_\pm^L(y) + B_\pm^G(y) + B_\pm^F(y). \quad (5.64)$$

For simplicity we numerically investigate Goldstone and anomaly modifications to the rainbow approximation Eqs.(5.5,5.6). Upon performing the momentum scaling $x \rightarrow \Lambda^2 x$, these equations are solved for A_\pm and $B_\pm(y)/\Lambda$ as before. The system of equations (5.63, 5.64) is highly sensitive to the value of α_X^\pm : the anomalous terms are either negligible, in which case the above-described 5.1.1 behaviour occurs, or rapidly come to dominate the iterative solution process, in which case $B_\pm(y) \rightarrow 0$. Thus as expected, the anomaly removes the discrepancy between the chiral sectors, viz the restoration of Eq.(5.9), however unfortunately not in the manner envisaged in [2]: the dynamical mass vanishes altogether!

Naturally a number of objections may be made to the above approximation. Firstly Eq.(5.55) is only strictly valid for "soft" photon legs. Once included in the loop shown in Fig. 5.4, large- q corrections in the fermion self energy would rapidly become dominant. Secondly if gauge-boson-Goldstone mixing is to be incorporated, it is necessary to move beyond the quenched approximation, in which case $Z_3 \neq 1$ and there are several diagrams (perturbatively, at least) more significant than the anomalous correction to the fermion self-energy.

Indeed, if an anomaly-matching condition, such as the sum over all fermions within a SM generation, is imposed the above discussion is rendered academic and the desirable features of the non-anomalous approximations

may be retained. Nonetheless, it represents the first, to our knowledge, attempt to quantitatively understand a possible role for the ABJ anomaly in the $D\chi$ SB context. Such considerations are clearly necessary to evaluate whether the anomaly-catalysed vacuum decay proposed in [2] can form the foundation for self-consistent introduction of three fermion generations.

Finally we note that the Goldstone terms are at least quadratic in the form factors B_{\pm} , therefore they will not contribute to the linearised bifurcation analysis in 5.1.2. Moreover the chosen form of the anomalous vertex, being independent of B_{\pm} also will not affect the lowest order result.

Chapter 6

Conclusion and outlook

In this thesis we have introduced two models in order to investigate a proposal [2] that dynamical chiral symmetry breaking in the hypercharge sector of the SM can give rise not only to fermion masses but also the three-generation structure and attendant CP violation.

6.0.2 Chiral-breaking model

The first theory, a toy fermion-pairing model, was demonstrated in chapter 4 to exhibit the qualitative features of the SM which remain poorly understood and therefore are desirable for the proposal [2]. The key feature was the observation that when both chiral-symmetric and -breaking scalar 4-fermi operators are permitted

$$2\tilde{x}\bar{\psi}_L\psi_R\bar{\psi}_R\psi_L + \tilde{r}(\bar{\psi}_L\psi_R)^2 - \tilde{r}^*(\bar{\psi}_R\psi_L)^2,$$

the form of the mean-field approximation differs depending on whether one of, or both, such types of term are present. In a 1-loop renormalisation group study of the running couplings this meant that there were three distinct renormalised flows depending upon whether \tilde{x} or \tilde{r} were non-vanishing. The low-energy effective 4-fermi operators were thus found, in terms of the dimensionless couplings $x = \Lambda_\chi\tilde{x}$, $r = \Lambda_\chi\tilde{r}$ to be

$$O_1 = (\bar{\psi}_L\psi_R)^2 + (\bar{\psi}_R\psi_L)^2; \quad x = 0, r - r^* = 0 \quad (6.1)$$

$$O_2 = \bar{\psi}_L\psi_R\bar{\psi}_R\psi_L; \quad r = 0 = r^*, x \neq 0 \quad (6.2)$$

$$O_3 = (\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)^2; \quad x + r + r^* = 0 \quad (6.3)$$

In each of these three cases a two-phase structure was suggested via solution of the analogues of the NJL “gap” equations, with the critical coupling

strengths given by

$$r > 4\pi \quad (6.4)$$

$$x > 4\pi \quad (6.5)$$

$$\ell \equiv \frac{x}{1 + \frac{rr^*}{x^2}} \frac{1}{1 - \text{Re}(r)(1 + rr^*/x^2)} > 4\pi \quad (6.6)$$

for the operators O_{1-3} respectively. In particular it was found that, while $O_{1,2}$ had trivial IR limits, the couplings x, r flowed to finite IR values for O_3 , making it a suitable “high energy” effective theory. In all cases a fixed point in the ultraviolet region was also found.

The next step was to introduce a “fundamental” fermion with three components which, while identical with respect to gauge interactions, participated in either symmetric, breaking or both types of fermion pairing, thereby living in distinct subspaces corresponding to the 4-fermi operators O_{1-3} . As each of the couplings x, r, ℓ attains the critical strength at separate scales the theory has three distinct phase transitions, each associated with dynamical mass generation for a component of the fundamental fermion.

This model was found to be similar to the 3-Higgs model of Kiselev[21] where the fermion families arise as a consequence of a Z_3 -symmetric vacuum. The latter [21] has been found to give good tree-level estimates of the CKM matrix elements.

The main qualitative difference is that here the auxiliary fields, which play the roles of the Higgs VEVs, are in general complex, chiral objects. Therefore in order to be left with the single CP-violating internal rotational uncertainty of [21] one requires the existence of a discrete $Z_3 \otimes Z_2$ symmetry which may, as pointed out in 6.1 be interesting in terms of left-right-symmetric matter-only models. Otherwise the model contains three extra (2 complex, 1 chiral) CP-violating phases.

While the proposed model was shown to have the required qualitative features several important issues remain to be resolved. Firstly, while the presence of chiral condensates is anticipated in the proposal [2] it is not immediately clear how such terms can be derived by integrating out the gauge-bosons from the hypercharge Lagrangian:

$$\mathcal{L}_{eff} \sim (c_L \psi_L \gamma^\mu \psi_L + c_R \psi_R \gamma^\mu \psi_R) \Delta_{\mu\nu} (c_L \psi_L \gamma^\nu \psi_L + c_R \psi_R \gamma^\nu \psi_R) \quad (6.7)$$

One possibility is to include the vacuum polarisation in the boson propagator, $\Delta_{\mu\nu}$, where mismatches between the right-left and left-right cross terms prevent the cancellation of parity-odd scalar terms in the subsequent Fierz re-ordering.

Secondly the question of vector 4-fermi interactions has been ignored. It has been demonstrated that for quenched QED [106], such terms remain irrelevant, even close to the fixed point. Here this is correlated with the fact that the 1-loop RGE equation for the coupling g_V does not contain a quadratic term $\sim g_V^2$. Similarly the corresponding terms are seen to be missing from the hypercharge equations for r^2 , ℓ^2 thus one can tentatively anticipate similar behaviour. While chiral symmetry guarantees the QED Wilson potential has next-to-leading order terms of degree eight in the fermion fields, in this model 6-fermi terms cannot be discounted. It is important to investigate the possible effect of these higher-order terms.

Finally while the explicit-breaking terms will break chiral symmetry at any scale the mean-field approximation suggests they only impart a fermion mass above a critical coupling strength. This result also requires an explanation, which may be outlined as follows. From high-temperature superconductivity the mean-field approximation is known to be deficient. In particular there exists an intermediate phase where violent phase fluctuations mean that while fermion pairing occurs, condensate formation is inhibited. Recently such a “pseudogap” has been demonstrated in the Gross-Neveu model [108] and is conjectured to exist in other relativistic fermionic theories.

6.0.3 Quenched hypercharge

The second model considered in this thesis is the hypercharge gauge interaction itself. Analysis of dynamical chiral symmetry breaking in quenched QED has an extensive literature. In particular we proceed by generalising the Schwinger-Dyson equation for the fermion self-energy.

In the simplest approximation with the quenched boson propagator and bare vertex it is found that for an arbitrary covariant gauge separate “gaps” for the left- and right-chirality fermions appeared, evidence for the chiral condensates considered in the 4-fermi model of chapter 4. Moreover they were found to arise at separate scales, as shown in Figure 5.2, indicating a multiple-phase structure.

The exception to this behaviour was the Landau gauge, whereupon the gaps and critical scales became degenerate. We therefore attempted to generalise the successful procedure [36], [91] developed for construction of a QED vertex with greatly-reduced sensitivity to gauge choice.

The breakdown of the chiral Ward-Takahashi identity Eq.(5.13) due to the ABJ anomaly and dynamically-generated Goldstone bosons meant that the procedure was not completely successful. In particular we constructed a vertex with longitudinal part satisfying the (non-anomalous) WTI Eq.(5.13)

and transverse term ensuring multiplicative renormalisability of the fermion SDE Eq.(4.83) to one loop. Diagrams due to the Goldstones and their anomalous mixing with the gauge-boson vertex were of at least quadratic order in the scalar form factors B_{\pm} in Eq.(2.2) and therefore the proposed vertex was found to be sufficient for a linearised bifurcation analysis, similar to that undertaken for QED [99]. Here it was found that the separation of chiral condensates persisted in all gauges, however with a difference in scaling exponents of $\sim 3\%$ the effect was greatly reduced. In particular in the neighbourhood of a critical coupling *both* mass terms acquired large anomalous dimensions, suggesting the splitting between the left- and right- gaps is significantly less than that obtained in rainbow approximation. These results were shown to be robust for a large range of covariant gauge parameter values.

Finally the effect of the composite scalars was included in the quenched rainbow approximation to investigate the effect upon the dynamical chiral symmetry breaking: the Goldstones are necessary to construct a unitary low-energy effective perturbation theory, while it was hypothesised that the anomaly enables the generation of physical fermion mass, by coupling left- (right-) fermions with the right- (left-) condensates. Inclusion of the bare Goldstone propagator and effective bare scalar-fermion couplings led to the appearance of mass terms depending upon all three couplings Eqs.(5.52, 5.53). Unfortunately the *ansatz* for the anomalous vertex term was found to dominate the fermion SDEs (5.63, 5.64) causing vanishing of all mass functions. There are numerous reasons for rejecting the *ansatz* Eq.(5.62), discussed in section 2.2. In particular a correct implementation would involve unquenching the boson propagator.

In summary we have shown in sections 5.1.1, 5.1.2 that a fermion propagating in a background of hypercharge gauge fields appears to couple differently to chiral scalar condensates, which become critical at different scales. These are necessary prerequisites for the dynamical generations hypothesis [2], however its viability depends upon whether an improved treatment of the anomaly can translate into physical fermion mass, with subsequent family structure and mixing behaviour.

6.1 Outlook

Until this stage, an appealing feature of the model is that no higher physics is required to qualitatively produce the SM features outlined in [2]. Once this requirement is lifted, one can speculate that the non-perturbative effects embodied in this model facilitate a range of new theories with the Standard Model as a low energy limit. The fields of condensed matter and quantum field theory have a long history of cross-pollination. Recently Volovik [113] has written extensively on similarities between the electroweak sector of the SM and the superfluid ^3HeB and it seems logical to seek QFT analogues of recent developments in condensed matter. Phenomena such as chiral surface states in unconventional superconductors Sr_2RuO_4 [114],[115] and $\text{Bi}_{1-x}\text{Ca}_x\text{MnO}_3$ [116], chiral superfluidity in ^3HeA [113], [117] or spin-charge separation (SCS) [118],[119] may well have relevance to the non-perturbative sector of the Standard Model.

We begin by noting the 4-fermi terms of Eq.(4.6), obtained upon integrating the hypercharge boson from the $U(1)$ sector of the SM, resemble a higher-dimensional analogue of the Luttinger liquid Lagrangian [119],[120],

$$\mathcal{L}_{Lut} = \bar{\psi}\partial\psi + g_2\bar{\psi}_L\lambda^\alpha\psi_R\lambda^\alpha\gamma^\mu\psi_R + g_4((\bar{\psi}_L\lambda^\alpha\gamma^\mu\psi_L)^2 + (\bar{\psi}_R\lambda^\alpha\gamma^\mu\psi_R)^2), \quad (6.8)$$

if one identifies the terms $\bar{\psi}_{L,R}\gamma^\mu\psi_{L,R}$ with left- and right-moving charge density operators, which in the Luttinger liquid are of the form $\psi_{L,R}\psi_{L,R}$, i.e. fermion-fermion condensates.

This model has, in 1+1 dimensions a number of remarkable features: the cross-term g_2 modifies the pole structure of the fermion propagator (indeed the Fierz-ordering of this term in 4 dimensions leads to the scalar term breaking chiral symmetry). The g_4 term lifts any residual degeneracies, similar to a hopping matrix element between spin chains [119], and leads in 1+1 dimensions to SCS. Here, the effect is signalled by the appearance of two poles in the fermion propagator. An attempt to inject a free fermion into the first unoccupied energy level above the Fermi surface causes in a hole excitation. The resulting hole-electron pair decomposes into spin and charge fluctuations which propagate through the medium with different velocities.

This, too reminds of the problem encountered in the SDE analysis of the previous chapter with the appearance of new poles related to each condensate. We then see a possible interpretation of this feature is for some kind of recombination of fermionic degrees of freedom. A qualitative argument, based on the Dirac sea picture of the anomaly [40] runs as follows. The two-dimensional (for simplicity) hypercharge theory has Lagrangian:

$$\mathcal{L}_{2D} = \bar{\psi}\gamma^\mu(i\partial_\mu - (c_L\chi_L + c_R\chi_R)Z_\mu)\psi \quad (6.9)$$

In the Dirac sea picture, second quantisation corresponds to filling all negative energy eigenmodes while leaving positive ones empty. Setting $Z_0 = 0$ and the potential $Z_1 = Z$ a space-time constant, the eigenmodes satisfy the 2-D Dirac equation

$$E = -\gamma^\mu(p_\mu - (c_L\chi_L - c_R\chi_R)Z_\mu) \quad (6.10)$$

For $Z = 0$ the energy-momentum dispersion relation

$$E = \pm p$$

is shown in the upper left of Figure 6.1. The left- and right-hand branches correspond to the separate fermion chiralities. If Z is adiabatically changed to a small (positive) value the relation is that of the upper right diagram,

$$E = \begin{cases} p + c_R\delta Z, \\ -(p + c_L\delta Z), \end{cases}$$

where we have assumed both c_L and c_R are positive. Gauge transformations in this case cause a nett production of right antiparticles and left particles. While the total number of states is conserved, the separate left and right numbers are not. Introducing now a mass term $m\bar{\psi}\psi$ to the Lagrangian Eq.(6.9) would lead to the “gapped” lower left diagram in Fig 6.1.

$$E = \pm \begin{cases} \sqrt{(p + c_R\delta Z)^2 + m^2} \\ \sqrt{(p + c_L\delta Z)^2 + m^2} \end{cases}$$

Increasing the value δZ has the effect of shifting the parabolas upwards and to the left (for negative couplings c_R, c_L , such as the hypercharge quantum numbers with $\sin^2\theta_W \rightarrow 1$) resulting in a nett production of right-handed particles at the expense of left ones, as expected in [2]. Using this analogy the necessary anomaly-induced collapse of the left vacuum appears to break down if $c_L = -c_R$ whereby the left and right parabolas shift in opposite directions, such that there is no net change in chirality. This happens for the hypercharge couplings for the electron, up- and down-type quarks at the values $\sin^2\theta_W = 1/4, 3/8$ and $3/4$ respectively. In this case an alternative scenario, such as the SCS type transition advocated here would be required to produce the fermion generations in a manner outlined below.

Of course these dispersion relations do not correspond to the eigenmodes of a full Dirac fermion; each branch contains only half the required number of degrees of freedom. Up until now we have considered a single fermion, however SCS is a many-body phenomenon. If we consider a superposition

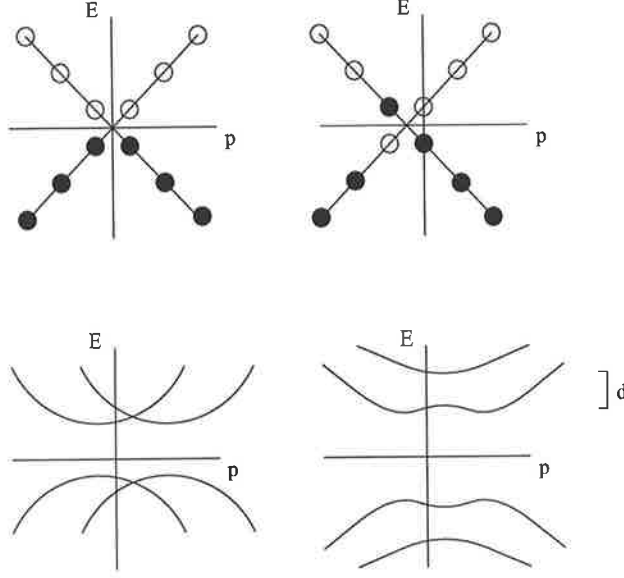


Figure 6.1: Dispersion relations for eigenmodes of Eq. (6.9). Black and white circles represent filled and empty states respectively.

of a fermion and antifermion to make up the requisite number of degrees of freedom it is apparent that this diagram could also correspond to dispersion relations for two distinct bosonic objects, as shown in the lower right picture, in the limit of a vanishing gap d . These relations, of the form

$$\begin{aligned} E_1^2 &= p^4 + m_1^2 \\ E_2^2 &= (p^2 + gS)^2 + (m_1^2 - d) \end{aligned}$$

have a natural interpretation as those of composite bosons. E_1 and E_2 would then represent fluctuations in “charge” and “spin”-type degrees of freedom respectively.

The hypothesised effect in 4-dimensions is that the SCS in the vector interactions results in condensates containing “charge” and “spin” degrees of freedom. Consider now a free fermion with global $U(2f)_L \otimes U(2f)_R$ “iso-flavour” symmetry. Addition of charge density interactions Eq.(6.8) breaks the symmetry

$$U(2f)_L \otimes U(2f)_R \supset (SU(2f)_L \otimes U(1))_L \otimes (SU(2f)_R \otimes U(1)). \quad (6.11)$$

For a single isoflavour, $f = 1$, the $U(2)$ fermions decouple into commuting $SU(2) \otimes U(1)$ sectors reminiscent of a LR-symmetric electroweak theory.

Upon breaking this to the QED scale, the degeneracy in the (iso)“spin” condensates is lifted, leading to fermion mass terms of the form

$$m = m_{\uparrow}(1 + \tau_3)/2 + m_{\downarrow}(1 - \tau_3)/2. \quad (6.12)$$

Note that a mass term of this form is used in Gribov’s calculation of the W and Z masses. The dominant contribution (from the heaviest fermion generation) to vacuum polarisation reproduces good approximations to both boson masses and the expected Higgs VEV.

Alternatively if instead of Abelian densities of left- and right-movers, as in the Luttinger lagrangian, interactions between “isospin” densities

$$\sigma^a = \bar{\psi}\tau^a\psi, \quad a = 1, \dots, 3 \quad (6.13)$$

where τ^a are the $SU(2)$ isospin matrices, the relevant decomposition into decoupled theories is [118]

$$U(2f) \otimes U(2f) \supset [SU(f)_2 \otimes SU(2)_f \otimes U(1)]_L \otimes SU(f)_2 \otimes SU(2)_f \otimes U(1)]_R \quad (6.14)$$

Here the integer subscript denotes the fact that the interaction is in fact described by a chiral Wess-Zumino-Witten [51] model, the value referring to the central charges.

If we identify f with the number of known fermion generations $f = 3$, then the model contains not only (iso)spin and (hyper)charge interactions but an $SU(3)$ “flavour” sector also. In $1 + 1$ dimensions [118] the analogous model contains a non-trivial fixed point which generates a mass gap for the fermion propagators. The bosonic spin fluctuations also acquire mass while the charge and flavour excitations remain gapless, strongly reminiscent of photons and gluons in the SM.

In this context we note the “dualised” standard model (DSM) [80] also associates the number of fermion generations with that of the fermionic colour degrees of freedom. This colour “dual”, analogous to the duality of electrodynamics under exchange of charge and magnetism, represents the *same* gauge symmetry as $SU(3)$, differing only by parity. The question of whether the decomposition (6.14) is equivalent to the DSM written in “LR” rather than “VA” notation certainly warrants further investigation.

The fact that the centre of the group $SU(3)$ is Z_3 also serves as a motivation for the self-consistent introduction of a Z_3 symmetric “fundamental” fermion in the generational model of Kiselev [21]. The Z_3 would then be interpreted as a relic of the broken dual $SU(3)$.

The model has a natural three-step decomposition, from the “Luttinger” phase, through the left-right symmetric SM, step (6.15), down to the

Standard Model in stage (6.16) before, finally, the conventional chiral symmetry breaking reproduces the familiar low-energy physics of stage (6.17):

$$U(6)_L \otimes U(6)_R \supset [SU(3)_2 \otimes SU(2) \otimes U(1)]_L \otimes [SU(3)_2 \otimes SU(2) \otimes U(1)]_R \quad (6.15)$$

$$\supset SU(3)_2 \otimes SU(2)_L \otimes U(1)_H \quad (6.16)$$

$$\supset SU(3)_c \otimes U(1)_{QED}. \quad (6.17)$$

This symmetry breaking pattern is consistent with the hierarchy of the three critical chiral scales shown to exist in the quenched hypercharge theory of the previous chapter.

In conclusion we observe that certain recent aspects of condensed matter, in particular the behaviour of chiral fluids, are potentially fertile ground for understanding how the behaviour of the scalar sector of the SM gives rise to fermion mass, generation number and flavour mixing. Regarding higher gauge theories, only one coupling and two six-dimensional fundamental fields (one “quark” and one “lepton”) would be required as input at the $U(6) \otimes U(6)$ level.

We argue that it appears logical and compelling to investigate whether a “technicolour” version of the left-right symmetric SM might be analogous to certain forms of SCS.

Appendix A

Fierz transformations

Consider an orthogonal basis Γ_a for the space of $n \times n$ matrices where

$$\text{tr}(\Gamma^a \Gamma^b) = \delta_{ab}/c. \quad (\text{A.1})$$

An arbitrary $n \times n$ matrix X has the expansion

$$X = x_a \Gamma^a = c \Gamma^a \text{tr}(X \Gamma_a), \quad (\text{A.2})$$

and from the completeness of Γ_a one obtains the general Fierz identity for any pair of matrices:

$$X_{ij} Y_{kl} = c \text{tr}(X \Gamma_a Y)_{il} \Gamma_{jk}^a. \quad (\text{A.3})$$

In the case of the 4-dimensional Lorentz group ($c = 1/4$) the identity is frequently used for manipulating 4-fermion scalar interactions. For the 16 Dirac matrices we define

$$\rho_{bc}^a = \frac{1}{4} \text{tr}(\Gamma^a \Gamma_b \Gamma_c), \quad (\text{A.4})$$

where a, b and c range over $S = I, V = \gamma^\nu, T = -i\sigma^{\mu\nu}/\sqrt{2}, A = \gamma^5 \gamma^\nu$ and $P = \gamma^5$. With the notation

$$(X.Y)_{42;31} \equiv \bar{\psi}_4 X^{42} \psi_2 \bar{\psi}_3 Y^{31} \psi_1, \quad (\text{A.5})$$

and using the identity

$$\psi_2 \psi_1 = \delta_{2\bar{2}} \delta_{1\bar{1}} \psi_{\bar{2}} \psi_{\bar{1}} = \frac{1}{4} \Gamma_{a2\bar{1}} \Gamma_{1\bar{2}}^a \psi_{\bar{2}} \psi_{\bar{1}}, \quad (\text{A.6})$$

we arrive at the general result

$$(\Gamma^a \cdot \Gamma_b)_{42;31} = \frac{1}{4} \rho_{cd}^a \rho_b^{ce} (\Gamma^d \cdot \Gamma^e)_{41;32}. \quad (\text{A.7})$$

In chapter 4 we encounter four-fermion interactions of the form

$$((V \pm A) \cdot (V \pm A))_{42;31}, \quad ((V - A) \cdot (V + A))_{42;31}.$$

The results for Lorentz scalars are commonplace in texts (for example [37],[17]) or are readily computed from the identity (A.7):

$$\begin{aligned} (V \cdot V)_{42;31} &= \rho_{VS}^V \rho_V^{VS} (S \cdot S)_{41;32} + (\rho_{SV}^V \rho_V^{SV} + \rho_{TV}^V \rho_V^{TV}) (V \cdot V)_{41;32} \\ &\quad + (\rho_{PA}^V \rho_V^{PA} + \rho_{TA}^V \rho_V^{TA}) (A \cdot A)_{41;32} + \rho_{AP}^V \rho_V^{AP} (P \cdot P)_{41;32} \\ &\equiv -(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2 + \frac{1}{2}((\bar{\psi}\gamma^\mu\psi)^2 + (\bar{\psi}\gamma^5\gamma^\mu\psi)^2), \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} (A \cdot A)_{42;31} &= \rho_{AS}^A \rho_A^{AS} (S \cdot S)_{41;32} + (\rho_{PV}^A \rho_A^{PV} + \rho_{TV}^A \rho_A^{TV}) (V \cdot V)_{41;32} \\ &\quad + (\rho_{SA}^A \rho_A^{SA} + \rho_{TA}^A \rho_A^{TA}) (A \cdot A)_{41;32} + \rho_{VP}^A \rho_A^{VP} (P \cdot P)_{41;32} \\ &\equiv (\bar{\psi}\psi)^2 - (\bar{\psi}i\gamma^5\psi)^2 + \frac{1}{2}((\bar{\psi}\gamma^\mu\psi)^2 + (\bar{\psi}\gamma^5\gamma^\mu\psi)^2). \end{aligned} \quad (\text{A.9})$$

The parity-violating, chiral symmetric terms, which cancel in 4, are included for completeness. The non-zero contributions from Eq.(A.7) are

$$\begin{aligned} (V \cdot A)_{42;31} &= (\rho_{SV}^V \rho_A^{SA} + \rho_{TV}^V \rho_A^{TA}) (V \cdot A)_{41;32} + (\rho_{TA}^V \rho_A^{TV} + \rho_{PA}^V \rho_A^{PV}) (A \cdot V)_{41;32} \\ &\quad + \rho_{VS}^V \rho_A^{VP} (S \cdot P)_{41;32} + \rho_{AP}^V \rho_A^{AS} (P \cdot S)_{41;32} \\ &\equiv \frac{1}{2}(\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^5\gamma^\mu\psi + \bar{\psi}\gamma_\mu\gamma^5\psi\bar{\psi}\gamma^\mu\psi) + (\bar{\psi}\psi\bar{\psi}\gamma^5\psi + \bar{\psi}\gamma^5\psi\bar{\psi}\psi) \end{aligned} \quad (\text{A.10})$$

$$(A \cdot V)_{42;31} = \frac{1}{2}(\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^5\gamma^\mu\psi + \bar{\psi}\gamma_\mu\gamma^5\psi\bar{\psi}\gamma^\mu\psi) - (\bar{\psi}\psi\bar{\psi}\gamma^5\psi + \bar{\psi}\gamma^5\psi\bar{\psi}\psi) \quad (\text{A.11})$$

Now the full expansion of 4-fermion terms in Eq.(4.6) is

$$\begin{aligned} &(c_R^2 + c_L^2 + c_X^2)(V \cdot V)_{42;31} + (c_R^2 + c_L^2 - c_X^2)(A \cdot A)_{42;31} \\ &+ (c_R^2 - c_L^2)(V \cdot A + A \cdot V)_{42;31} + c_X^2(VA - AV + AV - VA)_{42;31} \end{aligned}$$

Substituting the expressions (A.8, A.9, A.10), one arrives at the claimed form, Eq.(4.7), of the Lagrangian.

Appendix B

1-loop running couplings

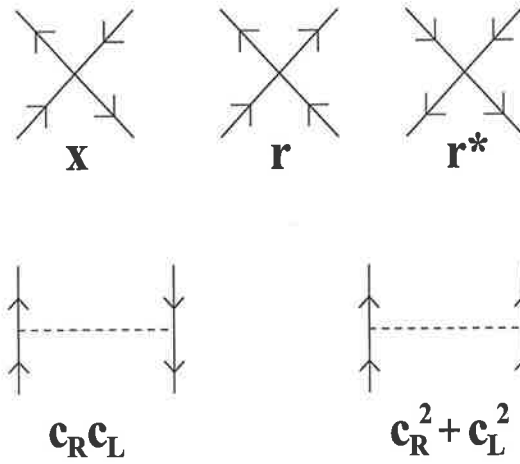


Figure B.1: 4-fermion and fermion-boson vertices in the gauged model.

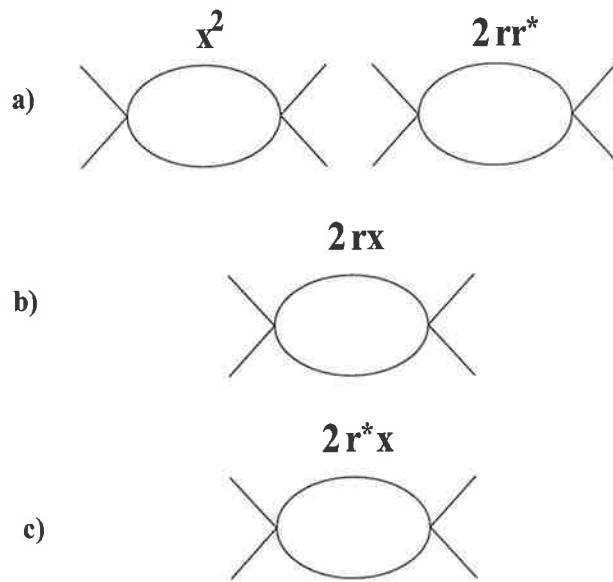


Figure B.2: 1-loop diagrams in the toy 4-fermion model contributing to the running of a) x , b) r and c) r^* respectively.

Appendix C

Transverse boson propagator

In this appendix we compute the 1-loop hypercharge boson propagator. Following Gribov [31] it is demonstrated that modifying the fermion-boson vertex to include a Goldstone-fermion coupling cancels the non-transverse contributions of massive fermion loops to the vacuum polarisation.

From the hypercharge Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^\mu(i\partial_\mu - m + (c_R\chi_R + c_L\chi_L))Z_\mu\psi + \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \quad (\text{C.1})$$

The 1-loop vacuum polarisation is computed to be

$$\begin{aligned} \Pi^{\mu\nu} &= - \int \frac{d^4p}{(2\pi)^4} \text{Tr}\{\Gamma^\mu S(p) g_j \Gamma^\nu S(p-k)\} \\ &= (c_R - c_L)^2 m^2 \delta^{\mu\nu} \int \frac{d^4p}{(2\pi)^4} X(k^2, p^2) \\ &\quad + (c_L + c_R)^2 (m^2 \int \frac{d^4p}{(2\pi)^4} X(k^2, p^2) \delta^{\mu\nu} + Y^{\mu\nu}) + i(c_R^2 - c_L^2) \epsilon^{\mu\rho\nu\sigma} Z_{\rho\sigma} \end{aligned}$$

where

$$\begin{aligned} X(k^2, p^2) &= \frac{1}{p^2 - m^2} \frac{1}{(p-k)^2 - m^2}, \\ Y^{\mu\nu} &= - \int \frac{d^4p}{(2\pi)^4} X(k^2, p^2) (p^\mu (p-k)^\nu + p^\nu (p-k)^\mu - \delta^{\mu\nu} p(p-k)), \\ Z_{\rho\sigma} &= - \int \frac{d^4p}{(2\pi)^4} X(k^2, p^2) p_\rho (p-k)_\sigma. \end{aligned}$$

For $c_L = c_R$ only the second term survives and yields the familiar QED result:

$$-\frac{i}{4\pi^2} (\delta^{\mu\nu} k^2 - k^\mu k^\nu) \Pi(k^2) \quad (\text{C.2})$$

where $\Pi(k^2)$ is the usual $U(1)$ scalar polarisation. More generally, following Gribov[31] we include the Goldstone contribution in each vertex in order to restore transversality to $\Pi^{\mu\nu}$:

$$\Gamma^\mu \rightarrow \Gamma^\mu - F.\eta \frac{k^\mu}{k^2} \quad (\text{C.3})$$

where F is the boson-Goldstone transition amplitude and the Goldstone-fermion coupling η is given by the Ward-Takahashi identity:

$$k_\mu(\Gamma^\mu - F.\eta \frac{k^\mu}{k^2}) = c_R \chi_R S^{-1}(p) - S^{-1}(p-k) c_L \chi_L. \quad (\text{C.4})$$

For the free fermion propagator it is

$$F.\eta = 2m(c_L - c_R)\gamma_5. \quad (\text{C.5})$$

Now Eq.(C.2) is replaced by 4 terms:

$$\Pi^{\mu\nu} = - \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \{ (\Gamma^\mu - F.\eta \frac{k^\mu}{k^2}) S(p) (\Gamma^\nu - F.\eta \frac{k^\nu}{k^2}) S(p-k) \} \quad (\text{C.6})$$

$$\equiv P_1 + P_2 + P_3 + P_4 \quad (\text{C.7})$$

At the one-loop level, the contraction $\epsilon_{\mu\nu\rho\sigma} Z^{\rho\sigma}$ vanishes. P_1 is just the bare result, Eq.(5.1) above. The second and third terms are equal (and vanish in this case) while

$$P_4 = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \{ F.\eta \frac{k^\mu}{k^2} S(p) F.\eta \frac{k^\nu}{k^2} S(p-k) \} \quad (\text{C.8})$$

$$= 2m^2(\alpha_S - \alpha_D) \frac{k^\mu k^\nu}{k^2} \int \frac{d^4 p}{(2\pi)^4} X(k^2, p^2) \frac{2}{k^2} (m^2 - p.(p-k)) \quad (\text{C.9})$$

Combining this with Eqs.(C.2) we see

$$\begin{aligned} \Pi^{\mu\nu} &= (\alpha_S - \alpha_D) m^2 (\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} (-\frac{4m^2}{k^2})) X(k^2) \\ &\quad + \alpha_D (\delta^{\mu\nu} k^2 - k^\mu k^\nu) \Pi(k^2) \end{aligned} \quad (\text{C.10})$$

which is transverse when the Goldstone self-energy

$$\Sigma_G = k_G^2 = -4m^2, \quad (\text{C.11})$$

a natural condition for composite bosons.

Bibliography

- [1] A. Pich, Lectures at the CERN Academic Training (Geneva, November, 1993), hep-ph/9412274.
- [2] S.D. Bass and A.W. Thomas, Mod. Phys. Lett. **A11**,(1996) 339.
- [3] A. Pich, Lectures at the Cargese '96 School –Masses of Fundamental Particles– Cargese, Corsica, 5-17 August 1996, hep-ph/9701263.
- [4] P.B. Renton, Eur. Phys. J. **C8** (1998) 585.
- [5] D.E. Groom *et al.*, Eur. Phys. J. **C15** (2000) 1.
- [6] C.D. Froggatt, H.B. Nielsen and D.J. Smith, Z.Phys. **C73** (1997) 333.
- [7] H. Harari and N. Seiberg, Phys. Lett. **B102** (1981) 263.
- [8] S.L. Adler, hep-th/9711393 (1997).
- [9] A. Davidson, T. Schwartz and R.R. Volkas, J.Phys. **G25** (1999) 1571.
- [10] E. Nardi, Talk given at 3rd Latin American Symposium on High Energy Physics (SILAFEA III), Cartagena de Indias, Colombia, April 2-8, 2000, hep-ph/0009329.
- [11] L.E. Ibáñez, Phys. Lett. **B303** (1993) 55.
- [12] L.E. Ibáñez and G.G. Ross, Phys. Lett. **B332** (1994) 100.
- [13] P. Binetruy, C. Deffayet, E. Dudas and P. Ramond, Phys.Lett. **B441** (1998) 163.
- [14] M. Green and J. Schwartz, Phys. Lett. **B149** (1984) 117.
- [15] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **127** (1962) 962.

- [16] T. Maskawa and H. Nakajima, Prog. Theor. Phys. **52**, (1974) 1326.
- [17] V.A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific 1993).
- [18] C.D. Roberts and A.G. Williams, in *Progress in Particle and Nuclear Physics* **33**, (Ed. Amand Faessler, Pergamon Press 1994).
- [19] S.L. Adler, Phys.Rev. D**59** (1999) 015012; *erratum-ibid.* D**59** (1999) 099902; Phys.Rev. D**60** (1999) 015002.
- [20] H. Chan and S.T. Tsou, Int. J. Mod. Phys. **A14** (1999) 2139.
- [21] V.V. Kiselev, hep-ph/9806523 (1998).
- [22] V. Visnjic, hep-ph/0002067 (2000).
- [23] V.V. Kiselev, hep-ph/9909545 (1998).
- [24] W.A. Bardeen, C.N. Leung and S.T. Love, Nucl. Phys. **B323**, (1989) 493.
- [25] M.A.B. Bég, Phys. Rev. D**39** (1989) 2373.
- [26] M. Göckeler, R. Horsely, V. Linke, P. Rakow, G. Schierholz and H. Stüben, Phys. Rev. Lett. **80**, (1998) 4119.
- [27] V. Bornyakov, A. Hoferichter and G. Schierholz, Nucl. Phys. Proc. Suppl. **94** (2001) 773-776.
- [28] M. Reenders, *The gauged Nambu-Jona Lasinio model*, (Ph.D thesis, University of Groningen 1999).
- [29] S.L. Adler, Phys. Rev. **177**, 2426 (1969).
- [30] J.S. Bell and R. Jackiw, Nuovo Cim. **60**, 47 (1969).
- [31] V.N. Gribov, Phys. Lett. **B336**, (1994) 243.
- [32] B-A. Li, hep-ph/0012051 (2000).
- [33] B-A. Li, Nucl. Phys **B** Proc. Suppl., (1999) 76,263.
- [34] K. Wilson, Phys Rep C **12**, (1974) 75.
- [35] K. Wilson and J. Kogut, Rev. Mod. Phys. **47**, (1975) 773.

- [36] J.S. Ball and T.W. Chiu, Phys. Rev. **D22**, 2542 (1980).
- [37] C. Itzykson and J-B. Zuber, *Quantum Field Theory*, (McGraw-Hill 1985).
- [38] T. Muta, *Foundations of Quantum Chromodynamics* (World Scientific 1987).
- [39] H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4**, 49 (1949).
- [40] R. Jackiw, Dirac Prize Lecture delivered at Trieste, Italy, in March 1999, hep-th/9903255.
- [41] V.N. Gribov, in Proceedings of non-perturbative Methods in Quantum Field Theories, Z. Horvath, L. Palla and A. Patkós (Eds.), (World Scientific 1987) 65.
- [42] N.D. Hari Dass Int. J. Mod. Phys. **B14** (2000) 1989-2010
- [43] K. Fujikawa, Phys. Rev **D21**, 2848 (1980).
- [44] M Swanson, *Path Integrals and Quantum Processes*, (Academic Press 1992).
- [45] W.A. Bardeen, Phys. Rev. **184**, 1848 (1969).
- [46] T.D. Kieu, Chin. J. Phys. **32** (1994) 1099-1108
- [47] T.D. Kieu, Mod. Phys. Lett. A **11** (1996) 2601-2610
- [48] R. Casana and S.A. Dias, Int. J. Mod. Phys. **A15** 4603, (2000).
- [49] R. Casana and S.A. Dias, J. Phys. **G27** (2001) 1501-1518.
- [50] J. Wess and B. Zumino, Phys. Lett. **37B** (1971), 95.
- [51] E. Witten, Nucl. Phys. **B223**, (1983) 422.
- [52] E. Coleman and S. Weinberg, Phys. Rev. D **7**, 1888 (1973).
- [53] L.L Salcedo, Eur. Phys. J. **C20** (2001) 161-184.
- [54] L.D. Landau, in *Niels Bohr and the Development of Physics*, (McGraw-Hill, 1955).
- [55] F. Wegner and J. Houghton, Phys. Rev. **A8**, 401 (1973).

- [56] K-I. Aoki, K. Morikawa, J-I. Sumi, H. Terao and M. Tomoyose, Prog. Theor. Phys. **102** 1151 (1999).
- [57] I.J.R. Aitchison and A.J.G. Hey, *Gauge theories in particle physics : a practical introduction*, (Hilger 1982)
- [58] S.L. Glashow, Nucl. Phys.**22**, 579 (1961).
- [59] S. Weinberg, Phys. Rev. Lett. **19** 1264 (1967).
- [60] A. Salam, in *Elementary Particle Theory*, Ed. N. Svartholm (Almqvist and Wiksells 1969) 367.
- [61] A. Doff and F. Pisano, Mod. Phys. Lett. **A15** (2000) 1471-1480
- [62] P.W. Higgs, Phys. Rev. **145**, 1156 (1966).
- [63] T.W.B. Kibble, Phys. Rev. **155**, 1554 (1967).
- [64] M. Krawczyk, Acta Phys. Polon. **B29**, 3543 (1998).
- [65] A.F. Nicholson and D.C. Kennedy, Int. J. Mod Phys. **A15** (2000) 1497.
- [66] D. Caenepeel and M. LeBlanc, hep-th/9404088 (1994).
- [67] S. Weinberg, Phys. Rev. **D19**, 1277 (1979).
- [68] L. Susskind, Phys. Rev. **D20**, 2619 (1979).
- [69] W.A. Bardeen, C.T. Hill and M.Lindner, Phys. Rev. **D41**, 1647 (1990).
- [70] V.A. Miransky, M. Tanabashi and K. Yamawaki, Phys. Lett. **221** (1989) 177.
- [71] M. Kobayashi and T. Maskawa, Prog. Theor. Phys **49**, 652 (1973).
- [72] J.H. Christenson, J.W. Cronin, V.L. Fitch and R. Turlay, Phys. Rev. Lett. **13** 138 (1964).
- [73] Y. Fukuda *et al.* (Super-Kamiokande collaboration), Phys. Lett. **B436**, 33 (1998).
- [74] Y. Fukuda *et al.* (Super-Kamiokande collaboration), Phys. Rev. Lett. **81**, 1562 (1998).

- [75] J.L. Rosner, invited talk presented at 2nd Tropical Workshop in Particle Physics and Cosmology, San Juan, Puerto Rico, May 1-6, 2000, hep-ph/0005258.
- [76] D. Espriu and J. Manzano, Phys. Rev. **D63** (2001) 073008.
- [77] M.B. Einhorn and J. Wudka, Phys. Rev. **D63** (2001) 073008.
- [78] L. Hall and S. Weinberg, Phys. Rev. **D48** (1993) 979-983.
- [79] V. Zarikas, Phys.Lett. **B384** (1996) 180-184.
- [80] H. Chan, Int. J. Mod. Phys. **A16** (2001) 163-178.
- [81] J. Bordes, H. Chan, and S.T. Tsou, Phys. Rev. **D65** (2002) 093006
- [82] P.I. Fomin and V.A. Miransky, Phys. Lett. **B64**, (1976) 166.
- [83] P.I. Fomin, V.P. Gusynin and V.A. Miransky, Phys. Lett. **B78**, (1978) 136.
- [84] R. Fukuda and T. Kugo, Nucl. Phys. **B117**, 250 (1976).
- [85] D. Atkinson and P.W. Johnson, J. Math. Phys. **28** (1987) 248.
- [86] K. Kondo, H. Mino and K. Yamawaki, Phys. Rev. **D39** (1989) 2430.
- [87] P.I. Fomin, V.P. Gusynin and V.A. Miransky, Phys. Lett. **B105**, (1981) 387.
- [88] V.A. Miransky, Nuovo Cimento **90A**, (1985) 149.
- [89] C.J. Burden and C.D. Roberts, Phys. Rev. **D47** (1993) 5581.
- [90] L.D. Landau and I.M. Khalatnikov, Sov. Phys. JETP **2** (1956) 69.
- [91] D.C. Curtis and M.R. Pennington, Phys. Rev. **D42**, 4165 (1990); Phys. Rev. **D48**, (1993) 4933.
- [92] A. Kizilersü, M. Reenders and M.R. Pennington, Phys. Rev. **D52** (1995) 1242.
- [93] A. Bashir, A. Kizilersü and M.R. Pennington, Phys. Rev. **D57** (1998) 1242.
- [94] Y. Takahashi, in "*Quantum Field Theory*", F. Mancini (Ed.), (Elsevier Science, 1986)

- [95] K. Kondo, Int. J. Mod. Phys. **A12** (1997) 5651-5686.
- [96] R. Delbourgo and G. Thompson, J. Phys. **G8**, L185.
- [97] J. Bernstein, *Elementary particles and their currents*, p 64 (Freeman, 1968).
- [98] D. Atkinson, V.P. Gusynin and P. Maris, Phys. Lett **303**, 157,1993.
- [99] D. Atkinson, J.C.R. Bloch, V.P. Gusynin, M.R. Pennington and M. Reenders, Phys. Lett. **B329**, (1994) 117.
- [100] J. Oliensis and P.W. Johnson, Phys. Rev. **D42**, (1990) 656.
- [101] V. Gusynin, Mod. Phys. Lett. **A5** (1990) 133.
- [102] K. Kondo and H. Nakatani, Prog. Theor. Phys. **88** (1992) 737.
- [103] P.E.L. Rakow, Nucl. Phys. **B356**, (1991) 27.
- [104] J. Gasser and H. Leutwyler, Phys. Ann. **158**, 142 (1984).
- [105] C. Bender, K. Milton and V Savage, Phys. Rev. **D62** (2000) 085001.
- [106] K-I. Aoki, K. Morikawa, J-I. Sumi, H. Terao and M. Tomoyose, Prog.Theor.Phys. **97** 479 (1997).
- [107] V. Branchina, Phys. Lett. **B549** (2002) 260-266.
- [108] E. Babaev, Int. J. Mod. Phys. **A16** (2001) 1175-1197.
- [109] M. Joyce, K. Kainulainen and T. Prokopec, JHEP 0010 (2000) 29
- [110] K. Matsuda, N. Takeda, T. Fukuyama and H, Nishiura, Phys. Rev. **D62**, 093001 (2000).
- [111] R. Pisarski and D. Rishcke, Phys.Rev. **D60** (1999) 094013.
- [112] H. Sonoda, hep-th/0005188 (2000).
- [113] G.E. Volovik, *Exotic properties of superfluid ^3He* , (World Scientific 1992).
- [114] G.M. Luke *et al.*, Nature **394**, (1998) 558.
- [115] T. Morinari and M. Sigirist, J. Phys. Soc. Jpn. 69, 2411 (2000).

- [116] S. Yoon *et al.*, cond-mat/0003250 (2000).
- [117] J. Goryo, J. Phys. Soc. Jap. 69 (2000) 3501-3504.
- [118] N. Andrei, M.R. Douglas and A. Jerez, cond-mat/9803134 (1998).
- [119] J. Voit, cond-mat/9510014 (2000).
- [120] F.D.M. Haldane, J. Phys. **C14**, 2585 (1981).