Head- and Flow-Based Formulations for the Frequency Domain Analysis of Fluid Transients in Arbitrary Pipe Networks

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Abstract

Applications of frequency-domain analysis in pipelines and pipe networks include resonance analysis, time-domain simulation and fault detection. Current frequency-domain analysis methods are restricted to series pipelines, single-branching pipelines and single-loop networks and are not suited to complex networks. This paper presents a number of formulations for the frequency-domain solution in pipe networks of arbitrary topology and size. The formulations focus on the topology of arbitrary networks and do not consider any complex network devices or boundary conditions, other than head and flow boundaries. The frequency-domain equations are presented for node elements and pipe elements, which correspond to the continuity of flow at a node and the unsteady flow in a pipe, respectively. Additionally, a pipe-node-pipe and reservoir-pipe pair set of equations are derived. A matrix-based approach is used to display the solution to entire networks in a systematic and powerful way. Three different formulations are derived based on the unknown variables of interest that are to be solved for, being the head-formulation, flow-formulation and the head-flow-formulation. These hold significant analogies to different steady-state network solutions. The frequency-domain models are tested against the method of characteristics (a commonly used time-domain model), with good result. The computational efficiency of each formulation is discussed with the most efficient formulation being the head-formulation.

Introduction

The use of time-domain or frequency-domain analyses depends upon the problem at hand. Suitable problems for frequency-domain analysis are those that are linear in nature or involve a small perturbation about a reference state. Frequency-domain analysis is used in applications such as resonance analysis (Chaudhry 1987; Wylie and Streeter 1993), leakage detection (Ferrante and Brunone 2003; Lee *et al*. 2005a, 2005b, 2006; Covas *et al*. 2006; Kim 2005, 2007, 2008) and blockage detection (Mohapatra *et al*. 2006a, 2006b; Sattar *et al*. 2008). Additionally, certain time-domain solutions can be calculated via the frequency-domain solution allowing many applications, which involve time-domain analyses, to utilise frequency-domain analyses. Suo and Wylie (1989) presented the impulse response method (IMPREM) where the frequency-domain response is transferred into a time-domain response. The technique assumes that the system is driven by a discharge perturbation at the downstream boundary and the solution requires a formulation of the impedance equations for the particular system. Kim (2007, 2008) presented a matrix-based implementation of the impedance method for a simple network, although the method is closely related to the transfer matrix method.

These applications, as described in the previous paragraph, have been limited to single pipelines, pipelines with single branches and single-loop networks. This paper derives different formulations for frequency-domain analysis for an arbitrary pipe network. For the purposes of clearly establishing the type of network considered in this paper the network elements considered include pipes, nodes, demands and reservoirs. Excitation to the system can be made through perturbations in either demand (or flow) at a junction or head at a reservoir. Analysis in the frequency domain, for a suitable problem, can be efficient and accurate provided that the nonlinearities involved are small. Additionally, frequency-domain analysis allows convenient inclusion of unsteady friction and viscoelastic behavior where their solution is efficient. The solution for a transient response, when calculated using frequency-domain analysis, requires the solution of the system response at many single frequency components, therefore, it is desirable that each frequency component be solved as efficiently as possible. Three sets of network equations are derived in this paper that are based on the continuity of flow at a node, the unsteady-state equations of continuity and motion for a pipe, and pipe-node-pipe and reservoir-pipe pairs. From those three sets of equations three formulations are derived based on solutions for the complex perturbations in heads and flow, heads only and flow only. The computational merits of each formulation and similarities to steady-state solution formulations are discussed.

Background

The analysis of pipelines in the frequency domain (which also includes Laplace domain analysis) began in the 1950s (summarised in Goodson and Leonard 1972, Stecki and Davis 1986). This work was typically limited to a single pipeline. The development of general frequency-domain solutions in more complicated pipelines involves two main methodological streams. The first method is the transfer matrix method (Chaudhry 1970, 1987). This method develops field matrices, which relate to the solution along the pipe, and point matrices that consider junctions, hydraulic devices and changes in pipe characteristics. A block-diagram is used to formulate the matrices, usually by hand, for more complicated systems like pipes in series, single-branches and single loops. While these units could be manipulated to solve small and restricted problems (limited to networks that do not have 2nd order loops), in a complex network the number of units required can quickly become overwhelming. The second method is the impedance method (Wylie 1965, Wylie and Streeter 1993). This method solves for the impedance which is equal to the complex head perturbations divided by the complex flow perturbations. Again, this method is usually formulated for each system by hand and, although is useful in forming explicit relationships in simple systems, is poorly suited to complex network analysis.

The behavior of various hydraulic devices and phenomena in the frequency-domain has been addressed by many authors. Chaudhry (1987) and Wylie and Streeter (1993) present a summary of solutions for different hydraulic elements, such as valves, orifices, junctions, and more. Suo and Wylie (1990a) present solutions for viscoelastic pipe material. Viscoelasticity was incorporated using a frequency-dependent wave speed. Similarly, Suo and Wylie (1990b) present frequency-domain solutions for rock-walled tunnels. Unsteady friction has been dealt with by, amongst many others, Brown (1962) and D'Souza and Oldenburger (1964). Vítkovský *et al*. (2003) present frequency-domain solutions for weighting function-type unsteady friction models. Finally, Tijsseling (1996) presents a number of studies where fluid-structure interaction has been considered in the frequency (or Laplace) domain.

In terms of network analysis, Wylie and Streeter (1993) present the frequency-domain solution for a simple network, although not expressed in an arbitrary way for general network analysis. Other network-type analyses do not directly consider the frequency-domain solution, but are nonetheless relevant. Ogawa *et al*. (1994) present frequency-domain solutions in networks with respect to the effect of earthquakes on water distribution networks. They used a matrix-based approach, but were solving for different response modes resulting from sinusoidal ground movement. Shimada *et al*. (2006) present an exploration into numerical error for time-line interpolations in pipe networks. Although this work relates to errors in time-domain methods, the errors are assessed in the frequency-domain where exact solutions exist. More recently, Kim (2007, 2008) presents a more generic approach to the application of the impedance method in networks, but with respect to a particular network. Recently, Zecchin *et al*. (2009) formulated a Laplace-domain network admittance matrix formulation of the fundamental network equations, which shares a similarity to the $\hat{h}$-formulation that is derived within this paper.

The remainder of this paper presents a systematic, matrix-based approach for frequency-domain analysis in arbitrary pipe networks.

Formulations for Frequency-Domain Analysis

The formulations for the frequency-domain solution investigated in this paper consider a simplified network. There is no consideration of hydraulic elements such as leaks, pumps, valves, etc. Additionally, there is no consideration of column separation, fluid structure interaction, minor losses or convective terms, etc. This paper is primarily concerned with the problem of finding the frequency-domain solution for an arbitrarily configured and basic network. As a matter of nomenclature, uppercase denotes a full variable in the time domain, lowercase denotes a perturbation variable in the time domain, and lowercase with a caret denotes a perturbation variable in the frequency domain.

Network Quantities

The network considered consists only of pipes, junctions, reservoirs and demand nodes. For an arbitrary network the quantities of each of these components are linked by

$$np = nn + nr + nl - nc \tag{1}$$

where np = number of pipes, nn = number of nodes, nr = number of reservoirs, nl = number of loops, and nc = number of (separate) components. This relationship is useful when considering the topology of an entire network. An arbitrary network consists of pipe (links) and node elements. The following sections define the relationships for these elements.

Frequency-Domain Equations for Node Elements

The head is common at a node and can be either known or unknown. Also, a node element represents a junction of pipes and demands. The continuity of flow is applied for pipes, p, connected to node k as

$$\sum_p Q_{p,k} = D_k \tag{2}$$

where $Q_{p,k}$ = flow into node k from pipe p and D_k is demand out of node k. Each pipe requires an arbitrarily set flow direction (not related to the actual flow direction). In terms of continuity, pipe flows are taken as positive into a node and demands are positive out of a node. Taking the perturbation of Q and D about steady-state conditions as $q = Q – Q_0$ and $d = D – D_0$ gives the continuity of perturbations at node k

$$\sum_p q_{p,k} = d_k \tag{3}$$

The Fourier transform gives the frequency-domain continuity at node k

$$\sum_p \hat{q}_{p,k} = \hat{d}_k \tag{4}$$

It should be noted that the relationship in (4) is now complex-valued and represents the continuity of flow at a node for different frequency components.

Frequency-Domain Equations for Pipe Elements

Each pipe element represents the behavior of unsteady pipe flow between two nodes. The equations of continuity and motion for unsteady pipe flow, including unsteady friction and a viscoelastic pipe material (Wylie and Streeter 1993, Gally *et al*. 1979, Vítkovský *et al*. 2006), are

$$\frac{\partial H}{\partial t} + \frac{a_0^2}{gA_0}\frac{\partial Q}{\partial x} + \frac{\alpha_0 \rho_0 D_0 a_0^2}{e_0}\left(\frac{\partial H}{\partial t} * \frac{\partial J_r}{\partial t}\right)(t) = 0 \tag{5}$$

$$\frac{\partial H}{\partial x} + \frac{1}{gA_0}\frac{\partial Q}{\partial t} + \frac{f_0 Q|Q|}{2gD_0 A_0^2} + \frac{16\nu}{gD_0^2 A_0}\left(\frac{\partial Q}{\partial t} * W_0\right)(t) = 0 \tag{6}$$

where H = head, a = wave speed, g = gravitational acceleration, D = pipe diameter, A = pipe cross-sectional area, e = pipeline thickness, ρ = fluid density, α = pipe restraint coefficient, ν = kinematic viscosity, J_r = retarded component of creep compliance function, W = unsteady friction weighting function, x = distance along pipe, and t = time. The subscript "0" on some variables denote that it is based on an initial or steady-state value. The operator "$*$" represents convolution. Taking a perturbation in flow ($q = Q - Q_0$) and head ($h = H - H_0$) and linearizing the steady friction term the equations of continuity and motion become

$$\frac{\partial h}{\partial t} + \frac{a_0^2}{gA_0}\frac{\partial q}{\partial x} + \frac{\alpha_0 \rho_0 D_0 a_0^2}{e_0}\left(\frac{\partial h}{\partial t} * \frac{\partial J_r}{\partial t}\right)(t) = 0 \tag{7}$$

$$\frac{\partial h}{\partial x} + \frac{1}{gA_0}\frac{\partial q}{\partial t} + \frac{f_0 |Q_0|}{gD_0 A_0^2} q + \frac{16\nu}{gD_0^2 A_0}\left(\frac{\partial q}{\partial t} * W_0\right)(t) = 0 \tag{8}$$

Taking the Fourier transform with respect to time and simplifying the resulting equation gives the frequency-domain equations for a pipe element.

$$\left(i\omega - \frac{\alpha_0 \rho_0 D_0 a_0^2 \omega^2 \hat{J}_r}{e_0}\right)\hat{h} + \frac{a_0^2}{gA_0}\frac{d\hat{q}}{dx} = 0 \tag{9}$$

$$\frac{d\hat{h}}{dx} + \left(\frac{i\omega}{gA_0} + \frac{f_0|Q_0|}{gD_0 A_0^2} + \frac{i\omega \nu \hat{W}_0 16}{gD_0^2 A_0}\right)\hat{q} = 0 \tag{10}$$

where i = imaginary unit and ω = angular frequency. Eqs. (9) and (10) are a set of coupled ordinary differential equations with full derivatives only in space (x). The transfer matrix solution for this system of coupled ODEs can be derived for a pipe element (p) relating the upstream (U) head and flow to the downstream (D) head and flow for given frequency perturbation as

$$\begin{Bmatrix} (\hat{q}_D)_p \\ (\hat{h}_D)_{j,p} \end{Bmatrix} = \begin{bmatrix} \cosh(\gamma_p L_p) & -Z_p^{-1}\sinh(\gamma_p L_p) \\ -Z_p \sinh(\gamma_p L_p) & \cosh(\gamma_p L_p) \end{bmatrix} \begin{Bmatrix} (\hat{q}_U)_p \\ (\hat{h}_U)_{k,p} \end{Bmatrix} \tag{11}$$

where L = pipe length and the propagation constant γ is

$$\gamma = \frac{i\omega}{a_0}\sqrt{(1+R_S+R_U)(1+R_V)} \tag{12}$$

and where the characteristic impedance Z is

$$Z = \frac{a_0}{gA_0}\sqrt{(1+R_S+R_U)(1+R_V)^{-1}} \tag{13}$$

and where the steady friction component R_S is

$$R_S = \begin{cases} \dfrac{32\nu}{i\omega D_0^2} & \text{for laminar flow} \\ \dfrac{f_0|Q_0|}{i\omega D_0 A_0} & \text{for turbulent flow} \end{cases} \tag{14}$$

and where the unsteady friction component R_U is

$$R_U = \frac{\nu 16 \hat{W}_0}{D_0^2} \tag{15}$$

and where the viscoelastic component R_V is

$$R_V = \frac{i\omega\alpha_0\rho_0 D_0 a_0^2 \hat{J}_r}{e_0} \tag{16}$$

The elastic wave speed is

$$a_0 = \sqrt{\frac{K}{\rho_0}\left(1+\frac{\alpha_0 D_0 J_e K}{2e_0}\right)^{-1}} \tag{17}$$

where J_e = elastic component of the creep compliance function ($J_e = 1/E$, where E = Young's modulus of elasticity) and α_0 = dimensionless pipe constraint coefficient which depends on the relative pipe wall thickness e_0/D_0, Poisson's ratio of the pipe wall material and the type of pipe anchoring. Note that for elastic pipe materials, such as steel, cast iron, copper, etc., the convolution term in Eq. (5) is removed making the term R_V in Eqs. (12) and (13) equal to zero and the constant $\alpha_0/2$ in Eq. (17) can be replaced by C_1 resulting in the more common form of the equations of continuity and motion for unsteady pipe flow (Wylie and Streeter 1993). Eq.

(11) can be directly compared to the field matrix for a pipe element in the transfer matrix method (Chaudhry 1970, 1987).

Frequency-Domain Equations for an Arbitrary Network

The previous sections have presented the relationships for individual node elements and pipe elements. This section outlines how those elements can be combined and organised for an arbitrary network of pipes. A topological matrix-based approach is considered allowing the presentation of relationships that apply to an entire network.

The organisation of all node elements is considered first, essentially specifying flow continuity at all nodes in a network. The complex unknown upstream and downstream flow perturbations for each pipe written as column vectors are

$$\begin{aligned} \hat{\mathbf{q}}_D &= \{\hat{q}_{D1}, \cdots, \hat{q}_{Dnp}\}^T \\ \hat{\mathbf{q}}_U &= \{\hat{q}_{U1}, \cdots, \hat{q}_{Unp}\}^T \end{aligned} \tag{18}$$

The complex demand perturbations at each node written a column vector are

$$\hat{\mathbf{d}} = \{\hat{d}_1, \cdots, \hat{d}_{nn}\}^T \tag{19}$$

Two topological matrices are required that define if a pipe is connected to a node by its downstream or upstream end. These pipe-node incidence matrices are defined as $\mathbf{B1}_D$ and $\mathbf{B1}_U$ respectively

$$\begin{aligned} (\mathbf{B1}_D)_{pk} &= \begin{cases} 1 & \text{if pipe } p \text{ enters node } k \\ 0 & \text{otherwise} \end{cases} \\ (\mathbf{B1}_U)_{pk} &= \begin{cases} 1 & \text{if pipe } p \text{ exits node } k \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{20}$$

Using Eqs. (18), (19) and (20), the frequency-domain nodal continuity equations (Eq. (4)) can be written in matrix form as

$$\mathbf{B1}_D^T \hat{\mathbf{q}}_D - \mathbf{B1}_U^T \hat{\mathbf{q}}_U = \hat{\mathbf{d}} \tag{21}$$

In a similar manner, the relationships for all pipe elements in a network can be written in matrix form. The complex unknown head perturbations at each node written as a column vector are

$$\hat{\mathbf{h}} = \{\hat{h}_1, \cdots, \hat{h}_{nn}\}^T \tag{22}$$

The complex known head perturbations at each reservoir written as a column vector are

$$\hat{\mathbf{r}} = \{\hat{r}_1, \cdots, \hat{r}_{nr}\}^T \tag{23}$$

An additional topological matrix is required to relate the connectivity of pipes and reservoirs. Two pipe-reservoir incidence matrices, $\mathbf{B2}_D$ and $\mathbf{B2}_U$ respectively, are defined for pipes that connect to a reservoir by its downstream or upstream end.

$$
\begin{aligned}
(\mathbf{B2}_D)_{pk} &= \begin{cases} 1 & \text{if pipe } p \text{ enters reservoir } k \\ 0 & \text{otherwise} \end{cases} \\
(\mathbf{B2}_U)_{pk} &= \begin{cases} 1 & \text{if pipe } p \text{ exits reservoir } k \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{24}
$$

Using Eqs. (20), (22), (23) and (24), the frequency-domain pipe element equations (Eq. (11)) for an entire network can be written in matrix form as

$$
\begin{aligned}
\hat{\mathbf{q}}_D &= \mathbf{c}\,\hat{\mathbf{q}}_U - \mathbf{z}^{-1}\mathbf{s}\left(\mathbf{B1}_U\,\hat{\mathbf{h}} + \mathbf{B2}_U\,\hat{\mathbf{r}}\right) \\
\left(\mathbf{B1}_D\,\hat{\mathbf{h}} + \mathbf{B2}_D\,\hat{\mathbf{r}}\right) &= -\mathbf{z}\,\mathbf{s}\,\hat{\mathbf{q}}_U + \mathbf{c}\left(\mathbf{B1}_U\,\hat{\mathbf{h}} + \mathbf{B2}_U\,\hat{\mathbf{r}}\right)
\end{aligned} \tag{25}
$$

The matrices **c** and **s** are diagonal matrices that represent the hyperbolic functions cosh and sinh for each pipe (for completeness **t** represents the tanh function which is used later), and the diagonal matrix **z** represents characteristic impedance for each pipe, that is

$$
\begin{aligned}
\mathbf{c} &= \mathrm{diag}\left[\cosh(\gamma_1 L_1), \ldots, \cosh(\gamma_p L_p)\right] \\
\mathbf{s} &= \mathrm{diag}\left[\sinh(\gamma_1 L_1), \ldots, \sinh(\gamma_p L_p)\right] \\
\mathbf{t} &= \mathrm{diag}\left[\tanh(\gamma_1 L_1), \ldots, \tanh(\gamma_p L_p)\right] \\
\mathbf{z} &= \mathrm{diag}\left[Z_1, \ldots, Z_p\right]
\end{aligned} \tag{26}
$$

Eqs. (21) and (25) define all of relationships for all of the node and pipe elements in an arbitrary pipe network. This set of equations can be solved for different frequency inputs allowing the development of the frequency response function. This paper considers three different formulations for the frequency-domain solution. All formulations are organised into the generic linear system **AX** = **B** that can be solved using existing complex matrix solvers. Comments relating to the solution efficiency of each formulation are discussed.

Frequency-Domain $\hat{q}\hat{h}$-Formulation

The first formulation is the $\hat{q}\hat{h}$-formulation, which solves for the complex flow and head perturbations. This is the most straightforward approach that uses Eqs (21) and (25) as they are. Putting this set of equations into matrix form gives

$$
\left[\begin{array}{c:c:c}
\mathbf{B1}_D^T & -\mathbf{B1}_U^T & \mathbf{0}_{nn} \\ \hdashline
-\mathbf{I}_{np} & \mathbf{c} & -\mathbf{z}^{-1}\mathbf{s}\,\mathbf{B1}_U \\ \hdashline
\mathbf{0}_{np} & -\mathbf{z}\,\mathbf{s} & \mathbf{c}\,\mathbf{B1}_U - \mathbf{B1}_D
\end{array}\right]
\left\{\begin{array}{c}
\hat{\mathbf{q}}_D \\ \hdashline \hat{\mathbf{q}}_U \\ \hdashline \hat{\mathbf{h}}
\end{array}\right\}
=
\left\{\begin{array}{c}
\hat{\mathbf{d}} \\ \hdashline \mathbf{z}^{-1}\mathbf{s}\,\mathbf{B2}_U\,\hat{\mathbf{r}} \\ \hdashline (\mathbf{B2}_D - \mathbf{c}\,\mathbf{B2}_U)\,\hat{\mathbf{r}}
\end{array}\right\} \tag{27}
$$

This can be written in a simplified form as

$$\mathbf{M}_{qh}\begin{Bmatrix}\hat{\mathbf{q}}_D\\ \hat{\mathbf{q}}_U\\ \hat{\mathbf{h}}\end{Bmatrix}=\mathbf{N}_{qh} \tag{28}$$

The matrix $\mathbf{M}_{qh}$ is complex, sparse, and asymmetric. Both $\mathbf{M}_{qh}$ and $\mathbf{N}_{qh}$ depend on frequency, although some elements of each are independent of frequency. The number of unknowns, and hence the size of $\mathbf{M}_{qh}$, is $2np+nn$.

Frequency-Domain $\hat{h}$-Formulation

The second formulation is the $\hat{h}$-formulation. This formulation begins by rearranging the pipe element equations from Eq. (25) in terms of the complex flow perturbations, that is

$$\begin{aligned}\hat{\mathbf{q}}_U&=(\mathbf{z}\,\mathbf{s})^{-1}\left[(\mathbf{c}\,\mathbf{B1}_U-\mathbf{B1}_D)\hat{\mathbf{h}}+(\mathbf{c}\,\mathbf{B2}_U-\mathbf{B2}_D)\hat{\mathbf{r}}\right]\\ \hat{\mathbf{q}}_D&=(\mathbf{z}\,\mathbf{s})^{-1}\left[(\mathbf{B1}_U-\mathbf{c}\,\mathbf{B1}_D)\hat{\mathbf{h}}+(\mathbf{B2}_U-\mathbf{c}\,\mathbf{B2}_D)\hat{\mathbf{r}}\right]\end{aligned} \tag{29}$$

Substituting the result into the node element equations (Eq. (21)) gives the solution of the complex head perturbations as

$$\begin{aligned}&\left[\mathbf{B1}_D^T(\mathbf{z}\,\mathbf{t})^{-1}\mathbf{B1}_D-\mathbf{B1}_D^T(\mathbf{z}\,\mathbf{s})^{-1}\mathbf{B1}_U-\mathbf{B1}_U^T(\mathbf{z}\,\mathbf{s})^{-1}\mathbf{B1}_D+\mathbf{B1}_U^T(\mathbf{z}\,\mathbf{t})^{-1}\mathbf{B1}_U\right]\hat{\mathbf{h}}\\ &\qquad=\left[\mathbf{B1}_D^T(\mathbf{z}\,\mathbf{s})^{-1}\mathbf{B2}_U+\mathbf{B1}_U^T(\mathbf{z}\,\mathbf{s})^{-1}\mathbf{B2}_D\right]\hat{\mathbf{r}}-\hat{\mathbf{d}}\end{aligned} \tag{30}$$

Written in a simplified form as

$$\mathbf{M}_h\,\hat{\mathbf{h}}=\mathbf{N}_h \tag{31}$$

The structure of the $\mathbf{M}_h$ matrix is of interest as it can affect how efficiently the linear solution can be solved. The $\mathbf{M}_h$ matrix is constructed as

$$\begin{aligned}(\mathbf{M}_h)_{jj}&=\sum_p^{nd}\left[Z_p\tanh(\gamma_pL_p)\right]^{-1}\text{ for all pipes }p\text{ connected to node }j\\ (\mathbf{M}_h)_{jk}&=\begin{cases}-\sum_p\left[Z_p\sinh(\gamma_pL_p)\right]^{-1}\text{ for all pipes }p\text{ connecting node }j\text{ to node }k\\ \text{zero if node }j\text{ is not connected to node }k\end{cases}\\ &=(\mathbf{M}_h)_{kj}\end{aligned} \tag{32}$$

The matrix $\mathbf{M}_h$ is complex, sparse and symmetric and shares these similarities with the *H*-formulation for the steady-state solution (as discussed later). The number of unknowns, and hence the size of $\mathbf{M}_h$, is *nn*. The $\mathbf{N}_h$ vector can be constructed as

$$(\mathbf{N}_h)_j=-\hat{d}_j+\sum_p\hat{r}_k\left[Z_p\sinh(\gamma_pL_p)\right]^{-1}\quad\begin{matrix}\text{for all pipes }p\text{ connecting}\\ \text{node }j\text{ with reservoir }k\end{matrix} \tag{33}$$

Both $\mathbf{M}_h$ and $\mathbf{N}_h$ are functions of frequency. Once the complex head perturbations have been determined the complex flow perturbations can be calculated using Eq. (29).

Frequency-Domain $\hat{q}$-Formulation

The final formulation is based on solving for the complex flow perturbations. The $\hat{q}$-formulation begins by rearranging the pipe element equations in Eq. (25) such that all known and unknown complex head perturbations are on the left side of the relationship and all unknown complex flow perturbations are on the right.

$$\begin{aligned} \mathbf{B1}_U \hat{\mathbf{h}} + \mathbf{B2}_U \hat{\mathbf{r}} &= \mathbf{z}\,\mathbf{s}^{-1}\left(\hat{\mathbf{q}}_D - \mathbf{c}\,\hat{\mathbf{q}}_U\right) \\ \mathbf{B1}_D \hat{\mathbf{h}} + \mathbf{B2}_D \hat{\mathbf{r}} &= \mathbf{z}\,\mathbf{s}^{-1}\left(\mathbf{c}\,\hat{\mathbf{q}}_D - \hat{\mathbf{q}}_U\right) \end{aligned} \tag{34}$$

Together with the node element equations, the above equation can be reformulated to link both the upstream and downstream complex flow perturbations between two pipes, provided they are connected by a common node or reservoir. There arises the need to generate all of the pipe-node-pipe (PNP) pairs and reservoir-pipe (RP) pairs in an arbitrary network.

The flows in pipes joined at a common node can be equated to form a set of equations representing pairs of pipes joined by a common head, i.e., the PNP pairs. Figure 1(a) shows an example of a node connected to four pipes. Also shown is a graph (in the mathematical sense) of all of the possible pipe pairings called the complete graph (see Figure 1(b)). If the degree of the node is dn then the total number of pipe pairings is ½(dn^2–dn). This complete set of pipe pairings would form an over-determined set of equations in terms of pipe pairs, whereas all that is required is a set of pipe-pairs that are non-degenerative when solving the linear system. A non-degenerative set of PNP pairs can be found by finding any spanning tree of the complete graph. In a pipe network sense, the set of PNP and RP pairs must form a continuous coverage across the whole network (no isolated areas). For a node with dn pipes connected to it the minimum number of non-degenerative PNP pairs is dn–1 from a total number of possible non-degenerative PNP-pair sets of dn^{dn-2}.

A logical method to generate a non-degenerative set of PNP pairs is to: (i) selectively consider each node in order of node number; (ii) determine the degree of the node (how many pipes are connected), (iii) select the pipe with the lowest pipe ID number and form a set of pairs with that pipe and all other pipes connected to the node; and then (iv) move to the next node and repeat. An example of this approach gives the selected spanning tree in Figure 1(c).

The total number of PNP pairs depends on the connectivity of the network as does the number of RP pairs; however, the sum of PNP and RP pairs must equal $2np - nn$. The PNP pairs can be defined in matrix form by first defining the following topological incidence matrices $\mathbf{B3}_D$, and $\mathbf{B3}_U$ for pipe pairs as

$$\left(\mathbf{B3}_D\right)_{pk} = \begin{cases} 1 & \text{if the } 1^{\text{st}} \text{ pipe } p \text{ in PNP pair } k \text{ enters common node} \\ -1 & \text{if the } 2^{\text{nd}} \text{ pipe } p \text{ in PNP pair } k \text{ enters common node} \\ 0 & \text{otherwise} \end{cases}$$

$$\left(\mathbf{B3}_U\right)_{pk} = \begin{cases} 1 & \text{if the } 1^{\text{st}} \text{ pipe } p \text{ in PNP pair } k \text{ exits common node} \\ -1 & \text{if the } 2^{\text{nd}} \text{ pipe } p \text{ in PNP pair } k \text{ exits common node} \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

The PNP pair equations (a rearrangement of Eq. (34)) can be written in the following form

$$\left(\mathbf{B3}_D^T \mathbf{c} + \mathbf{B3}_U^T\right)\mathbf{z}\,\mathbf{s}^{-1}\hat{\mathbf{q}}_D - \left(\mathbf{B3}_D^T + \mathbf{B3}_U^T \mathbf{c}\right)\mathbf{z}\,\mathbf{s}^{-1}\hat{\mathbf{q}}_U = \mathbf{0} \tag{36}$$

Similarly the RP pairs can be defined in matrix form by first defining the following topological incidence matrices $\mathbf{B4}_D$, $\mathbf{B4}_U$, and $\mathbf{B5}$ for RP pairs are

$$\left(\mathbf{B4}_D\right)_{pk} = \begin{cases} 1 & \text{if pipe } p \text{ in RP pair } k \text{ enters reservoir} \\ 0 & \text{otherwise} \end{cases}$$

$$\left(\mathbf{B4}_U\right)_{pk} = \begin{cases} 1 & \text{if pipe } p \text{ in RP pair } k \text{ exits reservoir} \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

$$\left(\mathbf{B5}\right)_{jk} = \begin{cases} 1 & \text{if pipe connects reservoir } j \text{ in RP pair } k \\ 0 & \text{otherwise} \end{cases}$$

The RP pair equations can be written (a rearrangement of Eq. (34)) in the following form

$$\left(\mathbf{B4}_D^T \mathbf{c} + \mathbf{B4}_U^T\right)\mathbf{z}\,\mathbf{s}^{-1}\hat{\mathbf{q}}_D - \left(\mathbf{B4}_D^T + \mathbf{B4}_U^T \mathbf{c}\right)\mathbf{z}\,\mathbf{s}^{-1}\hat{\mathbf{q}}_U = \mathbf{B5}^T\hat{\mathbf{r}} \tag{38}$$

The $\hat{q}$-formulation uses the node element equations, the PNP pair and RP pair equations, Eqs. (21), (36) and (38) respectively. Putting the set of equations into matrix form gives

$$\begin{bmatrix} \mathbf{B1}_D^T & -\mathbf{B1}_U^T \\ \left(\mathbf{B3}_D^T \mathbf{c} + \mathbf{B3}_U^T\right)\mathbf{z}\,\mathbf{s}^{-1} & -\left(\mathbf{B3}_D^T + \mathbf{B3}_U^T \mathbf{c}\right)\mathbf{z}\,\mathbf{s}^{-1} \\ \left(\mathbf{B4}_D^T \mathbf{c} + \mathbf{B4}_U^T\right)\mathbf{z}\,\mathbf{s}^{-1} & -\left(\mathbf{B4}_D^T + \mathbf{B4}_U^T \mathbf{c}\right)\mathbf{z}\,\mathbf{s}^{-1} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{q}}_D \\ \hat{\mathbf{q}}_U \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{d}} \\ \mathbf{0} \\ \mathbf{B5}^T\hat{\mathbf{r}} \end{Bmatrix} \tag{39}$$

Written in a simplified form as

$$\mathbf{M}_q \begin{Bmatrix} \hat{\mathbf{q}}_D \\ \hat{\mathbf{q}}_U \end{Bmatrix} = \mathbf{N}_q \tag{40}$$

The matrix $\mathbf{M}_q$ is complex, sparse, and asymmetric. The number of unknowns, and hence the size of $\mathbf{M}_q$, is $2np$. A difference between the $\hat{q}$-formulation and the other two formulations is that only the $\mathbf{M}_q$ matrix depends on frequency. The $\mathbf{N}_q$ matrix is independent of frequency and would only need to be calculated once for the full calculation of the transfer function. Once the complex upstream and downstream flow perturbations have been solved for Eq. (25) can be used to calculate the complex head perturbations.

Numerical Verification

The previous section presents three formations for the frequency-domain solution of an arbitrary pipe network. This section provides numerical verification of those formulations (Eqs. (27), (30) and (39)). All formulations produce exactly the same solution, thus no comparison in terms of accuracy can been made between the methods. However, the validity of the frequency-domain solution can be tested against a rigorously tested time-domain method. In this paper the Method of Characteristics (MOC) is used generate the frequency response function for validation. The perturbation size was kept small so as to not incur errors from the linearization of nonlinear terms. Additionally, a very finely discretised MOC diamond grid was used to reduce numerical error.

The first validation is performed on a simple pipeline (Figure 2) with parameters given in Vítkovský *et al* (2006). The pipeline is bounded by a known head at one end and a perturbed flow at the other end. Three cases are considered: (1) steady-state friction only, (2) steady and unsteady friction, and (3) steady friction, unsteady friction and a viscoelastic pipe material. The results are shown in Figure 3, Figure 4 and Figure 5, respectively, for the frequency response function at the flow boundary (node 2). The weighting function model for the unsteady friction is from Vardy and Brown (2003, 2004). The creep compliance function is for polyethylene at 25°C from Gally *et al*. (1979). As observed, the frequency-domain analysis and the time-domain analysis match.

The second validation considers a small pipe network from Liggett and Chen (1994), as shown in Figure 6. This network has 11 pipes and 7 nodes that are supplied from a single reservoir (node 1) and supplies two demands (nodes 4 and 6). The system is excited by a perturbation in the demand at node 6. Figure 7 shows the match between the frequency-domain and time-domain analyses for the head response at node 6.

Both validations show an excellent match between the frequency-domain and time-domain analyses. Of course, this is to be expected as both analyses are solving the same set of equations.

Discussion of Frequency-Domain Analysis

This section provides a further discussion of frequency-domain analysis in arbitrary networks. This includes properties of frequency-domain network matrices, comparison to steady-state analysis in arbitrary networks and efficiency of the frequency-domain formulations.

Properties of Frequency-Domain Network Matrices

During the formulation of the frequency-domain solution a number of matrices were defined. Selected properties of these matrices are now discussed. Consider the diagonal matrices that contain the hyperbolic functions for each pipe in a network, **c**, **s** and **t**, which are related by

$$\begin{aligned} \mathbf{c}^2 - \mathbf{s}^2 &= \mathbf{I}_{np} \\ \mathbf{t} &= \mathbf{s}\,\mathbf{c}^{-1} \end{aligned} \tag{41}$$

Many topological matrices share relationship based on basic system connectivity ideas. The matrices **B1**, **B2**, **B3**, and **B4** share relationships by noticing that no pipe can simultaneously enter a reservoir and enter a node at the same time, giving

$$\begin{aligned} \mathbf{B1}_D^T\mathbf{B2}_D &= \mathbf{0} \quad \mathbf{B1}_D^T\mathbf{B4}_D = \mathbf{0} \\ \mathbf{B3}_D^T\mathbf{B2}_D &= \mathbf{0} \quad \mathbf{B3}_D^T\mathbf{B4}_D = \mathbf{0} \end{aligned} \tag{42}$$

Similarly, no pipe can simultaneously exit a reservoir and exit a node, giving

$$\begin{aligned} \mathbf{B1}_U^T\mathbf{B2}_U &= \mathbf{0} \quad \mathbf{B1}_U^T\mathbf{B4}_U = \mathbf{0} \\ \mathbf{B3}_U^T\mathbf{B2}_U &= \mathbf{0} \quad \mathbf{B3}_U^T\mathbf{B4}_U = \mathbf{0} \end{aligned} \tag{43}$$

Additionally, the $\mathbf{B5}_U$, $\mathbf{B5}_D$ and **B5** matrices can be formed from existing matrices $\mathbf{B2}_U$, $\mathbf{B2}_D$, $\mathbf{B4}_U$, and $\mathbf{B4}_D$ as

$$\begin{aligned} \mathbf{B5}_D &= \mathbf{B2}_D^T\mathbf{B4}_D \\ \mathbf{B5}_U &= \mathbf{B2}_U^T\mathbf{B4}_U \end{aligned} \tag{44}$$

Similar relationships can be found in topological matrices for steady-state analysis (see Eqs. (89) and (90)).

Comparison to Steady-State Analysis

Given that both steady-state analysis and frequency-domain analysis can be performed in networks sharing the same topology, it comes as no surprise that some matrices from both analyses are related. Appendix A outlines three formulations (head, flow and loop) for the steady state solution in an arbitrary pipe network. The relationship between the $\mathbf{B1}_D$ and $\mathbf{B1}_U$ matrices and the steady-state topological node incidence matrix **A1** (see Eq. (53)) is

$$\mathbf{A1} = \mathbf{B1}_D - \mathbf{B1}_U \tag{45}$$

The relationship between the $\mathbf{B2}_D$ and $\mathbf{B2}_\mathrm{U}$ and the steady-state topological reservoir incidence matrix **A2** (see Eq. (57)) is

$$\mathbf{A2} = \mathbf{B2}_D - \mathbf{B2}_U \tag{46}$$

Other similarities occur in the shape of the linear systems formed when finding solutions in terms of heads (or complex head perturbations). The $\mathbf{M}_h$ matrix in the $\hat{h}$-

formulation (Eq. (30)) has elements in identical locations to the $\mathbf{J}_H$ of the steady-state *H*-formulation (see Eq. (83)) and the **P** matrix of the steady-state *QH*-formulation (see Eq. (74)). Zecchin *et al*. (2009) term the $\mathbf{J}_H$ matrix a hydraulic admittance matrix, as it maps from pressure to flow. A more in-depth comparison of the element locations common to the formulations can be observed in (32) and (84). The similarity occurs when a node-pipe incidence matrix is multiplied by its transpose. The resulting matrix is sparse and symmetric and in the case of the steady-state formulation is positive definite.

Other similarities are that the formulation for the frequency-domain $\hat{q}$-formulation (see Eq. (39)) and the steady-state *Q*-formulation (see Eq. (85)) are sparse and asymmetric. Both formulations require node element equations (continuity around a node); however, the steady-state formulation adds the loop equations (head loss corrections around a loop), whereas the frequency-domain formulation adds the pipe-node-pipe pair and reservoir-pipe pair equations. Both the frequency-domain $\hat{q}\hat{h}$-formulation (see Eq. (27)) and the basic steady-state *QH*-formulation (see Eq. (70)) are sparse and asymmetrical.

Computational Considerations

Given the three different formulations, a number of factors relate the linear solution to its computational efficiency, the most important being the number of unknowns of the linear system (see Table 1). In general, for a dense matrix the solution complexity is $O(n^3)$, whereas for a sparse matrix, the use of sparse matrix solvers will give a comparatively faster solution approaching $O(n^2)$. A small increase in the dimensionality of the problem results in a large increase in computational effort. This means that the $\hat{h}$-formulation, with the smallest number of unknowns, will be the computationally fastest formulation. Timing of the frequency-domain analysis for the network in Figure 6 gave the $\hat{h}$-formulation as the fastest, followed by the $\hat{q}$-formulation (43% slower), and the $\hat{q}\hat{h}$-formulation (60% slower), although this is generally problem dependent. (Note that the timings were performed by running 10,000 simulations on a PC with an Intel® Core™2 Duo CPU running Microsoft Vista ™. Relative measures are utilised to negate PC-specific results.)

For moderate and large networks the **M** matrices are sparse. Sparse matrix solvers should be used for efficient solution. Most sparse solvers have a pre-conditioning (or reordering) phase that would only need to be performed once as the topology of the **M** matrix does not change for different frequencies. An additional computational saving can be made for the $\hat{h}$-formulation which has a symmetric **M** matrix which could be exploited. Sparse matrix solvers also reduce the amount of memory required to solve large matrices. General relationships for the number of non-zero elements of **M** are shown in Table 1 (where *nrc* is number of reservoir-pipe connections and *nruc* is number of reservoir-pipe connections that connect at the upstream end of the pipe.). In terms of the network in Figure 6, the percentage of non-zero elements in **M** is 11%, 67% and 17% for the $\hat{q}\hat{h}$-, $\hat{h}$- and $\hat{q}$-formulations, respectively.

Another efficiency consideration is that some of the formulations, in particular the $\hat{q}\hat{h}$- and $\hat{q}$-formulations, have significant frequency-independent parts of their **M** matrix. These parts would only be required to be computed once when solving for different frequencies, thus making a time saving. Table 1 shows relationships for the number of frequency-independent and frequency-dependent elements of **M**. In terms of the network in Figure 6, the percentage of frequency-independent elements compared to the non-zero elements in **M** is 51%, 0.0% and 25% for the $\hat{q}\hat{h}$-, $\hat{h}$- and $\hat{q}$-formulations, respectively.

With regard to the solution of the linear equations, the condition number of **M** provides information about the computability of their solution using numerical methods. If the condition number is smaller than $\sim 10^6$ then the solution is computable using single precision variables and if the condition number if less than $\sim 10^{12}$ then the solution is computable with double precision variables. Figure 8 shows the condition number for each formulation across a range of frequencies for the network in Figure 6. The $\hat{h}$-formulation has the smallest condition numbers and should be most amenable to numerical solution. The $\hat{q}\hat{h}$-formulation has the largest condition numbers and should be computed using double precision variables.

It is trivial to solve for intermediate locations along a pipe from a known $\hat{h}$ and $\hat{q}$ using Eq. (11). Hence, it is only necessary to solve for points in a network where there is a change in the pipe's properties or there is a hydraulic device. Therefore trimming those intermediate points that do not represent a change in pipe properties (and their associated $\hat{h}$ and $\hat{q}$) from the linear system will reduce its size thus increasing computational efficiency. The intermediate points are then calculated using Eq. (11) after the linear system has been solved.

Conclusions

This paper presents formulations for the frequency-domain solution in arbitrary pipe networks. The formulations focus on the topology of arbitrary networks and do not consider any complex network devices or boundary conditions, other than head and flow boundaries. The frequency-domain equations are derived for pipe networks, including the effects of unsteady friction and viscoelastic pipe material. A topological-matrix-based approach is useful to organise the system of equations. Three sets of equations have been derived for (1) node element equations, (2) pipe element equations, and (3) pipe-node-pipe pair and reservoir-pipe pair equations. Three formulations, the $\hat{q}\hat{h}$-, $\hat{h}$- and $\hat{q}$-formulations, are derived and their various merits discussed. Of the three formulations the $\hat{h}$-formulation should be the most computationally efficient and accurate. The frequency-domain solution formulations share many characteristics with the steady-state solution formulations, allowing the re-use of some of the topological matrices. The systematic approach for the frequency-domain solution in pipes networks presented in this paper does not consider other hydraulic elements, such as valves, pumps, leaks, air vessels, etc., or other boundary condition types. It is envisaged that future research will consider these other hydraulic elements and boundary conditions, although their incorporation may not be straightforward. The calculation of the frequency response function is integral to other transient analysis applications, e.g. resonance studies, time-domain simulation (IMPREM) and fault detection methods, which will benefit from the methods presented in this paper.

Appendix A: Formulations for Steady-State Analysis

This section contains a basic derivation of different formulations for the steady-state solution for arbitrary pipe networks. The section's purpose is for comparison against the different frequency-domain solution formulations. The following sections outline the basic equations and three different solution formulations. Note that the loop flow correction formulation for steady-state analysis is not presented here.

Steady-State Basic Equations

The equations of WDS analysis are based on three relationships. The first considers flow continuity at a node, which is a statement of the conservation of mass. The sign convention adopted is that all flows entering a node are positive and flows exiting a node are negative. Given the sign convention, the summation of the flows entering and exiting a node must equal zero (no accumulation of mass). The continuity equation applied at a node k (or junction) for pipes p is

$$\sum_p \left(Q_0\right)_{p,k} = \left(D_0\right)_k \tag{47}$$

The second equation for WDS analysis describes the head loss due to friction along a pipe. For a particular pipe in a WDS, the Darcy-Weisbach head loss relationship (including reservoirs) for pipe p from node k to node j is

$$\left(H_0\right)_{k,p} - \left(H_0\right)_{j,p} = \frac{f_p L_p}{2gD_p A_p^2}\left(Q_0\right)_p \left|\left(Q_0\right)_p\right| \tag{48}$$

A third set of equations can be formulated based on the property that head loss around a loop is equal to zero. For a WDS there are two types of loops: simple loops and path loops. The simple loop is an internal loop of pipes. The equation that describes the summation of the head loss in pipes p around a simple loop l is

$$\sum_p \frac{f_{p,l} L_{p,l}}{2gD_{p,l} A_{p,l}^2}\left(Q_0\right)_{p,l} \left|\left(Q_0\right)_{p,l}\right| = 0 \tag{49}$$

There are many different simple loops that can be defined for a network; however, they form a non-degenerative set (sometimes called a fundamental cycle basis). Typically, the set of loops that contain the smallest number of pipes is most desirable, the number of which is nl. The path loop considers the head loss around a loop containing two reservoirs linked by a path. The head difference between the reservoirs acts like an additional head loss element. The head loss in pipes p between reservoirs k and j around a path loop l is

$$\sum_p \frac{f_{p,l} L_{p,l}}{2gD_{p,l} A_{p,l}^2}\left(Q_0\right)_{p,l} \left|\left(Q_0\right)_{p,l}\right| = \left(R_0\right)_{k,l} - \left(R_0\right)_{j,l} \tag{50}$$

There are many different combinations of reservoirs and pipe-paths that constitute a set of path loops. Again, the multiple path loops must form a non-degenerate set with the number of path loops equal to $np - nn - nl$.

Steady-State Equations for an Arbitrary Network

The three basic relationships (node elements, pipe elements, and loop elements) for the steady-state solution in an arbitrary pipe network are written in matrix-form in this section. The node elements, representing flow continuity are considered first. The unknown steady-state flows for each pipe are

$$\mathbf{Q}_0 = \left\{\left(Q_0\right)_1, \cdots, \left(Q_0\right)_{np}\right\}^T \tag{51}$$

The known steady-state demands at each node are

$$\mathbf{D}_0 = \{(D_0)_1, \cdots, (D_0)_{nn}\}^T \tag{52}$$

The topological node-pipe incidence matrix **A1** is defined as

$$\mathbf{A1}_{pk} = \begin{cases} 1 & \text{if pipe } p \text{ enters node } k \\ 0 & \text{if pipe } p \text{ and node } k \text{ are not connected} \\ -1 & \text{if pipe } p \text{ exits node } k \end{cases} \tag{53}$$

Using Eqs. (51), (52) and (53), the flow continuity around a node (Eq. (47)) for an arbitrary pipe network can be written in matrix-form as

$$\mathbf{A1}^T\, \mathbf{Q}_0 = \mathbf{D}_0 \tag{54}$$

In a similar manner, the head loss for all pipe elements in a network can be written in matrix form. The unknown steady-state heads at each node are

$$\mathbf{H}_0 = \{(H_0)_1, \cdots, (H_0)_{nn}\}^T \tag{55}$$

The known heads at each reservoir are

$$\mathbf{R}_0 = \{(R_0)_1, \cdots, (R_0)_{nr}\}^T \tag{56}$$

The topological reservoir-pipe incidence matrix **A2** is defined as

$$\mathbf{A2}_{pk} = \begin{cases} 1 & \text{if pipe } p \text{ enters reservoir } k \\ 0 & \text{if pipe } p \text{ and reservoir } k \text{ are not connected} \\ -1 & \text{if pipe } p \text{ exits reservoir } k \end{cases} \tag{57}$$

The head loss for each pipe (Eq. (48)) for an arbitrary pipe network can be written in matrix-form as

$$\mathbf{A1}\, \mathbf{H}_0 + \mathbf{A2}\, \mathbf{R}_0 = -\mathbf{G1}\, \mathbf{Q}_0 \tag{58}$$

where **G1** is a positive valued diagonal matrix that is dependent on $\mathbf{Q}_0$ and is defined as

$$\mathbf{G1} = \mathrm{diag}\left[\frac{f_p L_p}{2gD_p A_p^2} \left|(Q_0)_p\right| \right] \qquad p = 1, \ldots, np \tag{59}$$

Finally, the loop equations are considered. The loop-pipe incidence matrix (for both simple and path loops) for pipes p that belong to loop l is defined as

$$\mathbf{A3}_{pl} = \begin{cases} 1 & \text{if pipe } p \text{ belongs to loop } l \text{ and its direction is with the loop direction} \\ 0 & \text{if pipe } p \text{ does not belong to loop } l \\ -1 & \text{if pipe } p \text{ belongs to loop } l \text{ and its direction is against the loop direction} \end{cases} \tag{60}$$

The direction of the path linking two reservoirs in a loop is defined as identical to the simple loop's direction. The loop-reservoir incidence matrix for reservoirs k that belong to loop l is defined as

$$
\mathbf{A4}_{kl} = \begin{cases} 1 & \text{if reservoir } k \text{ belongs to loop } l \text{ and its path exits the reservoir} \\ 0 & \text{if reservoir } k \text{ does not belong to loop } l \\ -1 & \text{if reservoir } k \text{ belongs to loop } l \text{ and its path enters the reservoir} \end{cases} \tag{61}
$$

The head loss for both the simple and path loops are written in matrix form for an entire network as

$$
\mathbf{A3}^T \mathbf{G1Q}_0 + \mathbf{A4}^T \mathbf{R}_0 = \mathbf{0} \tag{62}
$$

The number of loop equations (including path loops) is equal to $np - nn$. Eqs. (54), (58) and (62) form the basis for formulation to solve the steady-state in an arbitrary network.

Steady-State Solution Algorithm

The three steady-state solution formulations are considered in this section: the *Q*-Formulation, the *H*-Formulation and the *QH*-Formulation. Unlike the frequency-domain equations, the set of steady-state equations are nonlinear. The Newton-Raphson algorithm can be used to determine a set of unknown variables from a set of non-linear equations. The iterative solution by the Newton-Raphson algorithm is derived by making a Taylor series expansion of a set of non-linear functions $\mathbf{Y}(\mathbf{X})$ about some initial vector of variables $\mathbf{X}_k$ (such that $\mathbf{Y}(\mathbf{X}_k)$ does not need to equal zero) as

$$
\mathbf{Y}(\mathbf{X}_k + \delta\mathbf{X}) = \mathbf{Y}(\mathbf{X}_k) + \mathbf{J}(\mathbf{X}_k)\delta\mathbf{X} + O(\delta\mathbf{X}^2) \tag{63}
$$

Ignoring the higher order terms and assuming that the perturbation of $\mathbf{X}_k$ by $\delta\mathbf{X}$ results in the correct steady-state solution (i.e., $\mathbf{Y}(\mathbf{X}_k+\delta\mathbf{X}) = \mathbf{0}$) produces

$$
\mathbf{0} = \mathbf{Y}(\mathbf{X}_k) + \mathbf{J}(\mathbf{X}_k)\delta\mathbf{X} \tag{64}
$$

where $\mathbf{J}$ is the Jacobian matrix that is defined as

$$
\mathbf{J}(\mathbf{X}) = \frac{\partial}{\partial \mathbf{X}} \mathbf{Y}(\mathbf{X}) \tag{65}
$$

Rearranging for $\delta\mathbf{X}$ gives

$$
\delta\mathbf{X} = -\mathbf{J}^{-1}(\mathbf{X}_k)\mathbf{Y}(\mathbf{X}_k) \tag{66}
$$

The final set of unknowns is calculated by addition of $\delta\mathbf{X}$ to $\mathbf{X}_k$ as

$$
\mathbf{X}_{solution} = \mathbf{X}_k + \delta\mathbf{X} \tag{67}
$$

If the vector of functions $\mathbf{Y}(\mathbf{X})$ is linear, then the solution vector of variables is

$$
\mathbf{X}_{solution} = \mathbf{X}_k - \mathbf{J}^{-1}(\mathbf{X}_k)\mathbf{Y}(\mathbf{X}_k) \tag{68}
$$

If the vector of functions $\mathbf{Y}(\mathbf{X})$ is non-linear, then the vector of variables $\mathbf{X}$ is iterated using the formula

$$\mathbf{X}_{k+1} = \mathbf{X}_k - \mathbf{J}^{-1}(\mathbf{X}_k)\mathbf{Y}(\mathbf{X}_k) \tag{69}$$

The Newton-Raphson algorithm exhibits quadratic convergence in the neighbourhood of the solution. The iterative solution procedure concludes when convergence criteria are met. The most computationally intensive component of the Newton-Raphson algorithm is dealing with the inversion or decomposition of the Jacobian matrix. The following sections consider the form of the Jacobian derived for each formulation.

Steady-State QH-Formulation

The first formulation considers the solution of both heads and flows simultaneously. The two relationships required to form a solvable system are Eqs. (54) and (58) which can be written in matrix form as

$$\left[\begin{array}{c:c} \mathbf{G1} & \mathbf{A1} \\ \hdashline \mathbf{A1}^T & \mathbf{0} \end{array}\right]\left\{\begin{array}{c} \mathbf{Q}_0 \\ \hdashline \mathbf{H}_0 \end{array}\right\} = \left\{\begin{array}{c} -\mathbf{A2}\,\mathbf{R}_0 \\ \hdashline \mathbf{D}_0 \end{array}\right\} \tag{70}$$

where both $\mathbf{Q}_0$ and $\mathbf{H}_0$ are required to be solved. Rearranging the above equation gives a set of non-linear equations $\mathbf{Y}$, the roots of which can be solved for using the Newton-Raphson algorithm.

$$\mathbf{Y}_{QH} = \left[\begin{array}{c:c} \mathbf{G1} & \mathbf{A1} \\ \hdashline \mathbf{A1}^T & \mathbf{0} \end{array}\right]\left\{\begin{array}{c} \mathbf{Q}_0 \\ \hdashline \mathbf{H}_0 \end{array}\right\} + \left\{\begin{array}{c} \mathbf{A2}\,\mathbf{R}_0 \\ \hdashline -\mathbf{D}_0 \end{array}\right\} \tag{71}$$

The Jacobian matrix is equal to

$$\mathbf{J}_{QH} = \left[\begin{array}{c:c} 2\mathbf{G1} & \mathbf{A1} \\ \hdashline \mathbf{A1}^T & \mathbf{0} \end{array}\right] \tag{72}$$

The Jacobian in Eq. (72) is sparse and symmetric for the Darcy-Weisbach head loss formulation used in this paper, but can be a difficult to invert or decompose. A more efficient way to deal with the Jacobian was shown by Todini and Pilati (1988), which was originally based on the Content Model (Collins *et al*. 1978). Todini and Pilati developed an efficient approach to the inversion of the Jacobian by partitioning as

$$\left[\begin{array}{c:c} \mathbf{T}^{-1} & \mathbf{A1} \\ \hdashline \mathbf{A1}^T & \mathbf{0} \end{array}\right]^{-1} = \left[\begin{array}{c:c} \mathbf{T} - \mathbf{T}\,\mathbf{A1}\,\mathbf{P}^{-1}\,\mathbf{A1}^T\,\mathbf{T} & \mathbf{T}\,\mathbf{A1}\,\mathbf{P}^{-1} \\ \hdashline \mathbf{P}^{-1}\,\mathbf{A1}^T\,\mathbf{T} & -\mathbf{P}^{-1} \end{array}\right] \tag{73}$$

where the positive diagonal matrix $\mathbf{G1}$ is dependent of $\mathbf{Q}_0$ and where the sub-matrices $\mathbf{T}$ and $\mathbf{P}$ are defined as

$$\mathbf{P} = \mathbf{A1}^T\,\mathbf{T}\,\mathbf{A1} \quad \text{and} \quad \mathbf{T} = (2\mathbf{G1})^{-1} \tag{74}$$

The critical and time-consuming step in the inversion of the Jacobian is inverting the sub-matrix $\mathbf{P}$. The matrix $\mathbf{P}$ is symmetric, diagonally dominant, has positive diagonal elements and has either zero or negative off-diagonal elements and is positive definite and of Stieltjes type. Also, for large networks **P** is sparse. Todini and Pilati (1988) suggest the use of the ICF/MCG algorithm for the efficient inversion of **P**.

$$
\begin{aligned}
(\mathbf{P})_{jj} &= n^{-1}\sum_i G1_{ii}^{-1} \text{ for all pipes } i \text{ connected to node } j \\
(\mathbf{P})_{jk} &= \begin{cases} -n^{-1}\sum_i G1_{ii}^{-1} \text{ for all pipes } i \text{ connecting node } j \text{ to node } k \\ \text{zero if node } j \text{ is not connected to node } k \end{cases} \\
&= (\mathbf{P})_{kj}
\end{aligned} \quad (75)
$$

The method of solution is applied in the following steps. First the following system of equations is solved for $(\mathbf{H}_0)_{k+1}$ as

$$\mathbf{P}_k\,(\mathbf{H}_0)_{k+1} = \tfrac{1}{2}\,\mathbf{A1}^T\,(\mathbf{Q}_0)_k - \mathbf{D}_0 - \mathbf{A1}^T\,\mathbf{T}_k\,\mathbf{A2}\,\mathbf{R}_0 \quad (76)$$

Then $(\mathbf{H}_0)_{k+1}$ is used to calculate $(\mathbf{Q}_0)_{k+1}$ by

$$(\mathbf{Q}_0)_{k+1} = \tfrac{1}{2}(\mathbf{Q}_0)_k - \mathbf{T}_k\,\left(\mathbf{A1}\,(\mathbf{H}_0)_{k+1} + \mathbf{A2}\,\mathbf{R}_0\right) \quad (77)$$

Steady-State H-Formulation

An alternative to the Q-formulation is to formulate the WDS equations in terms of the heads. In order to achieve this, Eq. (48) is rearranging to be in terms of the flows as

$$(Q_0)_p = \left(\frac{f_p L_p}{2gD_p A_p^2}\right)^{-0.5} \left((H_0)_{k,p} - (H_0)_{j,p}\right)\left|(H_0)_{k,p} - (H_0)_{j,p}\right|^{-0.5} \quad (78)$$

For an entire network, the matrix-based form of (48) in terms of **H** is

$$\mathbf{Q}_0 = -\mathbf{G2}\left(\mathbf{A1}\,\mathbf{H}_0 + \mathbf{A2}\,\mathbf{R}_0\right) \quad (79)$$

where the positive valued diagonal matrix **G2** is defined as

$$\mathbf{G2} = \mathrm{diag}\left[\left(\frac{f_p L_p}{2gD_p A_p^2}\left|(H_0)_{k,p} - (H_0)_{j,p}\right|\right)^{-0.5}\right] \qquad p = 1, \ldots, np \quad (80)$$

Now that a relationship exists for the flows in terms of the heads, this relationship can be substituted in the continuity equations (Eq. (54)) giving

$$\mathbf{A1}^T\,\mathbf{G2}\left(\mathbf{A1}\,\mathbf{H}_0 + \mathbf{A2}\,\mathbf{R}_0\right) + \mathbf{D}_0 = \mathbf{0} \quad (81)$$

The above set of equations represents the steady-state equations for a WDS in terms of the heads. Rearranging gives the set of functions **Y** for the Newton-Raphson algorithm as

$$\mathbf{Y}_H = \mathbf{A1}^T\,\mathbf{G2}\left(\mathbf{A1}\,\mathbf{H}_0 + \mathbf{A2}\,\mathbf{R}_0\right) + \mathbf{D}_0 \quad (82)$$

The Jacobian for the Newton-Raphson algorithm is

$$\mathbf{J}_H = \mathbf{A1}^T\left(\tfrac{1}{2}\mathbf{G2}\right)\mathbf{A1} \tag{83}$$

The matrix $\mathbf{J}_H$ is symmetric, diagonally dominant, has positive diagonal elements and has either zero or negative off-diagonal elements and is therefore positive definite and of Stieltjes type. Also, for large networks $\mathbf{J}_H$ is sparse. After solving for the heads, the solution flows can be calculated using Eq. (58). More directly, the matrix $\mathbf{J}_H$ is defined as

$$\begin{aligned}
(\mathbf{J}_H)_{jj} &= n^{-1}\sum_{p}^{nd}\mathbf{G2}_{pp} \text{ for all pipes } p \text{ connected to node } j \\
(\mathbf{J}_H)_{jk} &= \begin{cases} -n^{-1}\sum_{p}^{nd}\mathbf{G2}_{pp} \text{ for all pipes } p \text{ connecting node } j \text{ to node } k \\ \text{zero if node } j \text{ is not connected to node } k \end{cases} \\
&= (\mathbf{J}_H)_{kj}
\end{aligned} \tag{84}$$

It is to be noted that this matrix is similar to the Jacobian in the *QH*-formulation. In fact, both are of identical dimension and have identically located elements, which is obvious since both have similar components (i.e., $\mathbf{P} = \mathbf{A1}^T(½\mathbf{G1}^{-1})\,\mathbf{A1}$ and $\mathbf{J}_H = \mathbf{A1}^T(½\mathbf{G2})\mathbf{A1}$).

Steady-State Q-Formulation

Rearranging the basic WDS equations to be in terms of the flows only produces the *Q*-formulation. The *Q*-formulation considers the continuity equations (Eq. (54)) and the head loss around a loop equations (Eq. (62)), both of which are only dependent on $\mathbf{Q}_0$. Eqs. (54) and (62) can be written in a matrix form as

$$\begin{bmatrix} \mathbf{A1}^T \\ \hdashline \mathbf{A3}^T\mathbf{G1} \end{bmatrix}\mathbf{Q}_0 = \begin{Bmatrix} \mathbf{D}_0 \\ \hdashline -\mathbf{A4}^T\mathbf{R}_0 \end{Bmatrix} \tag{85}$$

In terms of the Newton-Raphson algorithm, the set of functions $\mathbf{Y}$ is

$$\mathbf{Y}_Q = \begin{bmatrix} \mathbf{A1}^T \\ \hdashline \mathbf{A3}^T\ \mathbf{G1} \end{bmatrix}\mathbf{Q}_0 + \begin{Bmatrix} -\mathbf{D}_0 \\ \hdashline \mathbf{A4}^T\ \mathbf{R}_0 \end{Bmatrix} \tag{86}$$

and the Jacobian is

$$\mathbf{J}_Q = \begin{bmatrix} \mathbf{A1}^T \\ \hdashline \mathbf{A3}^T\ 2\mathbf{G1} \end{bmatrix} \tag{87}$$

The Jacobian for the *Q*-formulation is sparse, but neither symmetric nor positive definite.

Steady-State Matrix Relationships

Some relationships exist between the steady-state topological matrices. Substituting Eq. (58) into Eq. (62) results in

$$\mathbf{A3}^T\,\mathbf{A1}\,\mathbf{H}_0 + \mathbf{A3}^T\,\mathbf{A2}\,\mathbf{R}_0 = \mathbf{A4}^T\,\mathbf{R}_0 \tag{88}$$

By observation the following relationships can be realised

$$\mathbf{A3}^T\;\mathbf{A2} = \mathbf{A4}^T \tag{89}$$

$$\mathbf{A3}^T\;\mathbf{A1} = \mathbf{0} \tag{90}$$

Although not presented here, other graph-theoretic relationships exist for topological matrices, such as derivation of the **A3** and **A4** pipe-loop incidence matrices from the pipe-node incidence matrices **A1** and **A2**.

Notation

A	=	cross-sectional pipe area;
a	=	wave speed;
A1	=	steady-state topological matrix (pipe-node incidence);
A2	=	steady-state topological matrix (pipe-reservoir incidence);
A3	=	steady-state topological matrix (pipe-loop incidence);
A4	=	steady-state topological matrix (reservoir-loop incidence);
B1	=	frequency-domain topological matrix (pipe-node incidence);
B2	=	frequency-domain topological matrix (pipe-reservoir incidence);
B3	=	frequency-domain topological matrix (common node pipe-pair incidence);
B4	=	frequency-domain topological matrix (reservoir pipe-pair incidence);
B5	=	frequency-domain topological matrix (reservoir pipe-pair incidence);
D	=	pipe diameter, demand;
d	=	perturbation in demand;
E	=	Young's modulus of elasticity;
e	=	pipe wall thickness;
f	=	Darcy-Weisbach friction factor;
G1, G2	=	steady-state diagonal matrices for steady friction components;
g	=	gravitational acceleration;
H	=	head (unknown head);
h	=	perturbation in head (unknown head);
I	=	identity matrix;
i	=	imaginary unit $\left(= \sqrt{-1}\right)$;
J	=	Jacobian matrix (Newton-Raphson algorithm);
J_e	=	elastic component of the creep compliance function;

J_r = retarded component of the creep compliance function;

K = bulk modulus of elasticity of fluid;

L = pipe length;

$\mathbf{M}$ = coefficient matrix;

$\mathbf{N}$ = right-hand-side vector/matrix;

nc = number of network components;

nd = degree of node;

nl = number of loops;

nn = number of nodes (unknown heads);

np = number of pipes;

nr = number of reservoirs (known heads);

nrc = number of reservoir-pipe connections;

$nruc$ = number of reservoir-pipe upstream connections;

Q = flow (unknown flow);

q = perturbation in flow (unknown flow);

R = reservoir head (known head);

r = perturbation in reservoir head (known head);

R_S = steady friction coefficient;

R_U = unsteady friction coefficient;

R_V = viscoelastic coefficient;

$\mathbf{s}$, $\mathbf{c}$, $\mathbf{t}$, $\mathbf{z}$ = frequency-domain diagonal matrices for sinh, cosh, tanh and Z components;

$\mathbf{T}$, $\mathbf{P}$ = steady-state matrices for Todini & Pilati algorithm;

t = time;

W = weighting function;

$\mathbf{X}$, $\mathbf{Y}$ = steady-state solution matrices (Newton-Raphson algorithm);

x = distance;

Z = characteristic impedance;

α = pipe constraint coefficient;

γ = propagation constant;

ρ = density of liquid;

ν = kinematic viscosity;

ω = angular frequency;

Subscripts:

0	=	initial or steady-state quantity;
U	=	upstream end of pipe;
D	=	downstream end of pipe;
QH	=	relating to the steady-state QH-formulation;
H	=	relating to the steady-state H-formulation;
Q	=	relating to then steady-state Q-formulation;
qh	=	relating to the frequency-domain $\hat{q}\hat{h}$-formulation;
h	=	relating to the frequency-domain $\hat{h}$-formulation;
q	=	relating to the frequency-domain $\hat{q}$-formulation;

Font Types:

Uppercase denotes a full quantity;

Lowercase denotes a perturbation quantity;

A caret denotes a Fourier transformed quantity;

Bold denotes a vector or matrix quantity;

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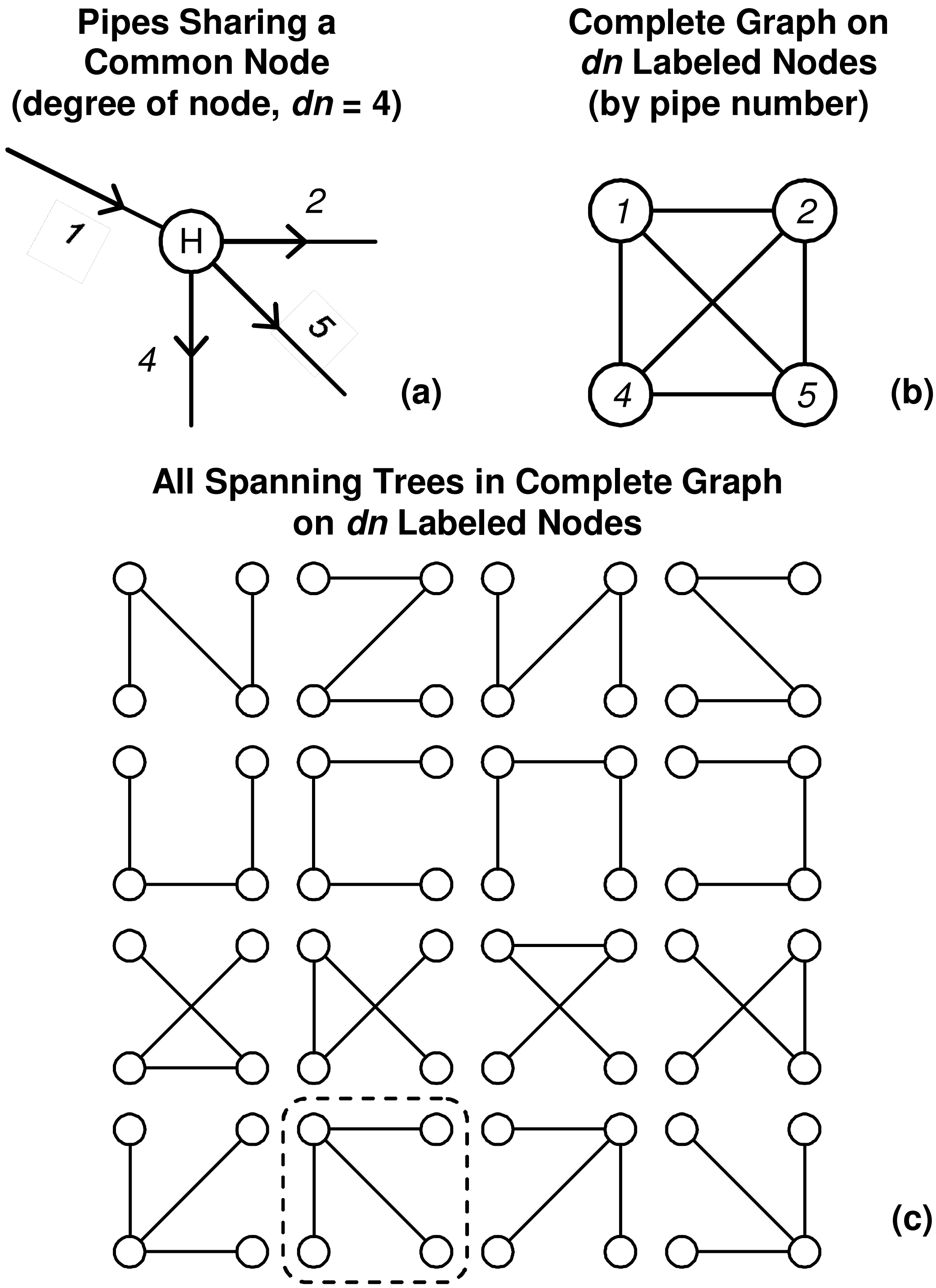
List of Figures and Tables

* Note that all figures are designed to be reduced and to fit within one column of the JHE.

** If these figure and table titles are deleted there will be errors from broken cross-references.



No. possible pipe pairs = $\frac{1}{2}(dn^2 - dn) = 6$

No. pipe pairs required to form a non-degenerative set = $dn - 1 = 3$

No. possible non-degenerative sets of pipe pairs = $dn^{dn-2} = 16$

Figure 1. Pipe pairings around a node in the example network

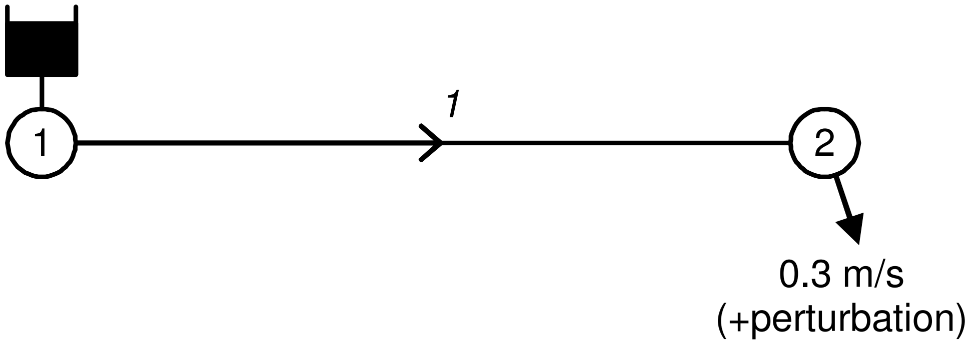


Figure 2. Example pipeline (from Vítkovský *et al*. 2006)

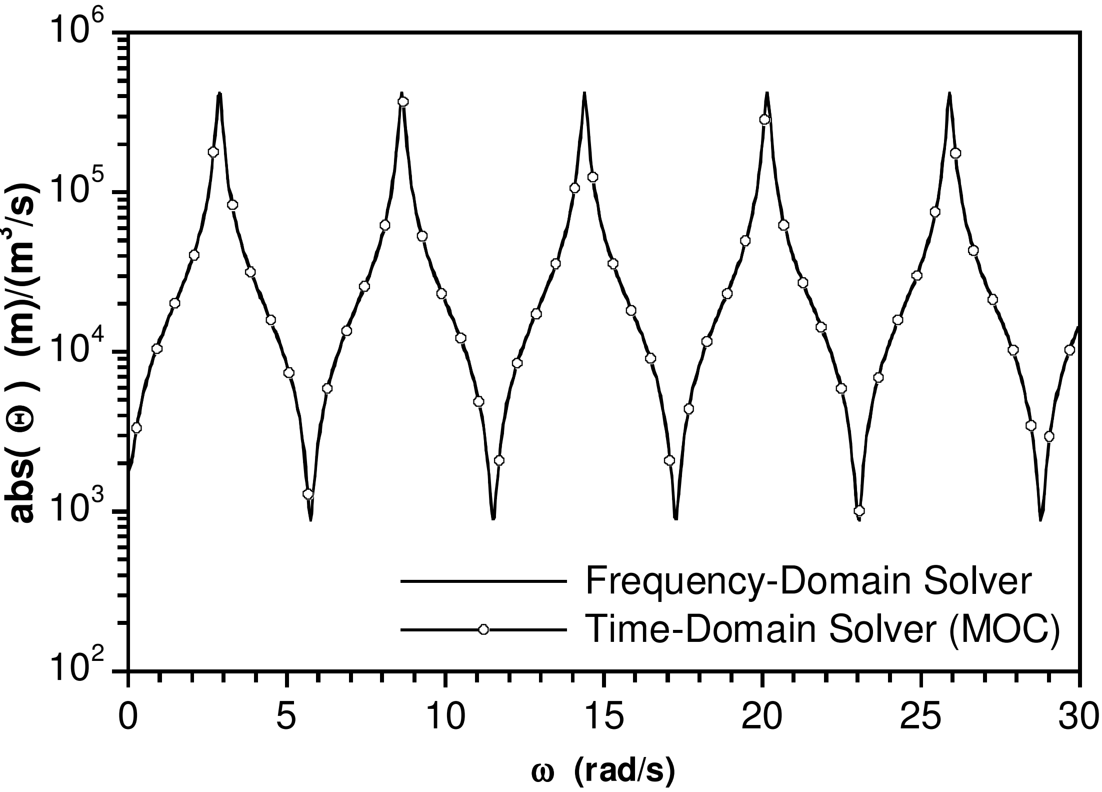


Figure 3. Frequency- and time-domain solutions for example pipeline with steady friction only

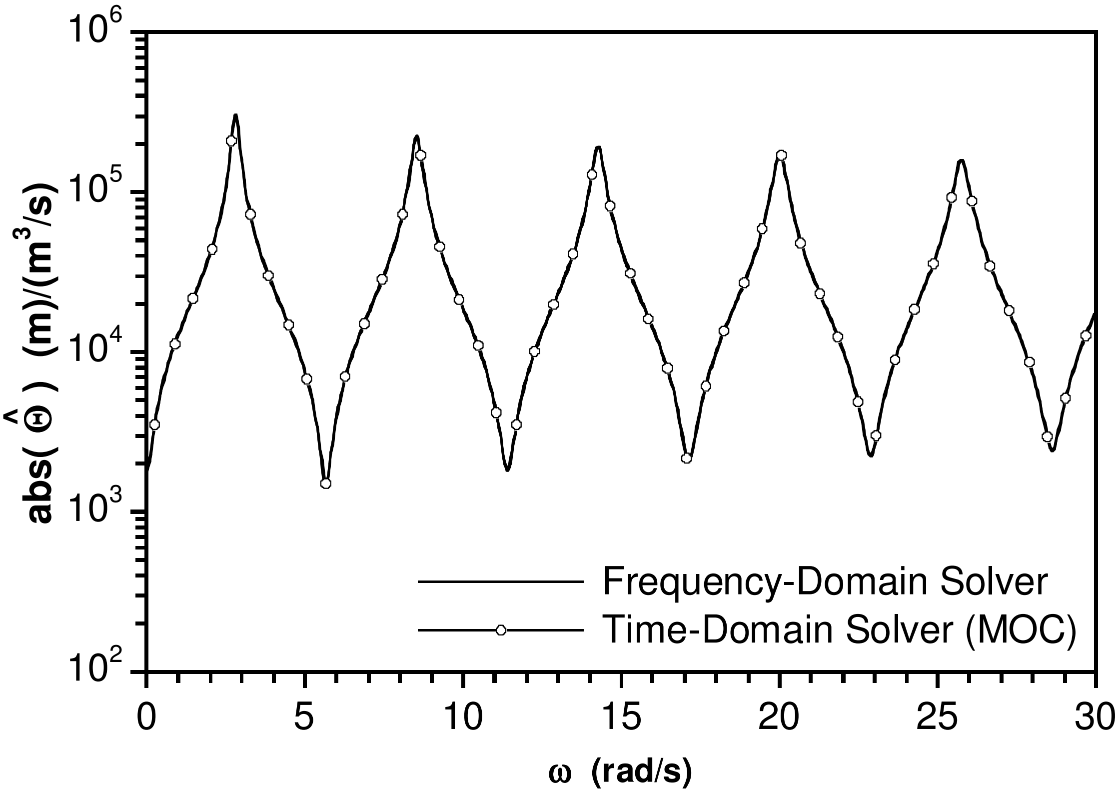


Figure 4. Frequency- and time-domain solutions for example pipeline with steady and unsteady friction only

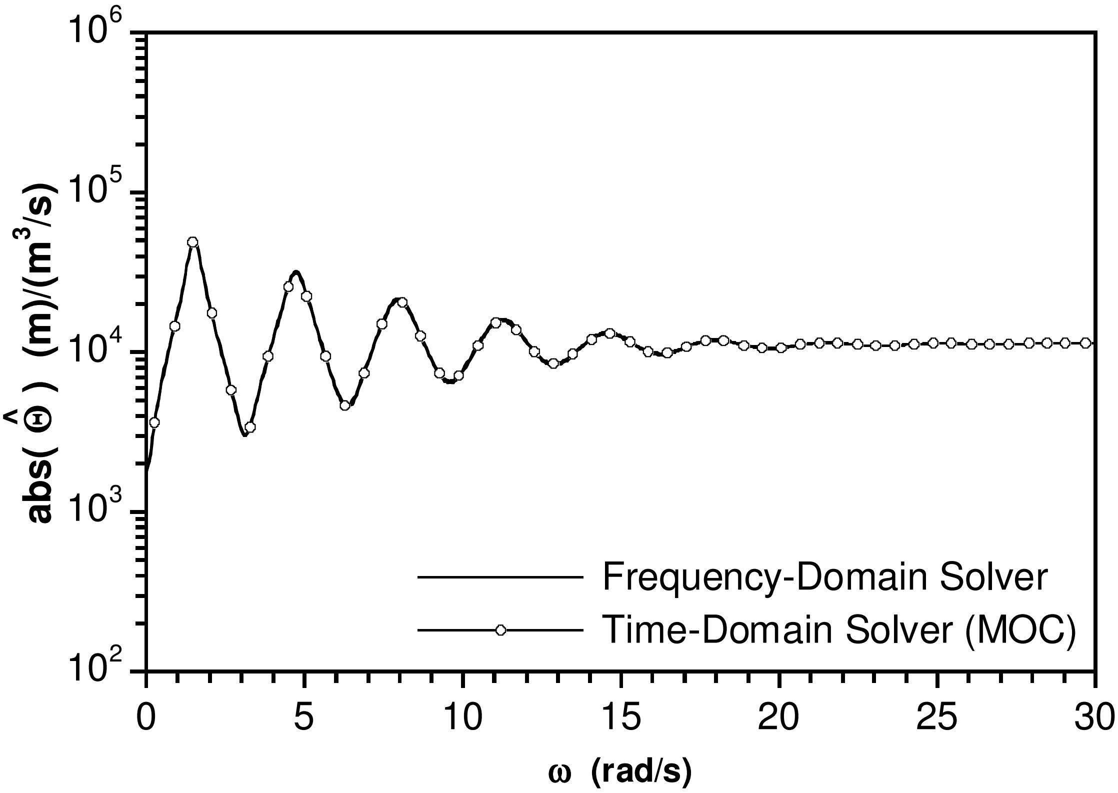


Figure 5. Frequency- and time-domain solutions for example pipeline with steady and unsteady friction and viscoelastic pipe material

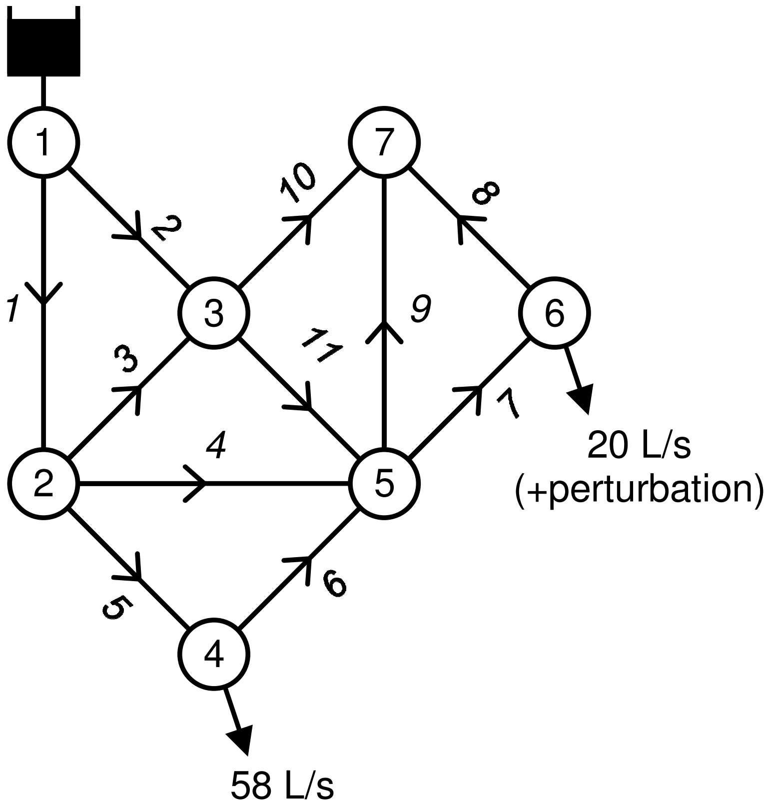


Figure 6. Example pipe network (from Liggett and Chen 1994)

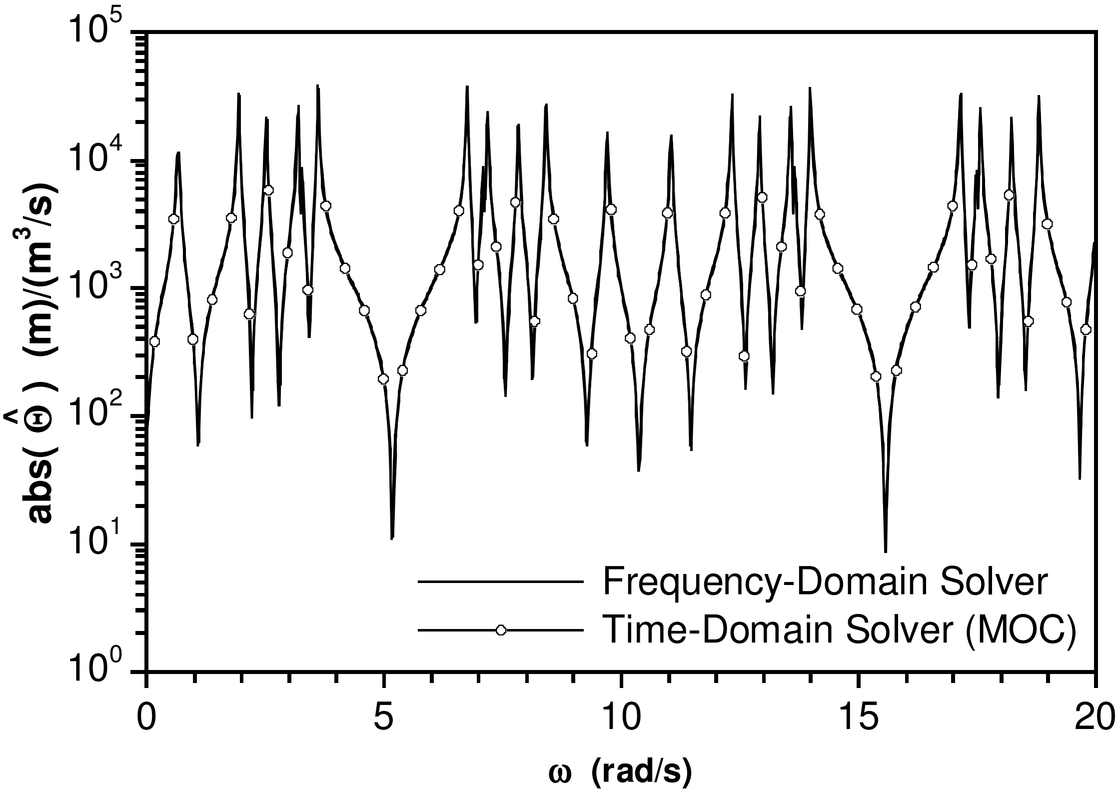


Figure 7. Frequency- and time-domain solutions for example pipe network

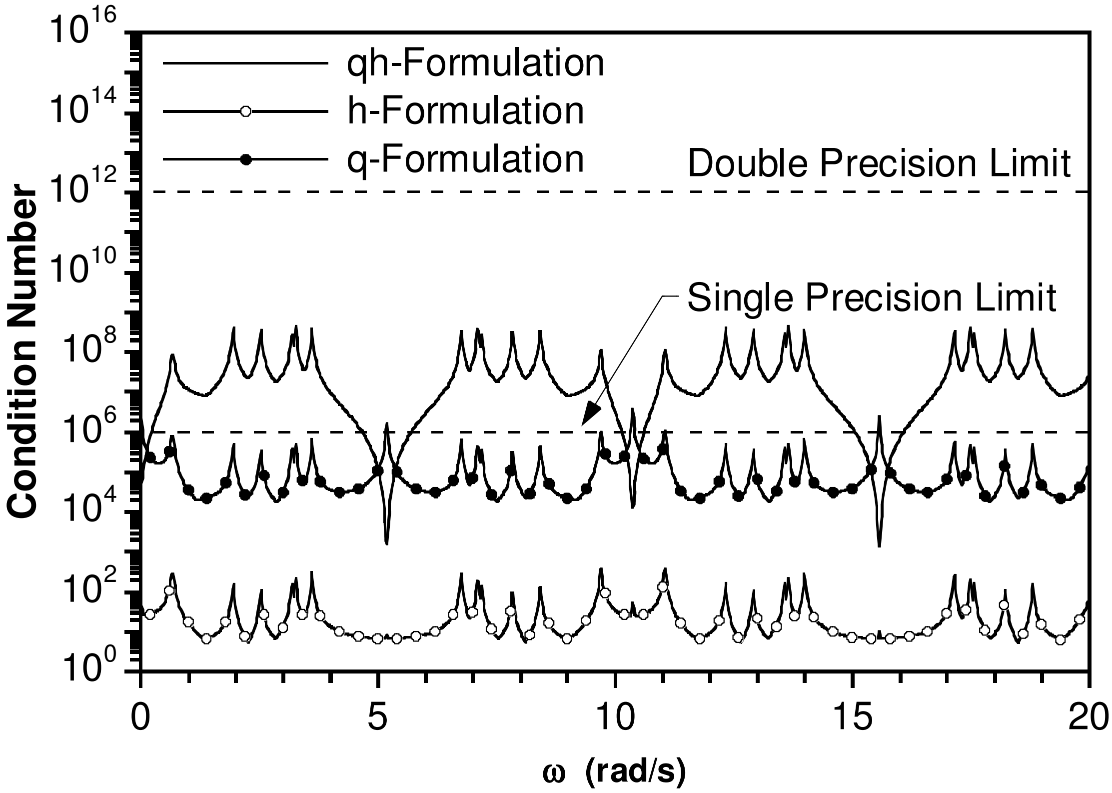


Figure 8. Condition number of coefficient matrix for example pipe network

Table 1. Properties of the coefficient matrix M

Matrix **M** Property	$\hat{q}\hat{h}$ -formulation	$\hat{h}$ -formulation	$\hat{q}$ -formulation
Unknowns	$2np+nn$	nn	$2np$
Total Elements	$(2np+nn)^2$	nn^2	$(2np)^2$
Non-Zero Elements	$8np-2nrc-nruc$	$2np+nn-2nrc$	$10np-4nn-3nrc$
Frequency-Independent Elements	$4np-2nrc+nruc$	0	$2np-nrc$
Frequency-Dependent Elements	$4np-2nruc$	$2np+nn-2nrc$	$8np-4nn-2nrc$