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Population Monotonic and Strategy-proof Mechanisms Respecting Welfare Lower Bounds*

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Abstract

The significance of *population monotonicity* and welfare bounds is well-recognized in the fair division literature. We consider the welfare bounds that are central to the fair allocation literature, namely, the *identical-preferences lower-bound, individual rationality,* the *stand-alone lower-bound,* and $k-$*fairness.* We characterize *population monotonic* and incentive compatible mechanisms which allocate an object efficiently and respect a welfare lower bound chosen in the fair allocation problem of allocating a collectively owned indivisible good or bad when monetary transfers are possible and preferences are private information.
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Key words: *welfare bounds, the identical-preferences lower-bound, individual rationality, the stand-alone lower-bound, k-fairness, population monotonicity, collective ownership, allocation of an indivisible good and money, NIMBY problems, imposition of a task, the Groves mechanisms, strategy-proofness.*

1 Introduction

One of the main distributive concerns of society is providing a safety net to its members. It is important for the functioning of a society that agents know their worst-case-scenario welfare levels. The minimal welfare guaranteed to the members of a society is generally an indicator of the level of solidarity and development of the society. Fairness also calls for the presence of welfare bounds that protect agents from factors for which they are not responsible, such as other agents' preferences or changes in population. We investigate welfare bounds in problems where population varies.

When a population varies, the resources may need to be reallocated. In doing so, two concerns could arise. First, since no agent is individually responsible for the population change, solidarity would call for the welfare levels of all agents in the initial population to change in the same direction (*population monotonicity*; Thomson, 1983). Second, the center that allocates the resources may wish to ensure that the welfare bound it implemented in the initial economy is still respected in the new economy. Our aim is to investigate whether these two goals are attainable simultaneously.

Population monotonicity is perhaps the most central axiom in the analysis of variable population models. This axiom has been analyzed in several problems such as bargaining theory, coalitional games, house allocation problems, as well as fair allocation problems in exchange economies; economies with single-peaked preferences, with public goods, with indivisible goods etc. (see Thomson, 1995a, for a survey). *Population monotonicity* is considered as a strong property frequently

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incompatible with efficiency and fairness criteria (see, for instance, Kim, 2003; Thomson, 1995b; Alkan, 1994; Tadenuma and Thomson, 1993). Even if, *population monotonic* mechanisms that meet fairness criteria such as *no-envy* and *egalitarian-equivalence*[1] exist in some models, such mechanisms may still generate "socially unacceptable" low levels of welfare to some agents in some economies. Hence, an allocation that respects an equitable welfare lower bound may be preferable to a, say, *envy-free* allocation.

The so called "fair allocation problem" arises when agents have equal rights or responsibilities over the allocated resources, that is, the resources are collectively owned. The literature on fair allocation proposed welfare bounds which not only act as a safety net, but also embody equity notions. A common way to incorporate notions of fairness in the design of a welfare bound is by carrying out the following thought experiment: First, the society agrees on a basic set of fairness notions that should be applied in a hypothetical "*reference economy*". These fairness notions determine an allocation and associated welfare levels in this reference economy. Then, these welfare levels are taken as a benchmark for the actual economy.

Perhaps, the oldest equitable welfare lower bound is the one that guarantees the welfare at the equal split of resources.[2] In economies where indivisible goods and money are allocated, since equal division is not well defined, an alternative fairness axiom is the following: Pick an agent and consider a reference economy where all agents have preferences identical to hers. Find the common welfare level enjoyed at a Pareto-efficient and *envy-free* allocation. Since each agent is only responsible for her own preference but not for the heterogeneity in the preferences in the actual economy, no agent should be worse off than she was in her reference economy (*identical-preferences lower-bound,* Moulin, 1990). Other welfare lower bounds that are mostly studied in the fair allocation literature include *individual rationality* (respecting the status-quo), the *stand-alone lower-bound* (respecting the autonomy of agents), and $k-$*fairness* (a welfare bound based on Rawlsian maxmin criterion, introduced by Porter, Shoham, and Tennenholtz, 2004). Except for $k-$*fairness,* these welfare bounds guarantee each agent a welfare level that depends only on her own preference. Hence, under the veil of ignorance about other agents' preferences, each agent knows beforehand the minimal welfare level she would attain once others' preferences are revealed and the allocation is realized.

The above-mentioned welfare bounds can be applied both in the fixed and the variable population cases. When the population varies, another appealing way to design a welfare bound is by utilizing notions of solidarity, which in general, give rise to welfare bounds: consider economic factors for which no agent is individually responsible but which affect the welfare levels of all agents (e.g. population, resources etc.). If there is a change in such a factor, then by solidarity, the welfare levels in the initial economy should be bounds on welfare in the new economy after the change. In our model, an increase in population is good news; hence, *population monotonicity* requires the welfares in the initial economy to be lower bounds on the welfares of the initial agents in the larger population.

Compatibility of *population monotonicity* with welfare bounds is not always guaranteed. For instance, when agents share a single, divisible, non-disposable task over which they have single-peaked preferences, *population monotonicity* is incompatible with the *equal-split welfare lower bound* (Thomson, 1995b). We investigate its compatibility with each of the aforementioned welfare lower bounds in a model where a central authority needs to assign an object, i.e., an indivisible

[1] *No-envy* (Foley, 1967) requires that no agent prefers another agent's bundle to her own. *Egalitarian-equivalence* (Pazner and Schmeidler, 1978) requires that each agent is indifferent between her bundle and a common reference bundle. In Yengin (2011), we show the existence of *population monotonic* selections from Groves mechanisms that satisfy *no-envy* and *egalitarian-equivalence.* See also Yengin (2012a).

[2] See for example Steinhaus, 1948; Dubins and Spanier, 1961; and Moulin, 1991.

good or bad (task), among a finite set of agents when monetary transfers are allowed and agents have quasilinear preferences. Some examples are auctions, imposition of a task as in government requisitions and eminent domain proceedings, and the siting problem of a discrete public good or bad[3] such as choosing the location of a desirable facility or event (Olympics event, park, international airport, etc.) or a noxious facility (waste disposal site, nuclear facility, prison etc.), also called as the "not-in-my-backyard, NIMBY" problem.[4] Without loss of generality, we consider the assignment of a task which is the case in most NIMBY problems such as determining which locality hosts a nuclear facility.

Unlike most of the previous studies on the compatibility of *population monotonicity* with welfare bounds, we analyze the case where agents' preferences are their private information; hence, an incentive compatibility requirement is needed in order to ensure that the allocation is efficient and respects welfare bounds with respect to the true preferences. We restrict our attention to the class of mechanisms that are *strategy-proof* (truthful reporting of preference is a weakly dominant strategy for all agents) and assign the task efficiently (*assignment-efficient*), i.e., the Groves mechanisms.

After we introduce the model in Section 2, we present our results in Section 3. As is well known, *assignment-efficient* and *strategy-proof* mechanisms do not balance the budget (total monetary transfer does not add up to zero in each economy). Hence, an important goal for the center is to pick a mechanism that generates the minimal possible deficit in each economy. For each of the aforementioned welfare lower bounds, our main result, Theorem 1, presents the characterizations of mechanisms that minimize the budget deficit in each economy among all *assignment-efficient*, *strategy-proof*, and *population monotonic* ones respecting the welfare bound. Proofs of the results in Section 3 are in the Appendix.

When the center is choosing a specific welfare bound to implement, several factors need to be considered: the choice of the welfare bound significantly affects not only the welfare levels attained by the agents, but also the budget deficit generated, and the simplicity of the transfer functions. Theorem 1 demonstrates how the complexity of the transfers of the *population monotonic* Groves mechanisms change as the center changes the imposed welfare lower bound. In Section 4, we informally discuss additional results on the budget deficits generated by Groves mechanisms respecting different welfare lower bounds. The formal presentation and proofs of these additional results are in the working paper version of our paper (Yengin, 2012c).

2 Model

There is an infinite set of "potential" agents indexed by the positive natural numbers $\mathbb{N} \equiv \{1, 2, ...\}$. In any given problem, only a finite number of them are present. Let $\mathcal{N}$ be the set of subsets of potential agents with at least two agents. Let $n \geq 2$ and N with $|N| = n$ be a typical element of $\mathcal{N}$.

A single task α is to be assigned by a "*center*" to one of the agents in a given population $N \in \mathcal{N}$. Each agent i has a cost function $c_i : \{\emptyset, \{\alpha\}\} \to \mathbb{R}_+$ with $c_i(\emptyset) = 0$.[5] Let $\mathcal{C}$ be the set of all such functions and $\mathcal{C}^N$ be the $n-$fold Cartesian product of $\mathcal{C}$.

[3] Assume that there is no question of whether the public good is to be provided and the only question is which locality will provide the public good and what the compensations are. In the NIMBY problem, we assume that all localities use the facility/event and derive the same benefit but only the hosting locality incurs a private cost. Monetary transfers facilitate the distribution of this cost among all localities. See Yengin (2013).

[4] Other examples include the following allocation problems: community housing, discrete resources in centrally planned economies, a commonly owned indivisible good in a cooperative enterprise such as cooperative supported agriculture, an inheritance among heirs, assignment of a job in a firm etc.

[5] As usual, $\mathbb{R}_+$ denotes the set of non-negative real numbers.

For each $N \in \mathcal{N}$, a *cost profile for* N is a list $c \equiv (c_1, ..., c_n)$. Let $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$ be the domain of cost profiles where for each $i \in \mathbb{N}$, $c_i \in \mathcal{C}$.

A cost profile defines an *economy.* Let $c, c', \widehat{c}$ be typical economies with associated agent sets $N, N', \widehat{N}$. For each $N \in \mathcal{N}$ and each $i \in N$, let c_{-i} be the cost profile of the agents in $N \backslash \{i\}$. For each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subseteq N$ and each $c \in \mathcal{C}^N$, let $c_{N'}$ be the restriction of c to N' : $c_{N'} \equiv (c_i)_{i \in N'}$.

For each $N \in \mathcal{N}$, each $k \in \{1, 2, ..., n\}$, each $c \in \mathcal{C}^N$, and each $i \in N$, let $c_{[k]}$ be the k-th cost in the ascending order of the costs in $\{c_1(\{\alpha\}), \ldots, c_n(\{\alpha\})\}$[6] and $(c_{-i})_{[k]}$ be the k-th cost in the ascending order of the costs in $\{c_1(\{\alpha\}), \ldots, c_{i-1}(\{\alpha\}), c_{i+1}(\{\alpha\}), ..., c_n(\{\alpha\})\}$.

There is a perfectly divisible good we call "money". Let t_i denote agent i's consumption of the good. We call t_i agent *i's transfer*: if $t_i > 0$, it is a transfer from the center to i; if $t_i < 0$, $|t_i|$ is a transfer from i to the center.

The center assigns the task and determines each agent's transfer. Agent i's utility when she is assigned $A_i \in \{\emptyset, \{\alpha\}\}$ and consumes $t_i \in \mathbb{R}$ is

$$u(A_i, t_i; c_i) = -c_i(A_i) + t_i.$$

For each $N \in \mathcal{N}$, let $\mathcal{A}(N) = \{(A'_i)_{i \in N} :$ for each $i \in N$, $A_i \in \{\emptyset, \{\alpha\}\}$, for each pair $\{i, j\} \subseteq N$, $A_i \cap A_j = \emptyset$, and $\bigcup\limits_{i \in N} A'_i = \{\alpha\}\}$ be the set of all possible assignments of α among the agents in N.

For each $N \in \mathcal{N}$, an *assignment for* N is a list $(A_i)_{i \in N} \in \mathcal{A}(N)$; and a *transfer profile for* N is a list $(t_i)_{i \in N} \in \mathbb{R}^N$. An *allocation for* $N \in \mathcal{N}$ is a list $(A_i, t_i)_{i \in N}$ where $(A_i)_{i \in N}$ is an assignment and $(t_i)_{i \in N}$ is a transfer profile for N.

A *mechanism* is a function $\varphi \equiv (A, t)$ defined over the union $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each economy an allocation: for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $\varphi_i(c) \equiv (A_i(c), t_i(c)) \in \{\emptyset, \{\alpha\}\} \times \mathbb{R}$.

For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, let $W(c)$ be the minimal total cost generated among all possible assignments for N. Note that $W(c) = c_{[1]}$.

Since an agent's transfer and total transfer can be of any size, every allocation is Pareto-dominated by some other allocation with higher transfers. Thus, there is no Pareto-efficient allocation in our model. However, we can define a notion of efficiency restricted to the assignment of the task. Since utilities are quasi-linear, given any economy c, an allocation that minimizes the total cost (i.e. that assigns the task to an agent with the minimal cost to perform the task) is Pareto-efficient for c among all allocations with the same, or smaller, total transfer. Our first axiom requires mechanisms to choose only such allocations.

Assignment-Efficiency: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum\limits_{i \in N} c_i(A_i(c)) = W(c)$.

Since costs are private information, an *assignment-efficient* mechanism assigns the task so that the actual total cost is minimal only if the agents report their true costs. Truthful reporting is also essential to determine the correct welfare bounds. Then, a desirable property for a mechanism is that no agent should ever benefit by misrepresenting her cost function (Gibbard, 1973; Satterthwaite, 1975).

Strategy-proofness[7]: For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $c'_i \in \mathcal{C}$, $u(\varphi_i(c); c_i) \geq u(\varphi_i(c'_i, c_{-i}); c_i)$.

[6] All ties are taken into account in this order. For instance, if there are two agents whose costs to perform the task are the lowest in c, then $c_{[1]} = c_{[2]}$.

[7] See Thomson (2005) for an extensive survey on strategy-proofness.

It is well-known that a mechanism is *assignment-efficient* and *strategy-proof* on $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ if and only if it is a Groves mechanism (follows from Holmström, 1979, since for each $N \in \mathcal{N}$, $\mathcal{C}^N$ is convex). The Groves mechanisms were introduced by Vickrey (1961), Clarke (1971), and Groves (1973). A Groves mechanism chooses, for each economy, an efficient assignment of the task. We work with single-valued Groves mechanisms and assume that each Groves mechanism is associated with a tie-breaking rule that determines which of the efficient assignments (if there are more than one) is chosen. Let $\mathcal{T}$ be the set of all possible tie-breaking rules and τ be a typical element of this set.

For each $i \in \mathbb{N}$, let h_i be a real-valued function defined over the union $\bigcup_{N\in\mathcal{N}:i\notin N} \mathcal{C}^N$. That is, for each $N \in \mathcal{N}$ with $i \in N$ and each $c \in \mathcal{C}^N$, h_i depends only on $c_{-i} \in \mathcal{C}^{N\setminus\{i\}}$. Let $h = (h_i)_{i\in\mathbb{N}}$ and $\mathcal{H}$ be the set of all such h.

The Groves mechanism associated with $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$, $\mathbf{G}^{h,\tau}$:
Let $G^{h,\tau} \equiv (A^\tau, t^{h,\tau})$ be such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $A^\tau(c)$ is an efficient assignment for c and for each $i \in N$,

$$t_i^{h,\tau}(c) \quad = \quad -\sum_{j\in N\setminus\{i\}} c_j(A_j^\tau(c)) + h_i(c_{-i}).$$

The transfer of each agent determined by a Groves mechanism has two parts. First, each agent pays the total cost incurred by all other agents at the assignment chosen by the mechanism. Second, each agent receives a constant sum of money that does not depend on her own cost.

Note that for each $h \in \mathcal{H}$, the mechanisms in $\{G^{h,\tau}\}_{\tau\in\mathcal{T}}$ are *Pareto-indifferent*[8] since for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(G_i^{h,\tau}(c); c_i) = -W(c) + h_i(c_{-i}). \tag{1}$$

3 Results

Suppose new agents join some initial population. The cost of an efficient assignment in the larger population is at most as large as the one in the smaller population[9], which is good news for the society. Since none of the agents in the initial population is responsible for the population growth, all of them should be at least as well off in the larger population as in the smaller one (Thomson, 1983). Hence, for the initially existing agents, their welfare levels in the smaller population are taken as lower bounds on their welfares in the larger population.

Population Monotonicity (PM) : For each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each $i \in N'$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_i(c_{N'}); c_i).$$

Population monotonicity together with equation (1) imply the following Lemma whose proof we omit.

[8] Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Allocations $(A_i, t_i)_{i\in N}$ and $(A'_i, t'_i)_{i\in N}$ are *Pareto-indifferent* for c if and only if for each $i \in N$, $u(A_i, t_i; c_i) = u(A'_i, t'_i; c_i)$. The mechanisms φ and φ' are *Pareto-indifferent* if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(\varphi_i(c); c_i) = u(\varphi'_i(c); c_i)$.

[9] The new agents may have lower costs than the existing ones and a reallocation of the task could lower the cost of an efficient assignment. In the worst case scenario, the initial allocation of the task is kept and the total cost does not change.

Lemma 1 *A Groves mechanism $G^{h,\tau}$ is population monotonic if and only if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each $i \in N'$, and each $c \in C^N$,*

$$h_i(c_{-i}) \geq h_i(c_{N' \setminus \{i\}}). \tag{2}$$

If participation is voluntary, then a natural requirement for a mechanism is to ensure that no agent experiences a welfare that is less than her status quo welfare when she didn't participate. Even if participation may not be voluntary, as in eminent domain proceedings, the center may still wish to ensure that agents have non-negative utilities.

Individual Rationality (IR): For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq 0.$$

Although *individual rationality* is desirable in many cases, it may not be required in situations where agents are collectively responsible for the associated cost of the assigned task. As an example, in times of a war or national emergency, the government can requisition the service of a civilian without fully compensating her cost. Since all agents are collectively responsible for the task, a fair distribution of this cost among all agents would lead to utility levels below status quo. Nevertheless, society would still be concerned with the equity of the distribution of welfare and guaranteeing a safety net to agents. Hence, welfare lower bounds which are weaker than *individual rationality* would be required.

One such welfare lower bound is the one that respects agents' autonomy. Imagine, there is only one agent in the society. Since she is the only one who is responsible for the completion of the task, she should bear all the cost. However, it would be unfair to tax this agent. Call her utility in this reference economy as her stand-alone utility. (For instance, in the NIMBY problem, this utility is the welfare when each locality autonomously builds its own facility.) In the actual economy, since all agents are collectively responsible for the cost of the task, no agent should end up worse than her stand-alone utility where she bore the cost alone.

The Stand-Alone Lower-Bound (SALB)[10]: For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq -c_i(\{\alpha\}).$$

For each $N \in \mathcal{N}$, each $i \in N$, and each $c_i \in \mathcal{C}$, let $c^i \in \mathcal{C}^N$ be agent i's "*reference*" economy in which all agents have the same cost function c_i. That is, $c^i \equiv (c^i_j)_{j \in N} \in \mathcal{C}^N$ is such that for each $j \in N$, $c^i_j \equiv c_i$. In c^i, since all agents have the same preference[11], fairness would require that they all experience the same welfare (fairness notions such as no-envy, anonymity, or equal treatment of equals would imply this result). Hence, under *assignment-efficiency* and *budget-balance*, each agent's utility would be $-\frac{W(c^i)}{n}$. Since, no agent is responsible for the preferences of the others, in the actual economy where preferences are heterogenous, no one should be worse than she is in her reference economy.

The Identical-Preferences Lower-Bound (IPLB): For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq -\frac{W(c^i)}{n}.$$

[10] Yengin (2012b) refers *SALB* as the "no-compensation lower-bound".

[11] Since utilities are quasi-linear, two agents have identical preferences if their cost functions are identical.

In exchange economies, variety in preferences is typically good news: exchange among agents who have different preferences benefit both parties of the trade. When there is only a single task to assign, then heterogeneity in preferences is never bad news (this is no longer the case in multiple-tasks setting, see Yengin, 2012c). To see this, note that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $W(c^i) = c_i(\{\alpha\})$. If there is $j \in N \backslash \{i\}$ such that $c_j(\{\alpha\}) < c_i(\{\alpha\})$, then $W(c) < W(c^i)$, otherwise $W(c) = W(c^i)$.

Consider an economy with n agents. Rawls' difference principle, also known as the *maximin criterion* requires that agents experience equal utilities unless it is possible to have a Pareto improvement on the equal utility distribution. If one requires that the center incurs no-deficit (total transfer is at most zero), then the maximin criterion implies an allocation that is *assignment-efficient, budget-balanced,* and *egalitarian* and the utility of each agent is $-\frac{1}{n}c_{[1]}$. Unfortunately, there is no *strategy-proof* mechanism which requires the utility achieved at such an allocation to be a welfare lower-bound ($1-$*fairness*) and generates no-deficit (see Corollary 1 of Porter et. al., 2004). To obtain no deficit, we can reduce the lower bound on the utilities: for each $k \in \{1, 2, ..., n\}$, consider the hypothetical economy where the task is assigned to an agent with the k-th lowest cost and utilities are equalized through transfers that balance the budget. The resulting utility of each agent is $-\frac{1}{n}c_{[k]}$. Let this reference utility be a lower bound on the actual utilities ($k-$*fairness*).[12] When the population varies, any given k will be greater than the number of people in some populations. Hence, for each $k \geq 1$, we generalize $k-$*fairness* to a variable population setting, by requiring that in each $n-$agent economy, the common reference utility is determined by assigning the task to an agent with the l-th lowest cost to perform that task where $l = \min\{k, n\}$.

Let $k \geq 1$.

k-Fairness: For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq -\frac{1}{n}c_{[\min\{k,n\}]}.$$

If the population was fixed, then the center would simply choose a welfare bound to apply in any given economy. When the population varies, the same chosen welfare bound can be applied in the new population; however, that would not guarantee that the welfares of all the initial agents are affected in the same direction. Thus, to maintain fairness, *population monotonicity* should be also satisfied. To see how much bite *population monotonicity* has in conjunction with welfare bounds, we first present the characterizations of the Groves mechanisms that respect each of the aforementioned welfare bounds without the requirement of *population monotonicity*.[13]

Lemma 2
(a) *A Groves mechanism $G^{h,\tau}$ is individually rational if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq W(c_{-i}). \tag{3}$$

(b) *A Groves mechanism $G^{h,\tau}$ respects the stand-alone lower-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq 0. \tag{4}$$

[12] Note that no deficit is compatible with $k-$*fairness* for $k \geq 3$; and for each $k \geq 1$, if a mechanism is $k-$*fair*, then it is $(k+1)-$*fair* as well.

[13] Lemma 2 generalizes to the multiple-tasks setting depending on the domain of the cost functions. For details, see Yengin (2012c).

(c) *A Groves mechanism $G^{h,\tau}$ respects the identical-preferences lower-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq \frac{n-1}{n} W(c_{-i}). \tag{5}$$

(d) *Let $k \geq 2$. A Groves mechanism $G^{h,\tau}$ is $k-$fair if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \quad \geq \quad W(c_{-i}) - \frac{1}{n}(c_{-i})_{[\min\{k,n\}-1]}. \tag{6}$$

Let us denote the class of Groves mechanisms satisfying axiom "X" as $\mathcal{G}^X$. For each axiom X, let $G^{h^X,\tau}$ be a Groves mechanism that generates the minimal deficit in each economy among all mechanisms in $\mathcal{G}^X$. Then, for each welfare bound X in Lemma 2, $G^{h^X,\tau}$ is characterized by the inequality presented in the relevant part of Lemma 2 holding as an equality. For instance, a Groves mechanism generates the minimal deficit in each economy among all *individually rational* Groves mechanisms if and only if (3) holds as an equality in each economy: the resulting mechanism $G^{h^{IR},\tau}$ is known as a Pivotal mechanism.

Under *assignment-efficiency* and *strategy-proofness*, certain logical relations exist between the above welfare bounds. For instance, by Theorem 2 and Corollary 3 in Atlamaz and Yengin (2008), $\mathcal{G}^{1-fair} \equiv \mathcal{G}^{2-fair}$. Similarly, even though the *identical-preferences lower-bound* and *$k-$fairness* stem from different intuitions and fairness ideas and in general, are not logically related, by Lemma 2 (c) and (d), $\mathcal{G}^{2-fair} \equiv \mathcal{G}^{IPLB}$.

Remark 1 *Under assignment-efficiency and strategy-proofness, the following are equivalent: the identical-preferences lower-bound, $1-$fairness, and $2-$fairness.*[14]

Remark 1 presents an interesting result: for Groves mechanisms, requiring that each agent i's utility is no less than the common utility at a Pareto-efficient and egalitarian allocation in i's reference economy c^i (i.e. *IPLB*) is equivalent to requiring that each agent's utility is no less than the common utility at a Pareto-efficient and egalitarian allocation at the actual economy c ($1-$*fairness*).

Population monotonicity can be incompatible with fairness notions and welfare bounds in other models (see, for instance, Thomson, 1995b). Fortunately, this is not the case for our model as the following result demonstrates:

Proposition 1 *If for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = \max_{j \in N \setminus \{i\}} \{c_j(\{\alpha\})\},$$

then the Groves mechanism $G^{h,\tau}$ satisfies population monotonicity, individual rationality, the identical-preferences lower-bound, the stand-alone lower-bound, and $k-$fairness for $k \geq 1$. $\Diamond$

Let $G^{h,\tau}$ be a *population monotonic* Groves mechanism. If for each pair $\{i,j\} \in \mathcal{N}$ and each $(c_i, c_j) \in \mathcal{C}^{\{i,j\}}$, $h_i(c_j) \geq 0$, then by *population monotonicity*, for each $N \in \mathcal{N}$ with $\{i,j\} \subset N$ and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) \geq h_i(c_j) \geq 0$. That is, if a *population monotonic* Groves mechanism $G^{h,\tau}$

[14] For further logical relations between the welfare lower bounds in the multiple-tasks setting, see the working paper.

respects *SALB* in two-agent economies, then it also respects *SALB* in any economy with more than two agents:

A Groves mechanism $G^{h,\tau}$ satisfies population monotonicity and the stand-alone lower-bound if and only if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each pair $\{i, j\} \subseteq N'$, and each $c \in \mathcal{C}^N$,

$$h_i(c_{-i}) \geq h_i(c_{N' \setminus \{i\}}) \text{ and } h_i(c_j) \geq 0. \tag{7}$$

So far, all the welfare bounds introduced and the results presented are easily generalizable to the setting where more than one task is to be distributed (see Yengin, 2012c). For instance, (7) characterizes the *population monotonic* Groves mechanisms respecting *SALB* even in the multiple-tasks setting. Unfortunately, if we strengthen the welfare bound to *individual rationality*, the *identical-preferences lower-bound*, or $k-fairness$, then the h functions associated with a *population monotonic* Groves mechanism do not have compact formulas unless we restrict our attention to the single-task case. Our main result next presents our characterizations for this case. Note that Theorem 1b also holds in the multiple-tasks case.

Theorem 1:

(a) *A Groves mechanism $G^{h,\tau}$ generates the minimal deficit in each economy among all Groves mechanisms that satisfy population monotonicity and individual rationality if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = (c_{-i})_{[n-1]}. \tag{8}$$

(b) *A Groves mechanism $G^{h,\tau}$ generates the minimal deficit in each economy among all Groves mechanisms that satisfy population monotonicity and the stand-alone lower-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = 0. \tag{9}$$

(c) *A Groves mechanism $G^{h,\tau}$ generates the minimal deficit in each economy among all Groves mechanisms that satisfy population monotonicity and the identical-preferences lower-bound* (or $1-$*fairness*) *if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = \max_{p \in \{1,2,\ldots,n-1\}} \left\{ \frac{p}{p+1} (c_{-i})_{[n-p]} \right\}. \tag{10}$$

(d) *Let $k \geq 2$. A Groves mechanism $G^{h,\tau}$ generates the minimal deficit in each economy among all Groves mechanisms that satisfy population monotonicity and $k-$fairness if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = \max_{t \in \{2,3,..,n\}} \max_{s \in \{1,..,n+1-t\}} \left\{ (c_{-i})_{[s]} - \frac{1}{t} (c_{-i})_{[s-2+\min(k,t)]} \right\}. \tag{11}$$

Several real life problems are concerned with the choice of the location of a single discrete public good or bad such as determining in which locality to build a nuclear facility. Allocation of a single task in a team of employees, choice of a representative or committee chair in an organization, provision of a multi-attribute public good[15], and allocation of a single estate among heirs are some of the other examples where Theorem 1 would provide useful suggestions.

[15] See Section 4 in Atlamaz and Yengin (2008).

In Theorem 1, suppose we drop the requirement of generating minimal-deficit. Without loss of generality, consider *individual rationality*. In the proof of Theorem 1a, we do not use the deficit minimization condition yet for inequalities (18) and (19). When applied jointly, Lemma 1 and Lemma 2a result in (18) and (19). That is, a Groves mechanism $G^{h,\tau}$ is *population monotonic* and *individually rational* if and only if for each $N \in \mathcal{N}$ with $|N| = 2$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i})$ is weakly greater than the right-hand-side of (18); and for each $N \in \mathcal{N}$ with $|N| \geq 3$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i})$ is weakly greater than the right-hand-side of (19). As the proof of Theorem 1a demonstrates, if for each population size and each economy with that population size, we recursively apply the deficit minimization condition on (18) and (19), then (18) and (19) must hold as equalities for each economy and we obtain (8). That is, for a *population monotonic* and *individually rational* Groves mechanism $G^{h,\tau}$, for each $N \in \mathcal{N}$ and each $i \in N$, equation (8) specifies the minimal value that $h_i(c_{-i})$ can take in each economy $c \in \mathcal{C}^N$. Hence, by (18) and (19), a Groves mechanism $G^{h,\tau}$ is *population monotonic* and *individually rational* if and only if Lemma 1 holds and for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i})$ is weakly greater than the right-hand-side of the equation (8).[16] A parallel argument applies to the other parts of Theorem 1 as well. Hence, if we drop the deficit minimization requirement, then agents would enjoy weakly higher utilities in each economy.[17]

Comparison of Theorem 1 with Lemma 2 shows that, under the deficit minimization condition, for each welfare lower bound X, additionally imposing *population monotonicity* modifies the form of the transfer functions of Groves mechanisms $G^{h^X,\tau}$ considerably and constricts the class $\mathcal{G}^X$.[18] When the deficit is minimized, it is possible to provide an intuitive interpretation of the Groves mechanisms respecting a welfare lower bound; for instance, Pivotal mechanisms grant each agent a utility equal to her externality (for the intuitive interpretation for $G^{h^{IPLB},\tau}$, see Section 4 in Yengin, 2013). On the other hand, such interpretations are hard to come up with for the transfers specified in Theorem 1. Comparison of Theorem 1 with Lemma 2 yields the following observations as well:

For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $W(c_{-i}) = (c_{-i})_{[1]} \leq (c_{-i})_{[n-1]}$. Hence, by (8) and (3), the minimal welfare levels and the minimal deficit generated in each economy among all mechanisms in $\mathcal{G}^{PM} \cap \mathcal{G}^{IR}$ are weakly higher than the minimal welfare levels and deficit generated in $\mathcal{G}^{IR}$. (See Yengin, 2012c, for a detailed analysis).

The above observation still holds if we replace *individual rationality* with the *identical-preferences lower-bound* or *$k-$fairness*. To see this, for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, when $k = 2$, the right-hand-sides of inequalities (5) and (6) are equal to $\frac{n-1}{n}(c_{-i})_{[1]}$ which is weakly smaller than the right-hand-side of (10); and the right-hand-side of (11) is weakly greater than the right-hand-side of (6) and equal to it for $t = n$.

On the other hand, for the *stand-alone lower-bound*, even though $\mathcal{G}^{PM} \cap \mathcal{G}^{SALB}$ is a strict subclass of $\mathcal{G}^{SALB}$, the minimal welfare levels and the minimal amount of deficit generated in each economy among all mechanisms in $\mathcal{G}^{PM} \cap \mathcal{G}^{SALB}$ is the same as the ones generated among all mechanisms in $\mathcal{G}^{SALB}$ due to (9) and (4).

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[16] Let $b > a > 1$ and consider $G^{\widetilde{h},\tau}$ such that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, if $|N| > 4$, then $\widetilde{h}_i(c_{-i}) = a(c_{-i})_{[n-1]}$ and if $|N| \leq 4$, then $\widetilde{h}_i(c_{-i}) = b(c_{-i})_{[n-1]}$, Then, $G^{\widetilde{h},\tau}$ is *individually rational* but not *population monotonic* (Lemma 1 is violated) even though for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $\widetilde{h}_i(c_{-i})$ is weakly greater than the right-hand-side of (8).

[17] For instance, consider $G^{\widehat{h},\tau}$ such that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $\widehat{h}_i(c_{-i}) = a(c_{-i})_{[n-1]} + b$, where $a \geq 1, b \geq 0$. Then, $G^{\widehat{h},\tau}$ is *population monotonic* and *individually rational*. The increase in deficit and total welfare when $G^{\widehat{h},\tau}$ is used instead of a mechanism as in Theorem 1a is $(a-1)((n-1)c_{[n]} + c_{[n-1]}) + nb \geq 0$.

[18] Note that adding *population monotonicity* would only affect the transfer functions when $n > 2$.

4 Remarks

In Section 4 of Yengin (2012c), we present several additional results which are structured into four subsections. Here, we briefly present and discuss these results and refer the interested readers to Yengin (2012c), for details and proofs.

In our model, by Green and Laffont (1977), no *assignment-efficient* and *strategy-proof* mechanism ensures that the *budget is balanced* (i.e., for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i \in N} t_i(c) = 0$). Fortunately, we can design ones that respect an upper bound on total transfer (budget deficit). In Subsection 4.2 of Yengin (2012c), we present upper bounds on deficit which together with particular welfare lower bounds characterize subclasses of the *assignment-efficient* and *strategy-proof* mechanisms.

Even without requiring *population monotonicity*, all the welfare lower bounds we considered are still compatible, since the Pivotal mechanisms respect each of them but violate *population monotonicity*. This result also holds in a multiple-task setting considered in Yengin (2012c). In general, even without requiring *assignment-efficiency, strategy-proofness*, or *population monotonicity, individual rationality* implies $k-$*fairness*. Similarly, in general, the *identical-preferences lower-bound* is weaker than *individual rationality* and stronger than the *stand-alone lower-bound*. Further logical relations are presented in Tables 3 and 4 of Yengin (2012c). Note that strengthening or weakening a welfare lower bound does not necessarily increase the complexity of the transfer functions as the form of the transfer functions in Theorem 1c are more complex than the ones in Theorem 1 (a) and (b).

Our goal in this paper is not to analyze by how much the budget deficit performance of a Groves mechanism would deteriorate if you strengthen the welfare lower bound. Before one can carry out in depth budget performance analysis, this paper provides the necessary first step of study, by presenting for each welfare bound, the *population monotonic* Groves mechanisms that would generate the minimal budget deficit in each economy. That being said, in Subsection 4.4 of Yengin (2012c), we present some results on the deficits generated under different welfare bounds. First, we compare the minimal deficits generated by Groves mechanisms respecting different welfare lower bounds. Our results confirm the usual trade-off between equity and efficiency: as the lower bound on welfare gets stronger (hence the welfares guaranteed become higher), the minimal possible deficit incurred by the center gets larger. An analogous comparison of deficits is also true under *population monotonicity*: in each economy, the deficit generated by mechanisms in Theorem 1a is the highest, Theorem 1b is the lowest, and Theorem 1c is in between.

Then, we investigate the amount of the increase in the minimal deficit and total welfare, when a welfare lower bound is strengthened. Such an analysis presents the trade-off the center would face when considering which welfare bound to implement. For instance, under the deficit-minimization condition, the gain in total welfare and the increase in budget deficit when *IPLB* is strengthened to *IR* is between the minimal and the second minimal cost in the economy. Finally, in Subsection 4.4.2 of Yengin (2012c), we show that for a welfare bound X among *IR, IPLB*, and $k-$*fairness*, in each economy, the difference between the minimal deficit generated among all mechanisms in $\mathcal{G}^{PM} \cap \mathcal{G}^X$ and the one generated among all mechanisms in $\mathcal{G}^X$ can be arbitrarily high and is not bounded above by any fixed amount $T \in \mathbb{R}$. We also show that for *population monotonic* Groves mechanisms, if we strengthen *SALB* to *IPLB*, or *IPLB* to *IR*, the increase in minimal deficit is at most the number of people times the maximal cost in the economy (i.e., $nc_{[n]}$).

An open question left for future research is to identify, for the multiple-tasks setting, alternative domains of cost functions or alternative cost environments in which we can find tractable expressions for *population monotonic* Groves mechanisms respecting welfare lower bounds and deficit upper bounds.

5 Appendix

Proof of Lemma 2:

(a) The proof is similar to the proof of Proposition 3 in Pápai (2003). Let $h \in \mathcal{H}$ be as in (3). Then, by equation (1), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(G_i^{h,\tau}(c); c_i) \geq W(c_{-i}) - W(c) \geq 0$. Hence, $G^{h,\tau}$ is *individually rational.*

Conversely, let $G^{h,\tau}$ be *individually rational.* Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) < W(c_{-i}). \tag{12}$$

Let $\widehat{c}_i(\{\alpha\}) = (c_{-i})_{[1]}$ and $\widehat{c} = (\widehat{c}_i, c_{-i})$. Since $c_{-i} = \widehat{c}_{-i}$ and $W(c_{-i}) = W(\widehat{c})$, by (12), $h_i(\widehat{c}_{-i}) < W(\widehat{c})$. Hence, by (1), $u(G_i^{h,\tau}(\widehat{c}); \widehat{c}_i) < 0$, which contradicts *individual rationality.*

(b) By Proposition 3a in Yengin (2012b).

(c) By Lemma 1 and Theorem 1 in Yengin (2013).

(d) The proof is implied by the proof of Theorem 1 in Atlamaz and Yengin (2008). Let $k \geq 2$ and h be as in (6). Note that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $W(c_{-i}) \geq W(c)$, and $(c_{-i})_{[\min\{k,n\}-1]} \leq c_{[\min\{k,n\}]}$. Hence, for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $W(c_{-i}) - \frac{1}{n}(c_{-i})_{[\min\{k,n\}-1]} \geq W(c) - \frac{1}{n}c_{[\min\{k,n\}]}$. By (1) and (6), $G^{h,\tau}$ is *k−fair.*

Conversely, for some $k \geq 2$, let $G^{h,\tau}$ be *k−fair.* Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) < W(c_{-i}) - \frac{1}{n}(c_{-i})_{[\min\{k,n\}-1]}. \tag{13}$$

Let $\widehat{c}_i(\{\alpha\}) = (c_{-i})_{[1]}$ and $\widehat{c} = (\widehat{c}_i, c_{-i}) \in \mathcal{C}^N$. By *k-fairness* and equation (1),

$$h_i(\widehat{c}_{-i}) \geq W(\widehat{c}) - \frac{1}{n}\widehat{c}_{[\min\{k,n\}]}. \tag{14}$$

Since $\widehat{c}_{-i} = c_{-i}$, by (13) and (14),

$$W(c_{-i}) - W(\widehat{c}) + \frac{1}{n}\left[\widehat{c}_{[\min\{k,n\}]} - (c_{-i})_{[\min\{k,n\}-1]}\right] > 0. \tag{15}$$

Since $\widehat{c}_i(\{\alpha\}) = (c_{-i})_{[1]}$, then $W(\widehat{c}) = W(c_{-i}) = (c_{-i})_{[1]}$ and $\widehat{c}_{[\min\{k,n\}]} = (c_{-i})_{[\min\{k,n\}-1]}$. These equalities together contradict (15). ■

Proof of Proposition 1: Let $k \geq 1$, $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$. Since $\max\limits_{j \in N \setminus \{i\}} c_j(\{\alpha\}) \geq W(c)$, by equation (1), $u(G_i^{h,\tau}(c); c_i) \geq 0 \geq \max\{-\frac{1}{n}W(c^i), -c_i(\{\alpha\}), -\frac{1}{n}c_{[\min\{k,n\}]}\}$. Thus, $G^{h,\tau}$ respects *IR, IPLB, SALB,* and *k-fairness.* Since for each $N' \subset N$ with $i \in N'$, $\max\limits_{j \in N \setminus \{i\}} c_j(\{\alpha\}) \geq \max\limits_{j \in N' \setminus \{i\}} c_j(\{\alpha\})$, then by (2), $G^{h,\tau}$ is *population monotonic.* ◊

For the next proofs, we need the following notation: Let $c, \widehat{c}, \widetilde{c}, ...$ denote typical economies associated with the agent sets $N, \widehat{N}, \widetilde{N}, ...$, respectively. For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $r \in \{2, 3, ..., n\}$, let $\mathcal{C}^r$ be the set of economies that have r number of agents and $\mathcal{D}^r(N, i, c)$ be the set of all economies obtained by removing the cost functions of any $(n-r)$ number of agents from c_{-i} :

For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $r \in \{2, 3, ..., n\}$,

$$\mathcal{D}^r(N, i, c) \equiv \{c' \in \mathcal{C}^r\text{: there exists } N' \subseteq N \text{ with } i \in N' \text{ and } |N'| = r \text{ such that } c' = c_{N'}\}.$$

Thus, Lemma 1 can be rephrased as follows: A Groves mechanism $G^{h,\tau}$ is population monotonic if and only if for each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq \max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \{h_i(\widetilde{c}_{-i})\}. \tag{16}$$

Note that *population monotonicity* does not impose any restriction on economies with two agents, so we take $r \geq 3$.

As an example, let $N = \{1, 2, 3, 4\}$ and $c \in \mathcal{C}^N$. Let $G^{h,\tau}$ be *population monotonic*. Then, (16) should be true for N, c, and $i = 1$. That is,

· for $r = n = 4$, $h_1(c_2, c_3, c_4) \geq \max\{h_1(c_2, c_3), h_1(c_2, c_4), h_1(c_3, c_4)\}$, and

· for $r = 3$, for each pair $\{j, k\} \subset \{2, 3, 4\}$, $h_1(c_j, c_k) \geq \max\{h_1(c_j), h_1(c_k)\}$.

Proof of Theorem 1:

(b) Let $h \in \mathcal{H}$ be as in (9). By inequality (2), $G^{h,\tau}$ is *population monotonic*. By Lemma 2b, $G^{h,\tau}$ satisfies *SALB*.

Conversely, let $G^{h,\tau}$ satisfy *population monotonicity* and *SALB*. For $G^{h,\tau}$ to generate the minimal deficit in each economy, (4) must hold as an equality in each economy, that is, h is as in (9). □

The proof for the rest of the parts is constructive.

Let $G^{h,\tau}$ be a Groves mechanism that generates the minimal deficit in each economy among all Groves mechanisms that satisfy *population monotonicity* and welfare lower bound X, where X is *IR* in part (a), *IPLB* in part (c), and $k-$*fairness* with $k \geq 2$ in part (d).

Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Note that for each $i \in N$, $W(c_{-i}) = (c_{-i})_{[1]}$.

Note that for each $i \in N$, each $r \in \{3, ..., n\}$, each $\widehat{c} \in \mathcal{D}^r(N, i, c)$, and each $s \in \{1, 2..., r-2\}$,

$$\max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} (\widetilde{c}_{-i})_{[s]} = (\widehat{c}_{-i})_{[s+1]}. \tag{17}$$

(Let c' be a maximizer of the left-hand-side of (17). Then, c' is obtained from $\widehat{c}$ by removing the cost function of an agent with the lowest cost to perform the task in $\widehat{c}_{-i}$. That is, $c' = \widehat{c}_{-j}$ where $\widehat{c}_j(\{\alpha\}) = (\widehat{c}_{-i})_{[1]}$.)

Since $\sum_{i \in N} t_i^{h,\tau}(c) = -(n-1)W(c) + \sum_{i \in N} h_i(c_{-i})$, to minimize the deficit we need to minimize $\sum_{i \in N} h_i(c_{-i})$. For each $i \in N$, *population monotonicity* and *welfare bound* X restrict the minimal value that $h_i(c_{-i})$ can take, which we investigate in the rest of the proof.

(a) By (16) and (3), *population monotonicity* and *IR* together imply, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^2(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq (\widehat{c}_{-i})_{[1]}, \tag{18}$$

(note that *population monotonicity* does not impose any restriction on economies with two agents; hence, the only restriction on economies in $\mathcal{D}^2(N,i,c)$ is by *IR*),
and for each $i \in N$, each $r \in \{3,...,n\}$, and each $\widehat{c} \in \mathcal{D}^r(N,i,c)$,

$$h_i(\widehat{c}_{-i}) \geq \max\left(\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\{h_i(\widetilde{c}_{-i})\},\ (\widehat{c}_{-i})_{[1]}\right) \tag{19}$$

where $\widehat{N}$ is the population associated with $\widehat{c}$.

To minimize the deficit, (18) and for $r = 3$, (19) should hold as an equality. Then, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N,i,c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c}\in\mathcal{D}^{2}(\widehat{N},i,\widehat{c})}\left\{(\widetilde{c}_{-i})_{[1]}\right\},\ (\widehat{c}_{-i})_{[1]}\right). \tag{20}$$

By (20) and (17), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N,i,c)$,

$$h_i(\widehat{c}_{-i}) = \max\left((\widehat{c}_{-i})_{[2]},\ (\widehat{c}_{-i})_{[1]}\right) = (\widehat{c}_{-i})_{[2]}. \tag{21}$$

Similarly, to minimize the deficit, for $r = 4$, (19) should hold as an equality. Then, by (17) and (21), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^4(N,i,c)$,

$$\begin{aligned} h_i(\widehat{c}_{-i}) &= \max\left(\max_{\widetilde{c}\in\mathcal{D}^{3}(\widehat{N},i,\widehat{c})}\left\{(\widetilde{c}_{-i})_{[2]}\right\},\ (\widehat{c}_{-i})_{[1]}\right), \\ &= \max\left((\widehat{c}_{-i})_{[3]},\ (\widehat{c}_{-i})_{[1]}\right) = (\widehat{c}_{-i})_{[3]}. \end{aligned}$$

By recursive substitution, at each step applying (17) and minimizing the deficit (i.e., (19) holding as an equality for each $r \in \{3,...,n\}$), we obtain, for each $i \in N$, each $r \in \{3,...,n\}$, and each $\widehat{c} \in \mathcal{D}^r(N,i,c)$,

$$h_i(\widehat{c}_{-i}) = \max\left((\widehat{c}_{-i})_{[r-1]},\ (\widehat{c}_{-i})_{[1]}\right) = (\widehat{c}_{-i})_{[r-1]}.$$

Note that $\widehat{c} \in \mathcal{D}^n(N,i,c)$ if and only if $\widehat{c} = c$. Hence, for $r = n$, we obtain the h function in (8). $\square$

(c) By (16) and (5), *population monotonicity* and *IPLB* together imply, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^2(N,i,c)$,

$$h_i(\widehat{c}_{-i}) \geq \frac{1}{2}(\widehat{c}_{-i})_{[1]}, \tag{22}$$

and for each $i \in N$, each $r \in \{3,...,n\}$, and each $\widehat{c} \in \mathcal{D}^r(N,i,c)$,

$$h_i(\widehat{c}_{-i}) \geq \max\left(\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\{h_i(\widetilde{c}_{-i})\},\ \frac{r-1}{r}(\widehat{c}_{-i})_{[1]}\right). \tag{23}$$

To minimize the deficit, (22) and for $r = 3$, (23) should hold as an equality. Then, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N,i,c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c}\in\mathcal{D}^{2}(\widehat{N},i,\widehat{c})}\left\{\frac{1}{2}(\widetilde{c}_{-i})_{[1]}\right\},\ \frac{2}{3}(\widehat{c}_{-i})_{[1]}\right). \tag{24}$$

By (24) and (17), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N,i,c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\frac{1}{2}(\widehat{c}_{-i})_{[2]},\ \frac{2}{3}(\widehat{c}_{-i})_{[1]}\right). \tag{25}$$

Similarly, to minimize the deficit, for $r=4$, (23) should hold as an equality. Then, by (25) and a similar argument to (17), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^4(N,i,c)$,

$$\begin{aligned} h_i(\widehat{c}_{-i}) &= \max\left(\max_{\widetilde{c}\in\mathcal{D}^3(\widehat{N},i,\widehat{c})}\left\{\max\left(\frac{1}{2}(\widetilde{c}_{-i})_{[2]},\ \frac{2}{3}(\widetilde{c}_{-i})_{[1]}\right)\right\},\ \frac{3}{4}(\widehat{c}_{-i})_{[1]}\right), \\ &= \max\left(\frac{1}{2}(\widehat{c}_{-i})_{[3]},\ \frac{2}{3}(\widehat{c}_{-i})_{[2]},\ \frac{3}{4}(\widehat{c}_{-i})_{[1]}\right). \end{aligned}$$

By recursive substitution, at each step applying (17) and minimizing the deficit (i.e., (23) holding as an equality for each $r \in \{3,...,n\}$), we obtain, for each $i \in N$, each $r \in \{3,...,n\}$, and each $\widehat{c} \in \mathcal{D}^r(N,i,c)$,

$$\begin{aligned} h_i(\widehat{c}_{-i}) &= \max\left(\frac{1}{2}(\widehat{c}_{-i})_{[r-1]},\ \frac{2}{3}(\widehat{c}_{-i})_{[r-2]},\ \frac{3}{4}(\widehat{c}_{-i})_{[r-3]},...,\frac{r-1}{r}(\widehat{c}_{-i})_{[1]}\right), \\ &= \max_{p\in\{1,2,...,r-1\}}\left\{\frac{p}{p+1}(\widehat{c}_{-i})_{[r-p]}\right\}. \end{aligned}$$

Note that $\widehat{c} \in \mathcal{D}^n(N,i,c)$ if and only if $\widehat{c} = c$. Hence, for $r=n$, we obtain the h function in (10).

Note that by Remark 1, we can replace the *identical-preferences lower-bound* with $1-$*fairness*, and the proof still holds true. $\square$

(d) Let $k \geq 2$. By (16) and (6), *population monotonicity* and $k-$*fairness* together imply, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^2(N,i,c)$,

$$h_i(\widehat{c}_{-i}) \geq (\widehat{c}_{-i})_{[1]} - \frac{1}{2}(\widehat{c}_{-i})_{[\min\{k,2\}-1]}, \tag{26}$$

and for each $i \in N$, each $r \in \{3,...,n\}$, and each $\widehat{c} \in \mathcal{D}^r(N,i,c)$,

$$h_i(\widehat{c}_{-i}) \geq \max\left(\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\{h_i(\widetilde{c}_{-i})\},\ (\widehat{c}_{-i})_{[1]} - \frac{1}{r}(\widehat{c}_{-i})_{[\min\{k,r\}-1]}\right). \tag{27}$$

To minimize the deficit, (26) and for $r=3$, (27) should hold as an equality. Then, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N,i,c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c}\in\mathcal{D}^2(\widehat{N},i,\widehat{c})}\left\{(\widetilde{c}_{-i})_{[1]} - \frac{1}{2}(\widetilde{c}_{-i})_{[\min\{k,2\}-1]}\right\}, (\widehat{c}_{-i})_{[1]} - \frac{1}{3}(\widehat{c}_{-i})_{[\min\{k,3\}-1]}\right). \tag{28}$$

Observation 1: For each $i \in N$, each $r \in \{2,...,n\}$, and each $\widehat{c} \in \mathcal{D}^r(N,i,c)$,

$$(\widehat{c}_{-i})_{[1]} - \frac{1}{r}(\widehat{c}_{-i})_{[\min\{k,r\}-1]} \equiv \max_{t\in\{r\}}\ \max_{s\in\{1,..,r+1-t\}}\left\{(\widehat{c}_{-i})_{[s]} - \frac{1}{t}(\widehat{c}_{-i})_{[s-2+\min(k,t)]}\right\}.$$

Let $r \in \{3, ..., n\}$ and $t \in \{2, 3, ..., r-1\}$.

Observation 2: For each $i \in N$, each $\widehat{c} \in \mathcal{D}^r(N, i, c)$, and each $m \in \{1, 2, ..., r-t\}$,

$$\max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \left\{ \max_{s \in \{m\}} \{(\widetilde{c}_{-i})_{[s]} - \frac{1}{t}(\widetilde{c}_{-i})_{[s-2+\min(k,t)]}\} \right\} = \max_{s \in \{m, m+1\}} \left\{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t}(\widehat{c}_{-i})_{[s-2+\min(k,t)]} \right\}. \tag{29}$$

Proof: Let $i \in N$, $\widehat{c} \in \mathcal{D}^r(N, i, c)$, and $m \in \{1, 2, ..., r-t\}$. Let $\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})$ be a maximizer of the left-hand-side of (29), $\widetilde{N}$ be the population associated with $\widetilde{c}$, and $\widehat{N}$ be the population associated with $\widehat{c}$. Since $|\widetilde{N}| = |\widehat{N}| - 1 = r - 1$ and $i \in \widetilde{N}$, then $\widetilde{c} = \widehat{c}_{-j}$ for some $j \in \widehat{N} \backslash \{i\}$. That is, we obtain $\widetilde{c}_{-i}$ by removing the cost function of one agent from $\widehat{c}_{-i}$. Hence, either (i) $(\widetilde{c}_{-i})_{[m]} = (\widehat{c}_{-i})_{[m]}$ or (ii) $(\widetilde{c}_{-i})_{[m]} = (\widehat{c}_{-i})_{[m+1]}$. In order to maximize the left-hand-side of (29), given $(\widetilde{c}_{-i})_{[s]}$, we need $(\widetilde{c}_{-i})_{[s-2+\min(k,t)]}$ as small as possible.
Note that since $\min(k, t) \geq 2$, then

$$(\widetilde{c}_{-i})_{[s]} \leq (\widetilde{c}_{-i})_{[s-2+\min(k,t)]}. \tag{30}$$

First, suppose (i) holds. That is, the cost function of an agent whose cost for performing α is more than $(\widehat{c}_{-i})_{[m]}$ has been removed from $\widehat{c}_{-i}$ to obtain $\widetilde{c}_{-i}$. Note that $\widehat{c}_{-i}$ contains $r-1$ cost functions and $\widetilde{c}_{-i}$ contains $r-2$ cost functions. Due to (30), to have $(\widetilde{c}_{-i})_{[s-2+\min(k,t)]}$ as small as possible for any $m \in \{1, 2, ..., r-t\}$, we need to obtain $\widetilde{c}_{-i}$ from $\widehat{c}_{-i}$ by removing the cost function of an agent who has the largest cost in $\widehat{c}_{-i}$, i.e., $\widetilde{c}_{-i} = ((\widehat{c}_{-i})_{[1]}, (\widehat{c}_{-i})_{[2]}, ..., (\widehat{c}_{-i})_{[\widehat{n}-2]})$. Hence, if $(\widetilde{c}_{-i})_{[m]} = (\widehat{c}_{-i})_{[m]}$, then by (30), $(\widetilde{c}_{-i})_{[m-2+\min(k,t)]} = (\widehat{c}_{-i})_{[m-2+\min(k,t)]}$. That is, we obtain the right-hand-side of (29) for $s = m$.

Next, suppose (ii) holds. That is, the cost function of an agent whose cost for performing α is less than $(\widehat{c}_{-i})_{[m]}$ has been removed from $\widehat{c}_{-i}$ to obtain $\widetilde{c}_{-i}$. Then, when $(\widetilde{c}_{-i})_{[m]} = (\widehat{c}_{-i})_{[m+1]}$, by (30), $(\widetilde{c}_{-i})_{[m-2+\min(k,t)]} = (\widehat{c}_{-i})_{[m-1+\min(k,t)]}$. That is, we obtain the right-hand-side of (29) for $s = m+1$. ◊

Observation 3: For each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$\max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \left\{ \max_{t \in \{2,..,r-1\}} \max_{s \in \{1,..,r-t\}} \{(\widetilde{c}_{-i})_{[s]} - \tfrac{1}{t}(\widetilde{c}_{-i})_{[s-2+\min(k,t)]}\} \right\}$$

$$= \max_{t \in \{2,..,r-1\}} \max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \left\{ \max_{s \in \{1,..,r-t\}} \{(\widetilde{c}_{-i})_{[s]} - \tfrac{1}{t}(\widetilde{c}_{-i})_{[s-2+\min(k,t)]}\} \right\},$$

$$= \max_{t \in \{2,..,r-1\}} \max_{s \in \{1,...,r+1-t\}} \left\{ (\widehat{c}_{-i})_{[s]} - \tfrac{1}{t}(\widehat{c}_{-i})_{[s-2+\min(k,t)]} \right\}.$$

Proof: The second equality follows from Observation 2. ◊

By (28) and Observations 1 and 3, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$\begin{aligned}
h_i(\widehat{c}_{-i}) &= \max\left(\max_{\widetilde{c} \in \mathcal{D}^2(\widehat{N}, i, \widehat{c})} \{ \max_{t \in \{2\}} \max_{s \in \{1,..,3-t\}} \{ (\widetilde{c}_{-i})_{[s]} - \frac{1}{t} (\widetilde{c}_{-i})_{[s-2+\min(k,t)]} \} \}, (\widehat{c}_{-i})_{[1]} - \frac{1}{3} (\widehat{c}_{-i})_{[\min\{k,3\}-1]} \right), \\
&= \max\left(\max_{t \in \{2\}} \max_{s \in \{1,..,4-t\}} \{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \},\ (\widehat{c}_{-i})_{[1]} - \frac{1}{3} (\widehat{c}_{-i})_{[\min\{k,3\}-1]} \right), \\
&= \max\left(\max_{t \in \{2\}} \max_{s \in \{1,..,4-t\}} \{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \}, \max_{t \in \{3\}} \max_{s \in \{1,..,4-t\}} \{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \} \right) \\
&= \max_{t \in \{2,3\}} \max_{s \in \{1,..,4-t\}} \left\{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \right\}. \qquad (31)
\end{aligned}$$

Note that the first three equalities follow from equality (28), Observations 1 and 3, respectively.

Similarly, to minimize the deficit, for $r = 4$, (27) should hold as an equality. Then, by (31), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^4(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c} \in \mathcal{D}^3(\widehat{N}, i, \widehat{c})} \left\{ \max_{t \in \{2,3\}} \max_{s \in \{1,..,4-t\}} \{ (\widetilde{c}_{-i})_{[s]} - \frac{1}{t} (\widetilde{c}_{-i})_{[s-2+\min(k,t)]} \} \right\}, (\widehat{c}_{-i})_{[1]} - \frac{1}{4} (\widehat{c}_{-i})_{[\min\{k,4\}-1]} \right). \qquad (32)$$

By Observation 1,

$$(\widehat{c}_{-i})_{[1]} - \frac{1}{4} (\widehat{c}_{-i})_{[\min\{k,4\}-1]} = \max_{t \in \{4\}} \max_{s \in \{1,..,5-t\}} \left\{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \right\}. \qquad (33)$$

Substituting (33) into (32) and applying Observation 3, we get for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^4(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max_{t \in \{2,3,4\}} \max_{s \in \{1,..,5-t\}} \left\{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \right\}.$$

By recursive substitution, at each step applying Observations 1 and 3, and minimizing the deficit (i.e., (27) holding as an equality for each $r \in \{3, ..., n\}$), we obtain, for each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max_{t \in \{2,3,..,r\}} \max_{s \in \{1,..,r+1-t\}} \left\{ (\widehat{c}_{-i})_{[s]} - \frac{1}{t} (\widehat{c}_{-i})_{[s-2+\min(k,t)]} \right\}.$$

Note that $\widehat{c} \in \mathcal{D}^n(N, i, c)$ if and only if $\widehat{c} = c$. Hence, for $r = n$, we obtain the h function in (11). □■

6 References

Atlamaz, M., Yengin, D., 2008. Fair Groves mechanisms. Social Choice and Welfare. 31, 573-587.

Bevia, C., 1996. Population monotonicity in economies with one indivisible good. Mathematical Social Sciences. 32, 125-138.

Clarke, EH., 1971. Multi-part pricing of public goods. Public choice. 11,17-33.

Dubins, L., Spanier, E., 1961. How to cut a cake fairly. American Mathematical Monthly, 68, 17.

Fleurbaey, M. 1995. Equality and responsibility. European Economic Review 39, 683-689.

Foley, D., 1967. Resource allocation and public sector. Yale Economic Essays. 7, 45-98.

Gibbard, A., 1973. Manipulation of voting schemes: A general result. Econometrica. 41, 587-601.

Green, J., Laffont, JJ., 1977. Characterization of satisfactory mechanisms for the revelation of preferences for public goods. Econometrica. 45, 427-438.

Groves, T., 1973. Incentives in teams. Econometrica. 41, 617-631.

Holmström, B., 1979. Groves' scheme on restricted domains. Econometrica. 47, 1137-1144.

Kim, H., 2004. Population monotonic rules for fair allocation problems. Social Choice and Welfare. 23, 59-70.

Moulin, H., 1990. Fair division under joint ownership: Recent results and open problems. Social Choice and Welfare. 7, 149-170.

Moulin, H., 1991. Welfare bounds in the fair division problem. Journal of Economic Theory. 54, 321–337.

Moulin, H., 2009. Auctioning or assigning an object: some remarkable VCG mechanisms. Social Choice and Welfare. 34, 193-216.

Ohseto ,S., 2006. Characterizations of strategy-proof and fair mechanisms for allocating indivisible goods. Economic Theory 29:111-121.

Pápai, S., 2003. Groves sealed bid auctions of heterogeneous objects with fair prices. Social Choice and Welfare 20:371-385.

Pazner, A., Schmeidler, D., 1978. Egalitarian equivalent allocations: A new concept of economic equity. The Quarterly Journal of Economics. 92, 671-687.

Porter, R., Shoham, Y., Tennenholtz, M., 2004. Fair imposition. Journal of Economic Theory. 118, 209–228.

Roemer J (1986): Equality of resources implies equality of welfare. Quarterly Journal of Economics 101:751-784.

Satterthwaite, M., 1975. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory. 10, 187-217.

Steinhaus, H., 1948. The Problem of Fair Division. Econometrica. 16, 101-104.

Thomson, W., 1983. The fair division of a fixed supply among a growing population. Mathematics of Operations Research. 8, 319-326.

Thomson, W., 1995a. Population monotonic allocation rules,. W.A. Barnett, H. Moulin, M. Salles and N.J. Schofield, eds., Social choice, welfare, and ethics. Cambridge University Press. pp.79-124.

Thomson, W., 1995b. Population-monotonic solutions to the problem of fair division when preferences are single-peaked. Journal of Economic Theory. 5, 229-246.

Thomson, W., 2004. On monotonicity in economies with indivisible goods. Social Choice and Welfare. 21, 195-205.

Thomson, W., 2005. Strategy-proof allocation rules on economic domains. Unpublished manuscript, University of Rochester.

Vickrey, W., 1961. Counterspeculation, auctions, and competitively sealed tenders. Journal of Finance. 16, 8-37.

Yengin, D., 2011. Axiomatizing political philosophy of distributive justice: equivalence of no-envy and egalitarian-equivalence with welfare-egalitarianism . University of Adelaide, School of Economics Research Paper No. 2011-19. https://economics.adelaide.edu.au/research/papers/doc/wp2011-19.pdf.

Yengin D., 2012a. Egalitarian-equivalent Groves mechanisms in the allocation of heterogeneous objects. Social Choice and Welfare, Volume 38, Number 1.

Yengin, D., 2012b. Characterizing welfare-egalitarian mechanisms with solidarity when valuations are private information. BE Journal in Theoretical Economics, Volume 12, Issue 1, Pages –, ISSN (Online) 1935-1704, DOI: 10.1515/1935-1704.1789.

Yengin, D., 2012c. Population Monotonic and Strategy-proof Mechanisms Respecting Welfare Lower Bounds. University of Adelaide, School of Economics Research Paper No. 2012-05. https://economics.adelaide.edu.au/research/papers/doc/wp2012-05.pdf.

Yengin, D., 2013. Identical preferences lower bound for allocation of heterogenous tasks and NIMBY problems. Forthcoming in Journal of Public Economic Theory.