# AN ANTIPLANE CRACK BETWEEN BONDED DISSIMILAR FUNCTIONALLY GRADED ISOTROPIC ELASTIC MATERIALS

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**Summary**

The problem of a plane crack along the interface of two dissimilar functionally graded isotropic half-spaces under antiplane strain is considered. The materials exhibit quadratic variation in the shear modulus. Numerical values for the stress intensity factors and crack displacement are obtained for some particular materials.

## 1. Introduction

The problem of determining the stress and displacement fields around a crack in a functionally graded material has been considered by a number of authors and a variety of such problems have been solved. For many of the problems considered the functional gradation is represented by exponential functions (see for example Erdogan (**1**), Chan *et al.* (**2**) and Chen and Erdogan (**3**)). Crack problems for materials in which the functional gradation is represented by polynomials or other analytic functions have been considered in several papers (see for example Clements *et al.* (**4**), Ang and Clements (**5**), Erdogan and Ozturk (**6**) and Clements (**7**)). Amongst the papers on cracks in functionally graded materials there exists a limited number of studies involving plane cracks lying along a plane bonded interface across which the functional gradation has a discontinuous derivative. For example in the paper by Chen and Erdogan (**3**) a problem is examined concerning a crack on the plane bonded interface between a functionally graded layer exhibiting exponential variation in the material parameters and a homogeneous half-space. Also Clements *et al.* (**4**) and Clements (**7**) considered problems involving a crack on the interface of two similar bonded functionally graded half-spaces under antiplane strain. In these two papers the graded shear modulus is continuous throughout the material but has a discontinuous derivative across the plane interface.

In this paper the problem of determining the displacement and stress around a crack along the interface between two dissimilar functionally graded isotropic half-spaces under antiplane strain is considered. The two materials have a shear modulus which varies quadratically with the Cartesian coordinates. Both the shear modulus and its derivative are discontinuous across the plane interface. To facilitate the solution of the problem the displacement over the crack faces is expressed in terms of a finite series of Chebyschev polynomials. A system of linear equations is obtained for the unknown coefficients in the series by imposing traction boundary condition on the crack faces. Expressions for

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the crack displacement and crack tip stress intensity factors are obtained in terms of the Chebyschev polynomials and these expressions are used to obtain numerical values of the crack displacement and stress intensity factors for some particular materials. The numerical results illustrate the effect of the inhomogeneity on the crack displacement and the stress intensity factors.

## 2. Statement of the problem

Referred to a Cartesian frame $Ox_1x_2x_3$ consider two dissimilar isotropic functionally graded half-spaces occupying the regions $x_2 > 0$ and $x_2 < 0$ and joined along the plane $x_2 = 0$ except in the region $|x_1| \leqslant a$, $x_2 = 0$, $-\infty < x_3 < \infty$ where there is a crack (Figure 1). The crack faces are subjected to a constant antiplane loading which is independent of the coordinate $x_3$. The problem is to find the displacement and stress throughout the material and, in particular, to determine the effect of the inhomogeneity on the crack displacement and the crack tip stress intensity factors.

In view of the geometry of the material and the nature of the applied loading a solution is sought in terms of the antiplane displacement $u_3$, the antiplane stresses $\sigma_{13}$ and $\sigma_{23}$ and the coordinates $x_1$ and $x_2$.

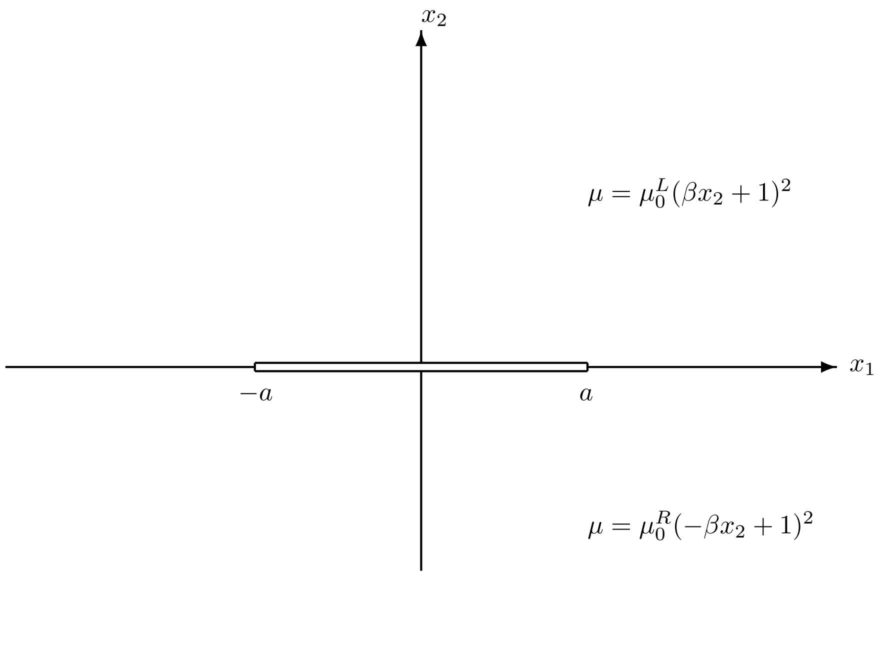


**Fig. 1** Crack geometry

## 3. Basic equations

The stress displacement relations for small antiplane deformations of a functionally graded isotropic elastic material take the form (Clements (**8**))

$$\sigma_{3j}(\mathbf{x}) = \mu(\mathbf{x})\frac{\partial u_3}{\partial x_j} \quad \text{for} \quad j = 1, 2, \tag{3.1}$$

where $\mu(\mathbf{x})$ denotes the shear modulus and $\mathbf{x} = (x_1, x_2)$. The equilibrium equation for antiplane deformations is

$$\frac{\partial \sigma_{3j}}{\partial x_j} = 0, \tag{3.2}$$

where the repeated summation convention (summing from 1 to 2) is employed for repeated Latin suffices. Substitution of (3.1) into (3.2) yields

$$\frac{\partial}{\partial x_j}\left[\mu(\mathbf{x})\,\frac{\partial u_3(\mathbf{x})}{\partial x_j}\right] = 0, \tag{3.3}$$

The shear modulus $\mu(\mathbf{x})$ is required to take the form

$$\mu(\mathbf{x}) = \mu_0\, g(\mathbf{x}), \tag{3.4}$$

where $\mu_0$ is a constant and $g(x_1, x_2) > 0$ is a twice differentiable function of the variables $x_1$ and $x_2$. Use of (3.4) in (3.3) provides

$$\mu_0\,\frac{\partial}{\partial x_j}\left(g\,\frac{\partial u_3}{\partial x_j}\right) = 0. \tag{3.5}$$

Let

$$u_3 = g^{-1/2}\,\psi(\mathbf{x}). \tag{3.6}$$

then (3.5) will be satisfied if $g$ and $\psi$ are solutions of the equations

$$\mu_0\,\frac{\partial^2 \psi}{\partial x_j \partial x_j} = 0 \tag{3.7}$$

and

$$\mu_0\,\frac{\partial^2 g^{1/2}}{\partial x_j \partial x_j} = 0. \tag{3.8}$$

A particular solution to equation (3.8) is chosen to take the form

$$g^{1/2} = \alpha x_2 + 1, \tag{3.9}$$

where $\alpha$ is a constant. A general solution to equation (3.7) may be written in the form

$$\psi = \Omega(z) + \overline{\Omega}(\overline{z}). \tag{3.10}$$

where $\Omega(z)$ is an arbitrary analytic function of the complex variable $z = x_1 + ix_2$ where $i$ denotes the square root of minus one and the bar denotes the complex conjugate.

From (3.1), (3.6) and (3.10) the displacement and stress in terms of the function $\Omega(z)$ may be written in the form

$$u_3 = 2g^{-1/2}\Re\left[\Omega(z)\right], \tag{3.11}$$
$$\sigma_{31} = 2g^{1/2}\mu_0\Re\left[\Omega'(z)\right], \tag{3.12}$$
$$\sigma_{32} = 2\mu_0\Re\Big[-\alpha\,\Omega(z) + g^{1/2}i\Omega'(z)\Big], \tag{3.13}$$

where $\Re$ denotes the real part of a complex number and primes denote differentiation of complex functions with respect to the argument in question.

**4. An interface crack between dissimilar materials.**

Suppose the material in the upper half-space $L$ contains a material with elastic moduli

$$\mu = \mu_0^L g(x_2) = \mu_0^L(\beta x_2 + 1)^2, \tag{4.1}$$

while the lower half-space $R$ contains a material with elastic moduli

$$\mu = \mu_0^R g(x_2) = \mu_0^R(-\beta x_2 + 1)^2, \tag{4.2}$$

where $\beta \geqslant 0$, $\mu_0^L > 0$ and $\mu_0^R > 0$ are constants. Hence the displacement and stress in the two half-spaces may be written in the form

$$u_3^L = g^{-1/2}\left[\psi(z) + \overline{\psi}(\overline{z})\right], \tag{4.3}$$
$$\sigma_{31}^L = \mu_0^L g^{1/2}\Big[\psi'(z) + \overline{\psi}'(\overline{z})\Big], \tag{4.4}$$
$$\sigma_{32}^L = \mu_0^L\Big[-\beta\,\psi(z) + g^{1/2}i\,\psi'(z)\Big] + \mu_0^L\Big[-\beta\,\overline{\psi}(\overline{z}) - g^{1/2}i\,\overline{\psi}'(\overline{z})\Big], \tag{4.5}$$

$$u_3^R = g^{-1/2}\left[\omega(z) + \overline{\omega}(\overline{z})\right], \tag{4.6}$$
$$\sigma_{31}^R = \mu_0^R g^{1/2}\left[\omega'(z) + \overline{\omega}'(\overline{z})\right], \tag{4.7}$$
$$\sigma_{32}^R = \mu_0^R\Big[\beta\,\omega(z) + g^{1/2}i\,\omega'(z)\Big] + \mu_0^R\Big[\beta\,\overline{\omega}(\overline{z}) - g^{1/2}i\,\overline{\omega}'(\overline{z})\Big], \tag{4.8}$$

where $\psi(z)$ and $\omega(z)$ are arbitrary analytic functions of the complex variable $z$. Outside the crack the displacement must be continuous across the interface $x_2 = 0$ for $|x_1| > a$ so that from (4.3) and (4.6)

$$\psi^+(x_1) + \overline{\psi}^-(x_1) = \omega^-(x_1) + \overline{\omega}^+(x_1) \quad \text{for} \quad |x_1| > a. \tag{4.9}$$

where

$$\lim_{x_2\to 0+}\psi(z) = \psi^+(x_1), \qquad \lim_{x_2\to 0-}\psi(z) = \psi^-(x_1), \tag{4.10}$$

$$\lim_{x_2\to 0+}\omega(z) = \omega^+(x_1), \qquad \lim_{x_2\to 0-}\omega(z) = \omega^-(x_1). \tag{4.11}$$

Hence

$$\psi^+(x_1) - \overline{\omega}^+(x_1) = \omega^-(x_1) - \overline{\psi}^-(x_1) \quad \text{for} \quad |x_1| > a. \tag{4.12}$$

Thus if a function $\chi(z)$ is defined by

$$\psi(z) - \overline{\omega}(z) = \chi(z) \quad \text{for} \quad z \in L, \tag{4.13}$$

$$\omega(z) - \overline{\psi}(z) = \chi(z) \quad \text{for} \quad z \in R \tag{4.14}$$

and the function $\chi(z)$ is required to be analytic in the whole plane cut along $[-a, a]$ then the displacement will be continuous on $x_2 = 0$ for $|x_1| > a$.

The stress $\sigma_{32}$ must be continuous across the interface $x_2 = 0$ for $|x_1| > a$ where the two half-spaces are joined so that, from (4.5) and (4.8)

$$\begin{aligned} & \mu_0^L \left[ -\beta\, \psi^+(x_1) + i\, \psi'^+(x_1) \right] + \mu_0^L \left[ -\beta\, \overline{\psi}^{\,-}(x_1) - i\, \overline{\psi}'^-(x_1) \right] \\ = \; & \mu_0^R \left[ \beta\, \omega^-(x_1) + i\, \omega'^-(x_1) \right] + \mu_0^R \left[ \beta\, \overline{\omega}^+(x_1) - i\, \overline{\omega}'^+(x_1) \right] \\ & \text{for} \quad |x_1| > a. \end{aligned} \tag{4.15}$$

Hence

$$\begin{aligned} & \mu_0^L \left[ -\beta\, \psi^+(x_1) + i\, \psi'^+(x_1) \right] - \mu_0^R \left[ \beta\, \overline{\omega}^+(x_1) - i\, \overline{\omega}'^+(x_1) \right] \\ = \; & \mu_0^L \left[ \beta\, \overline{\psi}^{\,-}(x_1) + i\, \overline{\psi}'^-(x_1) \right] + \mu_0^R \left[ \beta\, \omega^-(x_1) + i\, \omega'^-(x_1) \right] \\ & \text{for} \quad |x_1| > a. \end{aligned} \tag{4.16}$$

Thus if we define a function $\phi(z)$ by

$$\mu_0^L \left[ -\beta\, \psi(z) + i\, \psi'(z) \right] - \mu_0^R \left[ \beta\, \overline{\omega}(z) - i\, \overline{\omega}'(z) \right] = \phi(z) \quad \text{for} \quad z \in L, \tag{4.17}$$

$$\mu_0^L \left[ \beta\, \overline{\psi}(z) + i\, \overline{\psi}'(z) \right] + \mu_0^R \left[ \beta\, \omega(z) + i\, \omega'(z) \right] = \phi(z) \quad \text{for} \quad z \in R \tag{4.18}$$

and require $\phi(z)$ to be analytic in the whole plane cut along $[-a, a]$ then the stress will be continuous on $x_2 = 0$ for $|x_1| > a$.

Use of (4.13) and (4.14) to substitute for $\omega(z)$ and $\overline{\omega}(z)$ in (4.17) and (4.18) yields

$$-(\mu_0^L + \mu_0^R) \left[ \beta\, \psi(z) - i\, \psi'(z) \right] = -\mu_0^R \left[ \beta\, \chi(z) - i\, \chi'(z) \right] + \phi(z) \quad \text{for} \quad z \in L, \tag{4.19}$$

$$(\mu_0^L + \mu_0^R) \left[ \beta\, \overline{\psi}(z) + i\, \overline{\psi}'(z) \right] = -\mu_0^R \left[ \beta\, \chi(z) + i\, \chi'(z) \right] + \phi(z) \quad \text{for} \quad z \in R, \tag{4.20}$$

so that

$$-\beta\, \psi(z) + i\, \psi'(z) = \frac{-\mu_0^R}{\mu_0^L + \mu_0^R} \left[ \beta\, \chi(z) - i\, \chi'(z) \right] + \frac{1}{\mu_0^L + \mu_0^R}\, \phi(z) \quad \text{for} \quad z \in L, \tag{4.21}$$

$$\beta\, \overline{\psi}(z) + i\, \overline{\psi}'(z) = \frac{-\mu_0^R}{\mu_0^L + \mu_0^R} \left[ \beta\, \chi(z) + i\, \chi'(z) \right] + \frac{1}{\mu_0^L + \mu_0^R}\, \phi(z) \quad \text{for} \quad z \in R. \tag{4.22}$$

Hence denoting the antiplane stress over the crack faces by $\sigma_{32}(x_1, 0) = p(x_1)$ it follows from (4.5), (4.8), (4.21) and (4.22) that

$$\sigma_{32}^L = \mu_0^L \left[ -\beta\, \psi^+(x_1) + i\, \psi'^+(x_1) \right] + \mu_0^L \left[ -\beta\, \overline{\psi}^{\,-}(x_1) - i\, \overline{\psi}'^-(x_1) \right]$$

$$
\begin{aligned}
&= \frac{-\mu_0^L\mu_0^R}{\mu_0^L+\mu_0^R}\left[\beta\,\chi^+(x_1) - i\,\chi'^+(x_1)\right] + \frac{\mu_0^L}{\mu_0^L+\mu_0^R}\,\phi^+(x_1) \\
&\quad + \frac{\mu_0^L\mu_0^R}{\mu_0^L+\mu_0^R}\left[\beta\,\chi^-(x_1) + i\,\chi'^-(x_1)\right] - \frac{\mu_0^L}{\mu_0^L+\mu_0^R}\,\phi^-(x_1) = p(x_1) \\
&\qquad\qquad \text{for } |x_1| < a, \quad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{32}^R &= \mu_0^R\left[\beta\,\omega^-(x_1) + i\,\omega'^-(x_1)\right] + \mu_0^R\left[\beta\,\overline{\omega}^+(x_1) - i\,\overline{\omega}'^+(x_1)\right] \\
&= \mu_0^R\beta[\overline{\psi}^{\,-}(x_1) + \chi^-(x_1)] + i\mu_0^R[\overline{\psi}'^{\,-}(x_1) + \chi'^-(x_1)] \\
&\quad + \mu_0^R\beta[\psi^{\,+}(x_1) - \chi^+(x_1)] - i\mu_0^R[\psi'^{\,+}(x_1) - \chi'^+(x_1)] \\
&= \mu_0^R[\beta\overline{\psi}^{\,-}(x_1) + i\overline{\psi}'^{\,-}(x_1)] + \mu_0^R[(\chi^-(x_1) + i\chi'^-(x_1)] \\
&\quad + \mu_0^R\beta[\psi^{\,+}(x_1) - i\psi'^+(x_1)] - \mu_0^R[\chi'^{\,+}(x_1) - i\chi'^+(x_1)] \\
&= -\frac{\mu_0^L\mu_0^R}{\mu_0^L+\mu_0^R}\left[\beta\,\chi^+(x_1) - i\,\chi'^+(x_1)\right] - \frac{\mu_0^R}{\mu_0^L+\mu_0^R}\,\phi^+(x_1) \\
&\quad + \frac{\mu_0^L\mu_0^R}{\mu_0^L+\mu_0^R}\left[\beta\,\chi^-(x_1) + i\,\chi'^-(x_1)\right] + \frac{\mu_0^R}{\mu_0^L+\mu_0^R}\,\phi^-(x_1) = p(x_1) \\
&\qquad\qquad \text{for } |x_1| < a, \quad (4.24)
\end{aligned}
$$

where

$$\lim_{x_2\to 0+}\chi(z) = \chi^+(x_1), \qquad \lim_{x_2\to 0-}\chi(z) = \chi^-(x_1), \tag{4.25}$$

$$\lim_{x_2\to 0+}\phi(z) = \phi^+(x_1), \qquad \lim_{x_2\to 0-}\phi(z) = \phi^-(x_1). \tag{4.26}$$

Thus to solve the problem complex functions $\chi(z)$ and $\phi(z)$ are required which are analytic in the whole plane cut along $[-a, a]$ and satisfy equations (4.23) and (4.24). The stress is required to vanish at infinity so it is necessary that

$$\chi(z) = O(1/z), \qquad \phi(z) = O(1/z) \quad \text{as} \quad |z| \to \infty. \tag{4.27}$$

Subtracting (4.24) from (4.23) yields

$$\phi^+(x_1) = \phi^-(x_1) \;\text{ for }\; |x_1| < a. \tag{4.28}$$

Hence $\phi(z)$ is analytic in the whole plane and therefore, from the condition (4.27) and the maximum modulus principle for complex functions, must be identically zero. Thus equations (4.23) and (4.24) reduce to

$$-\beta[\chi^+(x_1) - \chi^-(x_1)] + i[\chi'^+(x_1) + \chi'^-(x_1)] = \frac{\mu_0^L + \mu_0^R}{\mu_0^L\mu_0^R}p(x_1) \;\text{ for }\; |x_1| < a. \tag{4.29}$$

From (4.3), (4.6), (4.13) and (4.14) the crack opening displacement across the boundary $x_2 = 0$ of the two regions $L$ and $R$ is given by

$$u_3^L(x_1, 0) - u_3^R(x_1, 0) = \chi^+(x_1) - \chi^-(x_1). \tag{4.30}$$

The function $\chi(z)$ is chosen to be analytic in the whole plane cut along the interval $|x_1| < a$, $x_2 = 0$. This ensures continuity of displacement on $x_2 = 0$, $|x_1| > a$ where the two materials are joined. Over the crack in $x_2 = 0$, $|x_1| < a$ the crack opening displacement across the crack faces is denoted by $\mathcal{U}(x_1)$ so that

$$\chi^+(x_1) - \chi^-(x_1) = \mathcal{U}(x_1) \quad \text{for} \quad |x_1| < a, \tag{4.31}$$

where $\mathcal{U}(x_1)$ is to be determined. Hence the function $\chi(z)$ is given by

$$\chi(z) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{\xi - z}. \tag{4.32}$$

On the crack faces $\chi^+(x_1)$ and $\chi^-(x_1)$ are given by

$$\chi^+(x_1) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{\xi - x_1} + \frac{1}{2}\mathcal{U}(x_1) \quad \text{for} \quad |x_1| < a, \tag{4.33}$$

$$\chi^-(x_1) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{\xi - x_1} - \frac{1}{2}\mathcal{U}(x_1) \quad \text{for} \quad |x_1| < a, \tag{4.34}$$

where the integrals in (4.33) and (4.34) are Cauchy principal value integrals.

Differentiation of (4.32) provides

$$\chi'(z) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{(\xi - z)^2}. \tag{4.35}$$

Hence

$$\chi'^+(x_1) = \frac{1}{2\pi i}\,⨎_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{(\xi - x_1)^2} + \frac{1}{2}[\mathcal{U}'(x_1)] \quad \text{for} \quad |x_1| < a, \tag{4.36}$$

$$\chi'^-(x_1) = \frac{1}{2\pi i}\,⨎_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{(\xi - x_1)^2} - \frac{1}{2}[\mathcal{U}'(x_1)] \quad \text{for} \quad |x_1| < a, \tag{4.37}$$

where the integrals in (4.36) and (4.37) are Hadamard finite part integrals. Addition of (4.36) and (4.37) yields

$$\chi'^+(x_1) + \chi'^-(x_1) = \frac{1}{\pi i}\,⨎_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{(\xi - x_1)^2} \quad \text{for} \quad |x_1| < a. \tag{4.38}$$

Substitution of (4.33), (4.34) and (4.38) into (4.29) gives

$$-\beta\,\mathcal{U}(x_1) + \frac{1}{\pi}\,⨎_{-a}^{a} \frac{\mathcal{U}(\xi)d\xi}{(\xi - x_1)^2} = \frac{\mu_0^L + \mu_0^R}{\mu_0^L\mu_0^R}\,p(x_1) \quad \text{for} \quad |x_1| < a. \tag{4.39}$$

It is convenient at this point to introduce the non-dimensional variables

$$\begin{aligned} &x = x_1/a, \quad t = \xi/a, \quad u = \mathcal{U}/a, \quad \overline{p} = p/\mathcal{C}, \quad \mu^L = \mu_0^L/\mathcal{C}, \quad \mu^R = \mu_0^R/\mathcal{C}, \\ &\beta' = \beta a, \quad \overline{u}_3^L = u_3^L/a, \quad \overline{u}_3^R = u_3^R/a, \quad \overline{\sigma}_{3j}^L = \sigma_{3j}^L/\mathcal{C}, \quad \overline{\sigma}_{3j}^R = \sigma_{3j}^R/\mathcal{C}, \end{aligned} \tag{4.40}$$

where $\mathcal{C}$ is a reference pressure. In terms of the non-dimensional variables equation (4.39) becomes

$$-\beta'\,u(x) + \frac{1}{\pi}\,⨎_{-1}^{1} \frac{u(t)dt}{(t - x)^2} = \frac{\mu^L + \mu^R}{\mu^L\mu^R}\,\overline{p}(x) \quad \text{for} \quad |x| < 1. \tag{4.41}$$

In view of the fact that the crack opening displacement in a homogeneous isotropic material under antiplane strain is proportional to $(1-x)^{1/2}$ we try for a solution to the equation (4.41) in terms of Chebyshev polynomials of the second kind $U_r(t)$ in the form

$$u(x) = (1-x^2)^{1/2}\sum_{r=1}^{J}\alpha_r U_{r-1}(x) \quad \text{for} \quad |x|<1, \tag{4.42}$$

where the $\alpha_r$ for $r = 1 \dots J$ are constants. Now from known results for Hadamard finite part integrals (see Erdogan and Ozturk (**6**))

$$\begin{aligned} \frac{1}{\pi}\,⨍_{-1}^{1}\frac{u(t)dt}{(t-x)^2} &= \sum_{r=1}^{J}\alpha_r\frac{1}{\pi}\,⨍_{-1}^{1}\frac{U_{r-1}(t)(1-t^2)^{1/2}dt}{(t-x)^2} \\ &= -\sum_{r=1}^{J}\alpha_r r U_{r-1}(x) \quad \text{for} \quad |x|<1. \end{aligned} \tag{4.43}$$

Use of (4.42) and (4.43) in (4.41) provides

$$\sum_{r=1}^{J}\left[-\beta'\,(1-x^2)^{1/2} - r\right]\alpha_r U_{r-1}(x) = \frac{\mu^L+\mu^R}{\mu^L\mu^R}\,\overline{p}(x) \quad \text{for} \quad |x|<1. \tag{4.44}$$

For a particular $J$ equation (4.44) contains $J$ unknowns $\alpha_r$ for $r = 1, 2, \dots J$. To generate $J$ linear equations which may be solved for the unknowns $\alpha_r$, $J$ distinct points are chosen in the interval $|x| < 1$. These points are chosen to be the zeros $\cos(k\pi/(J+1))$ for $k = 1, 2, \dots, J$ of the Chebyshev polynomial $U_J$. This choice ensures convergence of the sum in (4.44) as $J$ increases (see Mason and Handscomb (**9**)).

Once the $\alpha_r$ have been determined equations (4.42) and (4.40) give the crack opening displacement $\mathcal{U}(x_1, 0)$ and then $\chi(z)$ may be obtained through equation (4.32). Also since $\phi(z)$ is identically zero equations (4.19) and (4.20) will be satisfied if

$$\psi(z) = \frac{\mu_0^R}{\mu_0^L+\mu_0^R}\,\chi(z) \quad \text{for} \quad z \in L, \tag{4.45}$$

$$\overline{\psi}(z) = \frac{-\mu_0^R}{\mu_0^L+\mu_0^R}\,\chi(z) \quad \text{for} \quad z \in R \tag{4.46}$$

and hence from (4.13) and (4.14)

$$\overline{\omega}(z) = \psi(z) - \chi(z) = \frac{-\mu_0^L}{\mu_0^L+\mu_0^R}\,\chi(z) \quad \text{for} \quad z \in L, \tag{4.47}$$

$$\omega(z) = \overline{\psi}(z) + \chi(z) = \frac{\mu_0^L}{\mu_0^L+\mu_0^R}\,\chi(z) \quad \text{for} \quad z \in R, \tag{4.48}$$

so that once $\chi(z)$ has been determined $\psi(z)$ and $\omega(z)$ may be obtained from equations (4.45) to (4.48). Then substitution in equations (4.3) to (4.8) provides the displacement and stress throughout the material.

In particular the displacements $u_3^L$ and $u_3^R$ on $x_2 = 0$ are given by

$$\begin{aligned} u_3^L(x_1,0) &= \psi^+(x_1) + \overline{\psi}^-(x_1) = \frac{\mu_0^R}{\mu_0^L+\mu_0^R}\,[\chi^+(x_1) - \chi^-(x_1)] \\ &= \begin{cases} \frac{\mu_0^R}{\mu_0^L+\mu_0^R}\,\mathcal{U}(x_1) & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a, \end{cases} \end{aligned} \tag{4.49}$$

$$\begin{aligned} u_3^R(x_1,0) &= \omega^-(x_1) + \overline{\omega}^+(x_1) = \frac{-\mu_0^L}{\mu_0^L+\mu_0^R}\,[\chi^+(x_1) - \chi^-(x_1)] \\ &= \begin{cases} \frac{-\mu_0^L}{\mu_0^L+\mu_0^R}\,\mathcal{U}(x_1) & \text{for} \quad |x_1| < a, \\ 0 & \text{for} \quad |x_1| > a. \end{cases} \end{aligned} \tag{4.50}$$

Now since $\chi(z) = O(1/z)$ as $|z| \to \infty$ it follows from equations (4.45) to (4.48) that $\psi(z)$, $\overline{\psi}(z)$, $\omega(z)$ and $\overline{\omega}(z)$ are all $O(1/z)$ as $|z| \to \infty$. Therefore from equations (4.1) to (4.8) it is apparent that the displacement and stress tend to zero as $|z| \to \infty$. In particular if $\beta > 0$ then as $|x_2| \to \infty$ the displacement is $O(1/x_2^2)$ and the stress is $O(1/x_2)$.

## 5. Stress intensity factors (SIFs)

From equations (4.23), (4.24), (4.36), (4.37) and (4.40) it follows that for $|x| > 1$

$$\begin{aligned} \overline{\sigma}_{32}(x,0) &= \frac{\mu^L\mu^R}{\mu^L+\mu^R}\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)dt}{(t-x)^2} \\ &= \frac{\mu^L\mu^R}{\mu^L+\mu^R}\frac{1}{\pi}\sum_{r=1}^{J}\alpha_r\left[\int_{-1}^{1}\frac{U_{r-1}(t)\sqrt{1-t^2}dt}{(t-x)^2}\right]. \end{aligned} \tag{5.1}$$

Use of contour integration on the integral in (5.1) provides (Chan et al. (**2**))

$$\int_{-1}^{1}\frac{U_n(t)\sqrt{1-t^2}}{(t-x)^2}\,dt \to \frac{\pi U_n(1)}{\sqrt{2(x-1)}} \quad \text{as} \quad x \to 1+ \quad \text{for} \quad n = 0,1,2\ldots, \tag{5.2}$$

$$\int_{-1}^{1}\frac{U_n(t)\sqrt{1-t^2}}{(t-x)^2}\,dt \to \frac{\pi U_n(-1)}{\sqrt{-2(x+1)}} \quad \text{as} \quad x \to -1- \quad \text{for} \quad n = 0,1,2\ldots. \tag{5.3}$$

and use of these results in (5.1) yields

$$\overline{\sigma}_{32}(x,0) \to \frac{\mu^L\mu^R}{\mu^L+\mu^R}\sum_{r=1}^{J}\alpha_r\left[\frac{U_{r-1}(1)}{\sqrt{2(x-1)}}\right] \quad \text{as} \quad x \to 1+, \tag{5.4}$$

$$\overline{\sigma}_{32}(x,0) \to \frac{\mu^L\mu^R}{\mu^L+\mu^R}\sum_{r=1}^{J}\alpha_r\left[\frac{U_{r-1}(-1)}{\sqrt{-2(x+1)}}\right] \quad \text{as} \quad x \to -1-. \tag{5.5}$$

At the crack tip at $x = 1$ the stress intensity factor is given by

$$\mathcal{K}_+ = \lim_{x\to 1+}\sqrt{(x-1)}\,\overline{\sigma}_{32}(x,0) \tag{5.6}$$

$$= \frac{\mu^L\mu^R}{\mu^L+\mu^R}\frac{1}{\sqrt{2}}\sum_{r=1}^{J}\alpha_r U_{r-1}(1) \tag{5.7}$$

and at the crack tip at $x = -1$

$$\mathcal{K}_- = \lim_{x \to -1-} \sqrt{-(x+1)}\,\overline{\sigma}_{32}(x,0) \tag{5.8}$$

$$= \frac{\mu^L \mu^R}{\mu^L + \mu^R} \frac{1}{\sqrt{2}} \sum_{r=1}^{J} \alpha_r U_{r-1}(-1). \tag{5.9}$$

## 6. Two closed form solutions

At this point it is useful to consider two limiting cases when a closed form solution to the crack problem is readily obtained. These analytical solutions will provide a measure of verification for the numerical values calculated by employing equation (4.44).

Firstly for a homogeneous material $\beta = 0$ so that equation (4.29) with $p(x_1) = -p_0$ (constant) becomes

$$\left[\chi'^{+}(x_1) + \chi'^{-}(x_1)\right] = -\frac{p_0}{i}\left(\frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R}\right) \quad \text{for} \quad |x_1| < a. \tag{6.1}$$

The analytic function $\chi(z)$ satisfying (6.1) which is analytic in the whole plane cut along the $x_1$ axis from $x_1 = -a$ to $x_1 = a$ and also provides zero stress at infinity is available from the well-known solution to the Hilbert problem (see for example England (**10**)) in the closed form

$$\chi(z) = -\frac{p_0}{2i}\left(\frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R}\right)\left[z - \sqrt{z^2 - a^2}\right]. \tag{6.2}$$

Over the crack surface the crack opening displacement is given by

$$\mathcal{U}(x_1) = \chi^{+}(x_1) - \chi^{-}(x_1) \tag{6.3}$$

$$= p_0\left(\frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R}\right)\sqrt{a^2 - x_1^2} \quad \text{for} \quad |x_1| < a. \tag{6.4}$$

The stress $\sigma_{32}$ on $x_2 = 0$ for $|x_1| > a$ is given by

$$\sigma_{32} = i\left(\frac{\mu_0^L \mu_0^R}{\mu_0^L + \mu_0^R}\right)\left[\chi'^{+}(x_1) + \chi'^{-}(x_1)\right]$$

$$= -p_0\left[1 - \frac{|x_1|}{\sqrt{x_1^2 - a^2}}\right]. \tag{6.5}$$

In non-dimensional form equations (6.4) and (6.5) become

$$u(x,0) = p_0'\left(\frac{\mu^L + \mu^R}{\mu^L \mu^R}\right)\sqrt{1 - x^2} \quad \text{for} \quad |x| < 1, \tag{6.6}$$

$$\sigma_{32}' = -p_0'\left[1 - \frac{|x|}{\sqrt{x^2 - 1}}\right] \quad \text{for} \quad |x| > 1, \tag{6.7}$$

where $p_0' = p_0/\mathcal{C}$. In this case the stress intensity factors defined in (5.6) and (5.8) are given by

$$\mathcal{K}_+ = \mathcal{K}_- = \frac{p_0'}{\sqrt{2}}. \tag{6.8}$$

Secondly consider the case when $\beta$ is sufficiently large so that the terms involving $\beta$ in equations (4.23) and (4.24) with $\phi(z) \equiv 0$ are large compared to the remaining terms. Thus ignoring the remaining terms expressions for the stress on $x_2 = 0$ may be written in the form

$$\sigma_{32}^L = \sigma_{32}^R = -\beta \, \frac{\mu_0^L \mu_0^R}{\mu_0^L + \mu_0^R} [\chi^+(x_1) - \chi^-(x_1)] \tag{6.9}$$

so that setting $\sigma_{32}^L = \sigma_{32}^R = p(x_1) = -p_0$ yields

$$\chi^+(x_1) - \chi^-(x_1) = \frac{1}{\beta} \, \frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R} \, p_0 \qquad \text{for } |x_1| < a. \tag{6.10}$$

Use of Cauchy's theorem gives $\chi(z)$ in the form

$$\begin{aligned} \chi(z) &= \frac{p_0}{\beta} \, \frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R} \, \frac{1}{2\pi i} \int_{-a}^{a} \frac{dt}{t-z} \\ &= \frac{p_0}{\beta} \, \frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R} \, \frac{1}{2\pi i} \ln \left[ \frac{z-a}{z+a} \right] \end{aligned} \tag{6.11}$$

so that equations (4.31) and (6.9) provide

$$\mathcal{U}(x_1, 0) = \begin{cases} \frac{p_0}{\beta} \, \frac{\mu_0^L + \mu_0^R}{\mu_0^L \mu_0^R} & \text{for } |x_1| < a, \\ 0 & \text{for } |x_1| > a, \end{cases} \tag{6.12}$$

$$\sigma_{32}^L(x_1, 0) = \sigma_{32}^R(x_1, 0) = \begin{cases} -p_0 & \text{for } |x_1| < a, \\ 0 & \text{for } |x_1| > a. \end{cases} \tag{6.13}$$

In particular, on $x_2 = 0$ in non-dimensional form equations (6.12) and (6.13) yield

$$u(x, 0) = \begin{cases} \frac{p_0'}{\beta'} \, \frac{\mu^L + \mu^R}{\mu^L \mu^R} & \text{for } |x| < 1, \\ 0 & \text{for } |x| > 1, \end{cases} \tag{6.14}$$

$$\overline{\sigma}_{32}^L(x, 0) = \overline{\sigma}_{32}^R(x, 0) = \begin{cases} -p_0' & \text{for } |x| < 1, \\ 0 & \text{for } |x| > 1. \end{cases} \tag{6.15}$$

and hence, from (5.6) and (5.8), the stress intensity factors are given by

$$\mathcal{K}_+ = \mathcal{K}_- = 0 \tag{6.16}$$

## 7. Qualitative analysis of the crack displacement and SIFs

Once the constants $\mu^L$, $\mu^R$, $\beta'$ and the applied load $\overline{p}(x)$ are specified the equations (4.44), (4.41), (5.7) and (5.9) may be used to obtain numerical values for the crack opening displacement and stress intensity factors. These equations may also be employed to obtain some qualitative results regarding the crack opening displacement and stress intensity factors. To this end it is useful to make the substitution

$$\alpha_r = \left( \frac{\mu^L + \mu^R}{\mu_0^L \mu^R} \right) \alpha_r' \tag{7.1}$$

in these equations to provide

$$\sum_{r=1}^{J}\left[-\beta'\,(1-x^2)^{1/2}-r\right]\alpha'_r U_{r-1}(x)=\overline{p}(x)\qquad\text{for}\qquad |x|<1. \tag{7.2}$$

$$u(x)=\left(\frac{\mu^L+\mu^R}{\mu^L\mu^R}\right)(1-x^2)^{1/2}\sum_{r=1}^{J}\alpha'_r U_{r-1}(x)\qquad\text{for}\qquad |x|<1, \tag{7.3}$$

$$\mathcal{K}_+ \;=\; \frac{1}{\sqrt{2}}\sum_{r=1}^{J}\alpha'_r U_{r-1}(1), \tag{7.4}$$

$$\mathcal{K}_- \;=\; \frac{1}{\sqrt{2}}\sum_{r=1}^{J}\alpha'_r U_{r-1}(-1). \tag{7.5}$$

Equation (7.2) is employed to generate $J$ linear equations to determine the $\alpha'_r$ for $r = 1, 2 \ldots J$ and since this equation does not contain the constants $\mu^L$ and $\mu^R$ the $\alpha'_r$ are independent of these two constants. Thus from (7.3), (7.4) and (7.5) it follows that the stress intensity factors are independent of $\mu^L$ and $\mu^R$. Also for a given $\overline{p}(x)$, a fixed $\beta' \geqslant 0$ and a fixed $x = x_0$ with $|x_0| < 1$ it follows from equation (7.3) that

$$u(x_0)\propto\frac{\mu^L+\mu^R}{\mu^L\mu^R}. \tag{7.6}$$

Thus for a fixed $\beta'$ the displacement $u(x_0)$ becomes large if $\mu^L$ and/or $\mu^R$ tend to zero.

As $\beta'$ becomes large the coefficients in equation (7.2) become large and as a consequence the coefficients $\alpha'_r$ decrease in value with a resulting decrease in the crack opening displacement and stress intensity factors. When $\beta'$ becomes sufficiently large so that the term involving $r$ on the left hand side of equation (7.2) can be ignored then with $\overline{p}(x) = -p'_0$ the equation yields

$$(1-x^2)^{1/2}\sum_{r=1}^{J}\alpha'_r U_{r-1}(x)=\frac{p'_0}{\beta'}\qquad\text{for}\qquad |x|<1. \tag{7.7}$$

Hence

$$\sum_{r=1}^{J}\alpha'_r U_{r-1}(x)=\frac{p'_0}{\beta'}(1-x^2)^{-1/2}\qquad\text{for}\qquad |x|<1. \tag{7.8}$$

Thus using (7.8) to substitute for the sum in equations (7.3), (7.4) and (7.5) the crack opening displacement over the crack face and the stress intensity factors are given by

$$u(x)=\frac{p'_0}{\beta'}\left(\frac{\mu^L+\mu^R}{\mu^L\mu^R}\right)\qquad\text{for}\qquad |x|<1 \tag{7.9}$$

in agreement with equation (6.14).

When $\beta' = 0$ equation (7.2) with $\overline{p}(x) = -p_0'$ yields

$$\sum_{r=1}^{J} \alpha_r' r U_{r-1}(x) = p_0' \quad \text{for} \quad |x| < 1 \tag{7.10}$$

and since $U_0 = 1$ it follows that equation (7.10) will be satisfied if $\alpha_1' = p_0'$ and $\alpha_r' = 0$ for $r = 2, 3, \ldots J$. Thus substituting into equations (7.3), (7.4) and (7.5) the crack opening displacement and the crack tip stress intensity factors are given by

$$u(x) = p_0' \left( \frac{\mu^L + \mu^R}{\mu^L \mu^R} \right) \sqrt{1 - x^2} \quad \text{for} \quad |x| < 1, \tag{7.11}$$

$$\mathcal{K}_+ = \mathcal{K}_- = \frac{p_0'}{\sqrt{2}} \tag{7.12}$$

in agreement with (6.6) and (6.8).

In non-dimensional form equations (4.49) and (4.50) become

$$\overline{u}_3^L(x, 0) = \begin{cases} \frac{\mu^R}{\mu^L + \mu^R}\, u(x) & \text{for} \quad |x| < 1, \\ 0 & \text{for} \quad |x| > 1, \end{cases} \tag{7.13}$$

$$\overline{u}_3^R(x, 0) = \begin{cases} \frac{-\mu^L}{\mu^L + \mu^R}\, u(x) & \text{for} \quad |x| < 1, \\ 0 & \text{for} \quad |x| > 1. \end{cases} \tag{7.14}$$

These equations give the fractions of the crack opening displacement $u(x)$ which are made up by the displacement $\overline{u}_3^L(x, 0)$ in the upper half-space and the displacement $\overline{u}_3^R(x, 0)$ in the lower half-space. In particular if the ratio $\mu^L/\mu^R \to 0$ then $\overline{u}_3^R(x, 0) \to 0$ and $\overline{u}_3^L(x, 0) \to u(x)$. Similarly if the ratio $\mu^R/\mu^L \to 0$ then $\overline{u}_3^L(x, 0) \to 0$ and $\overline{u}_3^R(x, 0) \to -u(x)$.

## 8. Numerical results

In this section numerical results for the crack opening displacement and stress intensity factors are obtained for the crack problem considered in section 4.

To obtain the numerical values the procedure detailed in section 4 was applied to the equation (7.2) to obtain values of the unknown coefficients $\alpha_r'$ for $r = 1, 2 \ldots J$. To achieve convergence to two decimal places in the sum in equation (7.2) it was sufficient to take $J = 9$ and choose the values $x = \cos(k\pi/10)$ for $k = 1, 2, \ldots, 9$ to generate nine linear algebraic equations for the $\alpha_r'$ for $r = 1, 2 \ldots 9$. The linear equations were then solved and the values of $\alpha_r'$ substituted into equations (7.3) to (7.5) to provide numerical values for the crack opening displacement and the SIFs.

The numerical results obtained are consistent with the analytical results given by (6.6) to (6.8), (6.14) and (6.15) and the qualitative analysis in section 7. For the case when $p(x) = -p_0$ the values of the stress intensity factors obtained numerically using equations (7.2), (7.4) and (7.5) and displayed in Table 1 become small as $\beta'$ becomes large and tend to the value given by (6.8) for small $\beta'$. Also as $\beta'$ becomes large the crack opening displacement $u(x, 0)/p_0'$ obtained numerically using equation (7.3) and displayed in Figures 2 to 4 tends to the profile given by equation (6.14). Specifically when $\mu^R = 1$ and $\beta' = 10$ then

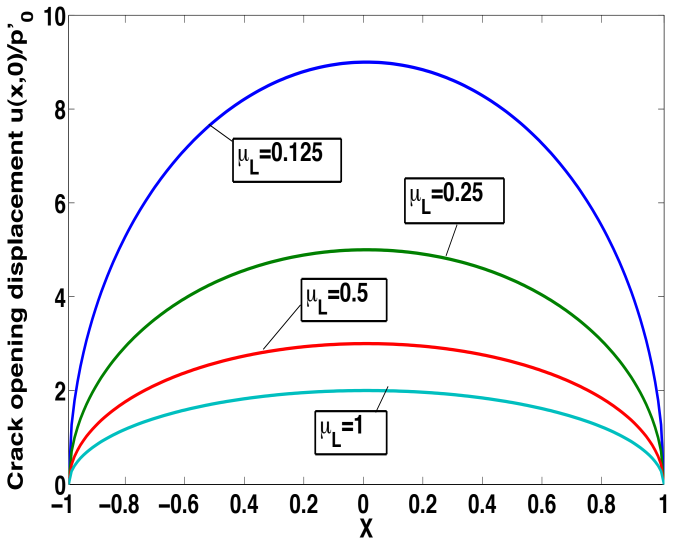


**Fig. 2** Crack opening displacement $u(x,0)/p'_0$ for $\beta' = 0$, $\mu^R = 1$ and various values of $\mu^L$.

**Table 1** The values of $\mathcal{K} = \mathcal{K}_+/p'_0 = \mathcal{K}_-/p'_0$ for various values of $\beta'$.

| $\beta'$ | $\mathcal{K}$ |
|---|---|
| 0 | 0.7071 |
| 0.01 | 0.7025 |
| 0.05 | 0.6848 |
| 0.1 | 0.6642 |
| 0.5 | 0.5417 |
| 1 | 0.4488 |
| 5 | 0.2179 |
| 10 | 0.1420 |
| 25 | 0.0721 |
| 50 | 0.0401 |
| 100 | 0.0213 |
| 500 | 0.0045 |

for $\mu^L = 1, 0.5, 0.25, 0.125$ the crack opening displacement calculated using (6.14) is given, respectively, by $u(x)/p'_0 = 0.2, 0.3, 0.5, 0.9$ while if $\beta' = 100$ the crack opening displacement calculated using (6.14) is given, respectively, by $u(x)/p'_0 = 0.02, 0.03, 0.05, 0.09$. A comparison of these displacement values with the corresponding displacement profiles in Figures 3 and 4 illustrates the movement of the crack opening displacement towards the values given by equation (6.14) as the value of $\beta'$ increases.

Figures 5 and 6 give the crack opening displacement for the cases $\beta' = 0$ and $\beta' = 10$ for

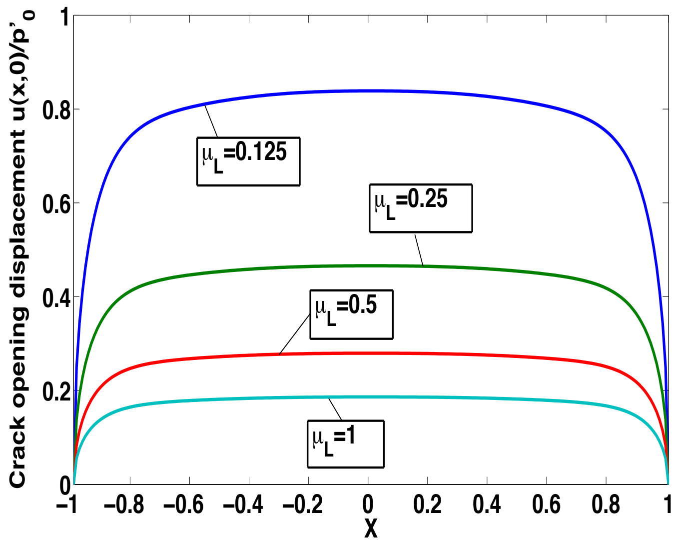


**Fig. 3** Crack opening displacement $u(x,0)/p'_0$ for $\beta' = 10$, $\mu^R = 1$ and various values of $\mu^L$.

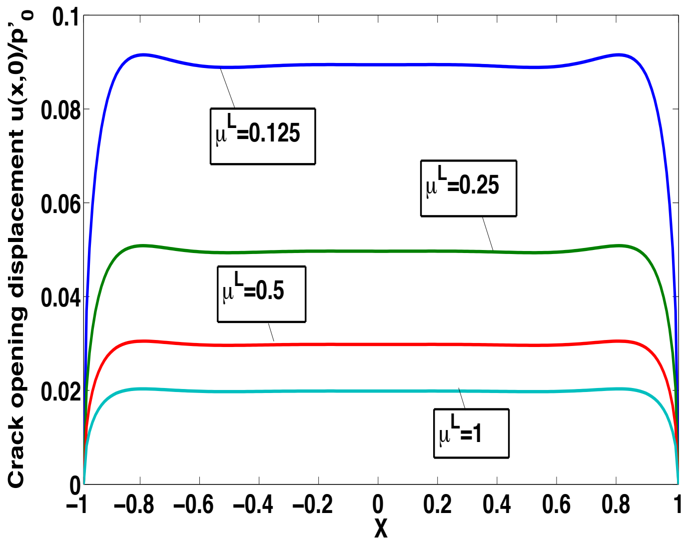


**Fig. 4** Crack opening displacement $u(x,0)/p'_0$ for $\beta' = 100$, $\mu^R = 1$ and various values of $\mu^L$.

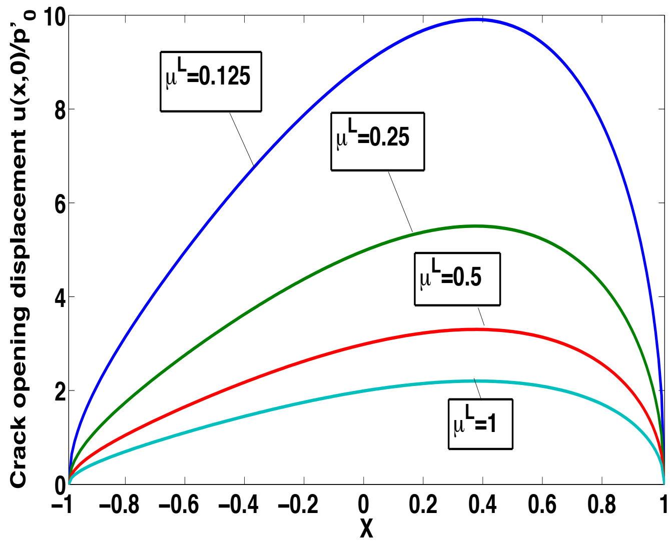


**Fig. 5** Crack opening displacement $u(x,0)/p_0'$ for $\beta' = 0$, $\mu^R = 1$ and $\overline{p}(x) = -p_0'(1+x)/2$.

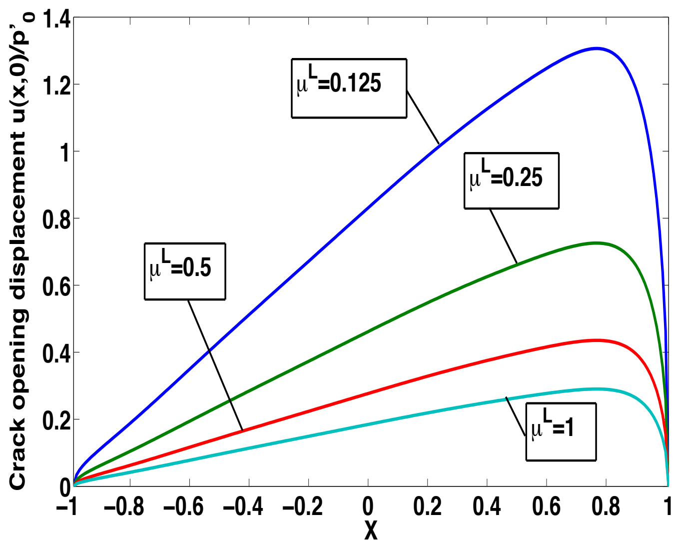


**Fig. 6** Crack opening displacement $u(x,0)/p_0'$ for $\beta' = 10$, $\mu^R = 1$ and $\overline{p}(x) = -p_0'(1+x)/2$.

various values of $\mu^L$ when the applied pressure on the crack faces varies according to the equation $\overline{p}(x) = -p_0'(1+x)/2$. The corresponding stress intensity factors for this load profile are $\mathcal{K}_+/p_0' = 1.061$ and $\mathcal{K}_-/p_0' = 0.354$ when $\beta' = 0$ and $\mathcal{K}_+/p_0' = 0.275$ and $\mathcal{K}_-/p_0' = 0.019$ when $\beta' = 10$.

As a consequence of the proportionality relationship (7.6) it follows that if $\mu^R = 1$ then for a given $\overline{p}(x)$, a fixed $\beta' \geqslant 0$ and a fixed $x_0 \in (0,1)$ the crack opening displacements $u(x_0)/p_0'$ for $\mu^L = 0.5$, $\mu^L = 0.25$, $\mu^L = 0.125$ are, respectively, 1.5, 2.5 and 4.5 times the crack opening displacement for $\mu^L = 1$. This is consistent with the displacement profiles in Figures 2 to 6.

By way of comparison with existing solutions to problems of the type considered in this paper it is noted that, for the special case of the problem considered in section 4 when $\mu^L = \mu^R$, Clements *et al.* (**4**) derived approximate explicit formulas for the stress intensity factors and the crack displacement. These approximate formulas are only relevant for small values of $\beta'$ and for such values they yield numerical values which are consistent with the corresponding values obtained from the formulas in the present paper. In particular the approximate formula obtained in (**4**) for the stress intensity factors for the case when the applied traction is $\overline{p}(x) = -p_0'$ takes the form

$$K_+ = K_- = \frac{p_0'}{\sqrt{2}}\left[1 - \frac{2\beta}{\pi}\right]. \tag{8.1}$$

It may be readily verified that for $0 \leqslant \beta' \leqslant 0.1$ the values generated by this formula are in close agreement with the corresponding values in Table 1.

## 9. Conclusion

An antiplane crack problem has been considered for a plane crack along the interface between two joined dissimilar functionally graded half-spaces in which the elastic parameters vary quadratically with the spatial coordinates.

Previous studies of crack problems of this type have been restricted to the case when the inhomogeneous half-spaces are the mirror images of each other in the plane interface. When such symmetry exists the problem can be reduced to consideration of a boundary value problem for a single half-space. The current study has investigated the more complicated case when the two half-spaces are different and hence are not symmetrical with respect to the plane interface. Expressions for the crack opening displacement and crack tip stress intensity factors have been obtained in terms of a finite sum of Chebyschev polynomials. These expressions have been used to obtain numerical values for the crack opening displacement and stress intensity factors for some particular materials. The results indicate how particular functional gradations of the material parameters give rise to a substantial reduction in the stress intensity factors and the crack opening displacement.

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