Essays on Continuous Time Diffusion Models

A DISSERTATION

SUBMITTED TO THE GRADUATE CENTRE
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Economics

By

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November 2013
Declaration

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ABSTRACT

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During the past few decades, continuous time diffusion models have become an integral part of financial economics. Especially, in certain core areas in finance, such as interest rate, asset pricing, option pricing, portfolio selection and volatility modelling, continuous time diffusion models have proved to be a very attractive way to conduct research and gain economic intuition. This thesis makes three main contributions to the field of continuous time diffusion models.

First, we propose regime-switching Heston, GARCH, and CEV stochastic volatility models where all parameters are allowed to vary depending on the state of the economy. Then we apply these models to describe the dynamics of S&P 500 and VIX. We find strong evidence of regime shifts for all models. The CEV model is statistically preferred to other two nested models in explaining dynamics of data.

Second, because the true transition density functions of regime-switching stochastic volatility models are unknown, the standard maximum likelihood estimation cannot be conducted. We first conduct the maximum likelihood estimation with closed-form likelihood expansions for regime-switching continuous time stochastic volatility models.

Third, to approximate a continuous time diffusion process, researchers often use the Euler approximation in the literature. Theoretically, the smaller the discretization interval is, the more accurate the Euler approximation is expected to be. However, even when the discretization interval is too small, the accuracy of the Euler approximation can get worse because of the roundoff error and random number generator bias. A variety of univariate and multivariate diffusion models from the literature are considered. We use the solution of a diffusion process when it is available and
usable as a benchmark. The Milstein approximation is also adopted to compare the accuracy of the Euler approximation. Depending on the problem of interest, different criteria are used to measure accuracy of approximation. The percentage error and strong convergence can be examined when a good approximation of sample path of a diffusion model is required. The weak convergence is preferred for the cases where approximation of moments of the process matters. In our Monte Carlo simulation studies of diverse diffusion models, we measure accuracy of the Euler approximation not only by using those criteria but also by looking at end point of the approximation. The simulation results show that an appropriate discretization interval must be picked for different diffusion models when applying the Euler approximation.
Acknowledgements

I would like to take this opportunity to express my sincere gratitude to a number of people, who have provided me with useful assistance and support during my PhD study.

First and foremost, I want to thank my principle supervisor Dr Seungmoon Choi. It has been an honour to be his first PhD student. He provided unreserved support during my PhD study and generously paved the way for my development as a research economist.

I would like to thank my co-supervisor Professor Jiti Gao for his kind assistance and support on academic and various other issues. I also wish to thank Associate Professor Ralph Bayer, who helped me book a computer lab to run simulations. I also thank Dr Nicholas Sim for his helpful suggestions when he attended my presentation. Special thanks go to Professor Christopher Findlay, Dr Mandar Oak, Dr Jacob Wong, Dr Tatyana Chesnokova and Mrs Anne Arnold for their assistance with administrative issues on various occasions.

I would like to thank all my friends for all the emotional support, camaraderie, entertainment, and caring they provided.

I wish to thank my brother Wang Yuan, who has been my best friend all my life and thank him for all his advice and support.

This thesis is dedicated to my mother Li Liu, who has provided me with absolute and unconditional support and encouragement throughout my PhD study. Without her, this thesis would not have been completed.
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