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Towards Renormalizable Gravity Without Negative-Energy States

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Towards renormalizable gravity without negative-energy states

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A second-derivative gauge theory with a massless spin-2 boson on flat spacetime is presented. The dynamical symmetry preserves the spacetime metric and follows from an alternative interpretation of the equivalence principle. Gauge ambiguity is eliminated by a choice of reference frame, and the gauge boson propagator is derived from an invariant four-parameter polynomial action involving only dimensionless couplings. It is deduced that the theory is power-counting renormalizable in this gauge for almost all configurations of parameters. Examination of the linearized radiation then shows that, for some of these configurations, all excitations have non-negative canonical energy density. The paper concludes with an analysis of the static isotropic solutions to the weak-field vacuum equations.

CONTENTS

I. INTRODUCTION

To conceive a consistent quantum theory compatible with observable gravitational phenomena is widely recognized as an outstanding challenge of theoretical physics. Unfortunately, applications of established quantum field theory techniques to general relativity fall short due to the form of the implied interactions. Their negative mass dimensionality suggests that ultraviolet divergences cannot be removed by renormalization of a finite number of parameters.

It so happens that pure gravity, as dictated by Einstein's theory, is finite at one loop [1] due to a cancellation particular to four spacetime dimensions. However, coupling to matter—whether it be of scalar [1], Maxwell [2], Yang-Mills [3], or Dirac [4] type—introduces nonrenormalizable divergences at that same order. And even pure gravity diverges at two loops [5].

Recent approaches to the subject involve (a) more general quantization procedures, often with perturbative notions discarded, or (b) exotic frameworks such as string theories. In many cases, one must relinquish ideas from conventional gauge theories that have contributed to the success of modern particle physics. In almost all cases, one must acquire a radically different point of view than that which is sufficient to effectively model the other forces of nature.

The divergences of general relativity can be curbed by adding four-derivative terms to the Einstein-Hilbert action [6]. The resulting theory is even asymptotically free [7]. Six additional degrees of freedom are introduced to the two excitations corresponding to Einstein's massless spin-2 graviton: one corresponds to a massive spin-0 particle and the remaining five to a massive spin-2 particle. The full particle content is required for renormalizability. Unfortunately, the massive spin-2 particle is a ghost—it has negative classical energy. This difficulty is characteristic of higher-derivative theories, and seems to be unavoidable if one requires both renormalizability and unitarity [8].

The appearance of ghosts poses an, as yet unresolved, obstacle towards the acceptance of presently known renormalizable theories of gravitation. One can surmise, however, that enhancing the particle content can aid renormalizability. It is generally accepted that macroscopic effects of gravity are primarily due to interactions with a massless spin-2 field. It is also known [9] that coupling a free field with these characteristics to its own energy-momentum tensor leads to Einstein's theory. But there is no fundamental reason for additional particles to be absent. In fact, phenomenology may indicate otherwise. The current standard cosmological model involves general relativity with a nonzero cosmological constant and cold dark matter. While it does fit experimental data well, its theoretical motivation is incomplete; it suffers from the well-known cosmological constant problem [10] and posits that most of the matter content of the universe is practically undetectable. Alternatives that aim to resolve difficulties at the galactic [11] and cosmological [12] scales all include additional degrees of freedom.

In this paper it is proposed that general covariance may not be the most appropriate symmetry for relativistic gravitation. Instead, physical laws are required to be invariant with respect to those local transformations that preserve the natural structure of Minkowski spacetime. That is, field equations, etc., should take the same form in inertial and rotating/accelerating frames, but need not be invariant with respect to local scale transformations. Then the volume form and metric can be constant and the field equations can be derived from a polynomial action, greatly simplifying renormalization.

After examining how reference frames transform with respect to gauge transformations in Section II, a complex-valued gauge field is introduced and used to construct various covariant quantities in Section III. Then, in Section IV, a second-order action that involves a weighted sum of four invariants is presented. These invariants are chosen so that the free-gauge-field action is sesquilinear quadratic in the gauge field (to promote positive kinetic energy) and so that the gauge field propagator behaves like k^{-2} for large momenta k. In Sections V and VI, a quantization via the Faddeev-Popov procedure is proposed and the gauge field propagator is derived. From the form of the propagator and the fact that all couplings have non-negative mass dimension, it is concluded that the theory is power-counting renormalizable. To further examine the particle content of the theory, and to check if any excitations have negative energy, Section VII explores plane wave solutions of the linearized vacuum equations. In addition to a massless spin-2 boson, various excitations of helicity 0 and 1 are present; depending on the parameters that appear in the action, there are either fourteen or sixteen dynamical degrees of freedom[1]. For certain values of these parameters, the linearized canonical energy is indeed non-negative. Finally, in Sections VIII and IX, weak-field static isotropic solutions are found and examined. The general equations of motion for a test particle within a spherically symmetric field prove to be difficult to solve directly. Several simplifying assumptions allow for a straightforward solution, however it is shown that the ensuing trajectories cannot correctly describe the delay of light passing by a spherically symmetric gravitational source. It remains to be investigated whether a more general approach can yield the correct result.

In the following: inactive indices are always omitted (contracted quantities are assigned unique symbols); the Minkowski metric is taken to be $[\eta] = \mathrm{diag}(1, -1, -1, -1)$; and the components of the antisymmetrization and symmetrization of a rank-2 tensor X are denoted by $X_{[\mu\nu]}$ and $X_{(\mu\nu)}$. Left- and right-handed spinor indices are denoted using unprimed and primed uppercase letters from the Latin alphabet like φ^A and $\chi^{A'}$; the two-dimensional antisymmetric symbol ε is used to (carefully) raise and lower spinor indices; and the vector-spinor translation symbol σ, given by the Pauli matrices, is used to construct the canonical double covering from the spin group $SL_2(\mathbb{C})$ to the connected Lorentz group (refer to Appendix A for details).

II. DYNAMICAL SYMMETRY

The equivalence principle can be interpreted as the requirement that gravitation be completely indistinguishable from the fictitious forces that appear in accelerated reference frames. It is therefore peculiar that general relativity postulates invariance with respect to local scale transformations—transformations that affect the spacetime metric and volume form. In fact, the resulting nonpolynomial couplings are one of the primary obstacles to the theory's renormalization. Another quirk of general covariance is that it is completely independent from Lorentz invariance: one can construct generally covariant theories that are Galilean in freely-falling frames. This section provides an alternative to general covariance that circumvents these issues while respecting the equivalence principle as stated above.

The theory in this paper is formulated on a flat Minkowski background. One can readily assume the existence of global coordinates. Nevertheless, certain notions from the theory of manifolds prove indispensable.

Let $\{x^\alpha\}_{\alpha=0,\dots,3}$ denote orthonormal (Cartesian) coordinates and $\{\partial/\partial x^\alpha\}$ the corresponding coordinate frame. Lowercase indices from the beginning of the Greek alphabet will be used to denote vector components with respect to these coordinates. Now let $\{\partial_\mu\}_{\mu=0,\dots,3}$ denote another frame related to $\{\partial/\partial x^\alpha\}$ by a smoothly varying proper orthochronous Lorentz transformation[2] $\Lambda = \Lambda(x)$,

$$\partial_\mu \equiv \left(\Lambda^{-1}\right)^\alpha_\mu \frac{\partial}{\partial x^\alpha} \,. \tag{2.1}$$

[1] Fourteen or sixteen degrees of freedom may seem excessive compared to the two degrees of freedom that Einstein's theory exhibits, but not unreasonable if one compares to the theories that govern electroweak and strong interactions: three massive weak bosons possess nine degrees of freedom between them and eight gluons boast a total of sixteen.

[2] Recall that a proper orthochronous Lorentz transformation Λ satisfies $\Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$, $\det\Lambda = 1$, and $\Lambda^0_0 \geq 1$, and resides in the connected Lorentz group.

By definition, the transformation Λ preserves η as well as time and space orientations. The (global) discrete transformations involving time and space inversions are neglected to make the treatment of spinor fields less convoluted.

One can try to find coordinates $\{y^\mu\}$ for which $\partial/\partial y^\mu = \partial_\mu$, however the relevant differential equation rarely admits solutions. Since, for any $\{y^\mu\}$, one has

$$\frac{\partial^2}{\partial y^{[\mu}\partial y^{\nu]}} = 0 \,, \tag{2.2}$$

a necessary condition for $\{\partial_\mu\}$ to be a coordinate frame is $\partial_{[\mu}\partial_{\nu]} = 0$. To demonstrate why this is not true in general, and to aid the forthcoming construction of covariant objects, it is useful to define the "object of anholonomity" ω, given by

$$\partial_{[\mu}\partial_{\nu]} = \omega^\rho_{\mu\nu}\,\partial_\rho \,. \tag{2.3}$$

Acting on some field Φ with the differential operator $\partial_{[\mu}\partial_{\nu]}$, and noting that (2.2) also applies for $\{x^\alpha\}$, one finds that

$$\partial_{[\mu}\partial_{\nu]}\Phi = \left[(\Lambda^{-1})^\alpha_{\ [\mu}\frac{\partial}{\partial x^{|\alpha|}}\right]\left[(\Lambda^{-1})^\beta_{\ \nu]}\frac{\partial}{\partial x^\beta}\right]\Phi \tag{2.4}$$

$$= (\Lambda^{-1})^\alpha_{\ [\mu}\left[\frac{\partial}{\partial x^{|\alpha|}}(\Lambda^{-1})^\beta_{\ \nu]}\right]\frac{\partial\Phi}{\partial x^\beta} \,, \tag{2.5}$$

and hence,

$$\omega^\rho_{\mu\nu} = \Lambda^\rho_\beta\,(\Lambda^{-1})^\alpha_{\ [\mu}\frac{\partial}{\partial x^{|\alpha|}}(\Lambda^{-1})^\beta_{\ \nu]} = -(\Lambda^{-1})^\alpha_{\ \mu}(\Lambda^{-1})^\beta_{\ \nu}\frac{\partial}{\partial x^{[\alpha}}\Lambda^\rho_{\ \beta]} \,. \tag{2.6}$$

The relevant condition for $\{\partial_\mu\}$ to be a coordinate frame is then [13] precisely $\omega = 0$, which requires that the curl in (2.6) vanishes. It is clear that this condition is satisfied if Λ is constant (i.e., a global Lorentz transformation), as one would expect. Frames $\{\partial_\mu\}$ for which ω is nonzero do not correspond to coordinates and are called "anholonomic." Orthonormal frames, such as those defined by vierbein in general relativity, are generally anholonomic.

With respect to an arbitrary (metric and orientation preserving) local transformation

$$\partial_\mu \to (\Lambda^{-1})^\nu_{\ \mu}\,\partial_\nu \,, \tag{2.7}$$

it can be seen from (2.6) that ω transforms like

$$\omega^\rho_{\mu\nu} \to (\Lambda^{-1})^\lambda_{\ \mu}(\Lambda^{-1})^\tau_{\ \nu}\left(\Lambda^\rho_\sigma\,\omega^\sigma_{\lambda\tau} - \partial_{[\lambda}\Lambda^\rho_{\ \tau]}\right) . \tag{2.8}$$

The affine nature of this transformation proves useful when seeking covariant quantities. Nevertheless, ω is devoid of physical content. It is merely an intrinsic measure of acceleration, an artefact of the inability to describe accelerating frames in terms of coordinates.

Working exclusively in frames which are related by local Lorentz transformations allows for a straightforward treatment of spinor fields[3]. However, a choice of orthonormal

[3] In generally covariant theories, on the other hand, one must introduce vierbein fields [4] as general linear groups do not admit spin representations [14].

frame does not unambiguously specify the components of such fields. This can be seen as follows[4]. Suppose that $\{\iota_A\}_{A=1,2}$ is a spin frame—a smoothly varying choice of basis for complex, 2-dimensional spinor space—such that

$$\partial_\mu = \sigma_\mu^{AA'}\, \iota_A \bar{\iota}_{A'}\ . \tag{2.9}$$

Then, with respect to a local transformation $\lambda = \lambda(x) \in SL_2(\mathbb{C})$, (2.9) transforms as required,

$$\partial_\mu \to \Lambda_\mu^\nu\, \partial_\nu = \sigma_\mu^{AA'}\, \lambda_A^{\ B}\, \bar{\lambda}_{A'}^{\ B'}\, \iota_B \bar{\iota}_{B'}\ , \tag{2.10}$$

where $\Lambda = \Lambda(\lambda)$ is given by the canonical double covering (A3) from $SL_2(\mathbb{C})$ to the connected Lorentz group. Since the relation (2.9) is invariant with respect to phase transformations

$$\iota \to e^{i\theta}\, \iota\ , \tag{2.11}$$

a choice of frame $\{\partial_\mu\}$ can at most fix the components of a spinor field $\varphi = \varphi^A \iota_A$ up to phase. The only way to avoid such ambiguities, so that the components of fields of any spin are well-defined, is to explicitly specify a spin frame.

The working in the remainder of this paper takes place in a spin frame $\{\iota_A\}$ that is orthonormal and oriented, in the sense that it induces a frame $\{\partial_\mu\}$, via (2.9), that is given by a local transformation (2.1) of the orthonormal coordinate frame $\{\partial/\partial x^\alpha\}$. A theory is constructed that is invariant with respect to gauge transformations $\lambda = \lambda(x) \in SL_2(\mathbb{C})$,

$$\iota_A \to \left(\lambda^{-1}\right)_A^{\ B} \iota_B\ . \tag{2.12}$$

In certain circumstances, it is useful to eliminate unphysical gauge degrees of freedom by requiring that $\{\iota_A\}$ lie within a specific class of frames. This can be accomplished by insisting that the object of anholonomity ω corresponding to $\{\partial_\mu\}$ vanishes,

$$\omega = 0\ . \tag{2.13}$$

It is always possible to move to a frame where (2.13) holds by applying a gauge transformation that agrees with the inverse of (2.1) up to a global Lorentz transformation.

The condition (2.13), unlike most gauge conditions, places no constraint upon any dynamical field. Nevertheless, (2.13) is not gauge covariant and does break gauge symmetry; from (2.8) it is evident that it only respects transformations $\Lambda = \Lambda(\lambda)$ that satisfy

$$\partial_{[\mu} \Lambda_{\nu]}^{\rho} = 0\ . \tag{2.14}$$

Since (2.14) amounts to an overdetermined linear system of 24 differential equations in 6 unknowns, only trivial solutions with $\partial\Lambda = 0$ exist. This means that, once one chooses a coordinate frame, the only further transformations that can be performed without violating (2.13) are global Lorentz transformations; special relativity re-emerges in Cartesian coordinates if gravitation is absent.

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[4] Notational conventions regarding spinors are established in Appendix A, which also serves as a brief review of chiral spinor algebra. The conventions used here agree, for the most part, with [15].

III. GAUGE COVARIANCE

In modern gauge theories, gauge fields are introduced to make the field equations that govern matter—typically fermionic matter—covariant in the simplest possible way. This is not the case with general relativity, which was developed before the significance of gauge theories was recognized. In fact, it is not even possible to construct a generally covariant description of spinor fields using only a dynamical spacetime metric. One must introduce unphysical degrees of freedom via vierbein [4] which respect, in addition to general covariance, a local Lorentz symmetry. This makes the treatment of fields with half-odd-integer spin in general relativity quite cumbersome.

In this paper, the fundamental gauge fields are chosen so as to couple to Dirac fields in a fashion similar to that of the familiar models of particle physics.

Let (φ,χ) be a Dirac bispinor field made up of left- and right-handed fields $\varphi=\varphi^A\iota_A$ and $\chi=\chi^{A'}\bar\iota_{A'}$. With respect to a global spacetime transformation $\lambda\in SL_2(\mathbb{C})$, it transforms like

$$(\varphi^A,\chi^{A'})\to(\lambda^A_B\,\varphi^B,\,\overline{\lambda}^{A'}_{B'}\chi^{B'})\,. \tag{3.1}$$

The Dirac Lagrangian is given by[5]

$$\begin{aligned}\mathcal{L}[\varphi,\chi]\equiv{}&\frac{i}{\sqrt{2}}\left[\sigma^\mu_{AA'}\,\overline{\varphi}^{A'}(\partial_\mu\varphi^A)+\sigma^\mu_{AA'}\,\overline{\chi}^A(\partial_\mu\chi^{A'})\right]\\&-\frac{i}{\sqrt{2}}\left[\sigma^\mu_{AA'}\,\varphi^A(\partial_\mu\overline{\varphi}^{A'})+\sigma^\mu_{AA'}\,\chi^{A'}(\partial_\mu\overline{\chi}^A)\right]+m\left(\varepsilon_{AB}\,\varphi^A\overline{\chi}^B+\varepsilon_{A'B'}\,\overline{\varphi}^{A'}\chi^{B'}\right).\end{aligned} \tag{3.2}$$

The easiest way to make the Lagrangian (3.2) invariant with respect to gauge transformations (2.12) is to introduce a (complex) gauge field $\mathcal{B}$ that transforms like

$$\begin{aligned}(\mathcal{B}_\mu)^A_B&\to(\Lambda^{-1})^\nu_\mu\left[\lambda^A_C\,(\lambda^{-1})^D_B\,(\mathcal{B}_\nu)^C_D+\frac{1}{K}\,(\lambda^{-1})^C_B\,\partial_\nu\lambda^A_C\right]\\&=\sigma^{EE'}_\mu\sigma^\nu_{FF'}\,(\lambda^{-1})^F_E\,(\overline{\lambda}^{-1})^{F'}_{E'}\left[\lambda^A_C\,(\lambda^{-1})^D_B\,(\mathcal{B}_\nu)^C_D+\frac{1}{K}\,(\lambda^{-1})^C_B\,\partial_\nu\lambda^A_C\right],\end{aligned} \tag{3.3}$$

where K is a (real) dimensionless coupling constant and $\Lambda=\Lambda(\lambda)$ is given by the double covering (A3). Then one can replace the derivatives $\partial\varphi$ and $\partial\chi$ with covariant derivatives given by

$$D_\mu\varphi^A\equiv\partial_\mu\varphi^A-K\,(\mathcal{B}_\mu)^A_B\,\varphi^B\qquad\text{and}\qquad D_\mu\chi^{A'}\equiv\partial_\mu\chi^{A'}-K\,(\overline{\mathcal{B}}_\mu)^{A'}_{B'}\,\chi^{B'}. \tag{3.4}$$

One can also covariantly differentiate 1-forms A like

$$D_\mu A_\nu\equiv\partial_\mu A_\nu-\Gamma^\rho_{\mu\nu}A_\rho\,, \tag{3.5}$$

where

$$\Gamma^\rho_{\mu\sigma}\equiv-K\,\sigma^\rho_{AA'}\,\sigma^{BA'}_\sigma\,(\mathcal{B}_\mu)^A_B-K\,\sigma^\rho_{AA'}\,\sigma^{AB'}_\sigma\,(\overline{\mathcal{B}}_\mu)^{A'}_{B'}=-2K\,\mathrm{Re}\left[\sigma^\rho_{AA'}\,\sigma^{BA'}_\sigma\,(\mathcal{B}_\mu)^A_B\right]. \tag{3.6}$$

[5] The bispinor representation commonly referred to as the chiral or Weyl representation is used here. If one sets

$$\psi\equiv[\varepsilon_{1A}\varphi^A,\varepsilon_{2A}\varphi^A,\chi^{1'},\chi^{2'}]^\top=[\varphi^2,-\varphi^1,\chi^{1'},\chi^{2'}]^\top,\qquad\overline{\psi}\equiv\psi^\dagger\begin{bmatrix}0&1\\1&0\end{bmatrix}=[\overline{\chi}^1,\overline{\chi}^2,\overline{\varphi}^{2'},-\overline{\varphi}^{1'}],$$

$$\text{and}\qquad\partial\!\!\!/=\begin{bmatrix}0&\partial_0\\\partial_0&0\end{bmatrix}+\begin{bmatrix}0&\boldsymbol{\sigma}\cdot\boldsymbol{\partial}\\-\boldsymbol{\sigma}\cdot\boldsymbol{\partial}&0\end{bmatrix},$$

then one can write (3.2) as

$$\mathcal{L}[\varphi,\chi]=\mathrm{Re}\left(i\,\overline{\psi}\partial\!\!\!/\psi\right)-m\,\overline{\psi}\psi\,.$$

Note that Γ is real and transforms like the affine connection of general relativity[6],

$$\Gamma^{\rho}_{\mu\sigma} \to \left(\Lambda^{-1}\right)^{\nu}_{\mu} \left[\Lambda^{\rho}_{\lambda} \left(\Lambda^{-1}\right)^{\tau}_{\sigma} \Gamma^{\lambda}_{\nu\tau} - \left(\Lambda^{-1}\right)^{\lambda}_{\sigma} \partial_{\nu} \Lambda^{\rho}_{\lambda}\right] . \tag{3.7}$$

It is reasonable to require compatibility with the metric η,

$$D_{\rho}\eta_{\mu\nu} = -\eta_{\lambda\nu}\Gamma^{\lambda}_{\rho\mu} - \eta_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu} = 2K\,\eta_{\mu\nu}\,\mathrm{Re}\left[\left(\mathcal{B}_{\rho}\right)^{A}_{A}\right] = 0 . \tag{3.8}$$

Taking the trace of (3.3) shows that the 1-form $\mathcal{B}^{A}_{A}$ transforms homogeneously. One can therefore satisfy (3.8), without ruining covariance, by setting

$$\mathcal{B}^{A}_{A} = 0 . \tag{3.9}$$

The condition (3.9) can be regarded in a similar fashion to the condition that the metric be symmetric in general relativity (i.e., it is imposed on the symbols themselves a priori). It also serves to eliminate $\mathrm{Im}\,\mathcal{B}^{A}_{A}$, which amounts to a superfluous electromagnetic potential.

Since the algebra $\mathfrak{sl}_2(\mathbb{C})$ consists of the traceless 2×2 complex matrices, the gauge field $\mathcal{B}$ may now be regarded as a 1-form taking values in its adjoint representation. However, decomposing $\mathcal{B}$ in terms of a basis for $\mathfrak{sl}_2(\mathbb{C})$ would only complicate the situation by introducing yet another set of indices. Nevertheless, one should keep in mind that $\mathcal{B}$ now only contains 12 independent complex components. The same information is contained in the 24 real components of Γ (satisfying (3.8)), as is evident from the existence of the formula for $\mathcal{B}$ in terms of Γ,

$$\left(\mathcal{B}_{\mu}\right)^{A}_{B} = -\frac{1}{2K}\,\sigma^{AA'}_{\rho}\,\sigma^{\sigma}_{BA'}\,\Gamma^{\rho}_{\mu\sigma} , \tag{3.10}$$

which emerges once (3.9) is imposed.

A general spin-tensor with the valence of $\mathcal{B}$ transforms, with respect to global Lorentz transformations, according to the 16-dimensional complex representation[7]

$$(\tfrac{1}{2},\tfrac{1}{2}) \otimes (\tfrac{1}{2},0)_{\mathbb{C}} \otimes (\tfrac{1}{2},0)_{\mathbb{C}} = (\tfrac{3}{2},\tfrac{1}{2})_{\mathbb{C}} \oplus (\tfrac{1}{2},\tfrac{1}{2})_{\mathbb{C}} \oplus (\tfrac{1}{2},\tfrac{1}{2})_{\mathbb{C}} . \tag{3.11}$$

The condition of tracelessness (3.9) eliminates exactly one of the vector factors $(\tfrac{1}{2},\tfrac{1}{2})_{\mathbb{C}}$, and so $\mathcal{B}$ and $\overline{\mathcal{B}}$ transform according to the representations

$$(\tfrac{3}{2},\tfrac{1}{2})_{\mathbb{C}} \oplus (\tfrac{1}{2},\tfrac{1}{2})_{\mathbb{C}} \qquad \text{and} \qquad (\tfrac{1}{2},\tfrac{3}{2})_{\mathbb{C}} \oplus (\tfrac{1}{2},\tfrac{1}{2})_{\mathbb{C}} . \tag{3.12}$$

It is clear that these fields contain components that rotate like objects of spin 2.

One can now proceed to construct covariant objects by examining how the antisymmetrized differential operator $[D,D]$ acts on various covariant fields. By applying it on a

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[6] Recall that, with respect to a general coordinate transformation dx'/dx in general relativity, the affine connection transforms like

$$\begin{aligned}\Gamma^{\rho}_{\mu\sigma} &\to \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial x'^{\rho}}{\partial x^{\lambda}}\frac{\partial x^{\tau}}{\partial x'^{\sigma}}\Gamma^{\lambda}_{\nu\tau} + \frac{\partial x'^{\rho}}{\partial x^{\lambda}}\frac{\partial^2 x^{\lambda}}{\partial x'^{\mu}\partial x'^{\sigma}} \\ &= \frac{\partial x^{\nu}}{\partial x'^{\mu}}\left[\frac{\partial x'^{\rho}}{\partial x^{\lambda}}\frac{\partial x^{\tau}}{\partial x'^{\sigma}}\Gamma^{\lambda}_{\nu\tau} - \frac{\partial x^{\lambda}}{\partial x'^{\sigma}}\frac{\partial^2 x'^{\rho}}{\partial x^{\nu}\partial x^{\lambda}}\right] .\end{aligned}$$

[7] The standard notation for specifying spin representations [16] is made use of here. For instance, the Dirac field (3.1) transforms according to $(\tfrac{1}{2},0)_{\mathbb{C}} \oplus (0,\tfrac{1}{2})_{\mathbb{C}}$, vectors fields transform according to $(\tfrac{1}{2},\tfrac{1}{2})$, and perturbations from the Minkowski metric in Einstein's theory (i.e., symmetric rank-2 Lorentz tensors) transform according to $(1,1)\oplus(0,0)$.

scalar ϕ, left-handed spinor φ, and 1-form A, and making sure to account for the object of anholonomity ω (2.3), one finds that

$$D_{[\mu}D_{\nu]}\phi = K\,\tau^{\rho}_{\mu\nu}\,\partial_\rho\phi\ , \tag{3.13}$$

$$D_{[\mu}D_{\nu]}\varphi^A = -K\left(\mathcal{G}_{\mu\nu}\right)^A_B\varphi^B + K\,\tau^{\rho}_{\mu\nu}\,D_\rho\varphi^A\ , \tag{3.14}$$

and

$$D_{[\mu}D_{\nu]}A_\rho = -R^{\sigma}_{\mu\nu\rho}\,A_\sigma + K\,\tau^{\sigma}_{\mu\nu}\,D_\sigma A_\rho\ , \tag{3.15}$$

where the torsion τ is given by

$$\tau^{\rho}_{\mu\nu} \equiv -\frac{1}{K}\,\Gamma^{\rho}_{[\mu\nu]} + \frac{1}{K}\,\omega^{\rho}_{\mu\nu}\ , \tag{3.16}$$

the field strength $\mathcal{G}$, by

$$\left(\mathcal{G}_{\mu\nu}\right)^A_B \equiv \partial_{[\mu}\left(\mathcal{B}_{\nu]}\right)^A_B - K\left(\mathcal{B}^A_C\right)_{[\mu}\left(\mathcal{B}_{\nu]}\right)^C_B - \omega^{\rho}_{\mu\nu}\left(\mathcal{B}_\rho\right)^A_B\ , \tag{3.17}$$

and the curvature R, by

$$R^{\rho}_{\mu\nu\sigma} \equiv \partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\sigma} - \omega^{\lambda}_{\mu\nu}\,\Gamma^{\rho}_{\lambda\sigma}\ . \tag{3.18}$$

Note that, as $\mathcal{B}$ is complex and traceless so is $\mathcal{G}$, whereas τ and R are real like Γ and ω.

It should be stressed that it is not possible to eliminate the torsion τ, as is done in Einstein's theory, by imposing further constraints on the affine connection. This is clear from the fact that both Γ and τ consist of 24 independent real components. In fact, it is not difficult to show directly from (3.16) and the metric compatibility condition (3.8) that, in a coordinate frame,

$$\Gamma^{\rho}_{\mu\nu} = 2K\,\eta^{\rho\sigma}\eta_{\lambda(\mu}\tau^{\lambda}_{\nu)\sigma} - K\,\tau^{\rho}_{\mu\nu}\ . \tag{3.19}$$

Therefore imposing the constraint $\tau = 0$ would eliminate all but pure gauge configurations. This is not surprising, as even in general relativity torsion permeates orthonormal frames.

Due to nonvanishing torsion, the curvature R does not share the same properties as the curvature in Riemannian geometry. In particular, the Bianchi identities must be modified to account for τ, and the contraction analogous to the Ricci tensor, $R^{\rho}_{\mu\rho\nu}$, is no longer symmetric.

IV. INVARIANT ACTION

An immediate concern with having a noncompact gauge group is the likelihood of indefinite classical energy. Recall that, even though standard general relativity (with and without a cosmological constant) is known to avoid such complications [17], they seem to be a characteristic flaw of the known renormalizable alternatives [6, 8].

The prospect of instability is here exemplified by the fact that the invariant analogous to the Yang-Mills Lagrangian,

$$\eta^{\mu\rho}\eta^{\nu\sigma}\left(\mathcal{G}_{\mu\nu}\right)^A_B\left(\mathcal{G}_{\rho\sigma}\right)^B_A + \text{h.c.}\ , \tag{4.1}$$

changes sign if one makes the replacement $\mathcal{G} \to i\,\mathcal{G}$. One can improve the situation somewhat by requiring that the free linearized system be symmetric with respect to global phase transformations

$$\mathcal{B} \to e^{i\theta}\,\mathcal{B}\,. \tag{4.2}$$

Unfortunately, couplings to Dirac fields and cubic self-interaction terms, which are required by gauge covariance, cannot be invariant with respect to such transformations. It is possible, however, to form a gauge invariant action that respects this symmetry in coordinate frames (where ω vanishes) in the limit $K \to 0$. Then one can say that this global U_1 symmetry that is present in the coordinate gauge (2.13) is broken by the coupling strength K, just as global chiral symmetries are broken by the fermion mass in (3.2). Imposing such a symmetry (while requiring that the free linearized system not depend on K) prohibits constructions like (4.1) from appearing in the Lagrangian. The allowed terms reduce to sesquilinear quadratics in $\mathcal{G}$ and certain contractions quadratic in the torsion and its covariant derivatives.

To aid the construction of invariants that are sesquilinear in $\mathcal{G}$, it is useful to define the covariant complex tensors

$$\mathcal{C}^{\rho}_{\mu\nu\sigma} \equiv \sigma^{\rho}_{AA'}\,\sigma^{BA'}_{\sigma}\,(\mathcal{G}_{\mu\nu})^{A}_{B}\,, \qquad \mathcal{H}_{\mu\nu} \equiv \mathcal{C}^{\rho}_{\mu\rho\nu}\,, \qquad \text{and} \qquad \mathcal{K} \equiv \eta^{\mu\nu}\,\mathcal{H}_{\mu\nu}\,. \tag{4.3}$$

Then the curvature R is proportional to the real part of $\mathcal{C}$,

$$-\frac{1}{K}\,R^{\rho}_{\mu\nu\sigma} = \mathcal{C}^{\rho}_{\mu\nu\sigma} + \overline{\mathcal{C}}^{\rho}_{\mu\nu\sigma}\,. \tag{4.4}$$

Due to the antisymmetry and tracelessness of $\mathcal{G}$, one has

$$\mathcal{C}^{\rho}_{\nu\mu\sigma} = -\mathcal{C}^{\rho}_{\mu\nu\sigma}\,, \qquad \mathcal{C}^{\rho}_{\mu\nu\rho} = 0\,, \qquad \text{and} \qquad |\mathcal{C}|^2 = 0\,, \tag{4.5}$$

where the last identity refers to the contraction of $\mathcal{C}$ with its conjugate. It can further be shown that $\mathcal{H}$ (resp. $\overline{\mathcal{H}}$) is imaginary (anti-)self-dual,

$$\mathcal{H}_{[\mu\nu]} = \frac{i}{2}\,\epsilon_{\mu\nu\rho\sigma}\,\mathcal{H}^{\rho\sigma}\,, \tag{4.6}$$

where ϵ denotes the 4-dimensional antisymmetric symbol with $\epsilon_{0123} = 1$. This implies that the invariant $\mathcal{H}_{[\mu\nu]}\overline{\mathcal{H}}^{[\mu\nu]}$ vanishes.

There also exist invariants that are not quadratic in the field strength $\mathcal{G}$, but still respect the approximate U_1 symmetry outlined above. For instance, in the coordinate gauge (2.13) one has

$$\mathcal{T} = -\varepsilon^{BC}\,\varepsilon^{B'C'}\,\sigma^{(\mu}_{AA'}\,\sigma^{\nu)}_{CC'}\,(\mathcal{B}_{\mu})^{A}_{B}\,(\overline{\mathcal{B}}_{\nu})^{A'}_{B'}\,, \tag{4.7}$$

where $\mathcal{T}$ is a linear combination of contracted quadratic products of the torsion (3.16),

$$\mathcal{T} \equiv \eta^{\mu\rho}\,\tau^{\nu}_{\mu\nu}\,\tau^{\sigma}_{\rho\sigma} + \frac{1}{4}\,\eta^{\mu\rho}\,\tau^{\sigma}_{\mu\nu}\,\tau^{\nu}_{\rho\sigma} - \frac{1}{8}\,\eta^{\mu\rho}\,\eta^{\nu\sigma}\,\eta_{\lambda\tau}\,\tau^{\lambda}_{\mu\nu}\,\tau^{\tau}_{\rho\sigma}\,. \tag{4.8}$$

Deriving the expression (4.7) involves manipulating the quantities

$$(\mathfrak{I}^{\mu\rho})^{BD}_{AC} \equiv \mathfrak{i}^{[\mu\nu][\rho\sigma]}_{\lambda\tau}\,\sigma^{\lambda}_{AA'}\,\sigma^{BA'}_{\nu}\,\sigma^{\tau}_{CC'}\,\sigma^{DC'}_{\sigma} \tag{4.9}$$

and
$$\left(\mathfrak{K}^{\mu\rho}\right)^{BB'}_{AA'} \equiv \mathfrak{i}^{[\mu\nu][\rho\sigma]}_{\lambda\tau} \left(\sigma^{\lambda}_{AC'}\,\sigma^{BC'}_{\nu}\,\sigma^{\tau}_{CA'}\,\sigma^{CB'}_{\sigma} + \sigma^{\tau}_{AC'}\,\sigma^{BC'}_{\sigma}\,\sigma^{\lambda}_{CA'}\,\sigma^{CB'}_{\nu}\right) , \tag{4.10}$$
where
$$\mathfrak{i}^{\mu\nu\rho\sigma}_{\lambda\tau} \equiv \delta^{\nu}_{\lambda}\,\delta^{\sigma}_{\tau}\,\eta^{\mu\rho} + \frac{1}{4}\,\delta^{\nu}_{\tau}\,\delta^{\sigma}_{\lambda}\,\eta^{\mu\rho} - \frac{1}{8}\,\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\lambda\tau} . \tag{4.11}$$
One finds that
$$\left(\mathfrak{J}^{\mu\rho}\right)^{BD}_{AC} = \frac{9}{16}\,\eta^{\mu\rho}\,\delta^{B}_{A}\,\delta^{D}_{C} + \frac{3}{8}\,\eta^{\nu\rho}\,\delta^{B}_{A}\,\sigma^{\mu}_{CA'}\,\sigma^{DA'}_{\nu} - \frac{3}{8}\,\eta^{\nu\rho}\,\delta^{D}_{C}\,\sigma^{\mu}_{AA'}\,\sigma^{BA'}_{\nu} \tag{4.12}$$
and
$$\left(\mathfrak{K}^{\mu\rho}\right)^{BB'}_{AA'} = -\varepsilon^{BC}\varepsilon^{B'C'}\,\sigma^{(\mu}_{AA'}\,\sigma^{\rho)}_{CC'} + \frac{11}{8}\,\eta^{\mu\rho}\,\delta^{B}_{A}\,\delta^{B'}_{A'} . \tag{4.13}$$
Since
$$\mathcal{T} = \left(\mathfrak{J}^{\mu\rho}\right)^{BD}_{AC}\left(\mathcal{B}_{\mu}\right)^{A}_{B}\left(\mathcal{B}_{\rho}\right)^{C}_{D} + \text{h.c.} + \left(\mathfrak{K}^{\mu\rho}\right)^{BB'}_{AA'}\left(\mathcal{B}_{\mu}\right)^{A}_{B}\left(\overline{\mathcal{B}}_{\rho}\right)^{A'}_{B'} , \tag{4.14}$$
ignoring the terms involving δ^B_A or δ^D_C in (4.12) and (4.13), as $\mathcal{B}$ is traceless (3.9), leads to (4.7). Similarly,
$$\begin{aligned}\mathcal{U} \equiv \eta^{\mu\rho}\left(D_{\lambda}\tau^{\nu}_{\mu\nu}\right)\left(D^{\lambda}\tau^{\sigma}_{\rho\sigma}\right) + \frac{1}{4}\,\eta^{\mu\rho}\left(D_{\lambda}\tau^{\sigma}_{\mu\nu}\right)\left(D^{\lambda}\tau^{\nu}_{\rho\sigma}\right)&\\ - \frac{1}{8}\,\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\lambda\tau}\left(D_{\eta}\tau^{\lambda}_{\mu\nu}\right)\left(D^{\eta}\tau^{\tau}_{\rho\sigma}\right)&\end{aligned} \tag{4.15}$$
is given, to leading order, by
$$\mathcal{U} = -\varepsilon^{BC}\varepsilon^{B'C'}\,\sigma^{(\mu}_{AA'}\,\sigma^{\nu)}_{CC'}\left[\partial_{\rho}\left(\mathcal{B}_{\mu}\right)^{A}_{B}\right]\left[\partial^{\rho}\left(\overline{\mathcal{B}}_{\nu}\right)^{A'}_{B'}\right] + O(K) , \tag{4.16}$$
in the coordinate gauge (2.13).

The complex tensors (4.3) and the invariants (4.8) and (4.15) are sufficient to form a Lagrangian that defines a power-counting renormalizable theory. The Lagrangian that is analysed in the remainder of this paper is
$$\mathcal{L}[\mathcal{B}] \equiv \alpha\,\mathcal{H}_{(\mu\nu)}\,\overline{\mathcal{H}}^{(\mu\nu)} + \beta\,\mathcal{K}\overline{\mathcal{K}} + \frac{\gamma}{G}\,\mathcal{T} + \delta\,\mathcal{U} , \tag{4.17}$$
where α, β, γ, and δ are dimensionless real parameters and G is the gravitational constant.

It should be emphasized that (4.17) is not the most general gauge invariant Lagrangian, or even the most general Lagrangian that respects the approximate U_1 symmetry mentioned above. For instance,
$$\mathcal{R} \equiv \frac{1}{K^2}\,\eta^{\mu\nu}\,R^{\rho}_{\mu\rho\nu} - \frac{2}{K}\,D^{\mu}\tau^{\nu}_{\mu\nu} + 6\,\eta^{\mu\rho}\,\tau^{\nu}_{\mu\nu}\tau^{\sigma}_{\rho\sigma} + 2\,\eta^{\mu\rho}\,\tau^{\sigma}_{\mu\nu}\tau^{\nu}_{\rho\sigma} \tag{4.18}$$
is linearly independent from the terms in (4.17) in general frames, but is proportional to $\mathcal{T}$ when ω vanishes globally. However, since the remainder of this paper regards the linearized theory in the coordinate gauge, terms like $\mathcal{R}$, $\mathcal{T}^2$, and $D^2\mathcal{T}$, which contribute at most boundary terms to the linearized theory when $\omega = 0$, are ignored.

The Lagrangian (4.17) is remarkably similar to that of the renormalizable higher-derivative model[8] discussed in [6], particularly if one replaces $\mathcal{T}$ with $\mathcal{R}$ and neglects torsion. However, (4.17) should imply a better behaved energy spectrum than if one were to replace $\mathcal{H}$ and $\mathcal{K}$ with contractions of the real curvature R and thereby introduce quadratic terms that are bilinear, as opposed to sesquilinear, in $\mathcal{B}$. In addition, (4.17) is polynomial with each term containing no more than four factors of the fundamental field $\mathcal{B}$. This should be contrasted to the Einstein-Hilbert and higher-derivative Lagrangians which, if expanded about a vacuum configuration, consist of infinite power series of metric perturbations due to their nonpolynomial nature.

A system of Dirac fields interacting gravitationally is then governed by the Lagrangian

$$\mathcal{L}[\varphi,\chi,\mathcal{B}] \equiv \mathcal{L}[\varphi,\chi] + \mathcal{L}[\mathcal{B}] + K\,(\mathcal{J}^{\mu})_{A}^{\;B}\,(\mathcal{B}_{\mu})_{B}^{\;A} + K\,(\overline{\mathcal{J}}^{\mu})_{A'}^{\;B'}\,(\overline{\mathcal{B}}_{\mu})_{B'}^{\;A'}\,, \tag{4.19}$$

where $\mathcal{L}[\varphi,\chi]$, $\mathcal{L}[\mathcal{B}]$ are given by (3.2), (4.17), and

$$(\mathcal{J}^{\mu})_{A}^{\;B} \equiv -\frac{i}{\sqrt{2}}\,\sigma^{\mu}_{AA'}\,\overline{\varphi}^{A'}\varphi^{B} + \frac{i}{2\sqrt{2}}\,\delta_{A}^{\;B}\,\sigma^{\mu}_{CC'}\,\overline{\varphi}^{C'}\varphi^{C} + \frac{i}{\sqrt{2}}\,\sigma^{\mu}_{AA'}\,\chi^{A'}\overline{\chi}^{B} - \frac{i}{2\sqrt{2}}\,\delta_{A}^{\;B}\,\sigma^{\mu}_{CC'}\,\chi^{C'}\overline{\chi}^{C}. \tag{4.20}$$

Additional terms proportional to $\delta_{A}^{\;B}$ have been included in (4.20) to eliminate redundant trace parts (which do not contribute to the Lagrangian as $\mathcal{B}$ is traceless).

V. ASPECTS OF QUANTIZATION

A thorough study of the quantized theory would be out of place in this work, but there are several immediate observations that can be made regarding quantization.

Consider the Lagrangian (4.17) in the coordinate gauge (2.13). By varying the free-gauge-field part with respect to $\partial_0\mathcal{B}$, one finds that the linearized canonical momentum conjugate to $\mathcal{B}$ is given by

$$(\Pi^{\mu})_{A}^{\;B} \equiv \left[\alpha\,\eta^{\sigma(\nu}\eta^{\tau)[0}\,\sigma^{\mu]}_{AC'} + \beta\,\eta^{\nu\tau}\eta^{\sigma[0}\,\sigma^{\mu]}_{AC'}\right]\sigma_{\sigma}^{BC'}\,\sigma^{\lambda}_{CA'}\,\sigma_{\nu}^{CB'}\,\partial_{[\tau}(\overline{\mathcal{B}}_{\lambda]})_{B'}^{\;A'} - \delta\,\varepsilon^{BC}\varepsilon^{B'C'}\,\sigma^{(\mu}_{AA'}\,\sigma^{\nu)}_{CC'}\,\partial_0(\overline{\mathcal{B}}_{\nu})_{B'}^{\;A'}\,. \tag{5.1}$$

As none of the components of Π vanish in general, this theory, unlike typical gauge theories, is without primary constraints. Nevertheless, the algebra is complicated enough that brute-force canonical quantization seems uninviting. Due to the nature of the dynamical symmetry, Lorentz covariant quantization is far more appealing.

Before one can write down correlation functions in terms of functional integrals, gauge fixing must be addressed. Let ω be the object of anholonomity corresponding to a frame $\{\partial_\mu\}$. With respect to a gauge transformation $\partial_\mu \to (\Lambda^{-1})_{\mu}^{\;\nu}\,\partial_\nu$,

$$\Lambda^{\mu}_{\;\nu} = \delta^{\mu}_{\;\nu} + \eta^{\mu\rho}\,\zeta_{\rho\nu} + O(\zeta^2)\,, \qquad \zeta_{\nu\mu} = -\zeta_{\mu\nu}\,, \tag{5.2}$$

---

[8] The Lagrangian for the renormalizable higher-derivative model of [6] can be written as

$$\mathcal{L}[g] \equiv \left(\alpha\,H_{\mu\nu}H^{\mu\nu} + \beta\,K^2 + \frac{\gamma}{G}\,K\right)\sqrt{-\det g}$$

where g is a dynamical spacetime metric, H and K are the corresponding (symmetric) Ricci tensor and curvature scalar, α, β and γ are real parameters, and G is the gravitational constant. Since H and K involve second derivatives of the metric g, the theory defined by $\mathcal{L}[g]$ is a fourth-order theory.

with ζ small, ω transforms according to (2.8) like

$$\omega^{\rho}_{\mu\nu} \to (\omega^{\rho}_{\zeta})_{\mu\nu} + O(\zeta^2)\,, \tag{5.3}$$

where

$$(\omega^{\rho}_{\zeta})_{\mu\nu} = \omega^{\rho}_{\mu\nu} - \eta^{\sigma\lambda}\,\zeta_{\lambda\mu}\,\omega^{\rho}_{\sigma\nu} - \eta^{\sigma\lambda}\,\zeta_{\lambda\nu}\,\omega^{\rho}_{\mu\sigma} + \eta^{\rho\lambda}\,\zeta_{\lambda\sigma}\,\omega^{\sigma}_{\lambda\nu} + \eta^{\rho\lambda}\partial_{[\mu}\zeta_{\nu]\lambda}\,. \tag{5.4}$$

Now suppose that $\{\partial_\mu\}$ is a coordinate frame (satisfying $\omega = 0$), so that (5.4) reduces to

$$(\omega^{\rho}_{\zeta})_{\mu\nu} = \eta^{\rho\lambda}\partial_{[\mu}\zeta_{\nu]\lambda}\,. \tag{5.5}$$

One can check that (5.5) agrees with (2.6). The coordinate gauge condition can then be enforced by inserting into functional integrals the unity

$$1 = \prod_{\rho} \int d\zeta\, \delta(\omega^{\rho}_{\zeta})\, \det\left[\frac{\partial\, \omega^{\rho}_{\zeta}}{\partial \zeta}\right]. \tag{5.6}$$

Explicitly, one has

$$\frac{\partial\,(\omega^{\rho}_{\zeta})_{\mu\nu}}{\partial \zeta_{\lambda\tau}} = \delta^{\tau}_{[\mu}\,\eta^{\rho\lambda}\,\partial_{\nu]}\,, \tag{5.7}$$

and can therefore write

$$\det\left[\frac{\partial\, \omega^{\rho}_{\zeta}}{\partial \zeta}\right] \propto \int d\varpi^*\, d\varpi\, \exp\left\{ i \int d^4x\, \eta^{\rho\lambda}\, (\varpi^*_x)_{\lambda[\mu}\, \partial_{\nu]}\varpi^{\mu\nu}_x \right\}, \tag{5.8}$$

where $\varpi^*_{\nu\mu} = -\varpi^*_{\mu\nu}$ and $\varpi^{\nu\mu} = -\varpi^{\mu\nu}$ are fermionic Faddeev-Popov ghost fields. As is the case with Yang-Mills theories in the axial gauge, this determinant is independent of the dynamical fields and therefore cancels out in correlation functions. Hence the Faddeev-Popov ghosts may be dropped in the coordinate gauge.

The vacuum expectation value of a gauge invariant time-ordered operator $\mathcal{O}$ is given, up to an unimportant normalization constant, by

$$\langle \mathcal{O} \rangle \propto \int d\varphi\, d\chi\, d\mathcal{B}\, d\zeta\, \mathcal{O}\, \exp\left\{ i \int d^4x\, \mathcal{L}[\varphi_x, \chi_x, \mathcal{B}_x] \right\} \left[\prod_{\rho} \delta(\omega^{\rho}_{\zeta}) \right]. \tag{5.9}$$

Using the fact that $d\varphi\, d\chi\, d\mathcal{B}$ and $\mathcal{L}[\varphi, \chi, \mathcal{B}]$ are gauge invariant and performing the integration with respect to ζ then yields

$$\langle \mathcal{O} \rangle \propto \int d\varphi\, d\chi\, d\mathcal{B}\, \mathcal{O}\, \exp\left\{ i \int d^4x\ \mathcal{L}[\varphi_x, \chi_x, \mathcal{B}_x]\big|_{\omega=0} \right\}. \tag{5.10}$$

Hence the Feynman rules in the coordinate gauge can be derived from the Lagrangian (4.17) with ω set to zero.

VI. PROPAGATOR

The first step taken to check whether a theory is renormalizable usually involves power-counting. This amounts to inspecting the asymptotic behaviour of propagators and the mass-dimensionality of couplings. Since the Lagrangian (4.19) involves only dimensionless couplings and the Dirac propagator is known, what remains to be found is the propagator for the gauge field.

As the coordinate gauge condition (2.13) does not place any constraint upon the gauge field, it is not clear how one should decompose the kinetic matrix and propagator. A standard approach is to make use of spin-projection operators [6, 18] that emphasize transversality. Unfortunately, there is no easy way to make use of projectors here since the two sides of the propagator possess different spacetime indices—one side has left-handed indices where the other has right-handed ones. However, one can still construct traceless symbols of the correct valence that are transverse in some indices and longitudinal in others.

Consider the transverse and longitudinal projectors for vector fields carrying momentum k,

$$\mathrm{t}_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{1}{k^2}\, k_\mu k_\nu \qquad \text{and} \qquad \mathrm{l}_{\mu\nu} \equiv \frac{1}{k^2}\, k_\mu k_\nu \ . \tag{6.1}$$

For the sake of constructing traceless quantities, it is useful to define the following square root of the transverse projector t,

$$(\theta_\mu)^A_{\ B} \equiv \frac{i\sqrt{2}}{\sqrt{k^2}} \left(\sigma^{AA'}_\mu \sigma^\nu_{BA'}\, k_\nu - \frac{1}{2}\, \delta^A_B\, k_\mu \right), \tag{6.2}$$

which satisfies

$$\theta^A_B\, \theta^B_A = \mathrm{t}\ , \quad \theta^A_A = 0\ , \quad k^\mu (\theta_\mu)^A_{\ B} = 0\ , \quad \mathrm{t}_{\mu\nu} (\theta^\nu)^A_{\ B} = (\theta_\mu)^A_{\ B}\ , \quad \mathrm{l}_{\mu\nu} (\theta^\nu)^A_{\ B} = 0\ . \tag{6.3}$$

It is now a straightforward matter to construct four linearly independent symbols

$$\begin{aligned} (\mathrm{P}_{\mu\nu})^{AA'}_{\ BB'} &\equiv (\theta^A_B)_{(\mu} (\overline{\theta}_{\nu)})^{A'}_{\ B'}\ , \qquad & (\mathrm{Q}_{\mu\nu})^{AA'}_{\ BB'} &\equiv (\theta^A_B)_{[\mu} (\overline{\theta}_{\nu]})^{A'}_{\ B'}\ , \\ (\mathrm{R}_{\mu\nu})^{AA'}_{\ BB'} &\equiv \mathrm{t}_{\mu\nu} (\theta_\rho)^A_{\ B} (\overline{\theta}^\rho)^{A'}_{\ B'}\ , \qquad & (\mathrm{S}_{\mu\nu})^{AA'}_{\ BB'} &\equiv \mathrm{l}_{\mu\nu} (\theta_\rho)^A_{\ B} (\overline{\theta}^\rho)^{A'}_{\ B'}\ . \end{aligned} \tag{6.4}$$

One can increase the span of (6.4) by including constructions of the form

$$\sigma^{AA'}_\rho \sigma^\sigma_{BB'}\, \mathrm{x}_{\mu\nu}\, \mathrm{y}^\rho_\sigma \qquad \text{and} \qquad \sigma^{AA'}_\rho \sigma^\sigma_{BB'}\, \mathrm{x}^\rho_{\{\mu}\, \mathrm{y}_{\nu\}\sigma}\ , \tag{6.5}$$

where $\mathrm{x}, \mathrm{y} = \mathrm{t}, \mathrm{l}$ and the braces refer to either symmetrization or antisymmetrization. This provides the combinations

$$\begin{aligned} (\mathrm{T}_{\mu\nu})^{AA'}_{\ BB'} &\equiv \sigma^{AA'}_\rho \sigma^\sigma_{BB'}\, \mathrm{t}^\rho_{(\mu}\, \mathrm{l}_{\nu)\sigma} + \sigma^{AA'}_\rho \sigma^\sigma_{BB'}\, \mathrm{l}^\rho_{(\mu}\, \mathrm{t}_{\nu)\sigma}\ , \\ (\mathrm{U}_{\mu\nu})^{AA'}_{\ BB'} &\equiv \sigma^{AA'}_\rho \sigma^\sigma_{BB'}\, \mathrm{t}^\rho_{[\mu}\, \mathrm{l}_{\nu]\sigma} - \sigma^{AA'}_\rho \sigma^\sigma_{BB'}\, \mathrm{l}^\rho_{[\mu}\, \mathrm{t}_{\nu]\sigma}\ , \end{aligned} \tag{6.6}$$

which are linearly independent to (6.4). Each of $\mathrm{X} = \mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}, \mathrm{T}, \mathrm{U}$ is traceless in the sense that

$$\mathrm{X}^{AA'}_{AB'} = 0 = \mathrm{X}^{AA'}_{BA'}\ . \tag{6.7}$$

To invert a matrix, one must first know the identity matrix. Here

$$(\mathrm{I}^{\rho}_{\mu})^{AD}_{BC} \equiv \delta^{\rho}_{\mu}\,\delta^{A}_{C}\,\delta^{D}_{B} - \frac{1}{2}\,\delta^{\rho}_{\mu}\,\delta^{A}_{B}\,\delta^{D}_{C}\,, \tag{6.8}$$

which satisfies

$$(\mathrm{I}^{\rho}_{\mu})^{AD}_{AC} = 0 = (\mathrm{I}^{\rho}_{\mu})^{AC}_{BC}\,, \tag{6.9}$$

serves as the identity. Now, consider the products

$$([\mathrm{X}\cdot\mathrm{Y}]^{\rho}_{\mu})^{AD}_{BC} \equiv (\mathrm{X}_{\mu\nu})^{AA'}_{BB'}\,(\mathrm{Y}^{\rho\nu})^{DB'}_{CA'}\,, \tag{6.10}$$

where $\mathrm{X},\mathrm{Y} = \mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S},\mathrm{T},\mathrm{U}$ and only vector indices are raised/lowered. One finds that P, Q, and R are orthogonal to S,

$$\mathrm{P}\cdot\mathrm{S} = \mathrm{Q}\cdot\mathrm{S} = \mathrm{R}\cdot\mathrm{S} = \mathrm{S}\cdot\mathrm{P} = \mathrm{S}\cdot\mathrm{Q} = \mathrm{S}\cdot\mathrm{R} = 0\,, \tag{6.11}$$

but the remaining pairs have nonzero overlap. If one decomposes an element of the span of (6.4) and (6.6) in terms of P, Q, R, S, T, and U, then finding the inverse of that element involves solving the linear system

$$\sum_{\mathrm{X},\mathrm{Y}} a_{\mathrm{XY}}\,\mathrm{X}\cdot\mathrm{Y} = \mathrm{I}\,. \tag{6.12}$$

Explicitly, (6.12) amounts to the six equations

$$\begin{aligned}
a_{\mathrm{PP}} - a_{\mathrm{PQ}} - a_{\mathrm{QP}} + a_{\mathrm{QQ}} + 4\,a_{\mathrm{RR}} + a_{\mathrm{TT}} + a_{\mathrm{TU}} + a_{\mathrm{UT}} + a_{\mathrm{UU}} &= 4\,,\\
2\,a_{\mathrm{SS}} + a_{\mathrm{TT}} - a_{\mathrm{TU}} - a_{\mathrm{UT}} + a_{\mathrm{UU}} &= 2\,,\\
5\,a_{\mathrm{PP}} + 3\,a_{\mathrm{PQ}} + 4\,a_{\mathrm{PR}} + 3\,a_{\mathrm{QP}} + a_{\mathrm{QQ}} + 4\,a_{\mathrm{RP}} - a_{\mathrm{TT}} - a_{\mathrm{TU}} - a_{\mathrm{UT}} - a_{\mathrm{UU}} &= 0\,,\\
5\,a_{\mathrm{PP}} + 3\,a_{\mathrm{PQ}} + 3\,a_{\mathrm{QP}} + a_{\mathrm{QQ}} + 4\,a_{\mathrm{QR}} + 4\,a_{\mathrm{RQ}} + a_{\mathrm{TT}} + a_{\mathrm{TU}} + a_{\mathrm{UT}} + a_{\mathrm{UU}} &= 0\,,\\
2\,a_{\mathrm{ST}} + 2\,a_{\mathrm{SU}} + a_{\mathrm{TP}} - a_{\mathrm{TQ}} - 2\,a_{\mathrm{TR}} - a_{\mathrm{UP}} + a_{\mathrm{UQ}} + 2\,a_{\mathrm{UR}} &= 0\,,\\
a_{\mathrm{PT}} - a_{\mathrm{PU}} - a_{\mathrm{QT}} + a_{\mathrm{QU}} - 2\,a_{\mathrm{RT}} + 2\,a_{\mathrm{RU}} + 2\,a_{\mathrm{TS}} + 2\,a_{\mathrm{US}} &= 0\,.
\end{aligned} \tag{6.13}$$

The action for $\mathcal{B}$, using the Lagrangian (4.17), can be written as

$$\int d^4x\,\mathcal{L}[\mathcal{B}_x] = \int d^4x\,d^4y\,(\mathcal{D}^{\mu\nu}_{xy})^{BB'}_{AA'}\,(\mathcal{B}^{x}_{\mu})^{A}_{B}\,(\overline{\mathcal{B}}^{y}_{\nu})^{A'}_{B'} + O(\mathcal{B}^3)\,, \tag{6.14}$$

where the kinetic matrix $\mathcal{D}$ is required to be traceless. By moving to momentum space,

$$(\mathcal{D}^{\mu\nu}_{xy})^{BB'}_{AA'} = \int \frac{d^4k}{(2\pi)^4}\,\exp[ik\cdot(x-y)]\,(\mathcal{D}^{\mu\nu}_{k})^{BB'}_{AA'}\,, \tag{6.15}$$

$\mathcal{D}$ can be decomposed like

$$(\mathcal{D}^{\mu\nu}_{k})^{BB'}_{AA'} = \sum_{\mathrm{X}} \mathcal{D}_{\mathrm{X}}\,(\mathrm{X}^{\mu\nu})^{BB'}_{AA'}\,, \tag{6.16}$$

where

$$\begin{aligned}
\mathcal{D}_{\mathrm{P}} &= \frac{1}{4}(\alpha + 2\beta + 4\delta)k^2 + \frac{\gamma}{G}\,, \qquad \mathcal{D}_{\mathrm{Q}} = \frac{1}{2}\beta k^2\,,\\
-2\mathcal{D}_{\mathrm{R}} = 2\mathcal{D}_{\mathrm{S}} = -\mathcal{D}_{\mathrm{T}} &= \delta k^2 + \frac{\gamma}{G}\,, \qquad \mathcal{D}_{\mathrm{U}} = 0\,.
\end{aligned} \tag{6.17}$$

The momentum space propagator Δ, which must satisfy

$$(\Delta_{\mu\nu})^{AA'}_{BB'}\,(\mathcal{D}^{\rho\nu})^{DB'}_{CA'} = (\mathrm{I}^{\rho}_{\mu})^{AD}_{BC} \qquad \text{and} \qquad (\Delta_{\mu\nu})^{AA'}_{BB'}\,(\mathcal{D}^{\mu\rho})^{BD'}_{AC'} = (\mathrm{I}^{\rho}_{\mu})^{A'D'}_{B'C'}\,, \tag{6.18}$$

can now be found by setting $a_{\mathrm{XY}} = \Delta_{\mathrm{X}}\,\mathcal{D}_{\mathrm{Y}}$ in (6.13) and solving for Δ_{X}. It is not difficult to see that this linear system is well-posed for most values of the parameters and a straightforward calculation yields

$$\begin{gathered}\Delta_{\mathrm{P}} = \frac{2\{G[3\alpha+8(\beta+\delta)]k^2+8\gamma\}}{\alpha k^2\{G[\alpha+3(\beta+\delta)]k^2+3\gamma\}}\,, \qquad \Delta_{\mathrm{Q}} = -\frac{2\{G[5\alpha+16(\beta+\delta)]k^2+16\gamma\}}{\alpha k^2\{G[\alpha+3(\beta+\delta)]k^2+3\gamma\}}\,, \\ \Delta_{\mathrm{R}} = 0\,, \qquad \Delta_{\mathrm{S}} = \frac{2[G(\alpha+8\delta)k^2+8\gamma]}{\alpha k^2(G\delta k^2+\gamma)}\,, \qquad \Delta_{\mathrm{T}} = \frac{16}{\alpha k^2}\,, \qquad \Delta_{\mathrm{U}} = 0\,,\end{gathered} \tag{6.19}$$

which completely determines the propagator given by

$$(\Delta^k_{\mu\nu})^{AA'}_{BB'} = \sum_{\mathrm{X}} \Delta_{\mathrm{X}}\,(\mathrm{X}_{\mu\nu})^{AA'}_{BB'}\,. \tag{6.20}$$

Separating the propagator coefficients (6.19) into partial fractions gives

$$\begin{gathered}\Delta_{\mathrm{P}} = \frac{16}{3\alpha k^2} + \frac{2G}{3\{G[\alpha+3(\beta+\delta)]k^2+3\gamma\}}\,, \qquad \Delta_{\mathrm{S}} = \frac{16}{\alpha k^2} + \frac{2G}{G\delta k^2+\gamma}\,, \\ \Delta_{\mathrm{Q}} = -\frac{32}{3\alpha k^2} + \frac{2G}{3\{G[\alpha+3(\beta+\delta)]k^2+3\gamma\}}\,, \qquad \Delta_{\mathrm{T}} = \frac{16}{\alpha k^2}\,.\end{gathered} \tag{6.21}$$

The quadratic form $\mathcal{D}$ is singular when $\alpha = 0$, $\alpha+3(\beta+\delta) = \gamma = 0$, or $\gamma = \delta = 0$.

From (6.21) it is evident that, as long as $\delta \neq 0$, the propagator behaves like $\Delta^k \sim k^{-2}$ for large momenta. This, together with the fact that all self-interactions and interactions with Dirac fields are dimensionless, signifies that the theory is power-counting renormalizable in the coordinate gauge.

Examining the poles of (6.21) suggests that there are excitations with masses μ_0 and μ_1 satisfying

$$\mu_0^2 = -\frac{3\gamma}{G[\alpha+3(\beta+\delta)]} \qquad \text{and} \qquad \mu_1^2 = -\frac{\gamma}{G\delta}\,. \tag{6.22}$$

Analysis of the linearized radiation shows that μ_0 and μ_1 correspond to spin-0 and spin-1 bosons, as their notation suggests. If either μ_0 or μ_1 is imaginary, the theory contains tachyons[9]. Vacuum stability then requires the occurrence of spontaneous symmetry breaking (tachyon condensation) to a configuration free of tachyons, if such a configuration exists. These circumstances have yet to be investigated, and will not be dealt with in this paper. Here only values of the parameters α, β, γ, and δ for which μ_0 and μ_1 can be chosen to be real and non-negative are considered.

It is interesting that, while analogues of the massive spin-0 excitation of the renormalizable higher-derivative modification of general relativity[10] [6] exist here, counterparts

[9] The term “tachyon” here refers to an instability rather than a violation of causality. A well-known example is the uncondensed Higgs boson.

[10] The renormalizable higher-derivative theory defined by the Lagrangian in Footnote 8 and discussed in [6] contains spin-0 and spin-2 particles with masses given by

$$\mu_0^2 = -\frac{\gamma}{2G(\alpha+3\beta)} \qquad \text{and} \qquad \mu_2^2 = \frac{\gamma}{G\alpha}\,.$$

The massive spin-2 particle is necessary for renormalizability, but poses a major obstacle towards the theory's acceptance because of its negative-definite energy.

of the problematic massive spin-2 ghost are conspicuously absent—in their place are massive spin-1 excitations.

Both of the masses (6.22) can be eliminated, without ruining renormalizability, by setting $\gamma = 0$. One configuration of parameters for which the propagator takes a particularly simple form is $\alpha = 16$, $\beta = -3$, $\gamma = 0$, and $\delta = -2$. Then one has

$$(\Delta^k_{\mu\nu})^{AA'}_{BB'} = \frac{1}{k^2}\,(\mathrm{P}_{\mu\nu})^{BB'}_{AA'} + \frac{1}{k^2}\,(\mathrm{T}_{\mu\nu})^{BB'}_{AA'}\,, \tag{6.23}$$

which suggests that there are no tachyons or ghosts.

VII. PLANE WAVES

Since the propagator cannot be written as a linear combination of spin projection operators, the most straightforward way to determine the particle content of the present theory is to analyse plane wave solutions. Gauge ambiguity is again eliminated by imposing the coordinate gauge condition (2.13), which also simplifies the algebra.

Varying the quadratic part of the Lagrangian (4.17) with respect to $\overline{\mathcal{B}}$, and making use of the formulae (4.7) and (4.16) for $\mathcal{T}$ and $\mathcal{U}$, yields the linearized vacuum field equations

$$(\mathfrak{L}^\mu)^B_A \equiv \left[\alpha\,\eta^{\tau(\lambda}\eta^{\sigma)[\rho}\,\sigma^{\mu]}_{CA'} + \beta\,\eta^{\sigma\lambda}\eta^{\tau[\rho}\,\sigma^{\mu]}_{CA'}\right]\sigma^{CB'}_{\tau}\,\sigma^{\nu}_{AC'}\,\sigma^{BC'}_{\lambda}\,\partial_\rho\partial_{[\sigma}(\mathcal{B}_{\nu]})^A_B$$
$$- \varepsilon^{BC}\varepsilon^{B'C'}\,\sigma^{(\mu}_{AA'}\sigma^{\nu)}_{CC'}\left[\delta\,\partial^2(\mathcal{B}_\nu)^A_B - \frac{\gamma}{G}\,(\mathcal{B}_\nu)^A_B\right] = 0\,. \tag{7.1}$$

One should note that the first term in (7.1) involves the linearized field strength in the coordinate gauge,

$$(\mathcal{G}^{(1)}_{\mu\nu})^A_B = \partial_{[\mu}(\mathcal{B}_{\nu]})^A_B\,, \tag{7.2}$$

and is reminiscent of the second-derivative terms that appear in Yang-Mills equations. The second term resembles a system of Klein-Gordon equations. It is natural to seek plane wave solutions of the form

$$(\mathcal{B}_\mu)^A_B = (\Sigma_\mu)^A_B\,\exp(i\,k\cdot x)\,, \qquad (\Sigma_\mu)^2_2 = -(\Sigma_\mu)^1_1\,, \tag{7.3}$$

where Σ and k are constant. Suppose that the momentum is directed in the direction of ∂_3, so that $k_1 = k_2 = 0$ and $k_0, k_3 > 0$.

Consider first the case when $\gamma = 0$. Then both of the masses (6.22) vanish and, from the form of the propagator (6.21), one can expect that all excitations are massless. One finds that nontrivial plane waves (7.3) satisfy the field equations (7.1) only if $k_3 = k_0$, as expected, and

$$(\Sigma_1)^1_2 = (\Sigma_2)^1_2 = 0\,, \qquad (\Sigma_2)^1_1 = i\,(\Sigma_1)^1_1\,, \qquad (\Sigma_3)^1_1 = (\Sigma_0)^1_1\,, \qquad (\Sigma_3)^1_2 = (\Sigma_0)^1_2\,. \tag{7.4}$$

Five coefficients are eliminated by (7.4) and the seven that remain can be chosen to be

$$(\Sigma_0)^1_1\,, \quad (\Sigma_0)^1_2\,, \quad (\Sigma_0)^2_1 \pm (\Sigma_3)^2_1\,, \quad (\Sigma_1)^1_1\,, \quad \text{and} \quad (\Sigma_1)^2_1 \pm i\,(\Sigma_2)^2_1\,. \tag{7.5}$$

With respect to a rotation of an angle θ about the direction of propagation[11], the coefficients (7.4, 7.5) transform like

$$\begin{gathered}
\left(\Sigma_0\right)^1_1 \to \left(\Sigma_0\right)^1_1\,, \qquad \left(\Sigma_1\right)^2_1 + i\left(\Sigma_2\right)^2_1 \to \left(\Sigma_1\right)^2_1 + i\left(\Sigma_2\right)^2_1\,, \\
\left(\Sigma_0\right)^1_2 \to e^{i\theta}\left(\Sigma_0\right)^1_2\,, \quad \left(\Sigma_0\right)^2_1 \pm \left(\Sigma_3\right)^2_1 \to e^{-i\theta}\left[\left(\Sigma_0\right)^2_1 \pm \left(\Sigma_3\right)^2_1\right], \quad \left(\Sigma_1\right)^1_1 \to e^{-i\theta}\left(\Sigma_1\right)^1_1\,, \\
\left(\Sigma_1\right)^2_1 - i\left(\Sigma_2\right)^2_1 \to e^{-2i\theta}\left[\left(\Sigma_1\right)^2_1 - i\left(\Sigma_2\right)^2_1\right].
\end{gathered} \tag{7.6}$$

Recalling that a plane wave with helicity h gains a factor of $e^{ih\theta}$ with respect to such a rotation, and that the conjugate field $\overline{\mathcal{B}}$ contributes excitations with opposite helicity, one can conclude that the theory with $\gamma = 0$ contains a total of fourteen dynamical degrees of freedom: it describes four massless spin-0 bosons, four massless spin-1 bosons, and a massless spin-2 boson.

To check whether any excitations are ghostlike, one can compute the linearized canonical energy density. Modulo a boundary term, the canonical energy density is given by

$$\mathcal{E} \equiv \left(\Pi^\mu\right)^B_A \partial_0 \left(\mathcal{B}_\mu\right)^A_B + \text{h.c.} + \left(\mathfrak{L}^\mu\right)^B_A \left(\mathcal{B}_\mu\right)^A_B\,, \tag{7.7}$$

where Π denotes the canonical momentum (5.1) conjugate to $\mathcal{B}$ and $\mathfrak{L}$ (7.1) vanishes on shell. Substituting the plane wave solution (7.3, 7.4) into (5.1) and (7.7) yields

$$\begin{aligned}
\mathcal{E} = -4\delta\, k_0^2 \left|\left(\Sigma_0\right)^1_2\right|^2 - 4\delta\, k_0^2 \left|\left(\Sigma_1\right)^1_1\right|^2 - \delta\, k_0^2 \left|\left(\Sigma_0\right)^2_1 - \left(\Sigma_3\right)^2_1\right|^2 \\
- \frac{1}{2}(\alpha + 8\delta)\, k_0^2\, \mathrm{Re}\left[\left(\left(\Sigma_0\right)^2_1 - \left(\Sigma_3\right)^2_1\right)\left(\overline{\Sigma}_1\right)^{1'}_{1'}\right].
\end{aligned} \tag{7.8}$$

This is not bounded from below unless $\delta = -\alpha/8$ and $\alpha > 0$, in which case

$$\mathcal{E} = \frac{\alpha}{2}\, k_0^2 \left|\left(\Sigma_0\right)^1_2\right|^2 + \frac{\alpha}{2}\, k_0^2 \left|\left(\Sigma_1\right)^1_1\right|^2 + \frac{\alpha}{8}\, k_0^2 \left|\left(\Sigma_0\right)^2_1 - \left(\Sigma_3\right)^2_1\right|^2 \geq 0\,, \tag{7.9}$$

which suggests that ghostlike excitations are absent. This agrees with the conclusions drawn from the propagator (6.23) when $\alpha = 16$, $\beta = -3$, $\gamma = 0$, and $\delta = -2$. However, not all of the excitations (7.5) possess nonvanishing canonical energy. From experience with general relativity [17, 19] one can infer that the total gravitational energy must involve a surface integral at spatial infinity, perhaps including the boundary term that was neglected in (7.7). A rigorous analysis of the energy within this theory has yet to be performed, however the situation looks promising due to the forms of the propagator (6.23) and canonical energy (7.9).

When $\gamma \neq 0$, exactly two of the spin-0 bosons and two of the spin-1 bosons gain mass, increasing the number of degrees of freedom from fourteen to sixteen. The masses of these spin-0 and spin-1 bosons are μ_0 and μ_1, as given by (6.22). Unfortunately, their canonical energy densities are

$$\mathcal{E}_0 = -\frac{18\gamma}{G}\left|\left(\Sigma_1\right)^1_2\right|^2 \qquad \text{and} \qquad \mathcal{E}_1 = \frac{\gamma}{G}\left(2\left|\left(\Sigma_0\right)^1_1\right|^2 + \left|\left(\Sigma_0\right)^2_1\right|^2 + \left|\left(\Sigma_0\right)^1_2\right|^2\right), \tag{7.10}$$

[11] With respect to a global Lorentz transformation $\Lambda(\lambda)$, the plane wave coefficients transform like

$$\left(\Sigma_\mu\right)^A_B \to \left(\Lambda^{-1}\right)^\nu_\mu\, \lambda^A_C \left(\lambda^{-1}\right)^D_B \left(\Sigma_\nu\right)^C_D\,.$$

For a rotation about ∂_3 by an angle θ, λ and Λ have nonvanishing components

$$\lambda^1_1 = e^{i\theta/2}\,, \qquad \lambda^2_2 = e^{-i\theta/2}\,, \qquad \Lambda^0_0 = \Lambda^3_3 = 1\,, \qquad \Lambda^1_1 = \Lambda^2_2 = \cos\theta\,, \qquad -\Lambda^1_2 = \Lambda^2_1 = \sin\theta\,.$$

respectively, and so for all values of $\gamma \neq 0$, either the massive spin-0 bosons or the massive spin-1 bosons have negative energy. Hence ghosts seem to be unavoidable unless $\gamma = 0$.

VIII. WEAK-FIELD POTENTIAL

If the present theory is to compete with general relativity as a theory of gravity, it must first pass the usual weak-field experimental tests. Since the formulation is quite different from that of Einstein's theory—it is not a metric theory and spacetime is flat—direct comparison of the field equations is difficult and can even be misleading. Therefore, the most straightforward way to check whether the theory can be accurate in the weak-field regime is to independently find and analyse spherically symmetric solutions of the linearized vacuum equations (7.1).

The most general (traceless) static isotropic form for $\mathcal{B}$ is

$$\begin{gathered}(\mathcal{B}_0)^A_{\ B} = g_1\, \sigma_j^{AA'}\, \sigma^0_{BA'}\, x^j\,, \\ (\mathcal{B}_i)^A_{\ B} = f_0\, \sigma_i^{AA'}\, \sigma^0_{BA'} + f_1 \left(\sigma_j^{AA'}\, \sigma^i_{BA'}\, x^j - \frac{1}{2}\, \delta^A_B\, x^i \right) + f_2\, \sigma_j^{AA'}\, \sigma^0_{BA'}\, x^i x^j\,,\end{gathered} \tag{8.1}$$

$i = 1,2,3$, where j is summed over 1 to 3 and g_1, f_0, f_1, and f_2 are functions of the rotational invariant $r = \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

It is easiest to solve the linearized field equations (7.1) for g_1, f_0, f_1, and f_2 in spherical coordinates,

$$x^1 = r \sin\theta \cos\phi\ , \qquad x^2 = r \sin\theta \sin\phi\ , \qquad x^3 = r\cos\theta\ . \tag{8.2}$$

In the following, quantities expressed in the coordinates $[\mathring{x}] = [t, r, \theta, \phi]$ are denoted with a ring over them like $\mathring{\mathcal{B}}$ and $\mathring{\Gamma}$. In particular, $\mathring{\eta}$ and $\mathring{\sigma}$ denote the Minkowski metric and vector-spinor translation symbols in these coordinates and $\mathring{\partial}$ denotes the Levi-Civita connection corresponding to $\mathring{\eta}$. Explicit formulas are given in Appendix B.

Substituting (8.1) into the linearized field equations and converting to spherical coordinates leads to the four ordinary differential equations

$$\begin{aligned}0 = {}& G[\alpha + 2(\beta+\delta)]\, r\, g_1'' + 4G[\alpha + 2(\beta+\delta)]\, g_1' + 2\gamma r g_1 \\ & - G[\alpha + 4(\beta+\delta)]\, r f_1'' - 4G[\alpha + 4(\beta+\delta)]\, f_1' - 4\gamma r f_1\,,\end{aligned} \tag{8.3a}$$

$$\begin{aligned}0 = {}& G[\alpha + 4(\beta+\delta)]\, r\, g_1'' + 4G[\alpha + 4(\beta+\delta)]\, g_1' + 4\gamma r g_1 \\ & - G[3\alpha + 8(\beta+\delta)]\, r f_1'' - 4G[3\alpha + 8(\beta+\delta)]\, f_1' - 8\gamma r f_1\,,\end{aligned} \tag{8.3b}$$

$$\begin{aligned}0 = {}& 6G\delta\, r f_0'' + G[\alpha + 12\delta] f_0' + 6\gamma r f_0 \\ & + 2G\delta\, r^3 f_2'' + 12 G\delta\, r^2 f_2' + r[2\gamma r^2 - G(\alpha - 12\delta)] f_2\,,\end{aligned} \tag{8.3c}$$

$$\begin{aligned}0 = {}& G(\alpha + 12\delta)\, r f_0'' + G(\alpha + 24\delta) f_0' + 12\gamma r f_0 \\ & + 4G\delta\, r^3 f_2'' - G(\alpha - 24\delta) r^2 f_2' + r[4\gamma r^2 - 2G(\alpha - 12\delta)] f_2\,.\end{aligned} \tag{8.3d}$$

Consider again the massless $\gamma = 0$ case. One finds that (8.3a) and (8.3b) are solved by

$$g_1 = \frac{C_1}{r^3} \qquad \text{and} \qquad f_1 = \frac{C_2}{r^3}\,, \tag{8.4}$$

and that (8.3c) and (8.3d) are solved by

$$f_2 = \frac{1}{r} f_0' \qquad \text{and} \qquad f_0 = \frac{C_3}{r^3} + \frac{C_4}{r}\,, \tag{8.5}$$

where C_1, C_2, C_3, and C_4 are constants. Additional constant terms have been neglected to obtain solutions that vanish at spatial infinity. One should note that, even though the Lagrangian (4.17) closely resembles that of higher-derivative generally covariant theories, the field equations (7.1) with $\gamma = 0$ do not admit solutions that increase with r.

If $\gamma \neq 0$, then the analysis is similar but a little more complicated. Eliminating f_1' and f_1'' from (8.3a) and (8.3b) gives

$$f_1 = -\frac{3}{2\mu_0^2}\frac{1}{r}\left(r g_1'' + 4 g_1'\right) + \frac{1}{2} g_1\,, \tag{8.6}$$

where μ_0^2 is given by (6.22). Inserting (8.6) into (8.3a) and (8.3b) then yields

$$G r^3 g_1^{(\mathrm{iv})} + 8 G r^2 g_1''' - \left(G \mu_0^2 r^3 - 8 G r\right) g_1'' - \left(4 G \mu_0^2 r^2 + 8 G\right) g_1' = 0\,, \tag{8.7}$$

which is solved by

$$g_1 = \frac{D_1}{r^3} + \frac{D_2}{r^3} e^{-\mu_0 r} + \frac{D_2 \mu_0}{r^2} e^{-\mu_0 r}\,. \tag{8.8}$$

Similarly, from (8.3c) and (8.3d), one obtains

$$f_2 = \frac{1}{r} f_0' \qquad \text{and} \qquad f_0 = \frac{D_3}{r^3} + \frac{D_4}{r^3} e^{-\mu_1 r} + \frac{D_4 \mu_1}{r^2} e^{-\mu_1 r}\,, \tag{8.9}$$

where μ_1^2 is the other mass from (6.22). Constant and rising exponential terms have been omitted, however it should be noted that, unlike the $\gamma = 0$ case, solutions which blow up at spatial infinity are now possible. This is common among theories containing massive excitations. It is also interesting that the mass μ_0 only appears in g_1 and f_1, while μ_1 only appears in f_0 and f_2. Taking the limit $\mu_0 \to 0$ ($\gamma \to 0$) in (8.8) and (8.6) reproduces the $\gamma = 0$ solutions (8.4). On the other hand, taking the limit $\mu_1 \to 0$ ($\gamma \to 0$) in (8.9) and comparing with (8.5) indicates a mass discontinuity: f_0 contains a term proportional to $1/r$ in the $\gamma = 0$ case that does not appear in the limit $\gamma \to 0$. So in both the $\gamma = 0$ and $\gamma \neq 0$ cases the static isotropic solutions that vanish at spatial infinity are four-parameter families, but in the limit $\gamma \to 0$ a parameter is lost.

IX. TEST PARTICLE TRAJECTORIES

One way to examine the physical consequences of the solutions found in Section VIII is to analyse the equations of motion for a test particle. Then one can determine whether the classical experimental tests of gravitational theory can be satisfied. In this section, only the $\gamma = 0$ solution (8.1, 8.4, 8.5) will be studied because it was shown in Section VII that ghosts are unavoidable in the $\gamma \neq 0$ theory, and the analysis in the $\gamma \neq 0$ case is merely more complicated. For the corresponding study of test particle trajectories in general relativity, see [20].

Let $u = u(s)$ be a family of vectors tangent to a test particle world line parametrized by s. Suppose that u is normalized so that, given a frame $\{\partial_\mu\}$, one has $d/ds = u^\mu\,\partial_\mu$

along the world line. If $\mathcal{B}$ were to vanish with respect to $\{\partial_\mu\}$ in a neighbourhood of some point along the particle's trajectory, then consistency with the special theory of relativity would require that, in that neighbourhood, the particle satisfies the equations of motion $du/ds = 0$. With respect to a gauge transformation $\partial_\mu \to \left(\Lambda^{-1}\right)^\nu_{\ \mu}\,\partial_\nu$, u transforms like $u^\mu \to \Lambda^\mu_\nu\, u^\nu$, and therefore

$$\frac{du^\mu}{ds} \to \Lambda^\mu_\nu \frac{du^\nu}{ds} + u^\rho \left(\partial_\rho \Lambda^\mu_\nu\right) u^\nu . \tag{9.1}$$

Inspecting the transformation law (3.7), one finds that

$$\Gamma^\mu_{(\rho\sigma)}\, u^\rho\, u^\sigma \to \Lambda^\mu_\lambda \Gamma^\lambda_{(\rho\sigma)}\, u^\rho\, u^\sigma - \left(\partial_\rho \Lambda^\mu_\sigma\right) u^\rho\, u^\sigma . \tag{9.2}$$

Hence the equations

$$\frac{du^\mu}{ds} + \Gamma^\mu_{(\rho\sigma)}\, u^\rho\, u^\sigma = 0 \tag{9.3}$$

are covariant while reducing to the correct relativistic equations of motion when $\mathcal{B} = 0$. Since curves in spacetime and their tangent vectors are geometric objects completely independent of coordinates, the equations (9.3) are valid even in anholonomic frames. One should also note that the equations (9.3) *do* depend on all of the dynamical degrees of freedom possessed by the gauge field since one can write $\Gamma^\rho_{(\mu\nu)}$ in terms of $\Gamma^\rho_{[\mu\nu]}$ (3.19) and Γ, $\mathcal{B}$ in terms of one another (3.6, 3.10).

In a coordinate frame where $\partial = \partial/\partial x$, some coordinate system $\{x^\mu\}$, one can refer to points on the trajectory like $x(s)$ and write $u = dx/ds$ to obtain equations of the same form as the familiar geodesic equations of Riemannian geometry. For a slowly moving particle one can neglect $d\mathbf{x}/ds$ with respect to dt/ds and divide (9.3) out by $(dt/ds)^2$ to obtain

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} . \tag{9.4}$$

One can now substitute the general static isotropic solution (8.1) into (9.4) and compare with the Newtonian result for a gravitating spherical body of mass M at the origin,

$$\frac{d^2x^i}{dt^2} = -\frac{GM}{r^3}\, x^i . \tag{9.5}$$

The correct Newtonian limit is achieved if, for large r,

$$\Gamma^i_{00} = -K\,\mathrm{Re}(g_1)\, x^i \approx \frac{GM}{r^3}\, x^i . \tag{9.6}$$

This can be used to fix the real part of the constant of integration C_1 in (8.4),

$$\mathrm{Re}(C_1) = -\frac{GM}{K} . \tag{9.7}$$

One can check that (9.7) is of the correct mass dimensionality by inspecting (8.1) and recalling that $\mathcal{B}$ has mass dimensionality 1.

Since the transformation from Cartesian to spherical coordinates $x \to \mathring{x}$ does not preserve the Minkowski metric, $\eta \to \mathring{\eta}$, the equations (9.3) must be modified accordingly. In spherical coordinates, the particle trajectory $\mathring{x}(s)$ must satisfy

$$\frac{d^2\mathring{x}^\mu}{ds^2} + \mathring{\boldsymbol{\Gamma}}^\mu_{(\rho\sigma)} \frac{d\mathring{x}^\rho}{ds} \frac{d\mathring{x}^\sigma}{ds} = 0 , \tag{9.8}$$

where

$$\mathring{\mathbf{\Gamma}} \equiv \mathring{\Xi} + \mathring{\Gamma} \;, \tag{9.9}$$

and $\mathring{\Xi}$ refers to the Christoffel symbols (B4) corresponding to the metric $\mathring{\eta}$. Analysis of the equations (9.8) within the static isotropic field configuration (8.1) is quite difficult for general g_1, f_0, f_1, and f_2. However, the equations simplify considerably if it is assumed that g_1, f_1 are real (i.e., $C_1, C_2 \in \mathbb{R}$) and that f_0, f_2 vanish (i.e., $C_3 = C_4 = 0$). In that case, $\mathring{\mathbf{\Gamma}}$ has the following nonvanishing symmetrized components[12]:

$$\mathring{\mathbf{\Gamma}}^t_{(tr)} = -\frac{1}{2}Krg_1 \;, \qquad \mathring{\mathbf{\Gamma}}^r_{tt} = -Krg_1 \;, \qquad \mathring{\mathbf{\Gamma}}^r_{\theta\theta} = \frac{1}{\sin^2\theta}\mathring{\mathbf{\Gamma}}^r_{\phi\phi} = -r - Kr^3 f_1 \;,$$
$$\mathring{\mathbf{\Gamma}}^\theta_{(r\theta)} = \frac{1}{r} + \frac{1}{2}Krf_1 \;, \qquad \mathring{\mathbf{\Gamma}}^\theta_{\phi\phi} = -\cos\theta\sin\theta \;, \qquad \mathring{\mathbf{\Gamma}}^\phi_{(r\phi)} = \frac{1}{r} + \frac{1}{2}Krf_1 \;, \qquad \mathring{\mathbf{\Gamma}}^\phi_{(\theta\phi)} = \cot\theta \;. \tag{9.10}$$

Inserting (9.10) into the equations (9.8) gives

$$\frac{d^2t}{ds^2} - Krg_1\frac{dt}{ds}\frac{dr}{ds} = 0 \;, \tag{9.11a}$$

$$\frac{d^2r}{ds^2} - Krg_1\left(\frac{dt}{ds}\right)^2 - (r + Kr^3 f_1)\left(\frac{d\Omega}{ds}\right)^2 = 0 \;, \tag{9.11b}$$

$$\frac{d^2\theta}{ds^2} + \left(\frac{2}{r} + Krf_1\right)\frac{dr}{ds}\frac{d\theta}{ds} - \cos\theta\sin\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \;, \tag{9.11c}$$

$$\frac{d^2\phi}{ds^2} + \left(\frac{2}{r} + Krf_1\right)\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \;, \tag{9.11d}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \;. \tag{9.12}$$

Since the field configuration is isotropic, one can assume that the particle trajectory governed by (9.11) is confined to some plane through the origin; $\theta = \pi/2$ automatically satisfies (9.11c). Substituting (8.4) into (9.11) and dividing (9.11a) and (9.11d) by dt/ds and $d\phi/ds$, one finds

$$\frac{d}{ds}\left(\ln\frac{dt}{ds} + \frac{KC_1}{r}\right) = 0 \;, \tag{9.13a}$$

$$\frac{d}{ds}\left(\ln\frac{d\phi}{ds} + \ln r^2 - \frac{KC_2}{r}\right) = 0 \;. \tag{9.13b}$$

From (9.13a) one can deduce that the parameter s may be normalized by choosing

$$\frac{dt}{ds} = \exp\left(-\frac{KC_1}{r}\right). \tag{9.14}$$

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[12] One should compare this to the Levi-Civita connection $\mathring{\Gamma}$ corresponding to a static isotropic metric g with nonvanishing components

$$\mathring{g}_{tt} = b \;, \qquad \mathring{g}_{rr} = -a \;, \qquad \mathring{g}_{\theta\theta} = -r^2 \;, \qquad \mathring{g}_{\phi\phi} = -r^2\sin^2\theta \;,$$

where $a = a(r)$ and $b = b(r)$. It has nonvanishing components

$$\mathring{\Gamma}^t_{tr} = \mathring{\Gamma}^t_{rt} = \frac{b'}{2b} \;, \qquad \mathring{\Gamma}^r_{tt} = \frac{b'}{2a} \;, \qquad \mathring{\Gamma}^r_{rr} = \frac{a'}{2a} \;, \qquad \mathring{\Gamma}^r_{\theta\theta} = \frac{1}{\sin^2\theta}\mathring{\Gamma}^r_{\phi\phi} = -\frac{r}{a} \;,$$
$$\mathring{\Gamma}^\theta_{r\theta} = \mathring{\Gamma}^\theta_{\theta r} = \frac{1}{r} \;, \qquad \mathring{\Gamma}^\theta_{\phi\phi} = -\cos\theta\sin\theta \;, \qquad \mathring{\Gamma}^\phi_{r\phi} = \mathring{\Gamma}^\phi_{\phi r} = \frac{1}{r} \;, \qquad \mathring{\Gamma}^\phi_{\theta\phi} = \mathring{\Gamma}^\phi_{\phi\theta} = \cot\theta \;.$$

The other equation (9.13b) yields a constant of the motion J,

$$r^2 \exp\left(-\frac{KC_2}{r}\right)\frac{d\phi}{ds} = J\,. \tag{9.15}$$

Finally, using (9.14) and (9.15) in (9.11b), multiplying by $2dr/ds$, and integrating yields the last constant of the motion E,

$$\left(\frac{dr}{ds}\right)^2 - \exp\left(-\frac{2KC_1}{r}\right) + \frac{J^2}{r^2}\exp\left(\frac{2KC_2}{r}\right) = -E\,. \tag{9.16}$$

Using (9.14), (9.15), and (9.16), the proper time is given by

$$\mathring{\eta}_{\mu\nu}\, d\mathring{x}^{\mu} d\mathring{x}^{\nu}\Big|_{\theta=\pi/2} = dt^2 - dr^2 - r^2 d\phi^2 = E\,ds^2\,, \tag{9.17}$$

so that for light $E = 0$, and for massive test particles $E > 0$. Dividing (9.16) by the square of (9.15) gives the equation governing the shape of orbits,

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{J^2}\exp\left[-\frac{2K(C_1+C_2)}{r}\right] + \frac{1}{r^2} = -\frac{E}{J^2}\exp\left(-\frac{2KC_2}{r}\right). \tag{9.18}$$

Taking the square root of (9.18) and integrating then yields

$$\phi = \pm\int \frac{dr}{r^2}\left\{\frac{1}{J^2}\exp\left[-\frac{2K(C_1+C_2)}{r}\right] - \frac{E}{J^2}\exp\left(-\frac{2KC_2}{r}\right) - \frac{1}{r^2}\right\}^{-1/2}. \tag{9.19}$$

Similarly, the equation governing the radial time history of orbits is given by dividing (9.16) by the square of (9.14),

$$\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2}\exp\left[\frac{2K(C_1+C_2)}{r}\right] - 1 = -E\exp\left(\frac{2KC_1}{r}\right). \tag{9.20}$$

Its solution is

$$t = \pm\int dr\left\{1 - \frac{J^2}{r^2}\exp\left[\frac{2K(C_1+C_2)}{r}\right] - E\exp\left(\frac{2KC_1}{r}\right)\right\}^{-1/2}. \tag{9.21}$$

It is now a simple matter to calculate two standard predictions of relativistic gravitation: the deflection and delay of light passing by a spherically symmetric gravitational source. From (9.17) it was deduced that for light $E = 0$. If r_0 is the distance of closest approach, where $dr/d\phi$ vanishes, then (9.18) yields

$$J^2 = r_0^2 \exp\left[-\frac{2K(C_1+C_2)}{r_0}\right]. \tag{9.22}$$

According to (9.19), the total angle swept by a lightlike trajectory passing by the origin is

$$\Delta\phi = 2\int_{r_0}^{\infty}\frac{dr}{r}\left\{\left(\frac{r}{r_0}\right)^2 \exp\left[(2KC_1 + 2KC_2)\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] - 1\right\}^{-1/2}. \tag{9.23}$$

Using the approximation

$$\left(\frac{r}{r_0}\right)^2 \exp\left[(2KC_1+2KC_2)\left(\frac{1}{r_0}-\frac{1}{r}\right)\right]-1$$
$$=\left[\left(\frac{r}{r_0}\right)^2-1\right]\left[1+\frac{2K(C_1+C_2)r}{r_0(r+r_0)}\right]+O(K^2)\,, \quad (9.24)$$

one finds that

$$\Delta\phi=\pi-\frac{2K(C_1+C_2)}{r_0}+O(K^2)\,. \quad (9.25)$$

The correct deflection is obtained if one sets

$$C_2=C_1 \quad (9.26)$$

and uses the value for C_1 obtained from the Newtonian limit (9.7),

$$\Delta\phi-\pi\simeq\frac{4GM}{r_0}\,. \quad (9.27)$$

The delay is found in a similar fashion. As $dr/dt=0$ at r_0, from (9.20) one has

$$J^2=r_0^2\exp\left[-\frac{2K(C_1+C_2)}{r_0}\right]. \quad (9.28)$$

The solution (9.21) then gives the time taken for light to travel from $r=r_0$ to $r=r_1$,

$$\delta t=\int_{r_0}^{r_1} dr\left\{1-\left(\frac{r_0}{r}\right)^2\exp\left[2K(C_1+C_2)\left(\frac{1}{r}-\frac{1}{r_0}\right)\right]\right\}^{-1/2}. \quad (9.29)$$

Using the approximation

$$1-\left(\frac{r_0}{r}\right)^2\exp\left[2K(C_1+C_2)\left(\frac{1}{r}-\frac{1}{r_0}\right)\right]=\left[1-\left(\frac{r_0}{r}\right)^2\right]\left[1+\frac{2K(C_1+C_2)r_0}{r(r+r_0)}\right]+O(K^2)\,, \quad (9.30)$$

one finds that

$$\delta t=\sqrt{r_1^2+r_0^2}-K(C_1+C_2)\left(\frac{r_1-r_0}{r_1+r_0}\right)^{1/2}+O(K^2)\,. \quad (9.31)$$

Inserting (9.26) and (9.7) then yields the delay

$$\delta t-\sqrt{r_1^2+r_0^2}\simeq 2GM\left(\frac{r_1-r_0}{r_1+r_0}\right)^{1/2}. \quad (9.32)$$

If $r_1\gg r_0$, then (9.32) suggests a constant delay of $2MG$, independent of r_1. In particular, if a radar signal were to graze the sun, reflect off Mercury, and then pass by the sun again on its way back to Earth, (9.32) would suggest a total delay of $8M_\odot G\simeq 12$ km. Unfortunately, this disagrees with the delay of $4M_\odot G(11+1)\simeq 72$ km predicted by the Einsteinian formula [20],

$$\delta t-\sqrt{r_1^2+r_0^2}\simeq 2GM\ln\left[\frac{r_1+\sqrt{r_1^2-r_0^2}}{r_0}\right]+GM\left(\frac{r_1-r_0}{r_1+r_0}\right)^{1/2}. \quad (9.33)$$

Since (9.33), and particularly its logarithmic dependence, has been experimentally verified [21], this discrepancy suggests that (9.32) is inaccurate. However, this could indicate a problem with the simplifying assumptions $C_3 = C_4 = 0$ and $C_1, C_2 \in \mathbb{R}$ rather than a shortcoming of the field equations (7.1) and equations of motion (9.3).

It remains to be seen if general solutions (8.1, 8.4, 8.5) with the correct Newtonian limit (9.7), but involving all of $\mathrm{Im}(C_1)$, C_2, C_3, and C_4, can model weak-field gravitational phenomena accurately. Since the general equations of motion are difficult to solve analytically, numerical methods could prove fruitful. Of course, one would first need to determine the constants of integration C_1, C_2, C_3, C_4 by, for instance, including a compact spherically symmetric source in the field equations (7.1).

X. CONCLUSION

All accepted theories of gravity up to now have been either nonrenormalizable, like Einstein's original theory, or renormalizable but involving higher derivatives and therefore negative energy excitations, such as the theory studied in [6]. In the present paper it has been demonstrated that there exist power-counting renormalizable gauge theories with massless spin-2 bosons that do not involve higher derivatives. A gauge invariant action with dimensionless couplings was proposed in Section IV and the gauge boson propagator was derived in the coordinate gauge (2.13) in Section VI to reach this conclusion. It has also been shown, in Section VII, that some of these theories have non-negative linearized canonical energy on shell. However, several theoretical difficulties have yet to be addressed before one can claim that these developments can lead to a consistent quantum field theory.

First of all, it must be checked that the theory presented here can actually be renormalized. After all, there is more to renormalizability than power-counting. Gauge invariance places restrictions on the allowed interactions and, typically, constrains the divergences that can occur in the full quantum theory. Investigating renormalizability in gauge theories entails checking that these constraints are strong enough that the allowed divergences can be cancelled by a renormalization of the fields in the action. This tends to work best if the action involves terms proportional to all possible power-counting renormalizable invariants. For instance, scalar electrodynamics requires a quartic scalar interaction for renormalizability. In Section IV, terms likely to produce ghostlike excitations were left out of the Lagrangian. It has yet to be proven that this constraint can be maintained under renormalization.

Another concern is that non-negative canonical energy does not necessarily guarantee the absence of negative-energy ghosts. In Section VII it was found that a number of excitations have vanishing canonical energy like the spin-2 boson of general relativity. This probably means that, as in Einstein's theory, the total energy depends on a surface integral at spatial infinity. It has yet to be decided whether this total energy of all the excitations is positive definite. And even if the linearized theory does have positive energy, the full nonlinear theory might not—the massive version of Einstein's theory, for example, suffers from a nonlinear ghost [22].

Finally, it remains to be seen whether the present theory can provide an accurate macroscopic description of gravitation. In Sections VIII and IX it was shown that spherically symmetric field configurations in the vacuum are compatible with Newton's theory in the weak-field nonrelativistic limit. It was also shown that the correct result for the deflection of light about the sun can be obtained. However, the test-particle trajectories examined in Section IX appear to fall short as far as some of the other standard relativistic tests of gravity are concerned—the predicted delay of light about the sun disagrees with that of Einstein's theory. A more general analysis of the field equations and/or test particle equations of motion is required to determine whether this is a consequence of the theory or the simplifying assumptions that were made to obtain the trajectories.

On the other hand, there are several clear advantages of the present theory over general relativity. For instance, it does not require curved spacetime. It is also derived from a polynomial action that suggests simple interactions between gravitation and fermionic matter. One can hope that this work provides insight towards the ultimate gravitational theory.

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Appendix A: Chiral representation

Recall that, as the connected Lorentz group is not simply connected, not all representations of the Lorentz algebra $\mathfrak{o}_{1,3}(\mathbb{R})$ correspond to linear representations of the group. However, every representation of $\mathfrak{o}_{1,3}(\mathbb{R})$ corresponds to a representation of the corresponding spin group, which is isomorphic to $SL_2(\mathbb{C})$. Since $SL_2(\mathbb{C})$ is endowed with a natural complex structure, one can construct spinors which transform according to both its standard (left-handed) and conjugate (right-handed) representations.

Let uppercase Latin indices denote spinor indices and range from 1 to 2, and let unprimed and primed indices denote left- and right-handed indices, respectively[13]. For instance, if φ is a left-handed spinor and χ is a right-handed spinor, then with respect to a transformation $\lambda \in SL_2(\mathbb{C})$, φ and χ transform like

$$\varphi^A \to \lambda^A_B\, \varphi^B \qquad \text{and} \qquad \chi^{A'} \to \overline{\lambda}^{A'}_{B'}\, \chi^{B'} , \tag{A1}$$

where overline denotes complex conjugation. Note that conjugation maps left-handed spinors to right-handed ones and vice versa.

Just as the invariant Minkowski metric and a choice of space/time orientation completely determine the connected Lorentz group, the invariance of the two-dimensional antisymmetric bilinear form $\varepsilon_{AB} = -\varepsilon_{BA}$, $\varepsilon_{12} = 1$ (and likewise on the right-handed space) completely determines the elements of $SL_2(\mathbb{C})$. If one also specifies the contravariant version of ε, here given by

$$\varepsilon^{AC}\varepsilon_{BC} = \delta^A_B , \tag{A2}$$

[13] This notation has become somewhat standard over the past few decades [15].

so that $\varepsilon^{12} = 1$, one can raise and lower spinor indices in an unambiguous and covariant way. However, since ε is not symmetric, care must be taken when doing so. With the convention (A2), spinor indices can be lowered (raised) by contracting with the first (resp. second) index of ε, like $\varphi_A = \varepsilon_{BA}\varphi^B$ (resp. $\varphi^A = \varepsilon^{AB}\varphi_B$). Although, to avoid confusion, spinor indices are rarely raised or lowered implicitly within this paper.

One final construction that is needed to make this formulation of chiral spinors useful is the canonical double covering from $SL_2(\mathbb{C})$ onto the connected Lorentz group. It is given by

$$\lambda \mapsto \Lambda(\lambda)^{\mu}_{\ \nu} = \sigma^{\mu}_{AA'}\,\sigma^{BB'}_{\nu}\,\lambda^{A}_{\ B}\,\overline{\lambda}^{A'}_{\ B'}\,, \tag{A3}$$

where σ encapsulates the Pauli matrices,

$$\begin{gathered}\begin{bmatrix}\sigma^{\mu}_{11'} & \sigma^{\mu}_{12'}\\ \sigma^{\mu}_{21'} & \sigma^{\mu}_{22'}\end{bmatrix} = [\sigma^{\mu}]\,,\\ [\sigma^0] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ [\sigma^1] = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ [\sigma^2] = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\ [\sigma^3] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\end{gathered} \tag{A4}$$

and indices have been raised and lowered as described above; explicitly,

$$\begin{gathered}\begin{bmatrix}\sigma^{11'}_{\mu} & \sigma^{12'}_{\mu}\\ \sigma^{21'}_{\mu} & \sigma^{22'}_{\mu}\end{bmatrix} = [\sigma_{\mu}]\,,\\ [\sigma_0] = [\sigma^0]\,,\ [\sigma_1] = [\sigma^1]\,,\ [\sigma_2] = -[\sigma^2]\,,\ [\sigma_3] = [\sigma^3]\,.\end{gathered} \tag{A5}$$

These symbols are Hermitian[14],

$$(\overline{\sigma})^{\mu}_{AA'} = \sigma^{\mu}_{AA'}\,, \tag{A6}$$

they satisfy

$$\sigma^{\mu}_{AA'}\,\sigma^{AA'}_{\nu} = \delta^{\mu}_{\nu} \qquad \text{and} \qquad \sigma^{AA'}_{\mu}\sigma^{\mu}_{BB'} = \delta^{A}_{B}\,\delta^{A'}_{B'}\,, \tag{A7}$$

and one can "pull" Lorentz transformations through them like

$$\sigma^{AA'}_{\mu}\,\Lambda(\lambda)^{\mu}_{\ \nu} = \sigma^{BB'}_{\nu}\,\lambda^{A}_{\ B}\,\overline{\lambda}^{A'}_{\ B'}\,. \tag{A8}$$

Another useful identity is

$$\varepsilon_{A'B'}\,\sigma^{AA'}_{(\mu}\sigma^{BB'}_{\nu)} = \frac{1}{2}\eta_{\mu\nu}\,\varepsilon^{AB}. \tag{A9}$$

It is also easy to see that, due to (A7), one has

$$\Lambda(\lambda)^{-1} = \Lambda\big(\lambda^{-1}\big)\,. \tag{A10}$$

Now one can identify global (metric and orientation preserving) spacetime transformations with elements of $SL_2(\mathbb{C})$ and relate spacetime indices with left- and right-handed index pairs using σ. Notably, one has

$$\varepsilon_{AB}\varepsilon_{A'B'}\,\sigma^{AA'}_{\mu}\sigma^{BB'}_{\nu} = \eta_{\mu\nu} \qquad \text{and} \qquad \eta_{\mu\nu}\,\sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'} = \varepsilon_{AB}\varepsilon_{A'B'}\,. \tag{A11}$$

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[14] Recall that conjugation takes left-handed indices to right-handed ones and vice versa. Due to the convention of keeping left-handed (unprimed) indices to the left of right-handed (primed) ones, the indices on the left-hand side of (A6) are conjugated and then reversed. So the left-hand side of (A6) effectively refers to the components of the conjugate transpose of σ^{μ}.

Appendix B: Spherical coordinates

In spherical coordinates,

$$x^1 = r\sin\theta\cos\phi\ , \qquad x^2 = r\sin\theta\sin\phi\ , \qquad x^3 = r\cos\theta\ , \tag{B1}$$

the Minkowski metric is given by[15]

$$[\mathring{\eta}] = \mathrm{diag}(1, -1, -r^2, -r^2\sin^2\theta)\ . \tag{B2}$$

In order to differentiate in a consistent manner, one must introduce the Christoffel symbols corresponding to (B2),

$$\mathring{\Xi}^{\rho}_{\mu\nu} = \frac{1}{2}\mathring{\eta}^{\rho\sigma}\left(\frac{\partial\mathring{\eta}_{\mu\sigma}}{\partial\mathring{x}^{\nu}} + \frac{\partial\mathring{\eta}_{\nu\sigma}}{\partial\mathring{x}^{\mu}} - \frac{\partial\mathring{\eta}_{\mu\nu}}{\partial\mathring{x}^{\sigma}}\right)\ . \tag{B3}$$

The nonvanishing components of $\mathring{\Xi}$ are

$$\begin{gathered} \mathring{\Xi}^{r}_{\theta\theta} = -r\ , \qquad \mathring{\Xi}^{r}_{\phi\phi} = -r\sin^2\theta\ , \qquad \mathring{\Xi}^{\theta}_{r\theta} = \mathring{\Xi}^{\theta}_{\theta r} = \frac{1}{r}\ , \\ \mathring{\Xi}^{\theta}_{\phi\phi} = -\cos\theta\sin\theta\ , \qquad \mathring{\Xi}^{\phi}_{r\phi} = \mathring{\Xi}^{\phi}_{\phi r} = \frac{1}{r}\ , \qquad \mathring{\Xi}^{\phi}_{\theta\phi} = \mathring{\Xi}^{\phi}_{\phi\theta} = \cot\theta\ . \end{gathered} \tag{B4}$$

For instance, the gradient of a 1-form A, with components in Cartesian coordinates $\partial_\mu A_\nu$, is given in spherical coordinates by

$$\mathring{\partial}_\mu \mathring{A}_\nu = \frac{\partial\mathring{A}_\nu}{\partial\mathring{x}^{\nu}} - \mathring{\Xi}^{\rho}_{\mu\nu}\mathring{A}_\rho\ . \tag{B5}$$

The components of the vector-spinor translation symbols σ, given in Cartesian coordinates by (A4) and (A5), are found by contracting with the vector basis like $\sigma_\mu\, dx^\mu$ and then substituting (B1). One finds that

$$\begin{gathered} [\mathring{\sigma}_t] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [\mathring{\sigma}_r] = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & -\cos\theta \end{bmatrix}, \\ [\mathring{\sigma}_\theta] = \frac{1}{\sqrt{2}}\begin{bmatrix} -r\sin\theta & re^{i\phi}\cos\theta \\ re^{-i\phi}\cos\theta & r\sin\theta \end{bmatrix}, \qquad [\mathring{\sigma}_\phi] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & ire^{i\phi}\sin\theta \\ -ire^{-i\phi}\sin\theta & 0 \end{bmatrix}, \end{gathered} \tag{B6}$$

so that

$$\varepsilon_{AB}\varepsilon_{A'B'}\mathring{\sigma}^{AA'}_{\mu}\mathring{\sigma}^{BB'}_{\nu} = \mathring{\eta}_{\mu\nu}\ . \tag{B7}$$

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