# Interval Markov chains: Performance measures and sensitivity analysis

Mingmei Teo

*Thesis submitted for the degree of*

*Master of Philosophy*

*in*

*Applied Mathematics*

*at*

*The University of Adelaide*

School of Mathematical Sciences

THE UNIVERSITY of ADELAIDE

December 2013

# Contents

# Abstract

There is a vast literature on Markov chains where point estimates of transition and initial probabilities are used to calculate various performance measures. However, using these point estimates does not account for the associated uncertainty in estimate. If these point estimates are used, then the best outcome possible would be an approximate solution. Hence, it would be beneficial if there was a way to allow for some uncertainty in the parameters and to carry this through the calculations.

One method of incorporating variation is to place bounds on the parameters and use these intervals rather than a single point estimate. By considering the intervals that contain point estimates, it is possible to control the amount of variation allowed. When these intervals are used in calculations, the results obtained are also intervals containing the true solution. Hence, allowing for an approximation of the result as well as a margin of error to be obtained.

One of the objectives of this thesis is to develop and investigate different methods of calculating intervals for various performance measures (for example, mean hitting times and expected total costs) for Markov chains when intervals are given for the parameters instead of point estimates. We develop a numerical method for obtaining intervals for the performance measures for general unstructured interval Markov chains through the use of optimisation techniques. During this development, we found a connection between interval Markov chains and Markov decision

processes and exploited it to obtain a form for our solution. Further, we also considered structured interval Markov chains, such as interval birth and death processes, and obtained analytic results for the classes of processes considered.

Following from the idea of structured Markov chains, we considered the Markovian SIR (*susceptible-infectious-recovered*) epidemic model and looked to extend the concepts developed for the unstructured interval Markov chains. Two important performance measures, namely the mean final epidemic size and mean epidemic duration, were of interest to us and we were able to prove analytic results for the mean final epidemic size. For the mean epidemic duration, we modified the numerical method for general unstructured interval Markov chains to calculate bounds on this performance measure.

The other objective of this thesis was to investigate if it was possible to use interval analysis as an alternative to sensitivity analysis. We explored this in the context of the SIR model, where the true value of the parameters of the model may not be known. Hence, if one were to be careful when using point estimates, one would consider using sensitivity analysis which explores the parameter space around the chosen estimates. We considered a distribution on the parameter estimates and used the methods developed in the early chapters of the thesis, to calculate intervals for performance measures. Using these intervals, we developed a method to obtain an approximate cumulative distribution function of the performance measure. This approximate cumulative distribution function was found to very closely resemble the cumulative distribution function obtained from extensive simulations.

# Signed Statement

I, Mingmei Teo, certify that this work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, by used in a submission for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

I also give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library catalogue and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

*S*ignature: ........................ *D*ate: ........................

# Acknowledgements

First, to my supervisors, Professor Nigel Bean and Dr Joshua Ross. Thank you for all your help, support and guidance throughout the past couple of years. I am immensely grateful for your patience in explaining concepts and ideas to me, whenever I knocked on your door, as well as the time and effort you have invested to make this project a thoroughly enjoyable experience. I really do not think I would have been able to achieve all of this without your help and for that, the both of you are the best supervisors one could have.

To everyone involved in the stochastic group, I thank you for your help during stochastic coffee. You have all contributed to my project in one way or another by allowing me to bounce ideas off you and providing helpful suggestions to solve my countless problems. In particular, I would like to thank Dr Giang Nguyen for her help with Markov decision processes. Without her help in identifying the connection with Markov decision processes, I think I would still be stuck with my proof.

I would like to thank everyone in the School of Mathematical Sciences for making these two years fun and enjoyable. There are undoubtably countless people to thank but here are a number I would like to thank specifically. To Kate, Jess, Vincent, Nic, Caitlin, Nicola and David A, thank you for all your help and also for the fun we have had the past couple of years. Further, I cannot forget the three people who have played a massive role over the past year in helping me get to this point. A big thank you to Ben, David and Hayden for their immense support and friendship, especially over the past six months, to enable me to not only finish my thesis but also have an amazing time doing so.

Finally, a big thank you to my parents for your unconditional love and support for the past 21 years of my life. Thank you for encouraging me to pursue my Masters and also being my chef, chauffeur and everything in between, especially over the last few months. This would all not have been possible without the both of you.