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Maximum principle for mean-field jump-diffusion stochastic delay differential equations and its application to finance

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Abstract

This paper investigates a stochastic optimal control problem with delay and of mean-field type, where the controlled state process is governed by a mean-field jump-diffusion stochastic delay differential equation. Two sufficient maximum principles and one necessary maximum principle are established for the underlying systems. As an application, a bicriteria mean-variance portfolio selection problem with delay is studied. Under certain conditions, explicit expressions are provided for the efficient portfolio and the efficient frontier, which are as elegant as those in the classical mean-variance problem without delays.

Keywords: Stochastic maximum principle; Mean-field model; Stochastic delay differential equation; Backward stochastic differential equation; Mean-variance portfolio selection.

1. Introduction

To develop sufficient and necessary conditions for optimality for the stochastic optimal control problems is not only an important theoretical problem, but also useful for applications in various areas, including engineering, finance, economics and operations research, and so on. The stochastic maximum principle is one of the major approaches to solve stochastic optimal control problems. Other approaches include, such as, the dynamic programming principle and the convex duality martingale method. The history of the stochastic maximum principle can be dated back to Pontryagin's maximum principle which is designed to find the best possible control for deterministic dynamical systems. This principle informally states that the optimal control is chosen by maximizing or minimizing the so-called Hamiltonian under appropriate conditions. By the stochastic maximum principle, solving optimal control problems is reduced to solving a system of forward-backward stochastic differential equations. The early contributions to the stochastic maximum principle approach were made by Kushner (1972), Bismut (1973) and Bensoussan (1982). In the last three decades, many extensions of the stochastic maximum principle have been made, see for example, Peng (1990), Tang and Li (1994), Framstad et al. (2004), Shi (2012), Haadem et al. (2013) and references therein. A systematic account on the subject can be found in Yong and Zhou (1999) and Øksendal and Sulem (2007). Particularly, the stochastic maximum principle has been widely adopted to solve the stochastic optimal control problems in finance, such as the mean-variance portfolio selection problem, the investment-consumption problem, the optimal insurance problem and others. Many examples were provided in Framstad et al. (2004).

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In recent years, stochastic optimal control problems for the mean-field stochastic differential equations (SDEs) have attracted an increasing attention. The mean-field SDEs can trace their roots to the Mckean-Valsov model, which was first introduced by Kac (1956) and Mckean (1966) to study physical systems with a large number of interacting particles. Lasry and Lions (2007) extended applications of the mean-field models to economics and finance. Intuitively speaking, the adjoint equation of a controlled state process driven by the mean-field SDE is a mean-field backward stochastic differential equation (BSDE). So it is not until Buckdahn et al. (2009a, 2009b) established the theory of the mean-field BSDEs that the stochastic maximum principle for the optimal control system of mean-field type has become a popular topic. Interested readers may refer to Andersson and Djehiche (2011), Buckdahn et al. (2011), Li (2012), Meyer-Brandis et al. (2012) and Shen and Siu (2013) for various versions of the stochastic maximum principles for the mean-field models.

The stochastic optimal control problems with delay have also received a lot of attention recently. Some examples can be found in Elsanosi et al. (2000). One of the reasons is that many real-world systems evolve according to not only their current state but also essentially their previous history. Indeed, the phenomenon of past path-dependence is common in the fields of both natural and social sciences, such as physics, chemistry, biology, finance and economics. In general, optimal control problems under delayed systems are very difficult to solve because of the infinite-dimensional state space structure. When only the distributed (average) and pointwise time delays are involved in the state process, however, optimal control problems are found to be solvable under certain conditions. Therefore, it pays us dividends to develop the stochastic maximum principle for delayed systems in certain cases. Current research on this topic can be divided into two directions. One direction involves a system of three-coupled adjoint equations, which consists of two BSDEs and one backward ordinary differential equation (ODE), see for example, Øksendal and Sulem (2000), David (2008), Agram et al. (2012) and Shi (2013). And another direction, the adjoint equation is given by a time-advanced BSDE. Some representative works in this direction, to name a few, include Chen and Wu (2010), Øksendal et al. (2011), Yu (2012) and Agram and Øksendal (2013).

In this paper, we consider a stochastic optimal control problem of a mean-field jump-diffusion delayed system, where the state process is governed by a mean-field jump-diffusion stochastic delay differential equation (SDDE). Under the Lipschitz continuity condition, we first prove the uniqueness and existence of a solution to the mean-field jump-diffusion SDDE. To develop our maximum principles, we then follow the aforementioned first direction to use the adjoint equations given by a system of three-coupled mean-field BSDEs, which is different from those consisting of two BSDEs and one backward ODE in previous literature. Suppose that the control domain is convex. We adopt the convex perturbation method in Bensoussan (1982) to show the necessary maximum principle. Under the convexity assumption of the Hamiltonian and the terminal cost, we provide two versions of the sufficient maximum principles for the problem. Particularly, it is only required that the terminal cost function is convex in an expected sense for the second sufficient maximum principle, which can be easily used to study the bicriteria mean-variance portfolio selection problem. Our mean-variance problem is different from the classical one in two ways. On the one hand, it is assumed that the investor's wealth process is modeled by a jump-diffusion SDDE due to various factors, such as the capital outflow/inflow, the financial market with bounded memory and the large investor effect. On the other hand, it is assumed that the investor aims at simultaneously maximizing the mean and minimizing the variance of the terminal wealth. This makes the mean-variance problem an optimal control problem of mean-field type or a time-inconsistent optimal control problem. Our second sufficient maximum principle is tailor-made for the problem. We represent the efficient portfolio as a feedback control and obtain a nice and neat expression for the efficient frontier. Inspired by our financial example, we finally discuss the solvability of the control problem with delay.

Our paper contributes to the literature in at least five aspects. Firstly, we establish both sufficient and necessary maximum principles for a control system with jump, delay and mean-field term. These features are very common in real-world systems. Secondly, our second sufficient maximum principle only requires the terminal cost function is convex in an expected sense, which is very flexible for applications. Thirdly, we propose a three-coupled system of BSDEs as the adjoint equations, where each of three BSDEs admits a unique adapted solution under suitable conditions. In previous literature, the three-coupled system of adjoint equations are given by two BSDEs and one backward ODE. However, in general, the uniqueness and existence of adapted solutions to such ODE-type adjoint equations may be problematic. Fourthly, we are the first to find an efficient frontier in a bicriteria mean-variance portfolio selection problem under a delayed control system, which is as elegant as the classical mean-variance problem without delays. Fifthly, we provide a sufficient condition under which the control problem with delay becomes finite-dimensional and hence solvable. This sufficient condition allows the control variable entering into both diffusion and jump parts of the control system. In addition, our paper is different from a recently published paper, Du et al. (2013), on the similar topic. Our control system incorporates jump and average delay in the state equation and both pointwise and average delay in the performance functional, which is more general than Du et al. (2013). The adjoint equations in our paper are given by a three-coupled system of BSDEs, which are totally different from that given by a time-advanced BSDE in Du et al. (2013). So the existence and uniqueness of related SDDEs and BSDEs are discussed under different conditions. Indeed, our paper follows the first direction to establish the stochastic maximum principles for delayed systems, which allows us to obtain the feedback control in the financial example, while Du et al. (2013) adopts the second direction.

The rest of this paper is structured as follows. Section 2 introduces the notation to be used and some preliminary results for the mean-field jump-diffusion SDDE and the mean-field jump-diffusion BSDE. In Section 3, we formulate the stochastic optimal control problem of a system driven by the mean-field jump-diffusion SDDE. Sections 4 and 5 provide the sufficient and necessary maximum principles for the problem, respectively. In Section 6, we use a version of our sufficient maximum principle to discuss a mean-variance portfolio selection problem with delay and of mean-field type. Closed-form expressions for the efficient portfolio and the efficient frontier are obtained. Section 7 is devoted to discussing when the control problem with delay is finite-dimensional. Finally some concluding remarks are given in Section 8.

2. Preliminaries

In this section, we discuss the uniqueness and existence results for the mean-field jump-diffusion SDDE and the mean-field jump-diffusion BSDE. Throughout this paper, we denote by $A^\top$ the transpose of a vector or matrix A, by $\mathrm{tr}(A)$ the trace of a square matrix A, by $\mathrm{diag}(y)$ the diagonal matrix with the elements of y on the diagonal, by $\|A\| := \sqrt{\mathrm{tr}(A^\top A)}$ the norm of a vector or a matrix A, by 0 the zero scalar, vector or matrix of appropriate dimensions. In addition, we adopt K and C as two positive generic constants, which may vary from line to line in this paper.

Let $\mathcal{T} := [0, T]$ denote a finite horizon, where $T < \infty$. We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which all randomness is defined. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous, $\mathbb{P}$-complete filtration $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$, which will be specified below. Furthermore, we assume that $\mathcal{F}(T) = \mathcal{F}$. Denote by $\mathbb{E}[\cdot]$ the expectation under $\mathbb{P}$. To simplify our notation, we will denote by

$$\overline{\varphi} := \mathbb{E}[\varphi] \ , \quad \overline{\varphi}(t) := \mathbb{E}[\varphi(t)] \ ,$$

for any random variable φ or random process $\varphi(\cdot)$ whenever no confusion arises. Let $\{W(t)|t \in \mathcal{T}\} = \{(W^1(t), W^2(t), \cdots, W^d(t))^\top | t \in \mathcal{T}\}$ be a d-dimensional standard Brownian motion with respect to its natural filtration under $\mathbb{P}$. We denote by $\Re_0 := \Re \backslash \{0\}$ and by $\mathcal{B}(E)$ the Borel σ-field generated by any set E. Let $N^i(dt, d\zeta), i = 1, 2, \cdots l$, be independent Poisson random measures on the product measurable space $(\mathcal{T} \times \Re_0, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(\Re_0))$, with compensators

$$\pi^i(dt, d\zeta) := \nu^i(d\zeta)dt \ ,$$

under $\mathbb{P}$ such that

$$\{(N^i - \pi^i)([0, t] \times A) | t \in \mathcal{T}\} \ ,$$

are $(\mathbb{F}, \mathbb{P})$-martingales for all $A \in \mathcal{B}(\Re_0)$ satisfying $\nu^i(A) < \infty$. Here $\nu^i(d\zeta)$ is the Lévy measure of the jump amplitude of the i-th Poisson random measure, which is assumed to be a σ-finite measure on $\Re_0$ satisfying

$$\int_{\Re_0} (1 \wedge \zeta^2) \nu^i(d\zeta) < \infty \ .$$

Write the l-dimensional Lévy measure as

$$\nu(d\zeta) := (\nu^1(d\zeta), \nu^2(d\zeta), \cdots, \nu^l(d\zeta))^\top ,$$

and the l-dimensional compensated Poisson random measure as

$$\widetilde{N}(dt, d\zeta) := (N^1(dt, d\zeta) - \nu^1(d\zeta)dt, \cdots, N^l(dt, d\zeta) - \nu^l(d\zeta)dt)^\top .$$

Suppose that the Brownian motion and the Poisson random measure are stochastically independent under $\mathbb{P}$. Let $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ denote the right-continuous, $\mathbb{P}$-complete, natural filtration generated by the Brownian motion and the Poisson random measure. We denote by $\mathscr{P}$ the predictable σ-field on $\Omega\times\mathcal{T}$.

On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we introduce the following spaces of processes which will be used later:

$\mathcal{L}^m(\mathcal{F}(T); H)$: the space of all H-valued, $\mathcal{F}(T)$-measurable random variables φ such that $\mathbb{E}[\|\varphi\|^m] < \infty$;
$\mathcal{S}^m(a, b; H)$: the space of all H-valued, $\mathbb{F}$-adapted càdlàg processes $\{\varphi(t) | a \le t \le b\}$ such that $\mathbb{E}[\sup_{a\le t\le b} \|\varphi(t)\|^m] < \infty$;
$\mathcal{L}^m(a, b; H)$: the space of all H-valued, $\mathbb{F}$-progressively measurable processes $\{\varphi(t) | a \le t \le b\}$ such that $\mathbb{E}[\int_a^b \|\varphi(t)\|^m dt] < \infty$;
$\mathcal{L}^2_\nu(a, b; \Re^{n\times l})$: the space of all $\Re^{n\times l}$-valued, $\mathscr{P}_{[a,b]}\otimes\mathcal{B}(\Re_0)$-measurable processes $\{\varphi(t, \zeta) | (t, \zeta) \in [a, b]\times\Re_0\}$ such that $\mathbb{E}[\int_a^b \int_{\Re_0} \mathrm{tr}[\varphi(t, \zeta)\mathrm{diag}(\nu(d\zeta))\varphi(t, \zeta)^\top]dt] < \infty$, where $\mathscr{P}_{[a,b]}$ denotes the σ-field of $\mathbb{F}$-predictable sets on $\Omega\times[a, b]$.

Furthermore, let $\mathcal{C}(a, b; H)$ denote the space of all continuous functions $\varphi : [a, b] \to H$, and $\mathcal{L}^2_\nu(\Re_0, \mathcal{B}(\Re_0); \Re^{n\times l})$ denote the Hilbert space of ν-almost sure equivalence classes formed by the functions from $\Re_0$ to the space of $\Re^{n\times l}$-valued matrices, with the norm $\|\cdot\|_\nu$ as follows

$$\|\varphi(\cdot)\|_\nu := \left(\int_{\Re_0} \mathrm{tr}[\varphi(\zeta)\mathrm{diag}(\nu(d\zeta))\varphi(\zeta)^\top]\right)^{\frac{1}{2}} .$$

In what follows, we shall discuss the uniqueness and existence of $\mathbb{F}$-adapted solutions to the mean-field jump-diffusion SDDE and the mean-field jump-diffusion BSDE. We first consider the following mean-field jump-diffusion SDDE

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t))dt \\ \qquad\qquad +\sigma(t, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t))dW(t) \\ \qquad\qquad +\int_{\Re_0} \gamma(t, \zeta, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t))\widetilde{N}(dt, d\zeta) , \\ X(t) = x_0(t) , \quad t \in [-\delta, 0] , \quad x_0(\cdot) \in \mathcal{C}(-\delta, 0; \Re^n) , \end{cases} \tag{1}$$

where

$$Y(t) := \int_{-\delta}^0 e^{\lambda s} X(t+s)ds , \quad Z(t) := X(t-\delta) ,$$

and

$$\overline{X}(t) := \mathbb{E}[X(t)] , \quad \overline{Y}(t) := \mathbb{E}[Y(t)] , \quad \overline{Z}(t) := \mathbb{E}[Z(t)] .$$

Note that $Y(t)$ and $Z(t)$ are given functionals of the path segment $\{X(t+s) | s \in [-\delta, 0]\}$ of X with the given averaging parameter $\lambda \in \Re$ and the given delay $\delta > 0$; $\overline{X}(t)$, $\overline{Y}(t)$ and $\overline{Z}(t)$ are the so-called mean-field terms. Here b, σ, γ are given mappings such that $b : \Omega\times\mathcal{T}\times(\Re^n\times\Re^n\times\Re^n)^2 \to \Re^n$, $\sigma : \Omega\times\mathcal{T}\times(\Re^n\times\Re^n\times\Re^n)^2 \to \Re^{n\times d}$, $\gamma : \Omega\times\mathcal{T}\times\Re_0\times(\Re^n\times\Re^n\times\Re^n)^2 \to \Re^{n\times l}$; b and σ are $\mathscr{P}\otimes(\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n))^2$-measurable, and γ is $\mathscr{P}\otimes\mathcal{B}(\Re_0)\otimes(\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n))^2$-measurable.

We suppose that the following conditions are satisfied

(A1) $x_0(\cdot) \in \mathcal{L}^2(-\delta,0;\Re^n)$, $b(\cdot,0,0,0,0,0,0) \in \mathcal{L}^2(0,T;\Re^n)$, $\sigma(\cdot,0,0,0,0,0,0) \in \mathcal{L}^2(0,T;\Re^{n\times d})$ and $\gamma(\cdot,\cdot,0,0,0,0,0,0) \in \mathcal{L}^2_\nu(0,T;\Re^{n\times l})$.

(A2) b, σ and γ are uniformly Lipschitz with respect to $\psi := (x,y,z)$ and $\overline{\psi} := (\overline{x},\overline{y},\overline{z})$, i.e. $\exists C>0$ such that $\forall t \in \mathcal{T}$, $\forall \psi_1 := (x_1,y_1,z_1), \psi_2 := (x_2,y_2,z_2), \overline{\psi}_1 := (\overline{x}_1,\overline{y}_1,\overline{z}_1), \overline{\psi}_2 := (\overline{x}_2,\overline{y}_2,\overline{z}_2) \in \Re^n\times\Re^n\times\Re^n$,

$$\begin{aligned}&\|b(t,\psi_1,\overline{\psi}_1)-b(t,\psi_2,\overline{\psi}_2)\|+\|\sigma(t,\psi_1,\overline{\psi}_1)-\sigma(t,\psi_2,\overline{\psi}_2)\|+\|\gamma(t,\cdot,\psi_1,\overline{\psi}_1)-\gamma(t,\cdot,\psi_2,\overline{\psi}_2)\|_\nu \\ &\quad \le C(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|+\|\overline{x}_1-\overline{x}_2\|+\|\overline{y}_1-\overline{y}_2\|+\|\overline{z}_1-\overline{z}_2\|)\ .\end{aligned}$$

Lemma 2.1. *Under Assumptions (A1) and (A2), the mean-field jump-diffusion SDDE (1) has a unique solution $X(\cdot) \in \mathcal{S}^2(0,T;\Re^n)$.*

Proof. See the Appendix □

Next we consider the following mean-field jump-diffusion BSDE

$$\begin{cases} dp(t) = -h(t,p(t),q(t),r(t,\cdot),\mathbb{E}[\Theta_1(t,p(t))],\mathbb{E}[\Theta_2(t,q(t))],\mathbb{E}[\Theta_3(t,r(t,\cdot))])dt \\ \qquad\qquad +q(t)dW(t)+\int_{\Re_0} r(t,\zeta)\widetilde{N}(dt,d\zeta)\ , \\ p(T) = \xi\ , \end{cases} \tag{2}$$

where $(\Theta_1,\Theta_2,\Theta_3)$ and (h,ξ) are given mappings such that $\Theta_1 : \Omega\times\mathcal{T}\times\Re^n \to \Re^n$, $\Theta_2 : \Omega\times\mathcal{T}\times\Re^{n\times d} \to \Re^n$, $\Theta_3 : \Omega\times\mathcal{T}\times\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l}) \to \Re^n$, $h : \Omega\times\mathcal{T}\times\Re^n\times\Re^{n\times d}\times\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l})\times\Re^n\times\Re^n\times\Re^n \to \Re^n$ and $\xi : \Omega \longrightarrow \Re^n$; Θ_1 is $\mathscr{P}\otimes\mathcal{B}(\Re^n)$-measurable, Θ_2 is $\mathscr{P}\otimes\mathcal{B}(\Re^{n\times d})$-measurable, Θ_3 is $\mathscr{P}\otimes\mathcal{B}(\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l}))$-measurable, h is $\mathscr{P}\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^{n\times d})\otimes\mathcal{B}(\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l}))\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)$-measurable and ξ is $\mathcal{F}(T)$-measurable.

We suppose that the following conditions are satisfied

(A3) $\Theta_i(\cdot,0) \in \mathcal{L}^2(0,T;\Re^n)$, $i=1,2,3$, $h(\cdot,0,0,0,0,0,0) \in \mathcal{L}^2(0,T;\Re^n)$ and $\xi \in \mathcal{L}^2(\mathcal{F}(T);\Re^n)$.

(A4) h is uniformly Lipschitz with respect to $\phi := (p,q,r)$ and $\theta := (\theta^1,\theta^2,\theta^3)$, i.e. $\exists C>0$ such that $\forall t \in \mathcal{T}$, $\forall \phi_1 := (p_1,q_1,r_1), \phi_2 := (p_2,q_2,r_2) \in \Re^n\times\Re^{n\times d}\times\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l})$, $\forall \theta_1 := (\theta^1_1,\theta^2_1,\theta^3_1), \theta_2 := (\theta^1_2,\theta^2_2,\theta^3_2) \in \Re^n\times\Re^n\times\Re^n$,

$$\begin{aligned}\|h(t,\phi_1,\theta_1)-h(t,\phi_2,\theta_2)\| \le\ & C(\|p_1-p_2\|+\|q_1-q_2\|+\|r_1-r_2\|_\nu \\ & +\|\theta^1_1-\theta^1_2\|+\|\theta^2_1-\theta^2_2\|+\|\theta^3_1-\theta^3_2\|)\ .\end{aligned}$$

(A5) $(\Theta_1,\Theta_2,\Theta_3)$ are uniformly Lipschitz with respect to $\phi := (p,q,r)$, respectively, i.e. $\exists C>0$ such that $\forall t \in \mathcal{T}$, $\forall \phi_1 := (p_1,q_1,r_1), \phi_2 := (p_2,q_2,r_2) \in \Re^n\times\Re^{n\times d}\times\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l})$,

$$\begin{aligned}&\|\Theta_1(t,p_1)-\Theta_1(t,p_2)\| \le C\|p_1-p_2\|\ , \\ &\|\Theta_2(t,q_1)-\Theta_2(t,q_2)\| \le C\|q_1-q_2\|\ , \\ &\|\Theta_3(t,r_1)-\Theta_3(t,r_2)\| \le C\|r_1-r_2\|_\nu\ .\end{aligned}$$

If $(\Theta_1,\Theta_2,\Theta_3)$ and (h,ξ) satisfy Assumptions (A3)-(A5), they are called the standard data of the mean-field jump-diffusion BSDE (2). The following existence and uniqueness result for the mean-field jump-diffusion BSDE extends Lemma 3.1 in Shen and Siu (2013).

Lemma 2.2. *Under Assumptions (A3)-(A5), the mean-field jump-diffusion BSDE (2) has a unique solution $(p(\cdot),q(\cdot),r(\cdot,\cdot)) \in \mathcal{S}^2(0,T;\Re^n)\times\mathcal{L}^2(0,T;\Re^{n\times d})\times\mathcal{L}^2_\nu(0,T;\Re^{n\times l})$.*

Proof. The proof is adapted from the proof of Lemma 3.1 in Shen and Siu (2013). So we do not repeat it here. □

3. Stochastic optimal control

In this section, we formulate the stochastic control problem under a mean-field, jump-diffusion, delayed system. We consider a controlled state process $\{X(t)|t \in \mathcal{T}\}$ given by the following mean-field jump-diffusion SDDE

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t), u(t))dt \\ \qquad\qquad +\sigma(t, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t), u(t))dW(t) \\ \qquad\qquad +\int_{\Re_0} \gamma(t, \zeta, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t), u(t))\widetilde{N}(dt, d\zeta) \ , \\ X(t) = x_0(t) \ , \quad t \in [-\delta, 0] \ , \quad x_0(t) \in \mathcal{C}(-\delta, 0; \Re^n) \ , \end{cases} \tag{3}$$

where b, σ, γ are given mappings such that $b : \Omega\times\mathcal{T}\times(\Re^n\times\Re^n\times\Re^n)^2\times U \to \Re^n$, $\sigma : \Omega\times\mathcal{T}\times(\Re^n\times\Re^n\times\Re^n)^2\times U \to \Re^{n\times d}$, $\gamma : \Omega\times\mathcal{T}\times\Re_0\times(\Re^n\times\Re^n\times\Re^n)^2\times U \to \Re^{n\times l}$; b and σ are $\mathscr{P}\otimes(\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n))^2\otimes\mathcal{B}(U)$-measurable, and γ is $\mathscr{P}\otimes\mathcal{B}(\Re_0)\otimes(\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n))^2\otimes\mathcal{B}(U)$-measurable. Here the control domain U is a nonempty convex subset of $\Re^k$. Furthermore, we require that the control process $\{u(t)|t \in \mathcal{T}\}$ is $\mathbb{F}$-predictable and has right limits.

Definition 3.1. *A control process $u(\cdot)$ is said to be admissible if $u(\cdot) \in \mathcal{L}^2(0, T; \Re^k)$ and $u(t) \in U$, a.e. $t \in [0, T]$, $\mathbb{P}$-a.s. Write $\mathcal{A}$ for the set of all admissible control processes.*

Suppose the following conditions hold

(A6) $x_0(\cdot) \in \mathcal{L}^2(-\delta, 0; \Re^n)$, $b(\cdot, 0, 0, 0, 0, 0, 0, 0) \in \mathcal{L}^2(0, T; \Re^n)$, $\sigma(\cdot, 0, 0, 0, 0, 0, 0, 0) \in \mathcal{L}^2(0, T; \Re^{n\times d})$ and $\gamma(\cdot, \cdot, 0, 0, 0, 0, 0, 0, 0) \in \mathcal{L}^2_\nu(0, T; \Re^{n\times l})$.

(A7) For almost all $(\omega, t) \in \Omega\times\mathcal{T}$, b, σ and γ are continuously differentiable in $(x, y, z, \overline{x}, \overline{y}, \overline{z}, u)$. Moreover, all partial derivatives $\bigtriangledown_a\rho$ are uniformly bounded, where $a = x, y, z, \overline{x}, \overline{y}, \overline{z}, u$ and $\rho = b, \sigma, \gamma$.

Under Assumptions (A6)-(A7), Assumptions (A1)-(A2) are satisfied. By Lemma 2.1, we can see that the mean-field jump-diffusion SDDE (3) admits a unique solution $X(\cdot) \in \mathcal{S}^2(0, T; \Re^n)$ associated with any $u(\cdot) \in \mathcal{A}$.

The performance functional of our stochastic control problem has the following mean-field and delay type:

$$J(u(\cdot)) = \mathbb{E}\bigg[\int_0^T f(t, X(t), Y(t), Z(t), \overline{X}(t), \overline{Y}(t), \overline{Z}(t), u(t))dt + g(X(T), Y(T), \overline{X}(T), \overline{Y}(T))\bigg] \ , \tag{4}$$

where f and g are given mappings such that $f : \Omega\times\mathcal{T}\times(\Re^n\times\Re^n\times\Re^n)^2\times U \to \Re$ and $g : \Omega\times(\Re^n\times\Re^n)^2 \to \Re$; f is $\mathscr{P}\otimes(\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n))^2\otimes\mathcal{B}(U)$-measurable and g is $\mathcal{F}(T)\otimes(\mathcal{B}(\Re^n)\otimes\mathcal{B}(\Re^n))^2$-measurable.

Suppose that the following conditions hold

(A8) For almost all $(\omega, t) \in \Omega\times\mathcal{T}$, f is differentiable in $(x, y, z, \overline{x}, \overline{y}, \overline{z}, u)$ with continuous derivatives $\bigtriangledown_a f$, where $a = x, y, z, \overline{x}, \overline{y}, \overline{z}, u$. Moreover, for almost all $(\omega, t) \in \Omega\times\mathcal{T}$, there exists a constant $C > 0$ such that for all $(x, y, z, \overline{x}, \overline{y}, \overline{z}, u) \in (\Re^n\times\Re^n\times\Re^n)^2\times U$,

$$|f(t, x, y, z, \overline{x}, \overline{y}, \overline{z}, u)| \leq C(1+\|x\|^2+\|y\|^2+\|z\|^2+\|\overline{x}\|^2+\|\overline{y}\|^2+\|\overline{z}\|^2+\|u\|^2) \ ,$$

and

$$\|\bigtriangledown_a f(t, x, y, z, \overline{x}, \overline{y}, \overline{z}, u)\| \leq C(1+\|x\|+\|y\|+\|z\|+\|\overline{x}\|+\|\overline{y}\|+\|\overline{z}\|+\|u\|) \ ,$$

where $a = x, y, z, \overline{x}, \overline{y}, \overline{z}, u$.

(A9) For almost all $\omega \in \Omega$, g is differentiable in $(x, y, \overline{x}, \overline{y})$ with continuous derivatives $\bigtriangledown_a g$, where $a = x, y, \overline{x}, \overline{y}$. Moreover, for almost all $\omega \in \Omega$, there exists a constant $C > 0$ such that for all $(x, y, \overline{x}, \overline{y}) \in (\Re^n\times\Re^n)^2$,

$$|g(x, y, \overline{x}, \overline{y})| \leq C(1+\|x\|^2+\|y\|^2+\|\overline{x}\|^2+\|\overline{y}\|^2) \ ,$$

and

$$\|\bigtriangledown_a g(x, y, \overline{x}, \overline{y})\| \leq C(1+\|x\|+\|y\|+\|\overline{x}\|+\|\overline{y}\|) \ ,$$

where $a = x, y, \overline{x}, \overline{y}$.

Here f and g represent the running cost and the terminal cost of the control problem, respectively. It is clear that under Assumptions (A8)-(A9), the performance functional (4) is well defined for any $u(\cdot) \in \mathcal{A}$.

The problem is to find a control process $u^*(\cdot) \in \mathcal{A}$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot)) \ . \tag{5}$$

The admissible control $u^*(\cdot)$ satisfying (5) is called an optimal control process. Correspondingly, the state process $X^*(\cdot)$ associated with $u^*(\cdot)$ is called an optimal state process.

We define the Hamiltonian $\mathcal{H} : \Omega\times\mathcal{T}\times(\Re^n\times\Re^n\times\Re^n)^2\times U\times(\Re^n)^3\times(\Re^{n\times d})^3\times(\mathcal{L}^2_\nu(\Re_0,\mathcal{B}(\Re_0);\Re^{n\times l}))^3 \to \Re$ of the problem by

$$\begin{aligned}&\mathcal{H}(t,x,y,z,\overline{x},\overline{y},\overline{z},u,p,q,r)\\ &= f(t,x,y,z,\overline{x},\overline{y},\overline{z},u)+b(t,x,y,z,\overline{x},\overline{y},\overline{z},u)^\top p_1+(x-\lambda y-e^{-\lambda\delta}z)^\top p_2\\ &\quad+\mathrm{tr}[\sigma(t,x,y,z,\overline{x},\overline{y},\overline{z},u)^\top q_1]+\int_{\Re_0}\mathrm{tr}[\gamma(t,\zeta,x,y,z,\overline{x},\overline{y},\overline{z},u)\mathrm{diag}(\nu(d\zeta))r_1(t,\zeta)^\top] \ ,\end{aligned} \tag{6}$$

where $p := (p_1,p_2,p_3)$, $q := (q_1,q_2,q_3)$, $r := (r_1,r_2,r_3)$. From Assumptions (A8) and (A9), it is clear that the Hamiltonian $\mathcal{H}$ is also continuously differentiable with respect to $(x,y,z,\overline{x},\overline{y},\overline{z},u)$. Write $\bigtriangledown_a\mathcal{H}$ for the corresponding partial derivatives, where $a = x,y,z,\overline{x},\overline{y},\overline{z},u$.

To derive the maximum principle in the next two sections, we introduce the adjoint equations of the controlled system (3)-(4) governing the unknown $\mathbb{F}$-adapted processes $\{p(t)|t\in\mathcal{T}\} = \{(p_1(t),p_2(t),p_3(t))|t\in\mathcal{T}\}$, $\{q(t)|t\in\mathcal{T}\} = \{(q_1(t),q_2(t),q_3(t))|t\in\mathcal{T}\}$ and $\{r(t,\zeta)|(t,\zeta)\in\mathcal{T}\times\Re_0\} = \{(r_1(t,\zeta),r_2(t,\zeta),r_3(t,\zeta))|(t,\zeta)\in\mathcal{T}\times\Re_0\}$ as follows:

$$\begin{cases} dp_1(t) = -\{\bigtriangledown_x\mathcal{H}(t)+\mathbb{E}[\bigtriangledown_{\overline{x}}\mathcal{H}(t)]\}dt+q_1(t)dW(t)+\displaystyle\int_{\Re_0} r_1(t,\zeta)\widetilde{N}(dt,d\zeta) \ ,\\ p_1(T) = \bigtriangledown_x g(T)+\mathbb{E}[\bigtriangledown_{\overline{x}}g(T)] \ ,\end{cases} \tag{7}$$

$$\begin{cases} dp_2(t) = -\{\bigtriangledown_y\mathcal{H}(t)+\mathbb{E}[\bigtriangledown_{\overline{y}}\mathcal{H}(t)]\}dt+q_2(t)dW(t)+\displaystyle\int_{\Re_0} r_2(t,\zeta)\widetilde{N}(dt,d\zeta) \ ,\\ p_2(T) = \bigtriangledown_y g(T)+\mathbb{E}[\bigtriangledown_{\overline{y}}g(T)] \ ,\end{cases} \tag{8}$$

and

$$\begin{cases} dp_3(t) = -\{\bigtriangledown_z\mathcal{H}(t)+\mathbb{E}[\bigtriangledown_{\overline{z}}\mathcal{H}(t)]\}dt+q_3(t)dW(t)+\displaystyle\int_{\Re_0} r_3(t,\zeta)\widetilde{N}(dt,d\zeta) \ ,\\ p_3(T) = 0 \ ,\end{cases} \tag{9}$$

where we denote by

$$\mathcal{H}(t) := \mathcal{H}(t,X(t),Y(t),Z(t),\overline{X}(t),\overline{Y}(t),\overline{Z}(t),u(t),p(t),q(t),r(t,\cdot)) \ ,$$

and

$$g(T) := g(X(T),Y(T),\overline{X}(T),\overline{Y}(T)) \ ,$$

whenever no confusion arises. Indeed, the three-coupled system of (7)-(9) is a linear mean-field jump-diffusion BSDE. Under Assumptions (A6)-(A9), Assumptions (A3)-(A5) are satisfied. By Lemma 2.2, we can see that the mean-field jump-diffusion BSDEs (7)-(9) admit unique solutions $(p_i(\cdot),q_i(\cdot),r_i(\cdot,\cdot)) \in \mathcal{S}^2(0,T;\Re^n)\times\mathcal{L}^2(0,T;\Re^{n\times d})\times\mathcal{L}^2_\nu(0,T;\Re^{n\times l})$, for $i = 1,2,3$.

4. Sufficient maximum principle

In this section, we consider the sufficient condition for optimality for the problem. For any control processes $u(\cdot)$ and $u^*(\cdot) \in \mathcal{A}$, let $\Psi(\cdot) = (X(\cdot), Y(\cdot), Z(\cdot))$, $\Psi^*(\cdot) = (X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ and $\Phi(\cdot) = (p(\cdot), q(\cdot), r(\cdot,\cdot))$, $\Phi^*(\cdot) = (p^*(\cdot), q^*(\cdot), r^*(\cdot,\cdot))$ be the corresponding solutions to the state equation (3) and the adjoint equations (7)-(9), respectively. To unburden our notation, we write

$$\begin{aligned}
&\eta(\cdot) = X(\cdot) - \lambda Y(\cdot) - e^{-\lambda\delta} Z(\cdot) \ , \quad \eta^*(\cdot) = X^*(\cdot) - \lambda Y^*(\cdot) - e^{-\lambda\delta} Z^*(\cdot) \ , \\
&\rho(t) = \rho(t, \Psi(t), \overline{\Psi}(t), u(t)) \ , \quad \rho^*(t) = \rho(t, \Psi^*(t), \overline{\Psi}^*(t), u^*(t)) \ , \quad \rho = b, \sigma, \gamma, f \ , \\
&g(T) = g(X(T), \overline{X}(T), Y(T), \overline{Y}(T)) \ , \quad g^*(T) = g(X^*(T), \overline{X}^*(T), Y^*(T), \overline{Y}^*(T)) \ , \\
&\mathcal{H}(t) = \mathcal{H}(t, \Psi(t), \overline{\Psi}(t), u(t), \Phi^*(t)) \ , \quad \mathcal{H}^*(t) = \mathcal{H}(t, \Psi^*(t), \overline{\Psi}^*(t), u^*(t), \Phi^*(t)) \ .
\end{aligned}$$

Next, we derive a representation of the difference $J(u(\cdot)) - J(u^*(\cdot))$ in terms of the Hamiltonian $\mathcal{H}$ and the terminal cost g as well as the state process.

Lemma 4.1. *Suppose that Assumptions (A6)-(A9) are satisfied. If $(p_3^*(t), q_3^*(t), r_3^*(t,\cdot)) = (0,0,0)$, for a.e. $t \in [0,T]$, $\mathbb{P}$-a.s., then we have*

$$\begin{aligned}
J(u(\cdot)) - J(u^*(\cdot)) \;=\; & \mathbb{E}\Big[\int_0^T \Big(\mathcal{H}(t) - \mathcal{H}^*(t) - (X(t) - X^*(t))^\top \{\triangledown_x \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{x}} \mathcal{H}^*(t)]\} \\
&\qquad - (Y(t) - Y^*(t))^\top \{\triangledown_y \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{y}} \mathcal{H}^*(t)]\} \\
&\qquad - (Z(t) - Z^*(t))^\top \{\triangledown_z \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{z}} \mathcal{H}^*(t)]\}\Big) dt\Big] \\
& + \mathbb{E}\big[g(T) - g^*(T) - (X(T) - X^*(T))^\top \{\triangledown_x g^*(T) + \mathbb{E}[\triangledown_{\overline{x}} g^*(T)]\} \\
&\qquad - (Y(T) - Y^*(T))^\top \{\triangledown_y g^*(T) + \mathbb{E}[\triangledown_{\overline{y}} g^*(T)]\}\big] \ , \qquad (10)
\end{aligned}$$

for any $u(\cdot), u^*(\cdot) \in \mathcal{A}$.

Proof. By the definitions of the Hamiltonian $\mathcal{H}$ and the performance functional $J(u(\cdot))$, it is easy to check that

$$\begin{aligned}
J(u(\cdot)) - J(u^*(\cdot)) \;=\; & \mathbb{E}\Big[\int_0^T \Big\{\mathcal{H}(t) - \mathcal{H}^*(t) - (b(t) - b^*(t))^\top p_1^*(t) \\
& - \mathrm{tr}[(\sigma(t) - \sigma^*(t))^\top q_1^*(t)] - (\eta(t) - \eta^*(t))^\top p_2^*(t) \\
& - \int_{\Re_0} \mathrm{tr}[\{\gamma(t,\zeta) - \gamma^*(t,\zeta)\} \mathrm{diag}(\nu(d\zeta)) r_1^*(t,\zeta)^\top]\Big\} dt\Big] \\
& + \mathbb{E}\big[g(T) - g^*(T)\big] \ .
\end{aligned}$$

Since $(p_3^*(t), q_3^*(t), r_3^*(t,\cdot)) = (0,0,0)$, applying Itô's formula to

$$(X(t) - X^*(t))^\top p_1^*(t) + (Y(t) - Y^*(t))^\top p_2^*(t) + (Z(t) - Z^*(t))^\top p_3^*(t) \ ,$$

results in

$$\begin{aligned}
& \mathbb{E}\big[(X(T) - X^*(T))^\top \{\triangledown_x g^*(T) + \mathbb{E}[\triangledown_{\overline{x}} g^*(T)]\} + (Y(T) - Y^*(T))^\top \{\triangledown_y g^*(T) + \mathbb{E}[\triangledown_{\overline{y}} g^*(T)]\}\big] \\
& = \mathbb{E}\Big[\int_0^T (b(t) - b^*(t))^\top p_1^*(t) + \mathrm{tr}[(\sigma(t) - \sigma^*(t))^\top q_1^*(t)] + (\eta(t) - \eta^*(t))^\top p_2^*(t) \\
& \quad + \int_{\Re_0} \mathrm{tr}[\{\gamma(t,\zeta) - \gamma^*(t,\zeta)\} \mathrm{diag}(\nu(d\zeta)) r_1^*(t,\zeta)^\top] - (X(t) - X^*(t))^\top \{\triangledown_x \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{x}} \mathcal{H}^*(t)]\} \\
& \quad - (Y(t) - Y^*(t))^\top \{\triangledown_y \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{y}} \mathcal{H}^*(t)]\} - (Z(t) - Z^*(t))^\top \{\triangledown_z \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{z}} \mathcal{H}^*(t)]\} dt\Big] \ .
\end{aligned}$$

Therefore, combining the above two equations yields the desired result. $\square$

Theorem 4.1. **[Sufficient maximum principle I]** *Suppose that Assumptions (A6)-(A9) are satisfied. If the following conditions hold*

1. *the Hamiltonian* $\mathcal{H}$ *is convex in* $(x, y, z, \overline{x}, \overline{y}, \overline{z}, u)$,
2. *the terminal cost function* g *is convex in* $(x, y, \overline{x}, \overline{y})$,
3. $\mathcal{H}(t, \Psi^*(t), \overline{\Psi}^*(t), u^*(t), \Phi^*(t)) = \min_{u \in U} \mathcal{H}(t, \Psi^*(t), \overline{\Psi}^*(t), u, \Phi^*(t))$, *for a.e.* $t \in [0, T]$, $\mathbb{P}$*-a.s.,*
4. $(p_3^*(t), q_3^*(t), r_3^*(t, \cdot)) = (0, 0, 0)$, *for a.e.* $t \in [0, T]$, $\mathbb{P}$*-a.s.,*

then $u^*(\cdot)$ *is an optimal control process and* $\Psi^*(\cdot) = (X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ *is the corresponding optimal state process.*

Proof. For any control process $u(\cdot) \in \mathcal{A}$ and the corresponding $\Psi(\cdot) = (X(\cdot), Y(\cdot), Z(\cdot))$, we can see from Lemma 4.1 that

$$\begin{aligned} J(u(\cdot)) - J(u^*(\cdot)) \;\; = \;\; & \mathbb{E}\bigg[\int_0^T \bigg(\mathcal{H}(t) - \mathcal{H}^*(t) - (X(t) - X^*(t))^\top \{ \triangledown_x \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{x}} \mathcal{H}^*(t)] \} \\ & \qquad - (Y(t) - Y^*(t))^\top \{ \triangledown_y \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{y}} \mathcal{H}^*(t)] \} \\ & \qquad - (Z(t) - Z^*(t))^\top \{ \triangledown_z \mathcal{H}^*(t) + \mathbb{E}[\triangledown_{\overline{z}} \mathcal{H}^*(t)] \} \bigg) dt \bigg] \\ & + \mathbb{E}\big[g(T) - g^*(T) - (X(T) - X^*(T))^\top \{ \triangledown_x g^*(T) + \mathbb{E}[\triangledown_{\overline{x}} g^*(T)] \} \\ & \qquad - (Y(T) - Y^*(T))^\top \{ \triangledown_y g^*(T) + \mathbb{E}[\triangledown_{\overline{y}} g^*(T)] \} \big] \; . \end{aligned}$$

Then using the convexity of $\mathcal{H}$ and g gives

$$J(u(\cdot)) - J(u^*(\cdot)) \geq \mathbb{E}\left[\int_0^T (u(t) - u^*(t))^\top \triangledown_u \mathcal{H}^*(t) dt \right] \; .$$

Furthermore, we deduce from Condition 3 that

$$\begin{aligned} & (u(t) - u^*(t))^\top \triangledown_u \mathcal{H}^*(t) \\ & = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \big[\mathcal{H}(t, \Psi^*(t), \mathbb{E}[\Psi^*(t)], (1-\epsilon) u^*(t) + \epsilon u(t), \Phi^*(t)) - \mathcal{H}(t, \Psi^*(t), \mathbb{E}[\Psi^*(t)], u^*(t), \Phi^*(t)) \big] \geq 0 \; . \end{aligned}$$

Combining the above two inequalities gives the desired result, i.e. $u^*(\cdot)$ is an optimal control process and $X^*(\cdot)$ is the corresponding optimal state process. □

The convexity of the terminal cost g is sometimes too strict to be satisfied. It may be violated in applications (see the bicriteria mean-variance portfolio selection problem in Section 6). This will limit the applicability of our sufficient maximum principle. To overcome this limitation, we note that the proof of Theorem 4.1 still holds as long as the terminal cost g is convex in an expected sense. Therefore, relaxing the convexity of the terminal cost g, we provide the second sufficient maximum principle, which is a corollary of Theorem 4.1.

Corollary 4.1. **[Sufficient Stochastic Maximum principle II]** *Suppose that Assumptions (A6)-(A9) are satisfied. If the following conditions hold*

1. *the Hamiltonian* $\mathcal{H}$ *is convex in* $(x, y, z, \overline{x}, \overline{y}, \overline{z}, u)$,
2. *the terminal cost function* g *is convex in an expected sense, i.e.*

$$\begin{aligned} & \mathbb{E}\big[g(X_1, \overline{X}_1, Y_1, \overline{Y}_1) - g(X_2, \overline{X}_2, Y_2, \overline{Y}_2) \big] \\ & \geq \mathbb{E}\big[(X_1 - X_2)^\top \big\{ \triangledown_x g(X_2, \overline{X}_2, Y_2, \overline{Y}_2) + \mathbb{E}[\triangledown_{\overline{x}} g(X_2, \overline{X}_2, Y_2, \overline{Y}_2)] \big\} \big] \\ & \quad + \mathbb{E}\big[(Y_1 - Y_2)^\top \big\{ \triangledown_y g(X_2, \overline{X}_2, Y_2, \overline{Y}_2) + \mathbb{E}[\triangledown_{\overline{y}} g(X_2, \overline{X}_2, Y_2, \overline{Y}_2)] \big\} \big] \; , \end{aligned}$$

for any random variables $X_1, Y_1, X_2, Y_2 \in \mathcal{L}^2(\mathcal{F}(T); \Re^n)$,

3. $\mathcal{H}(t,\Psi^*(t),\overline{\Psi}^*(t),u^*(t),\Phi^*(t))=\min_{u\in U}\mathcal{H}(t,\Psi^*(t),\overline{\Psi}^*(t),u,\Phi^*(t))$, *for a.e.* $t\in[0,T]$, $\mathbb{P}$*-a.s.,*
4. $(p_3^*(t),q_3^*(t),r_3^*(t,\cdot))=(0,0,0)$, *for a.e.* $t\in[0,T]$, $\mathbb{P}$*-a.s.,*

then $u^*(\cdot)$ *is an optimal control process and* $\Psi^*(\cdot)=(X^*(\cdot),Y^*(\cdot),Z^*(\cdot))$ *is the corresponding controlled state process.*

Proof. The proof is similar to that of Theorem 4.1. So we do not repeat it here. □

5. Necessary maximum principle

In this section, we consider the necessary condition for optimality for the problem. Suppose that $u^*(\cdot)$ is the optimal control with the corresponding optimal state process and adjoint process denoted by $\Psi^*(\cdot)=(X^*(\cdot),Y^*(\cdot),Z^*(\cdot))$ and $\Phi^*(\cdot)=(p^*(\cdot),q^*(\cdot),r^*(\cdot,\cdot))$. Since the control domain U is convex, for any given admissible control $u(\cdot)\in\mathcal{A}$, the following perturbed control process

$$u^\epsilon(\cdot)=u^*(\cdot)+\epsilon(u(\cdot)-u^*(\cdot))\ ,\quad 0\le\epsilon\le 1\ ,$$

is also an element of $\mathcal{A}$. We denote by $\Psi^\epsilon(\cdot):=(X^\epsilon(\cdot),Y^\epsilon(\cdot),Z^\epsilon(\cdot))$ the corresponding perturbed state process with the initial value $X^\epsilon(t)=x_0(t)$, $t\in[-\delta,0]$. To simplify our notation, we write

$$g^\epsilon(T):=g(X^\epsilon(T),\overline{X}^\epsilon(T),Y^\epsilon(T),\overline{Y}^\epsilon(T))\ ,$$

and

$$\mathcal{H}^\epsilon(t):=\mathcal{H}(t,\Psi^\epsilon(t),\overline{\Psi}^\epsilon(t),u^\epsilon(t),\Phi^*(t))\ .$$

First of all, we give the estimate of the perturbed state process $\Psi^\epsilon(\cdot)=(X^\epsilon(\cdot),Y^\epsilon(\cdot),Z^\epsilon(\cdot))$.

Lemma 5.1. *Suppose that Assumptions (A6)-(A9) are satisfied. Then we have*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\|X^\epsilon(t)-X^*(t)\|^2\Big]+\mathbb{E}\Big[\sup_{0\le t\le T}\|Y^\epsilon(t)-Y^*(t)\|^2\Big]+\mathbb{E}\Big[\sup_{0\le t\le T}\|Z^\epsilon(t)-Z^*(t)\|^2\Big]=O(\epsilon^2)\ . \tag{11}$$

Proof. Applying Itô's formula to $\|X^\epsilon(t)-X^*(t)\|^2$, integrating from 0 to T and taking expectations on both sides, we can derive as in the proof of Lemma 2.1 (see the Appendix) that

$$\mathbb{E}[\|X^\epsilon(t)-X^*(t)\|^2]\le K\mathbb{E}\Big[\int_0^t\|X^\epsilon(s)-X^*(s)\|^2ds\Big]+K\mathbb{E}\Big[\int_0^t\|u^\epsilon(s)-u^*(s)\|^2ds\Big]\ .$$

Using Grönwall's inequality and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}\|X^\epsilon(t)-X^*(t)\|^2\Big]&\le K\mathbb{E}\Big[\int_0^T\|u^\epsilon(t)-u^*(t)\|^2dt\Big]\\&=K\epsilon^2\mathbb{E}\Big[\int_0^T\|u(t)-u^*(t)\|^2dt\Big]=O(\varepsilon^2)\ ,\end{aligned}$$

where K is a positive constant independent of ϵ.

Moreover, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}\|Y^\epsilon(t)-Y^*(t)\|^2\Big]&=\mathbb{E}\Big[\sup_{0\le t\le T}\|\int_{-\delta}^0e^{\lambda s}(X^\epsilon(t+s)-X^*(t+s))ds\|^2\Big]\\&\le\mathbb{E}\Big[\sup_{0\le t\le T}\int_{-\delta}^0e^{2\lambda s}ds\int_{-\delta}^0\|X^\epsilon(t+s)-X^*(t+s)\|^2ds\Big]\end{aligned}$$

$$
\begin{aligned}
&\le K\mathbb{E}\bigg[\sup_{0\le t\le T}\sup_{-\delta\le s\le 0}\|X^\epsilon(t+s)-X^*(t+s)\|^2\bigg]\\
&\le K\mathbb{E}\bigg[\sup_{0\le t\le T}\|X^\epsilon(t)-X^*(t)\|^2\bigg]=O(\epsilon^2)\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\bigg[\sup_{0\le t\le T}\|Z^\epsilon(t)-Z^*(t)\|^2\bigg] &= \mathbb{E}\bigg[\sup_{0\le t\le T}\|X^\epsilon(t-\delta)-X^*(t-\delta)\|^2\bigg]\\
&\le \mathbb{E}\bigg[\sup_{0\le t\le T}\|X^\epsilon(t)-X^*(t)\|^2\bigg]=O(\epsilon^2)\ .
\end{aligned}
$$

Therefore, combining above three inequalities gives the desired result. □

Based on Lemma 4.1 and Lemma 5.1, we derive the variational formula for the performance functional $J(u(\cdot))$ in terms of the Hamiltonian $\mathcal{H}$.

Lemma 5.2. *Suppose that Assumptions (A6)-(A9) are satisfied. If $(p_3^*(t),q_3^*(t),r_3^*(t,\cdot))=(0,0,0)$, for a.e. $t\in[0,T]$, $\mathbb{P}$-a.s., then for any control process $u(\cdot)\in\mathcal{A}$, the directional derivative of the performance functional $J(u(\cdot))$ at $u^*(\cdot)$ in the direction $u(\cdot)-u^*(\cdot)$ is given by*

$$
\frac{d}{d\epsilon}J(u^*(\cdot)+\epsilon(u(\cdot)-u^*(\cdot)))|_{\epsilon=0}=\mathbb{E}\bigg[\int_0^T(u(t)-u^*(t))^\top\bigtriangledown_u\mathcal{H}^*(t)dt\bigg]\ . \tag{12}
$$

Proof. By Lemma 4.1, we have

$$
J(u^\epsilon(\cdot))-J(u^*(\cdot))=I_1^\epsilon+I_2^\epsilon+\mathbb{E}\bigg[\int_0^T\epsilon(u(t)-u^*(t))^\top\bigtriangledown_u\mathcal{H}^*(t)dt\bigg]\ , \tag{13}
$$

where

$$
\begin{aligned}
I_1^\epsilon \ :=\ \mathbb{E}\bigg[\int_0^T\Big(&\mathcal{H}^\epsilon(t)-\mathcal{H}^*(t)-(X^\epsilon(t)-X^*(t))^\top\{\bigtriangledown_x\mathcal{H}^*(t)+\mathbb{E}[\bigtriangledown_{\overline{x}}\mathcal{H}^*(t)]\}\\
&-(Y^\epsilon(t)-Y^*(t))^\top\{\bigtriangledown_y\mathcal{H}^*(t)+\mathbb{E}[\bigtriangledown_{\overline{y}}\mathcal{H}^*(t)]\}\\
&-(Z^\epsilon(t)-Z^*(t))^\top\{\bigtriangledown_z\mathcal{H}^*(t)+\mathbb{E}[\bigtriangledown_{\overline{z}}\mathcal{H}^*(t)]\}-\epsilon(u(t)-u^*(t))^\top\bigtriangledown_u\mathcal{H}^*(t)\Big)dt\bigg]\ .
\end{aligned}
$$

and

$$
\begin{aligned}
I_2^\epsilon \ :=\ \mathbb{E}\big[&g^\epsilon(T)-g^*(T)-(X^\epsilon(T)-X^*(T))^\top\{\bigtriangledown_x g^*(T)+\mathbb{E}[\bigtriangledown_{\overline{x}}g^*(T)]\}\\
&-(Y^\epsilon(T)-Y^*(T))^\top\{\bigtriangledown_y g^*(T)+\mathbb{E}[\bigtriangledown_{\overline{y}}g^*(T)]\}\big]\ .
\end{aligned}
$$

Under Assumptions (A6)-(A9), combining the Taylor expansions, Lemma 5.1 and the dominated convergence theorem, we conclude that

$$
I_1^\epsilon=o(\epsilon)\ , \tag{14}
$$

and

$$
I_2^\epsilon=o(\epsilon)\ . \tag{15}
$$

Hence, putting (14)-(15) into (13), we obtain

$$
\begin{aligned}
\frac{d}{d\epsilon}J(u^*(\cdot)+\epsilon(u(\cdot)-u^*(\cdot)))|_{\epsilon=0} &= \lim_{\epsilon\to0^+}\frac{J(u^\epsilon(\cdot))-J(u^*(\cdot))}{\epsilon}\\
&= \mathbb{E}\bigg[\int_0^T(u(t)-u^*(t))^\top\bigtriangledown_u\mathcal{H}^*(t)dt\bigg]\ .
\end{aligned}
$$

The proof is complete. □

We now are ready to give the necessary condition for optimality for the problem.

Theorem 5.1. **[Necessary Stochastic Maximum principle]** *Suppose that Assumptions (A6)-(A9) are satisfied. If* $(p_3^*(t), q_3^*(t), r_3^*(t,\cdot)) = (0,0,0)$*, for a.e.* $t \in [0,T]$*,* $\mathbb{P}$*-a.s., then we have*

$$(u - u^*(t))^\top \nabla_u \mathcal{H}(t, \Psi^*(t), \overline{\Psi}^*(t), u^*(t), \Phi^*(t)) \geq 0 \ , \quad \forall u \in U, \quad \textit{for a.e. } t \in [0,T], \quad \mathbb{P}\textit{-a.s.}. \tag{16}$$

Proof. From Lemma 5.2, we have

$$\mathbb{E}\left[\int_0^T (u(t) - u^*(t))^\top \nabla_u \mathcal{H}^*(t) dt\right] = \lim_{\epsilon \to 0^+} \frac{J(u^*(\cdot) + \epsilon(u(\cdot) - u^*(\cdot))) - J(u^*(\cdot))}{\epsilon} \geq 0 \ .$$

Then we could follow Benssousan (1982) to prove that (16) holds. The proof is complete. □

6. Application to a bicriteria mean-variance problem with delay

In this section, we apply the second version of the sufficient maximum principle (i.e. Corollary 4.1) to discuss a bicriteria mean-variance portfolio selection problem with delay, where the objective is to maximize the return and minimize the risk at the same time. David (2008) considered a single-objective mean-variance problem (i.e. minimizing the variance of the terminal wealth for a given mean) under a delayed system. However, David (2008) only provided the optimal portfolio for a quadratic-loss minimization problem related to the single-objective mean-variance problem. The so-called single-objective mean-variance portfolio selection problem is indeed a constrained optimization problem. Using the Lagrangian duality approach can transform the constrained problem to an equivalent min-max problem without constraint. More importantly, the inner quadratic-loss minimization problem is time-consistent, which can be easily solved by either the dynamic programming principle or the stochastic maximum principle. The bicriteria mean-variance portfolio selection problem introduces the mean-variance tradeoff into the optimality criteria. Indeed, it is a multi-objective optimization problem, which is equivalent to a single-objective optimization problem with a weighted average of the two competing criteria. However, since the weighted average performance functional involves a nonlinear (quadratic) function of the expected term, even this equivalent single-objective optimization problem is time-inconsistent, where both Bellman's optimality principle and Pontryagin's maximum principle do not work. So the bicriteria mean-variance problem has long been investigated by the stochastic LQ theory with an ingenious embedding technique (see Zhou and Li, 2000). Only until recently, the stochastic maximum principle of mean-field type was found useful to solve the bicriteria mean-variance problem (see Anderson and Djehiche, 2011). It is worth mentioning that the time-inconsistency issue in the bicriteria problem motivates recent enthusiastic pursuit of time-consistent optimal strategies in time-inconsistent control problems. Our paper will consider the bicriteria mean-variance problem with delay and derive not only the efficient portfolio but also the efficient frontier for the problem.

Consider a continuous-time financial market where a risk-free bond and k risky shares are traded. The dynamics of the bond price process $\{S_0(t) | t \in \mathcal{T}\}$ evolves over time as

$$dS_0(t) = a(t) S_0(t) dt \ , \quad S_0(0) = 1 \ ,$$

where $a(t)$ is the risk-free interest rate at time t and $a : \mathcal{T} \to \Re$ is a uniformly bounded, deterministic function of t.

The price processes of the other k risky shares, $\{S_i(t) | t \in \mathcal{T}\}$, $i = 1, 2, \cdots, k$, are governed by the following SDEs:

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sigma_i(t)dW(t) + \int_{\Re_0} \gamma_i(t,\zeta)\widetilde{N}(dt, d\zeta)\right] \ , \quad S_i(0) = s_i > 0 \ ,$$

where $b_i(t)$, $\sigma_i(t)$ and $\gamma_i(t,\zeta)$ are the appreciation rate, the volatility vector and the jump ratio vector of the i-th share at time t and $b_i : \mathcal{T} \to \Re$, $\sigma_i : \mathcal{T} \to (0, +\infty)^d$, $\gamma_i : \mathcal{T} \times \Re_0 \to (-1, +\infty)^l$ are all uniformly bounded, deterministic functions of t. To exclude arbitrage opportunities, we assume that $b(t) > a(t)$, for each $t \in \mathcal{T}$.

Here W and $\widetilde{N}$ are the d-dimensional standard Brownian motion and the l-dimensional compensated Poisson random measure defined in Section 2. Write

$$\sigma(t) := \begin{pmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \vdots \\ \sigma_n(t) \end{pmatrix} \in \Re^{n\times d} \quad \text{and} \quad \gamma(t,\zeta) := \begin{pmatrix} \gamma_1(t,\zeta) \\ \gamma_2(t,\zeta) \\ \vdots \\ \gamma_n(t,\zeta) \end{pmatrix} \in \Re^{n\times l} \ ,$$

for the volatility matrix and the jump ratio matrix of risky shares, respectively. By convention, we assume that the following non-degeneracy condition is satisfied, i.e.

$$\Theta(t) := \sigma(t)\sigma(t)^\top + \int_{\Re_0} \gamma(t,\zeta)\mathrm{diag}(\nu(d\zeta))\gamma(t,\zeta)^\top \geq CI_n \ ,$$

for all $t \in \mathcal{T}$. Here C is a positive constant and I_n is the $(n\times n)$-identity matrix.

In what follows, we denote by $u_i(t)$, $i = 1, 2, \cdots, k$, the amount of an investor's wealth allocated in the i-th share at time t. We call $u(\cdot) := (u_1(\cdot), u_2(\cdot), \cdots, u_k(\cdot))^\top$ a portfolio strategy of the investor. Denote by $X(t) := X^u(t)$ the wealth process, i.e. the total wealth of the investor at time t corresponding to the portfolio strategy $u(\cdot)$. Then the amount of the wealth invested in the risk-free bond is $X(t) - \sum_{i=1}^k u_i(t)$. If we suppose that (1) the shares are infinitely divisible and can be traded continuously over time, (2) there are no transaction costs, taxes, and short-selling constraints in trading and (3) the trading strategies are self-financing, then the wealth process of the investor follows

$$dX(t) = [a(t)X(t) + u(t)^\top B(t)]dt + u(t)^\top \sigma(t)dW(t) + \int_{\Re_0} u(t)^\top \gamma(t,\zeta)\widetilde{N}(dt, d\zeta) \ , \tag{17}$$

where the risk premium vector is defined by

$$B(t) := \big(b_1(t) - a(t), b_2(t) - a(t), \cdots, b_k(t) - a(t)\big)^\top \ .$$

Note that (17) is a jump-diffusion SDE without delay and the mean-variance problem under this model can be solved using the classical stochastic sufficient maximum principle for control systems without delay (see, for example, Framstad et al., 2004). In what follows, we shall formulate a wealth process with delay, which may arise in various situations in practice.

With a little abuse of notation, the modified wealth process of the investor is still denoted by $X(\cdot)$. Suppose that the modified wealth process is governed by the following jump-diffusion stochastic delay differential equation:

$$\begin{cases} dX(t) = [\mu(t)X(t) + \alpha(t)Y(t) + \beta Z(t) + u(t)^\top B(t)]dt \\ \qquad\qquad + u(t)^\top \sigma(t)dW(t) + \displaystyle\int_{\Re_0} u(t)^\top \gamma(t,\zeta)\widetilde{N}(dt, d\zeta) \ , \quad t \in \mathcal{T} \ , \\ X(t) = x > 0 \ , \quad t \in [-\delta, 0] \ , \end{cases} \tag{18}$$

where $\mu(\cdot)$ and $\alpha(\cdot) : \mathcal{T} \to \Re$ are two uniformly bounded, deterministic function of t; $\beta \in \Re$ is a constant; and

$$Y(t) = \int_{-\delta}^0 e^{\lambda s} X(t+s)ds \ , \quad Z(t) = X(t-\delta) \ ,$$

which represent the average and pointwise delay information of the wealth process in the past period $[t-\delta, t]$, respectively. Here $X(t) = x > 0$, $t \in [-\delta, 0]$, can be interpreted that the investor is endowed with the initial wealth x at time $-\delta$, holds the wealth and invests nothing before time 0, and makes the investment of x at time 0.

Remark 6.1 ($\alpha(t)>0$ and $\beta>0$)**.** *In practice, there may exist capital outflow from or inflow into the investor's current wealth, which is related to the past investment performance of the wealth. For instance, individual investors are usually structurally inferior in the market due to lack of professional knowledge, information and experience. To improve investment performance, the investor may appoint some financial institutions, such as mutual funds and hedge funds, to manage his wealth. If that were the case, there might be capital outflow from the investor's wealth when paying performance-related reward to the asset manager. Other cases may cause capital outflow/inflow include hedonic consumption, cost-cutting and capital injection, and so on. To be more specific:*

(a) if the past investment performance is good, the investor will pay an incentive reward to the asset manager;
(b) if the past investment performance is good, the investor will use a part of his wealth for hedonic consumption;
(c) if the past investment performance is bad, the investor will cut unnecessary costs to save the loss and create the internal cash flow;
(d) if the past investment performance is bad, the investor will inject some money to make sure that the final investment goal is achievable.

Let us consider a sample linear capital outflow/inflow structure. Suppose that the instantaneous capital outflow/inflow at time t is given by

$$f_0(t,X(t)-Z(t),X(t)-Y(t))=\beta(X(t)-Z(t))+\alpha(t)(X(t)-Y(t))\ , \tag{19}$$

where the capital outflow/inflow accounts for both the absolute investment performance between time t and time $t-\delta$ and the average investment performance over the period $[t-\delta,t]$. Since $\alpha(t)>0$ and $\beta>0$, a capital outflow is expected as in Case (a) or (b) when the past investment performance is good. Whereas, a capital inflow corresponding to Case (c) or (d) is expected when the past investment performance is bad. Taking into account the capital outflow/inflow (19) in the original wealth process (17), we obtain the modified wealth process

$$\begin{aligned} dX(t) &= [a(t)X(t)+u(t)^\top B(t)-f_0(t,X(t)-Z(t),X(t)-Y(t))]dt \\ &\quad +u(t)^\top\sigma(t)dW(t)+\int_{\Re_0}u(t)^\top\gamma(t,\zeta)\widetilde{N}(dt,d\zeta) \\ &= [(a(t)-\alpha(t)-\beta)X(t)+\alpha(t)Y(t)+\beta Z(t)+u(t)^\top B(t)]dt \\ &\quad +u(t)^\top\sigma(t)dW(t)+\int_{\Re_0}u(t)^\top\gamma(t,\zeta)\widetilde{N}(dt,d\zeta)\ ,\quad t\in\mathcal{T}\ . \end{aligned} \tag{20}$$

Here we write $\mu(t):=a(t)-\alpha(t)-\beta$ and assume that $a(t)>\alpha(t)+\beta$ to ensure that $\mu(t)>0$, for each $t\in\mathcal{T}$.

Remark 6.2 ($\alpha(t)<0$ and $\beta<0$)**.** *On the other hand, rearranging (18) yields*

$$\begin{aligned} dX(t) &= \left\{a(t)X(t)+\sum_{i=1}^k u_i(t)\left[b_i(t)+\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}-a(t)\right]\right\}dt \\ &\quad +\sum_{i=1}^k u_i(t)\sigma_i(t)dW(t)+\sum_{i=1}^k\int_{\Re_0}u_i(t)\gamma_i(t,\zeta)\widetilde{N}(dt,d\zeta) \\ &= \frac{X(t)-\sum_{i=1}^k u_i(t)}{S_0(t)}dS_0(t)+\sum_{i=1}^k\frac{u_i(t)}{\widetilde{S}_i(t)}d\widetilde{S}_i(t)\ , \end{aligned} \tag{21}$$

where the modified share price processes $\{\widetilde{S}_i(t)|t\in\mathcal{T}\}$, $i=1,2,\cdots,k$, are defined by

$$\begin{aligned} d\widetilde{S}_i(t) &= \widetilde{S}_i(t)\left\{\left[b_i(t)+\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}\right]dt\right. \\ &\quad \left.+\sigma_i(t)dW(t)+\int_{\Re_0}\gamma_i(t,\zeta)\widetilde{N}(dt,d\zeta)\right\}\ ,\quad \widetilde{S}_i(0)=\ s_i>0\ , \end{aligned} \tag{22}$$

with $\sum_{i=1}^{k}\alpha_i(t)=-\alpha(t)$, $\sum_{i=1}^{k}\beta_i=-\beta$ *and* $\alpha_i(\cdot):\mathcal{T}\to\Re_+$, $\beta_i>0$*. This situation was related to the financial market with bounded memory considered in Chang et al. (2011). Although* $u_i(\cdot)$ *may take zero values, we require that* $\frac{0}{0}=1$ *by convention. So the terms* $\frac{u_i(t)}{u_i(t)}$ *in the first line of (21) still make sense even if* $u_i(t)=0$*, for some* i *and* t*. Thus, the modified wealth process can be considered as the wealth process of the investor who adopts the portfolio strategy* $u(\cdot)$ *in a modified financial market with bounded memory consisting of the risk-free bond* S_0 *and the modified risky shares* $\widetilde{S}_i$*.*

In practice, the investor may look at the performance of his wealth in the past before allocating the money into risky assets.

(i) If the investor is holding a long position in risky shares ($u_i(t)>0$)*, a good performance tends to drive the investor to buy more risky shares, and hence pushes the price of risky shares even higher, i.e.*

$$\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}>0\Longrightarrow b_i(t)+\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}>b_i(t)\ ;$$

(ii) on the contrary, the investor will buy less risky shares in case of a bad performance, which may pull down the price of risky shares, i.e.

$$\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}<0\Longrightarrow b_i(t)+\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}<b_i(t)\ .$$

(iii) If the investor is holding a short position in risky shares ($u_i(t)<0$)*, a good performance tends to drive the investor to short-sell more risky shares, and hence pulls down the price of risky shares even further, i.e.*

$$\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}<0\Longrightarrow b_i(t)+\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}<b_i(t)\ ;$$

(vi) on the contrary, the investor will short-sell less risky shares in case of a bad performance, which may drive the price of risky shares higher, i.e.

$$\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}>0\Longrightarrow b_i(t)+\alpha_i(t)\frac{X(t)-Y(t)}{u_i(t)}+\beta_i\frac{X(t)-Z(t)}{u_i(t)}>b_i(t)\ .$$

Remark 6.3. *In addition, we may naturally associate the modified share price processes (22) with the concept of 'large investor' (see Cvitanić and Ma, 1996), whose portfolio choices and wealth affects the appreciate rates of risky shares. The difference between our paper and existing literature is that both the current and the past wealth rather than only the current wealth have impacts on the appreciation rates. This may be more realistic in practice. The investor may rely on not only the absolute level of the wealth but also the relative change of the wealth to make investment decision. Indeed, it is not unreasonable to regard institutional investors with trillion-dollar assets under management (for example, asset managers: BlackRock and Fidelity, and insurers: AIG and Allianz, etc.) as large investors. Any decisions on portfolio choices of these investors may have significant impacts on asset prices in the market.*

We now consider a bicriteria mean-variance portfolio selection problem where the investor aims at minimizing the variance of the terminal wealth and the average wealth over the period $[T-\delta,T]$ while maximizing the expected return of them. Our performance functional is then given by

$$\begin{aligned} J(u(\cdot)) &= \frac{\eta}{2}\mathrm{Var}[X(T)+\theta Y(T)]-\mathbb{E}[X(T)+\theta Y(T)] \\ &= \mathbb{E}\left[\frac{\eta}{2}(X(T)+\theta Y(T))^2-(X(T)+\theta Y(T))\right]-\frac{\eta}{2}(\mathbb{E}[X(T)+\theta Y(T)])^2 \\ &= \mathbb{E}[g(X(T),\mathbb{E}[X(T)],Y(T),\mathbb{E}[Y(T)])]\ , \end{aligned} \tag{23}$$

where

$$g(x,\overline{x},y,\overline{y})=\frac{\eta}{2}(x+\theta y)^2-(x+\theta y)-\frac{\eta}{2}(\overline{x}+\theta\overline{y})^2\ . \tag{24}$$

Here the constant $\theta \in \Re$ is the weight between $X(T)$ and $Y(T)$, and the constant $\eta > 0$ is the weight balancing criteria of minimizing the variance and maximizing the mean. It is noted that using the weight θ in the performance functional, we incorporate both the terminal wealth $X(T)$ and the average (delayed) wealth $Y(T)$ over the period $[T-\delta, T]$ when defining the final mean-variance performance measure. However, if we only take into account the wealth at a single point, namely, the terminal wealth $X(T)$, it is very likely that the asset manager will be tempted to adopt short-term risk-taking behavior so as to manipulate the final performance measure and achieve shining performance at an instant. Such short-term risk-taking behavior may harm sustainable operation of the investment fund. Incorporating the average wealth $Y(T)$ into the final performance measure can shift the asset manager's focus from a single point to a period, hence alleviating the imprudent short-term risk-taking behavior and resolving the principal-agent problem (please refer to Remark 6.5 below for more details about the principal-agent problem). Finding a portfolio $u^*(\cdot) \in \mathcal{A}$ that minimizes the performance functional (23) is referred to as the bicriteria mean-variance portfolio selection problem with delay, where the admissible set $\mathcal{A}$ is defined in Definition 3.1. In particular, we formulate the problem as follows:

Definition 6.1. *The bicriteria mean-variance portfolio selection problem with delay is the following stochastic optimal control problem*

$$\begin{cases} \textit{minimize} \quad J(u(\cdot)) = \mathbb{E}[g(X(T), \mathbb{E}[X(T)], Y(T), \mathbb{E}[Y(T)])] \ , \\ \textit{subject to} \quad \begin{cases} (X(\cdot), u(\cdot)) \ \textit{satisfy} \ (18) \ , \\ u(\cdot) \in \mathcal{A} \ . \end{cases} \end{cases} \tag{25}$$

Although the controlled state equation (18) is not of mean-field type, the performance functional (23) is apparently of mean-field type. So the problem (25) is a special mean-field control problem, which raises time-inconsistency issue in the problem. Therefore, Bellman's dynamic programming principle does not work here. To overcome this difficulty, we tailor the sufficient maximum principle II to solve the problem (25) in the rest of this section.

Since g is not convex in $\overline{x}$ and $\overline{y}$, Condition 2 in Theorem 4.1 is not satisfied. So the sufficient maximum principle I can not be applied to the problem. Using the necessary maximum principle, Andersson and Djehiche (2011) obtained a candidate of the optimal control in a simplified control system without jump and delay. Although this candidate of the optimal control coincided with the optimal portfolio strategy found in Zhou and Li (2000)'s pioneering work, only the sufficient condition for optimality can verify that the candidate is indeed an optimal control of the problem. This motivates us to investigate again whether there exists a version of the sufficient maximum principle, which can be used to solve the bicriteria mean-variance problem. Indeed, relaxing the convexity condition of the terminal cost g to an expected sense, we can apply the second version of the sufficient maximum principle (Corollary 4.1) to solve the problem (25) completely.

First of all, we verify that Conditions 1 and 2 in Corollary 4.1 are satisfied. From (6), the Hamiltonian of the bicriteria mean-variance problem has the following form

$$\begin{aligned} \mathcal{H}(t,x,y,z,u,p,q,r) \quad = \quad & [\mu(t)x+\alpha(t)y+\beta z+u^\top B(t)]p_1+[x-\lambda y-e^{-\lambda\delta}z]p_2 \\ & +u^\top \sigma(t)q_1^\top + \int_{\Re_0} u^\top \gamma(t,\zeta)\mathrm{diag}(\nu(d\zeta))r_1(t,\zeta)^\top \ . \end{aligned} \tag{26}$$

It is clear that the Hamiltonian is a linear function of (x, y, z, u), and thereby is convex in (x, y, z, u). Furthermore, for any $X_1, X_2, Y_1, Y_2 \in \mathcal{L}^2(\mathcal{F}(T); \Re)$, it can be verified that

$$\begin{aligned} & \mathbb{E}\big[(X_1-X_2)\big\{\triangledown_x g(X_2,\overline{X}_2,Y_2,\overline{Y}_2)+\mathbb{E}[\triangledown_{\overline{x}} g(X_2,\overline{X}_2,Y_2,\overline{Y}_2)]\big\}\big] \\ & \quad +\mathbb{E}\big[(Y_1-Y_2)\big\{\triangledown_y g(X_2,\overline{X}_2,Y_2,\overline{Y}_2)+\mathbb{E}[\triangledown_{\overline{y}} g(X_2,\overline{X}_2,Y_2,\overline{Y}_2)]\big\}\big] \\ & = \mathbb{E}\big[g(X_1,\overline{X}_1,Y_1,\overline{Y}_1)-g(X_2,\overline{X}_2,Y_2,\overline{Y}_2)\big]-\frac{\eta}{2}\mathrm{Var}\big[(X_1+\theta Y_1)-(X_2+\theta Y_2)\big] \\ & \leq \mathbb{E}\big[g(X_1,\overline{X}_1,Y_1,\overline{Y}_1)-g(X_2,\overline{X}_2,Y_2,\overline{Y}_2)\big] \ . \end{aligned}$$

With the Hamiltonian given by (26), the system of the adjoint equations becomes

$$\begin{cases} dp_1(t) = -[\mu(t)p_1(t)+p_2(t)]dt+q_1(t)dW(t)+\int_{\Re_0} r_1(t,\zeta)\widetilde{N}(dt,d\zeta) \ , \\ p_1(T) = \eta\{(X(T)-\mathbb{E}[X(T)])+\theta(Y(T)-\mathbb{E}[Y(T)])\}-1 \ , \end{cases} \tag{27}$$

$$\begin{cases} dp_2(t) = -[\alpha(t)p_1(t)-\lambda p_2(t)]dt+q_2(t)dW(t)+\int_{\Re_0} r_2(t,\zeta)\widetilde{N}(dt,d\zeta) \ , \\ p_2(T) = \theta\Big(\eta\{(X(T)-\mathbb{E}[X(T)])+\theta(Y(T)-\mathbb{E}[Y(T)])\}-1\Big) \ , \end{cases} \tag{28}$$

and

$$\begin{cases} dp_3(t) = -[\beta p_1(t)-e^{-\lambda\delta}p_2(t)]dt+q_3(t)dW(t)+\int_{\Re_0} r_3(t,\zeta)\widetilde{N}(dt,d\zeta) \ , \\ p_3(T) = 0 \ . \end{cases} \tag{29}$$

Let $u^*(\cdot) \in \mathcal{A}$ be an control process such that Condition 3 in Corollary 4.1 holds. Denote by $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ and $p^*(\cdot) = (p_1^*(\cdot), p_2^*(\cdot), p_3^*(\cdot))$, $q^*(\cdot) = (q_1^*(\cdot), q_2^*(\cdot), q_3^*(\cdot))$, $r^*(\cdot,\cdot) = (r_1^*(\cdot,\cdot), r_2^*(\cdot,\cdot), r_3^*(\cdot,\cdot))$ the corresponding state processes and adjoint processes, respectively. So applying the first order condition to the Hamiltonian with respect to u gives

$$B(t)p_1^*(t)+\sigma(t)q_1^*(t)^\top+\int_{\Re_0}\gamma(t,\zeta)\mathrm{diag}(\nu(d\zeta))r_1^*(t,\zeta)^\top = 0 \ . \tag{30}$$

Using Condition 4 in Corollary 4.1 (i.e. $(p_3^*(t), q_3^*(t), r_3^*(t,\cdot)) = (0,0,0)$) to (29) implies that

$$\beta p_1^*(t)-e^{-\lambda\delta}p_2^*(t) = 0 \ , \quad \forall t \in \mathcal{T} \ . \tag{31}$$

Setting $t = T$ in (31), we obtain that

$$\theta = \beta e^{\lambda\delta} \ .$$

Then Conditions 1-4 in Corollary 4.1 are all satisfied. Therefore, $u^*(\cdot)$ is the optimal control process. In what follows, we derive a closed-form expression for the efficient portfolio $u^*(\cdot)$ via solving the system of the adjoint equations (27)-(29).

Define a set of transformed processes $\{\widehat{p}_2^*(t)|t \in \mathcal{T}\}$, $\{\widehat{q}_2^*(t)|t \in \mathcal{T}\}$, $\{\widehat{r}_2^*(t,\zeta)|(t,\zeta) \in \mathcal{T}\times\Re_0\}$ by putting

$$\widehat{p}_2^*(t) = \beta^{-1}e^{-\lambda\delta}p_2^*(t) \ , \quad \widehat{q}_2^*(t) = \beta^{-1}e^{-\lambda\delta}q_2^*(t) \ , \quad \widehat{r}_2^*(t) = \beta^{-1}e^{-\lambda\delta}r_2^*(t) \ . \tag{32}$$

Then, the adjoint equations (27) and (28) associated with $u^*(\cdot)$ can be reformulated as

$$\begin{cases} dp_1^*(t) = -[\mu(t)p_1^*(t)+\beta e^{\lambda\delta}\widehat{p}_2^*(t)]dt+q_1^*(t)dW(t)+\int_{\Re_0} r_1^*(t,\zeta)\widetilde{N}(dt,d\zeta) \ , \\ p_1^*(T) = \eta\{(X^*(T)-\mathbb{E}[X^*(T)])+\theta(Y^*(T)-\mathbb{E}[Y^*(T)])\}-1 \ , \end{cases} \tag{33}$$

and

$$\begin{cases} d\widehat{p}_2^*(t) = -[\alpha(t)\beta^{-1}e^{-\lambda\delta}p_1^*(t)-\lambda\widehat{p}_2^*(t)]dt+\widehat{q}_2^*(t)dW(t)+\int_{\Re_0} \widehat{r}_2^*(t,\zeta)\widetilde{N}(dt,d\zeta) \ , \\ \widehat{p}_2^*(T) = \eta\{(X^*(T)-\mathbb{E}[X^*(T)])+\theta(Y^*(T)-\mathbb{E}[Y^*(T)])\}-1 \ . \end{cases} \tag{34}$$

From (31) and (32), we can see that $p_1^*(t) = \widehat{p}_2^*(t)$, for each $t \in \mathcal{T}$. So,

$$\mu(t)+\beta e^{\lambda\delta} = \alpha(t)\beta^{-1}e^{-\lambda\delta}-\lambda \ ,$$

or

$$\alpha(t)=\beta e^{\lambda\delta}[\mu(t)+\beta e^{\lambda\delta}+\lambda]\ .$$

Evidently, both (33) and (34) are equivalent to

$$\begin{cases} dp_1^*(t)=-[\mu(t)+\beta e^{\lambda\delta}]p_1^*(t)dt+q_1^*(t)dW(t)+\int_{\Re_0} r_1^*(t,\zeta)\widetilde{N}(dt,d\zeta)\ , \\ p_1^*(T)=\eta\{(X^*(T)-\mathbb{E}[X^*(T)])+\theta(Y^*(T)-\mathbb{E}[Y^*(T)])\}-1\ . \end{cases} \tag{35}$$

From the terminal condition of (35), we try the following solution

$$p_1^*(t)=\varphi(t)\{(X^*(t)-\mathbb{E}[X^*(t)])+\beta e^{\lambda\delta}(Y^*(t)-\mathbb{E}[Y^*(t)])\}+\phi(t)\ . \tag{36}$$

where $\varphi:\mathcal{T}\to\Re$ and $\phi:\mathcal{T}\to\Re$ are two deterministic functions of t. Let φ_t and ϕ_t denote derivatives of φ and ϕ with respect to t, respectively.

Applying Itô's differentiation rule to $p_1^*(t)$ gives

$$\begin{aligned} dp_1^*(t) &= \Big\{\varphi(t)\Big[\mu(t)(X^*(t)-\mathbb{E}[X^*(t)])+\alpha(t)(Y^*(t)-\mathbb{E}[Y^*(t)])+\beta(Z^*(t)-\mathbb{E}[Z^*(t)]) \\ &\quad +(u^*(t)-\mathbb{E}[u^*(t)])^\top B(t)\Big]+\varphi(t)\beta e^{\lambda\delta}\Big[X^*(t)-\lambda Y^*(t)-e^{-\lambda\delta}Z^*(t) \\ &\quad -\mathbb{E}[X^*(t)-\lambda Y^*(t)-e^{-\lambda\delta}Z^*(t)]\Big]+\varphi_t(t)\Big[(X^*(t)-\mathbb{E}[X^*(t)])+\beta e^{\lambda\delta}(Y^*(t)-\mathbb{E}[Y^*(t)])\Big] \\ &\quad +\phi_t(t)\Big\}dt+\varphi(t)u^*(t)^\top\sigma(t)dW(t)+\varphi(t)\int_{\Re_0} u^*(t)^\top\gamma(t,\zeta)\widetilde{N}(dt,d\zeta) \\ &= \Big\{\varphi(t)\Big[\mu(t)(X^*(t)-\mathbb{E}[X^*(t)])+\alpha(t)(Y^*(t)-\mathbb{E}[Y^*(t)])+(u^*(t)-\mathbb{E}[u^*(t)])^\top B(t)\Big] \\ &\quad +\varphi(t)\beta e^{\lambda\delta}\Big[(X^*(t)-\mathbb{E}[X^*(t)])-\lambda(Y^*(t)-\mathbb{E}[Y^*(t)])\Big] \\ &\quad +\varphi_t(t)\Big[(X^*(t)-\mathbb{E}[X^*(t)])+\beta e^{\lambda\delta}(Y^*(t)-\mathbb{E}[Y^*(t)])\Big]+\phi_t(t)\Big\}dt \\ &\quad +\varphi(t)u^*(t)^\top\sigma(t)dW(t)+\varphi(t)\int_{\Re_0} u^*(t)^\top\gamma(t,\zeta)\widetilde{N}(dt,d\zeta)\ . \end{aligned} \tag{37}$$

Comparing the coefficients of (35) with those of (37), we must have

$$\begin{aligned} &-[\mu(t)+\beta e^{\lambda\delta}]\Big\{\varphi(t)\Big[(X^*(t)-\mathbb{E}[X^*(t)])+\beta e^{\lambda\delta}(Y^*(t)-\mathbb{E}[Y^*(t)])\Big]+\phi(t)\Big\} \\ &=\varphi(t)\Big[\mu(t)(X^*(t)-\mathbb{E}[X^*(t)])+\alpha(t)(Y^*(t)-\mathbb{E}[Y^*(t)])+(u^*(t)-\mathbb{E}[u^*(t)])^\top B(t)\Big] \\ &\quad +\varphi(t)\beta e^{\lambda\delta}\Big[(X^*(t)-\mathbb{E}[X^*(t)])-\lambda(Y^*(t)-\mathbb{E}[Y^*(t)])\Big] \\ &\quad +\varphi_t(t)\Big[(X^*(t)-\mathbb{E}[X^*(t)])+\beta e^{\lambda\delta}(Y^*(t)-\mathbb{E}[Y^*(t)])\Big]+\phi_t(t)\ , \end{aligned} \tag{38}$$

and

$$q_1^*(t)=\varphi(t)u^*(t)^\top\sigma(t)\ ,\quad r_1^*(t,\zeta)=\varphi(t)u^*(t)^\top\gamma(t,\zeta)\ . \tag{39}$$

Substituting (36) and (39) into (30), we obtain the efficient portfolio

$$u^*(t) = -\Theta(t)^{-1}B(t)\left[(X^*(t){-}\mathbb{E}[X^*(t)]){+}\beta e^{\lambda\delta}(Y^*(t){-}\mathbb{E}[Y^*(t)]){+}\frac{\phi(t)}{\varphi(t)}\right] . \tag{40}$$

It is easy to see that the expected value of the efficient portfolio is given by

$$\mathbb{E}[u^*(t)] = -\Theta(t)^{-1}B(t)\frac{\phi(t)}{\varphi(t)} . \tag{41}$$

Denote by $\rho(t) := B(t)^\top\Theta(t)^{-1}B(t)$, for each $t \in \mathcal{T}$. Substituting (40) into (38) and setting the coefficients of $X^*(t){-}\mathbb{E}[X^*(t)]$ and $Y^*(t){-}\mathbb{E}[Y^*(t)]$ to be zeros yield that

$$\begin{cases} \varphi_t(t){+}[2(\mu(t){+}\beta e^{\lambda\delta}){-}\rho(t)]\varphi(t) = 0 , \\ \varphi(T) = \eta , \end{cases}$$

and

$$\begin{cases} \phi_t(t){+}[\mu(t){+}\beta e^{\lambda\delta}]\phi(t) = 0 , \\ \phi(T) = -1 . \end{cases}$$

Solving gives

$$\varphi(t) = \eta \exp\left\{\int_t^T [2(\mu(s){+}\beta e^{\lambda\delta}){-}\rho(s)]ds\right\} ,$$

and

$$\phi(t) = -\exp\left\{\int_t^T [\mu(s){+}\beta e^{\lambda\delta}]ds\right\} .$$

Next we derive an explicit expression for the efficient frontier of the problem. By Itô's formula,

$$\begin{aligned} d(X^*(t){+}\theta Y^*(t)) &= [(\mu(t){+}\theta)(X^*(t){+}\theta Y^*(t)){+}u^*(t)^\top B(t)]dt \\ &\quad {+}u^*(t)^\top\sigma(t)dW(t){+}\int_{\Re_0} u^*(t)^\top\gamma(t,\zeta)\widetilde{N}(dt,d\zeta) . \end{aligned}$$

Taking expectations on both sides leads to

$$\begin{aligned} d\mathbb{E}[X^*(t){+}\theta Y^*(t)] &= \{(\mu(t){+}\theta)\mathbb{E}[X^*(t){+}\theta Y^*(t)]{+}\mathbb{E}[u^*(t)^\top]B(t)\}dt \\ &= \left\{(\mu(t){+}\theta)\mathbb{E}[X^*(t){+}\theta Y^*(t)]{-}\rho(t)\frac{\phi(t)}{\varphi(t)}\right\}dt , \end{aligned}$$

where the initial value is

$$\mathbb{E}[X^*(0){+}\theta Y^*(0)] = x\left[1{+}\frac{\theta}{\lambda}(1{-}e^{-\lambda\delta})\right] .$$

Solving gives

$$\begin{aligned} \mathbb{E}[X^*(t){+}\theta Y^*(t)] &= x\left[1{+}\frac{\theta}{\lambda}(1{-}e^{-\lambda\delta})\right]e^{\int_0^t[\mu(s)+\theta]ds}{-}\int_0^t e^{\int_0^s[\mu(v)+\theta]dv}\rho(s)\frac{\phi(s)}{\varphi(s)}ds \\ &= x\left[1{+}\frac{\theta}{\lambda}(1{-}e^{-\lambda\delta})\right]e^{\int_0^t[\mu(s)+\theta]ds}{+}\frac{1}{\eta}e^{\int_t^T[\rho(s)-(\mu(s)+\theta)]ds}\left[e^{\int_0^t\rho(s)ds}{-}1\right] . \end{aligned} \tag{42}$$

Again by Itô's formula,

$$
\begin{aligned}
& d\Big(X^*(t)+\theta Y^*(t)-\mathbb{E}[X^*(t)+\theta Y^*(t)]\Big)^2 \\
&= \Big\{2(\mu(t)+\theta)\Big(X^*(t)+\theta Y^*(t)-\mathbb{E}[X^*(t)+\theta Y^*(t)]\Big)^2 \\
&\quad +2\Big(X^*(t)+\theta Y^*(t)-\mathbb{E}[X^*(t)+\theta Y^*(t)]\Big)(u^*(t)-\mathbb{E}[u^*(t)])^\top B(t)+u^*(t)^\top \Theta(t)u^*(t)\Big\}dt \\
&\quad +2\Big(X^*(t)+\theta Y^*(t)-\mathbb{E}[X^*(t)+\theta Y^*(t)]\Big)\Big[u^*(t)^\top\sigma(t)dW(t)+\int_{\Re_0}u^*(t)^\top\gamma(t,\zeta)\widetilde{N}(dt,d\zeta)\Big] .
\end{aligned}
$$

Again taking expectations on both sides leads to

$$
d\mathrm{Var}[X^*(t)+\theta Y^*(t)] = \Big\{[2(\mu(t)+\theta)-\rho(t)]\mathrm{Var}[X^*(t)+\theta Y^*(t)]+\rho(t)\frac{\phi^2(t)}{\varphi^2(t)}\Big\}dt \ , \tag{43}
$$

where the initial value is given by

$$
\mathrm{Var}[X^*(0)+\theta Y^*(0)] = 0 \ .
$$

Then the solution of (43) is

$$
\begin{aligned}
\mathrm{Var}[X^*(t)+\theta Y^*(t)] &= \int_0^t e^{\int_0^s[2(\mu(v)+\theta)-\rho(v)]dv}\rho(s)\frac{\phi^2(s)}{\varphi^2(s)}ds \\
&= \frac{1}{\eta^2}e^{\int_t^T[\rho(s)-2(\mu(s)+\theta)]ds}\Big[e^{\int_0^T\rho(s)ds}-e^{\int_t^T\rho(s)ds}\Big] \ ,
\end{aligned} \tag{44}
$$

Comparing the values of the mean (42) and the variance (44) at time $t=T$ results in

$$
\begin{aligned}
\frac{\mathrm{Var}[X^*(T)+\theta Y^*(T)]}{\Big\{\mathbb{E}[X^*(T)+\theta Y^*(T)]-x\Big[1+\frac{\theta}{\lambda}(1-e^{-\lambda\delta})\Big]e^{\int_0^T[\mu(s)+\theta]ds}\Big\}^2} &= \frac{1}{e^{\int_0^T\rho(s)ds}-1} \\
&= \frac{e^{-\int_0^T\rho(s)ds}}{1-e^{-\int_0^T\rho(s)ds}} \ .
\end{aligned}
$$

Consequently, it follows immediately that the efficient frontier of the problem is given by

$$
\mathrm{Var}[X^*(T)+\theta Y^*(T)] = \frac{e^{-\int_0^T\rho(s)ds}}{1-e^{-\int_0^T\rho(s)ds}}\Big\{\mathbb{E}[X^*(T)+\theta Y^*(T)]-x\Big[1+\frac{\theta}{\lambda}(1-e^{-\lambda\delta})\Big]e^{\int_0^T[\mu(s)+\theta]ds}\Big\}^2 \ ,
$$

or

$$
\mathbb{E}[X^*(T)+\theta Y^*(T)] = x\Big[1+\frac{\theta}{\lambda}(1-e^{-\lambda\delta})\Big]e^{\int_0^T[\mu(s)+\theta]ds}+\sqrt{\frac{1-e^{-\int_0^T\rho(s)ds}}{e^{-\int_0^T\rho(s)ds}}}\sqrt{\mathrm{Var}[X^*(T)+\theta Y^*(T)]} \ .
$$

Remark 6.4. *The dynamic mean-variance portfolio selection problem has been well explored using different methods. Please see Zhou and Li (2000) for the stochastic linear-quadratic method, Framstad et al. (2004) for the maximum principle, Shen and Siu (2013) for the mean-field maximum principle. Although the existing literature has provided mathematically elegant results for the mean-variance problem, the focus is still on financial models without delay. The mean-variance problem with delay is an open problem. Our paper is the first to find both the efficient portfolio and the efficient frontier under a jump-diffusion model with delay. It is worth mentioning that if there is no jump and delay, the solution we obtained for the bicriteria mean-variance problem coincides with that obtained by Zhou and Li (2000).*

Remark 6.5. *In the bicriteria mean-variance problem, we find the following conditions*

$$\theta = \beta e^{\lambda\delta} \ ,$$

and

$$\alpha(t) = \beta e^{\lambda\delta}[\mu(t)+\beta e^{\lambda\delta}+\lambda] \ ,$$

under which the third adjoint equation admits a zero solution. This makes the problem finite-dimensional and hence solvable. Note that $\mu(t) = a(t)-\alpha(t)-\beta$*, for each* $t \in \mathcal{T}$*. Here the investor and his asset manager can be considered as the principal and the agent, respectively. The best interests of the investor is to maximize the return while minimize the risk. Parameters including* θ*,* λ*,* δ*,* β *and* α *are exogenously predetermined by the investor such that the asset manager will act in his best interests. That is, the investor first select the averaging parameter* λ *and the delay time* δ *to calculate the delayed wealth; the investor next choose the weight* θ *between* X *and* Y *in the final mean-variance performance measure of his wealth, where* $\theta \neq -1$*; finally he must set* $\beta = \theta e^{-\lambda\delta}$ *and* $\alpha(t) = \frac{\theta}{1+\theta}[a(t)-\beta+\theta+\lambda]$ *as the weights proportional to* $X(t)-Y(t)$ *and* $X(t)-Z(t)$ *and adjust capital outflow/inflow (see Remark 6.1) according to the past investment performance. Otherwise, the principal (investor) may face up to the agency dilemma that the agent (asset manager) is unable to act in the investor's best interests to achieve the efficient frontier of the investment portfolio. This is one possible rationale for such conditions. Indeed, they are considered as mechanisms to help the investor overcome the principal-agent problem and reduce the agency costs to some extent. Note that we only consider the control problem of the agent and take* θ*,* λ*,* δ*,* β *and* α *as exogenously determined. It may be interesting to investigate the control problem of the principal. In that case, some parameters among* θ*,* λ*,* δ*,* β *and* α *should be considered as control variables, whose optimal values together with the optimal portfolio strategy is endogenously solved through the principal-agent problem.*

7. Discussions on the solvability of the problem

Inspired by our financial example, in this section, we discuss the solvability of the control problem with delay. We now return to the performance functional given by (4). In general, the optimal control problems are infinite-dimensional since the value function may depend on the initial path in a complicated way. The conditions under which optimal control problems with delay become finite-dimensional have been extensively discussed in the dynamic programming principle approach (see, for example, Elsanosi et al., 2000, Lassen and Risebro, 2003, David, 2008, Shi, 2013 and etc.). Simply speaking, to make the problems finite-dimensional, it is required that the value function depends only on the initial path $x_0(\cdot)$ through the following two functionals

$$x = X(x_0(\cdot)) = x_0(0) \ , \quad y = Y(x_0(\cdot)) = \int_{-\delta}^{0} e^{\lambda s} x_0(s) ds \ ,$$

rather than the whole initial path.

In our maximum principles, a zero solution assumption is imposed for the third adjoint equation (9). However, the control system given by (3)-(4) may not guarantee that the third adjoint equation (9) has a zero solution. In what follows, we provide a sufficient condition to ensure that the third adjoint equation (9) has a zero solution and the control problem is finite-dimensional. Consider the following coefficients

$$\begin{aligned}
b(t,x,y,z,\overline{x},\overline{y},\overline{z},u) &= \mu(t,x+\beta e^{\lambda\delta}y,\overline{x}+\beta e^{\lambda\delta}\overline{y},u)-\beta e^{\lambda\delta}(x-\lambda y-e^{-\lambda\delta}z) \ , && (45)\\
\sigma(t,x,y,z,\overline{x},\overline{y},\overline{z},u) &= \widetilde{\sigma}(t,x+\beta e^{\lambda\delta}y,\overline{x}+\beta e^{\lambda\delta}\overline{y},u) \ , && (46)\\
\gamma(t,\zeta,x,y,z,\overline{x},\overline{y},\overline{z},u) &= \widetilde{\gamma}(t,\zeta,x+\beta e^{\lambda\delta}y,\overline{x}+\beta e^{\lambda\delta}\overline{y},u) \ , && (47)\\
f(t,x,y,z,\overline{x},\overline{y},\overline{z},u) &= \widetilde{f}(t,x+\beta e^{\lambda\delta}y,\overline{x}+\beta e^{\lambda\delta}\overline{y},u) \ , && (48)\\
g(x,y,\overline{x},\overline{y}) &= \widetilde{g}(x+\beta e^{\lambda\delta}y,\overline{x}+\beta e^{\lambda\delta}\overline{y}) \ , && (49)
\end{aligned}$$

where $\beta \in \Re$ is a constant; $\mu, \widetilde{\sigma}, \widetilde{f}, \widetilde{\gamma}, \widetilde{g}$ are functions of appropriate dimensions and satisfy suitable measurability conditions such that Assumptions (A6)-(A9) holds (please refer to Section 3). Denote by

$$\begin{aligned}\rho(t) &:= \rho(t, X(t)+\beta e^{\lambda\delta}Y(t), \overline{X}(t)+\beta e^{\lambda\delta}\overline{Y}(t), u(t)) \ , \quad \rho = \mu, \widetilde{\sigma}, \widetilde{f}, \widetilde{\gamma} \ , \\ \widetilde{g}(T) &:= \widetilde{g}(X(T)+\beta e^{\lambda\delta}Y(T), \overline{X}(T)+\beta e^{\lambda\delta}\overline{Y}(T)) \ ,\end{aligned}$$

and by $\bigtriangledown_{xy}$ and $\bigtriangledown_{\overline{xy}}$ the partial derivatives with respect to the arguments evaluated at $X(t)+\beta e^{\lambda\delta}Y(t)$ and $\overline{X}(t)+\beta e^{\lambda\delta}\overline{Y}(t)$, respectively. It is not difficult to see that the first triple of adjoint processes $(p_1(\cdot), q_1(\cdot), r_1(\cdot,\cdot))$ is governed by the following mean-field jump-diffusion BSDE:

$$\begin{cases} dp_1(t) = -\Big\{ \bigtriangledown_{xy}\mu(t)^\top p_1(t) + \mathrm{tr}[\bigtriangledown_{xy}\widetilde{\sigma}(t)^\top q_1(t)] + \int_{\Re_0} \mathrm{tr}[\bigtriangledown_{xy}\widetilde{\gamma}(t,\zeta)\mathrm{diag}(\nu(d\zeta))r_1(t,\zeta)^\top] \\ \qquad + \mathbb{E}\Big[\bigtriangledown_{\overline{xy}}\mu(t)^\top p_1(t) + \mathrm{tr}[\bigtriangledown_{\overline{xy}}\widetilde{\sigma}(t)^\top q_1(t)] + \int_{\Re_0} \mathrm{tr}[\bigtriangledown_{\overline{xy}}\widetilde{\gamma}(t,\zeta)\mathrm{diag}(\nu(d\zeta))r_1(t,\zeta)^\top] \Big] \Big\} dt \\ \qquad + q_1(t)dW(t) + \int_{\Re_0} r_1(t,\zeta)\widetilde{N}(dt,d\zeta) \ , \\ p_1(T) = \bigtriangledown_{xy}\widetilde{g}(T) + \mathbb{E}[\bigtriangledown_{\overline{xy}}\widetilde{g}(T)] \ , \end{cases} \tag{50}$$

the second triple of adjoint processes satisfies $(p_2(\cdot), q_2(\cdot), r_2(\cdot,\cdot)) = \beta e^{\lambda\delta}(p_1(\cdot), q_1(\cdot), r_1(\cdot,\cdot))$, and the third triple of adjoint processes is $(p_3(\cdot), q_3(\cdot), r_3(\cdot,\cdot)) = (0,0,0)$. From Lemma 2.2, $(p_i(\cdot), q_i(\cdot), r_i(\cdot,\cdot))$ are unique solutions to the three adjoint equations, for each $i = 1,2,3$. Therefore, under (45)-(49), the third adjoint equation (9) always has a zero solution.

To see why the control problem becomes finite-dimensional, we regard $X(\cdot)+\beta e^{\lambda\delta}Y(\cdot)$ as the state process. Then our control system can be transformed to the following state process and performance functional

$$\begin{cases} d(X(t)+\beta e^{\lambda\delta}Y(t)) = \mu(t, X(t)+\beta e^{\lambda\delta}Y(t), \overline{X}(t)+\beta e^{\lambda\delta}\overline{Y}(t), u(t))dt \\ \qquad + \widetilde{\sigma}(t, X(t)+\beta e^{\lambda\delta}Y(t), \overline{X}(t)+\beta e^{\lambda\delta}\overline{Y}(t), u(t))dW(t) \\ \qquad + \int_{\Re_0} \widetilde{\gamma}(t,\zeta, X(t)+\beta e^{\lambda\delta}Y(t), \overline{X}(t)+\beta e^{\lambda\delta}\overline{Y}(t), u(t))\widetilde{N}(dt,d\zeta) \ , \quad t \in \mathcal{T} \ , \\ X(0)+\beta e^{\lambda\delta}Y(0) = x_0(0)+\beta e^{\lambda\delta}\int_{-\delta}^0 e^{\lambda s}x_0(s)ds \ , \end{cases} \tag{51}$$

and

$$\begin{aligned} J(u(\cdot)) &= \mathrm{E}\Big[\int_0^T \widetilde{f}(t, X(t)+\beta e^{\lambda\delta}Y(t), \overline{X}(t)+\beta e^{\lambda\delta}\overline{Y}(t), u(t))dt \\ &\quad + \widetilde{g}(X(T)+\beta e^{\lambda\delta}Y(T), \overline{X}(T)+\beta e^{\lambda\delta}\overline{Y}(T)) \Big] \ . \end{aligned} \tag{52}$$

So the transformed control system is a mean-field jump-diffusion control system without delay and the control problem is finite-dimensional. Then classical control problems, such as the stochastic linear quadratic control, the investment-consumption problem and the mean-variance portfolio selection problem can be solved under the transformed control system. Interested readers may refer to Shen and Siu (2013) for the stochastic maximum principles under the mean-field jump-diffusion control systems without delay as (51)-(52).

8. Conclusion

We investigated the sufficient and necessary stochastic maximum principles for a mean-field jump-diffusion SDDE. The sufficient maximum principle was applied to solve a bicriteria mean-variance portfolio selection problem with delay and of mean-field type. Although the mean-variance problem in our paper is much more complicated than the classical one, we still obtained closed-form expressions for the efficient portfolio and the efficient frontier. A sufficient condition under which the control problem with delay becomes finite-dimensional was also provided.

Appendix

Proof of Lemma 2.1. Consider $\mathscr{P}$-measurable functions $b_0 : \Omega\times\mathcal{T} \to \Re^n$, $\sigma_0 : \Omega\times\mathcal{T} \to \Re^{n\times d}$ and $\mathscr{P}\otimes\mathcal{B}(\Re_0)$-measurable function $\gamma_0 : \Omega\times\mathcal{T}\times\Re_0 \to \Re^{n\times l}$ such that $b_0(\cdot) \in \mathcal{L}^2(0, T; \Re^n)$, $\sigma_0(\cdot) \in \mathcal{L}^2(0, T; \Re^{n\times d})$ and $\gamma_0(\cdot, \cdot) \in \mathcal{L}^2_\nu(0, T; \Re^{n\times l})$. It is clear that the following ordinary jump-diffusion SDE

$$\begin{cases} dX(t) = b_0(t)dt+\sigma_0(t)dW(t)+\displaystyle\int_{\Re_0} \gamma_0(t, \zeta)\widetilde{N}(dt, d\zeta) \ , \\ X(t) = x_0(t) \ , \quad t \in [-\delta, 0] \ , \quad x_0(t) \in \mathcal{C}(-\delta, 0; \Re^n) \ , \end{cases} \tag{A1}$$

has a unique adapted solution $X(\cdot) \in \mathcal{S}^2(0, T; \Re^n)$. So for any given $\mathbb{F}$-adapted random process $x(\cdot) \in \mathcal{S}^2(0, T; \Re^n)$ with $x(t) = x_0(t)$, $\forall t \in [-\delta, 0]$, the following ordinary jump-diffusion SDE

$$\begin{cases} dX(t) = b(t, x(t), y(t), z(t), \overline{x}(t), \overline{y}(t), \overline{z}(t))dt \\ \qquad\qquad +\sigma(t, x(t), y(t), z(t), \overline{x}(t), \overline{y}(t), \overline{z}(t))dW(t) \\ \qquad\qquad +\displaystyle\int_{\Re_0} \gamma(t, \zeta, x(t), y(t), z(t), \overline{x}(t), \overline{y}(t), \overline{z}(t))\widetilde{N}(dt, d\zeta) \ , \\ X(t) = x_0(t) \ , \quad t \in [-\delta, 0] \ , \end{cases} \tag{A2}$$

where

$$y(t) := \int_{-\delta}^0 e^{\lambda s}x(t+s)ds \ , \quad z(t) := x(t-\delta) \ ,$$

and

$$\overline{x}(t) := \mathbb{E}[x(t)] \ , \quad \overline{y}(t) := \mathbb{E}[y(t)] \ , \quad \overline{z}(t) := \mathbb{E}[z(t)] \ ,$$

also admits a unique adapted solution $X(\cdot) \in \mathcal{S}^2(0, T; \Re^n)$. Therefore, we can define a mapping $\mathcal{I}$ from $\mathcal{S}^2(0, T; \Re^n)$ into itself such that $X(\cdot) = \mathcal{I}(x(\cdot))$. Next we prove that $\mathcal{I}$ is a strict contraction mapping on $\mathcal{S}^2(0, T; \Re^n)$ equipped with the norm $\|\cdot\|_\beta$

$$\|X(\cdot)\|_\beta := \mathbb{E}\left[\sup_{0\le t\le T} e^{-\beta t}\|X(t)\|^2\right] \ , \tag{A3}$$

for a suitable $\beta \ge 0$.

Given any $x_1(\cdot), x_2(\cdot) \in \mathcal{S}^2(0, T; \Re^n)$ with $x_1(t) = x_2(t) = x_0(t)$, $\forall t \in [-\delta, 0]$, we set $X_1(\cdot) = \mathcal{I}(x_1(\cdot))$ and $X_2(\cdot) = \mathcal{I}(x_2(\cdot))$. To simplify our notation, we write

$$\rho_i(t) = \rho(t, x_i(t), y_i(t), z_i(t), \overline{x}_i(t), \overline{y}_i(t), \overline{z}_i(t)) \ , \quad \rho = b, \sigma, \gamma \ , \quad i = 1, 2 \ .$$

Applying Itô's formula to $e^{-\beta t}\|X_1(t)-X_2(t)\|^2$, integrating from 0 to T and taking expectations on both sides give that

$$\begin{aligned} &\mathbb{E}[e^{-\beta t}\|X_1(t)-X_2(t)\|^2] \\ &= -\beta\mathbb{E}\left[\int_0^t e^{-\beta s}\|X_1(s)-X_2(s)\|^2 ds\right]+2\mathbb{E}\left[\int_0^t e^{-\beta s}(X_1(s)-X_2(s))^\top(b_1(s)-b_2(s))ds\right] \\ &\quad +\mathbb{E}\left[\int_0^t e^{-\beta s}\|\sigma_1(s)-\sigma_2(s)\|^2 ds\right]+\mathbb{E}\left[\int_0^t e^{-\beta s}\|\gamma_1(s, \cdot)-\gamma_2(s, \cdot)\|^2_\nu ds\right] \\ &\le (-\beta+1)\mathbb{E}\left[\int_0^t e^{-\beta s}\|X_1(s)-X_2(s)\|^2 ds\right]+\mathbb{E}\left[\int_0^t e^{-\beta s}\|b_1(s)-b_2(s)\|^2 ds\right] \\ &\quad +\mathbb{E}\left[\int_0^t e^{-\beta s}\|\sigma_1(s)-\sigma_2(s)\|^2 ds\right]+\mathbb{E}\left[\int_0^t e^{-\beta s}\|\gamma_1(s, \cdot)-\gamma_2(s, \cdot)\|^2_\nu ds\right] \end{aligned}$$

$$
\begin{aligned}
&\le (-\beta+1)\mathbb{E}\Big[\int_0^t e^{-\beta s}\|X_1(s)-X_2(s)\|^2 ds\Big]+C\mathbb{E}\Big[\int_0^t e^{-\beta s}\|x_1(s)-x_2(s)\|^2 ds\Big]\\
&\quad +C\mathbb{E}\Big[\int_0^t e^{-\beta s}\|\overline{x}_1(s)-\overline{x}_2(s)\|^2 ds\Big]+C\mathbb{E}\Big[\int_0^t e^{-\beta s}\|y_1(s)-y_2(s)\|^2 ds\Big]\\
&\quad +C\mathbb{E}\Big[\int_0^t e^{-\beta s}\|\overline{y}_1(s)-\overline{y}_2(s)\|^2 ds\Big]+C\mathbb{E}\Big[\int_0^t e^{-\beta s}\|z_1(s)-z_2(s)\|^2 ds\Big]\\
&\quad +C\mathbb{E}\Big[\int_0^t e^{-\beta s}\|\overline{z}_1(s)-\overline{z}_2(s)\|^2 ds\Big]\\
&\le (-\beta+1)\mathbb{E}\Big[\int_0^t e^{-\beta s}\|X_1(s)-X_2(s)\|^2 ds\Big]+K\mathbb{E}\Big[\int_0^t e^{-\beta s}\|x_1(s)-x_2(s)\|^2 ds\Big]\ ,
\end{aligned}
\tag{A4}
$$

where the first inequality is due to

$$
2A^\top B \le \|A\|^2+\|B\|^2\ , \quad \forall A, B \in \Re^n\ , \tag{A5}
$$

and the last inequality is due to Jensen's inequality and the following two inequalities:

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t e^{-\beta s}\|y_1(s)-y_2(s)\|^2 ds\Big]\\
&= \mathbb{E}\Big[\int_0^t e^{-\beta s}\|\int_{-\delta}^0 e^{\lambda r}(x_1(s+r)-x_2(s+r))dr\|^2 ds\Big]\\
&\le \mathbb{E}\Big[\int_{-\delta}^0 e^{2\lambda r}dr\int_0^t e^{-\beta s}\int_{-\delta}^0 \|x_1(s+r)-x_2(s+r)\|^2 dr ds\Big]\\
&\le \frac{1-e^{-2\lambda\delta}}{2\lambda}\mathbb{E}\Big[\int_0^t\int_{s-\delta}^s e^{-\beta v}\|x_1(v)-x_2(v)\|^2 dv ds\Big]\\
&\le \frac{1-e^{-2\lambda\delta}}{2\lambda}\mathbb{E}\Big[\int_0^t\int_{-\delta}^t e^{-\beta v}\|x_1(v)-x_2(v)\|^2 dv ds\Big]\\
&\le K\mathbb{E}\Big[\int_0^t e^{-\beta s}\|x_1(s)-x_2(s)\|^2 ds\Big]\ ,
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^t e^{-\beta s}\|z_1(s)-z_2(s)\|^2 ds\Big] &= \mathbb{E}\Big[\int_0^t e^{-\beta s}\|x_1(s-\delta)-x_2(s-\delta)\|^2 ds\Big]\\
&= e^{-\beta\delta}\mathbb{E}\Big[\int_0^{t-\delta} e^{-\beta s}\|x_1(s)-x_2(s)\|^2 ds\Big]\\
&\le \mathbb{E}\Big[\int_0^t e^{-\beta s}\|x_1(s)-x_2(s)\|^2 ds\Big]\ .
\end{aligned}
\tag{A7}
$$

By Grönwall's inequality to (A4), we have

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T} e^{-\beta t}\|X_1(t)-X_2(t)\|^2\Big] &\le Ke^{(-\beta+1)T}\mathbb{E}\Big[\int_0^T e^{-\beta t}\|x_1(t)-x_2(t)\|^2 dt\Big]\\
&\le Ke^{(-\beta+1)T}\mathbb{E}\Big[\sup_{0\le t\le T} e^{-\beta t}\|x_1(t)-x_2(t)\|^2\Big]\ ,
\end{aligned}
\tag{A8}
$$

where K is a positive constant independent of β. If we set $\beta = (1+\frac{1}{T}\log\frac{K}{2})\vee 0$, then

$$
\mathbb{E}\Big[\sup_{0\le t\le T} e^{-\beta t}\|X_1(t)-X_2(t)\|^2\Big] \le \frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T} e^{-\beta t}\|x_1(t)-x_2(t)\|^2\Big]\ . \tag{A9}
$$

Therefore, we can conclude that $\mathcal{I}$ is a strict contraction mapping on $\mathcal{S}^2(0,T;\Re^n)$ equipped with the norm $\|\cdot\|_\beta$, where $\beta=(1+\frac{1}{T}\log\frac{K}{2})\vee 0$. Then $\mathcal{I}$ has a unique fixed point $X(\cdot)\in\mathcal{S}^2(0,T;\Re^n)$, which is the unique solution to the mean-field jump-diffusion SDDE (3). $\square$

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