Modelling environmental turbulent fluids and multiscale modelling couples patches of wave-like system

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Abstract

Many environmental flows of water have large lateral extent compared to the thickness, such as rivers, floods, tides and tsunamis. This dissertation firstly develops a 2D model to more appropriately model large scale simulations, derived from a 3D turbulence model based on the Smagorinski large eddy closure. We explore the implications of changing the theoretical base from depth-averaging to a slow manifold of the turbulent Smagorinski large eddy closure. Centre manifold theory suggests the existence of slow manifold in the system. Embedding the physical problem into a family of problems, computer algebra constructs the slow manifold of the flow in terms of fluid depth and depth-averaged lateral velocities. The model includes the effects and interactions of inertia, advection, bed drag, gravitational forcing and turbulent dissipation with minimal assumptions. Numerical simulations, implemented on staggered grids in space, of channel flows show that the model is reasonable and reliable to describe the dynamics of large scale environmental turbulent fluids.

Sediment transport is important in the environment. Then the dissertation adapts the turbulent modelling and dynamics to include the suspended sediment transport. A slow manifold exists in the system. The evolution of the depth-averaged concentration on the slow manifold governs the dynamics of the suspended sediment in the turbulent fluid flows. The sediment model includes the effects of sediment erosion, advection, dispersion, and also the interactions between the sediment and the turbulent flow. The applications of the suspended sediment model on concentration distributions in channel flows and under large waves indicate that this model is reasonable.

The dissertation secondly develops a gap-tooth scheme to significantly reduce the expensive numerical simulations of complicated waves over large spatial domains. We aim to develop and explore the methodologies for wave dominant dynamics. The gap-tooth scheme is built from given microscale simulations of complicated physical processes such as sea ice or turbulent shallow water. Our long term aim is to enable macroscale simulations obtained by coupling small patches of simulations together over large physical distances.
A staggered grid is used for the microscale simulation of the fields of depth $h$ and velocity $u$ in the wave-like systems. We introduce a macroscale staggered grid to couple the microscale patches. Linear or cubic or quintic interpolation provides boundary conditions on the field in each patch. Linear analysis of the whole coupled multiscale system establishes that the resultant macroscale dynamics is appropriate. Numerical simulations support the linear analysis.

Eigenvalue analysis suggests that the gap-tooth scheme empowers feasible computation of large scale simulations of wave-like dynamics with complicated underlying physics. As an pilot study, the dissertation implements numerical simulations of dam-breaking waves by the gap-tooth scheme. Comparison among the gap-tooth simulation, the microscale simulation over the whole domain and the published experimental data shows the gap-tooth scheme is feasible to compute large scale wave-like dynamics.

Viscous thin fluid flow has long wave dynamics. The dissertation primarily attempts to use the gap-tooth scheme to explore viscous flow of a layer of fluid. An outstanding issue is the need to create microscale details for each patch appropriate to the macroscale information. The dissertation develops a two-layer model for this viscous layer of fluid, which will have more microscale modes than classic one-layer models, but without the full complexity of fully resolved vertical structures. The two layers are artificial and have no distinguishing physical feature. Linear analysis indicates that an unphysical instability appears for high wavenumber. We introduce a regularising operator to stabilise the model.

The gap-tooth scheme is used to model the viscous layer of fluid with the microscale simulator of the developed two-layer model. To create microscale details for each patch appropriate to the macroscale information, computer algebra leads to the classic one-layer model from the developed two-layer model. The coupling conditions of the gap-tooth scheme are developed by interpolating the macroscale value at the centre of a patch to the boundaries of each neighbouring patch. Numerical simulations of the viscous layer of fluid are successfully implemented by such gap-tooth scheme. Results show that the developed gap-tooth scheme is feasible to model the viscous layer of fluid.

**Keywords:** Navier–Stokes equation, Smagorinski, suspended sediment, channel flows, gap-tooth scheme, dam-breaking waves, thin fluid flow, two-layer model
Certification of dissertation

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Chapter 1

Introduction

Environmental turbulent flow fields, including rivers, floods, tides and tsunamis, are highly irregular. How to appropriately model these turbulent flows is still a great challenge. This dissertation, using centre manifold theory instead of conventional depth-averaging flow equations, develops a 2D model for more appropriately modelling large scale simulations of these environmental turbulent flows. The 2D model is systematically derived from a 3D turbulence model based on the Smagorinski large eddy closure. Then the dissertation adapts the turbulent modelling and dynamics to include the transport of suspended sediment.

Another challenge of turbulent flows is to reduce the expensive numerical simulations of these environmental waves in large spatial domains. Thus, the dissertation develops a gap-tooth scheme for wave-like systems. The gap-tooth scheme is applied to model thin fluid film flow with significant inertia.

1.1 Introduction to modelling environmental fluids and suspended sediment

The first aim of this dissertation is to develop a two dimensional model to more appropriately simulate the environmental large scale turbulent fluids, such as floods, tides and tsunamis. The model resolves three dimensional structures to provide a mathematically sound closure.

1.1.1 Modelling environmental fluids

Environmental turbulent fluids, such as floods, tides and tsunamis, yield fascinating fluid dynamics (Callander 1978, Burchard & Petersen 1999, Roberts
Figure 1.1: The ocean waves on the Newcastle beach in Newcastle in Australia. These ocean waves have large wavelength and relative small depth when approaching the shore.

Figure 1.2: The Yellow River in China. This river carries vast amounts of suspended sediment.
et al. 2002, Carrier et al. 2003, Battjes 2006, Li et al. 2011, e.g.). Figure 1.1 shows ocean waves on a beach, a typical environmental turbulent flow. Such environmental turbulent flows have large horizontal spatial extent and a relatively small flow depth.

Conventional models of such environmental flows are typically based on depth averaged flow equations, including the Navier–Stokes equation, the $k$-$\varepsilon$ equation and the $k$-$\omega$ equation (Duan 2004, Wu 2004, Leveque et al. 2011, Yu 2012, e.g.). Roberts (1997) conjectured that these conventional depth-averaging dynamical equations are quantitatively unsound except perhaps for low Reynolds number flows. Instead, centre manifold theory has the potential to put such dynamical equations on a firm basis (Roberts 1988, e.g.).

Chapter 2 explores the implications of changing the theoretical base from depth-averaging to a slow manifold of the turbulent Smagorinski large eddy closure. The aim is to more appropriately model the fluid dynamics of such complex environmental fluids using such a turbulent closure based on the nondimensional flow equations,

$$
\nabla \cdot \mathbf{q} = 0, \quad (1.1)
$$
$$
\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = -\nabla p + \nabla \cdot \mathbf{\tau} - \mathbf{g}, \quad (1.2)
$$

where $\mathbf{q}(x,y,z,t)$ is the turbulent mean velocity field, the field $\mathbf{\tau}(x,y,z,t)$ is the turbulent mean deviatoric stress tensor, $p(x,y,z,t)$ is the pressure field and the vector $\mathbf{g}$ is the forcing from gravity. Section 2.2 details the nondimensional flow fields, flow equations, and the boundary conditions on the free surface and on the mean bed.

We aim to model turbulent flow when there are large changes in fluid depth to be applicable to extreme events. However, we restrict attention to flow where lateral spatial gradients are relatively small. In this scenario, section 2.3 shows that centre manifold theory suggests there is a slow manifold in the system (Roberts 1988, Potzsche & Rasmussen 2006, e.g.). Section 2.3 embeds the physical problem into a family of problems using an artificial parameter $\gamma$ in the tangential stress boundary conditions on the free surface, where $\gamma = 1$ recovers the original physical boundary conditions. Computer algebra in Appendix A.1 then constructs the slow manifold of the flow in terms of the fluid depth $h(x,y,t)$ and the depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$. Evaluating the resulting slow manifold model at the physical case of parameter $\gamma = 1$, section 2.4 provides a nondimensional model for the fluid dynamics,

$$
\frac{\partial}{\partial t} \begin{pmatrix} h \\ \bar{u} \\ \bar{v} \end{pmatrix} = \mathbf{G}(h, b, \bar{u}, \bar{v}, \bar{q}, \tan \theta), \quad (1.3)
$$
where $b(x, y)$ represents the mean bed topography, $\bar{q} = \sqrt{u^2 + v^2}$ is the local mean flow speed, and $\tan \theta$ is the slope of the mean bed. The function vector $G_t$ is given in equations (2.1)–(2.3), which includes the effects and interactions of inertia, advection, bed drag, gravitational forcing and turbulent dissipation with minimal assumptions. Although this model is expressed in terms of depth-averaged lateral velocities, they are derived not by depth-averaging, but instead by systematically accounting for interaction between vertical profiles and horizontal gradients of the velocity, the stress, bed drag, lateral space variations and bed topography. The dominant terms in the model (1.3) agree with established modelling (Rodi 1984, Yulistiyanto et al. 1998, Bousmar 2002, e.g.) reflecting the well established importance of mass and momentum conservation. The model (1.3) includes more subtle effects that can be important in complex flow regimes.

Section 2.5 linearly analyses the derived model and explores the stability of the model. Then, section 2.6 builds a staggered grid scheme to implement numerical solutions of the model. As a practical trial of the validity of the modelling, section 2.7 implements the numerical simulations using centred difference approximations to the spatial derivatives in the turbulence model (1.3) on staggered grids in space. Section 2.7 applies the model (1.3) to the straight, meandering and river-like channel flows. Numerical results indicate our model is reasonable and reliable to describe the dynamics of the environmental large scale turbulent flows.

1.1.2 Modelling suspended sediment in turbulent flow

Sediment erosion and sediment transport are important in the environment. Figure 1.2 shows the Yellow River in China, which carries vast amount of suspended sediment. We build on the turbulent modelling and dynamics to include sediment. For simplicity at this stage, we only consider the suspended sediment and neglect the bed load.

Chapter 3 aims to more appropriately model the suspended sediment in complex environmental flows. Section 3.2 details the dimensional 3D advection-diffusion equation,

$$\frac{\partial c}{\partial t} + \nabla \cdot (q c) = \nabla \cdot (w_f c) + \nabla \cdot (\epsilon_s \nabla c),$$

where $c(x, y, z, t)$ is the turbulent mean suspended concentration (volume fraction), $q(x, y, z, t)$ is the turbulent mean fluid velocity, and $w_f$ is the constant falling velocity of the particles, $\epsilon_s(x, y, z, t)$ is the turbulent mixing. Equation (3.2), together with the boundary conditions on the free surface
and on the mean bed in section 3.2, describes the dynamics of the suspended sediment.

Chapter 3 restricts attention to suspended sediment in turbulent fluids where lateral spatial gradients are relatively small. Section 3.3 shows that centre manifold theory (Roberts 1988, Potzsche & Rasmussen 2006, e.g.) again supports the existence of a slow manifold in the system. To form the slow manifold, section 3.3 embeds the physical problem into a family of problems using an artificial parameter $\gamma_c$ in the flux boundary conditions on the free surface and on the mean bed, where $\gamma_c = 1$ recovers the original physical boundary conditions. Computer algebra in Appendix A.1 establishes the slow manifold. Section 3.4 records the depth-averaged concentration $\bar{c}(x, y, t)$ on the slow manifold evolving according to a nondimensional model of the form

$$\frac{\partial \bar{c}}{\partial t} = g_c(\bar{c}, h, b, \bar{u}, \bar{v}, q, w_f, \tan \theta).$$

(1.5)

The function $g_c$ is detailed in equation (3.1) and includes the effects of sediment erosion, advection, and dispersion. The model also includes interactions between the sediment and the turbulent flow, for example, some buoyancy driven effects.

Section 3.5 describes the vertical distribution of the suspended sediment. The good agreement between the numerical and analytical vertical distribution in steady flow indicates our model is reasonable. Section 3.6 uses the derived model (1.5) explore the suspended sediment in channel flows and in large waves. Numerical simulations agree with published findings (Lin & Falconer 1996, Demuren & Rodi 1986, Ribberink & Al-Salem 1995, Zedler & Street 2006, e.g.), which indicates our model (1.5) is reasonable, although much more work needs to be done.

1.2 Introduction of gap-tooth scheme and its application on viscous thin film flow

The second aim of the dissertation is to develop a gap-tooth scheme to significantly reduce the expensive numerical simulations of complicated waves over large spatial domains. Then the dissertation initiates the use of the gap-tooth scheme to explore viscous flow of a layer of fluid.

1.2.1 Gap-tooth simulation of wave-like system

Kevrekidis et al. (2003) proposed the gap-tooth scheme for equation-free computation. The key idea of the gap-tooth scheme is to apply microscopic
The equation free approach provides on the fly closure methods which constitute critical components of, for example, mathematical homogenization (e.g., [21, 8, 1]), renormalization group techniques (e.g., [6, 13, 4]), and multiscale finite elements (e.g., [9, 3]). These closure methods not only need to be computationally efficient but also need to be capable of reproducing the physical dynamics with high fidelity. Our approach avoids inefficiencies in the oversampling methods through using correct coupling conditions, to some order of accuracy, between the macroscale separated teeth. That is, we seek a methodology that can be systematically refined.

Using microscopic simulators of the one-dimensional Burgers equation, Roberts and Kevrekidis [19] demonstrated the possibility of achieving high order accuracy in the gap-tooth scheme for macroscale dynamics. The particular microsimulator we use is a fine scale discretization of the PDE which we execute only in the interior of the teeth (see Figure 1). At each time step during execution, the microsimulator within each tooth requires boundary values which must be continuously updated. If the microsimulator was to be executed over the entire macrodomain, these boundary values would naturally come from the immediately neighboring fine grid; this grid is missing in gap-tooth simulation. That pilot study considered only microsimulators which had boundary conditions of specified flux at the edges of their simulation teeth. Here we generalize the analysis to consider microsimulators with

- Dirichlet boundary conditions of specified field \( u \) at the tooth edges (section 2),
- mixed boundary conditions of specified \( \alpha v_j + \beta \partial_x v_j \) at the tooth edges (section 3), or
- nonlocal two-point boundary conditions such as those arising in a microscale discretization of a PDE (section 4).

Consider the gap-tooth scheme (e.g., [7, 20]) illustrated in Figure 1. Let \( v_j(x, t) \) be the fine scale, microscopic field in the \( j \)th tooth and \( U_j \) the \( j \)th coarse grid value, that is, the value at the center of each tooth. Let the tooth width be \( h \). Then the edge of simulations to a number of well separated subdomains and couple them by interpolation to simulate the macroscopic behaviour. Figure 1.3 shows an example of gap-tooth simulation of Burgers’ equation on \([0, 2\pi]\) through microsimulation on eight teeth, each of small width \( \pi/20 \) by Roberts & Kevrekidis (2005). The scheme only implements the microscale simulation on a patch, and then couple the patches together to describe the dynamics over the whole domain. Thus, the challenges are to explore the microscale simulator on a patch and explore the coupling conditions to couple the patches.

The gap-tooth scheme has to date been applied only to dissipative dynamics, whereas chapter 4 aims to develop and explore the methodologies for wave-dominated dynamics. The gap-tooth scheme is built from given microscale simulations of complicated physical processes such as sea ice or turbulent shallow water. The long term aim of chapter 4 is to enable macroscale simulations obtained by coupling small patches of simulations together over large physical distances.

However, chapter 4 initially explores the coupling of patch simulations of wave-like PDEs. With the line of development being to water waves section 4.2 discusses the dynamics of two complementary fields called the ‘depth’ \( h \) and ‘velocity’ \( u \). Section 4.2 uses a staggered grid for the microscale simulation.
of the depth $h$ and velocity $u$. The microscale simulation is a pared down version of the nonlinear equations of the shallow water dynamics because our aim is to adapt the approach to problems such as turbulent floods and tsunamis.

Section 4.3 introduces a macroscale staggered grid to couple the microscale patches. The patches are coupled by interpolating information from neighbouring patches into boundary values for each of the microscale patch simulators. Section 4.4 reports algebraic analysis and numerical determination of eigenvalues that both confirm the accuracy of the proposed gap-tooth scheme for wave-like dynamics.

Section 4.5 explores the gap-tooth simulation with a nonlinear microscale simulator on a patch. Eigenvalue analysis suggests that the gap-tooth scheme empowers feasible computation of large scale simulations of wave-like dynamics with complicated underlying physics. As an pilot study, section 4.6 implements numerical simulations of dam-breaking waves by the gap-tooth scheme. Comparison among the gap-tooth simulation, the microscale simulation over the whole domain and the published experimental data (Stansby et al. 1998, e.g.) shows the gap-tooth scheme is feasible for this field.

1.2.2 Gap-tooth scheme explores viscous flow of a layer of fluid

Viscous fluid film flow has long wave dynamics. Chapter 5 primarily attempts to use the gap-tooth scheme to explore viscous flow of a layer of fluid. The layer is thick enough so that inertia is important and the dominant behaviour is of slightly damped waves. An outstanding issue is the need to create microscale details for each patch appropriate to the macroscale information.

Chapter 5 also develops a new two-layer model for this viscous layer of fluid, which will have more microscale modes than classic one-layer models, but without the full complexity of fully resolved vertical structures. Section 5.1 considers that the viscous fluid film flow has two artificial layers, which have no distinguishing physical feature. Then section 5.1 describes the continuity and modified Navier-Stokes equations of the fluid film flow with two artificial layers. The boundary conditions on the free surface, on the plate and on the interface of the two artificial layers are considered.

Section 5.2 shows the modelling and theoretical support for existing a semi-slow manifold in the system. Computer algebra in Appendix A.3 constructs the semi-slow manifold in the two-layer system. Then section 5.3 records the nondimensional two-layer model for the 1D fields of flow depth $h(x, t)$ and
two layer mean velocities $\bar{u}_1(x, t)$ and $\bar{u}_2(x, t)$:

$$\frac{\partial}{\partial t} \begin{pmatrix} h \\ \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = G_g(h, \bar{u}_1, \bar{u}_2, \text{Re}, \tan \theta),$$

(1.6)

where $\text{Re}$ is the Reynolds number and $\tan \theta$ is the slope of the plate. The function $G_g$ is detailed in equations (5.26)–(5.28) and includes the effects of gravity, drag, advection and dispersion. Eigenvalue analysis in section 5.4 shows that an unphysical instability appears for high wavenumber. Section 5.4 then introduces an asymptotically sound regularising operator to stabilise the model.

Section 5.5 uses the gap-tooth scheme to model the viscous layer of fluid with the microscale simulator of the developed two-layer model (1.6). To create microscale details for each patch appropriate to the macroscale information, computer algebra in Appendix A.3 leads to the classic one-layer model from the developed two-layer model. The coupling conditions of the gap-tooth scheme are developed by interpolating macroscale value from the centre of each patch to the boundaries of neighbouring patches. Section 5.5 successfull implements the numerical simulations of the viscous layer of fluid by such gap-tooth scheme. Results show that the developed gap-tooth scheme is feasible to model the viscous layer of fluid.

Finally, Appendix A lists the computer algebras for deriving the models in chapter 2–5, and Appendix B lists the matlab codes used to implement numerical simulations of the models in chapter 2–5.
Chapter 2

Modelling 3D turbulent fluids based upon the Smagorinski large eddy closure

2.1 Introduction

Modelling environmental turbulent fluids, such as rivers, floods and tsunamis, is one of the major challenges in fluid dynamics. This kind of environmental turbulent flow typically have large wavelength compared with the fluid depth, such as the ocean waves in Figure 1.1. For example, Gisler (2008) introduced tsunamis and reviewed the tsunami simulations. Tsunamis have very long wavelengths and very small amplitudes out in the open ocean. Conventional models of such environmental fluids are often carried out by depth-averaging flow equations (Demuren 1993, Bousmar 2002, Nezu 2005, Rodi 2010, e.g.). This chapter derives a three dimensional Smagorinski model for large scale turbulent fluids by avoiding the depth-averaged method.

Depth-averaged flow equations contribute much to our understanding and simulation of the turbulent fluids. Based on depth-averaging the Navier–Stokes equation, Bousmar (2002) and Bousmar & Zech (2004) proposed an exchange discharge model (EDM) to study the dynamics of fluid in compound channels. The EDM solves the momentum transfers through the turbulent exchange and geometrical transfer between channel subsections, the channel and the shallow regions. Fredsoe & Deigaard (1992) in their book also reviewed the depth-averaged Navier–Stokes equations and the depth-averaged kinetic energy and dissipation equations (k–ε equation), and the application of such depth-averaged flow equations on modelling turbulent fluids.

Nevertheless, Roberts (1997) discussed evidence that the depth-averaging
in such models is quantitatively unsound for turbulent fluids. Based on centre manifold theory, Roberts (1997) derived a dynamical model for the thin film flow from the Navier–Stokes equations and compared with other models based on depth-averaging flow equations. The differences suggest that the depth-averaging is quantitatively unsound. Subsequently, to avoid depth-averaging flow equations, Roberts et al. (2002) and Georgiev et al. (2007, 2009) used centre manifold theory built dynamical models to explore the dynamics of turbulent fluids. For example, Georgiev et al. (2009) proposed a two dimensional turbulent fluid model with a straightforward Smagorinski eddy viscosity and applied to simulate the dam breaking waves. The aim of the works (Roberts 1997, Roberts et al. 2002, Georgiev et al. 2009, e.g.) was to use the centre manifold theory derive a simpler and more robust class of models to study fluid dynamics.

The main purpose of the present chapter is to extend the two dimensional Smagorinski turbulence model of Roberts (2008b) to three dimensions and then study the dynamics of three dimensional environmental turbulent fluids with the new model. Based on centre manifold theory, I derive the following 3D coupled equations in the fluid fields of depth $h(x, y, t)$ and depth-averaged lateral velocities $\bar{u}(x, y, t)$ and $\bar{v}(x, y, t)$ to model turbulent fluids:

$$\frac{\partial h}{\partial t} = - \frac{\partial h \bar{u}}{\partial x} - \frac{\partial h \bar{v}}{\partial y}, \quad (2.1)$$

$$\frac{\partial \bar{u}}{\partial t} = - 0.00283 \frac{\bar{u} q}{h} + 0.993 \left( \tan \theta - \frac{\partial h}{\partial x} - \frac{\partial b}{\partial x} \right)
- 1.025 \bar{u} \frac{\partial \bar{u}}{\partial x} - 1.017 \bar{v} \frac{\partial \bar{u}}{\partial y}
+ 0.094 \frac{\bar{q}}{h} \left[ \frac{\partial}{\partial x} \left( h^2 \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^2 \frac{\partial \bar{u}}{\partial y} \right) \right]
+ 0.084 \bar{u}^2 - \bar{v}^2 \left[ \frac{\partial}{\partial x} \left( h^2 \frac{\partial \bar{u}}{\partial x} \right) - \frac{\partial}{\partial y} \left( h^2 \frac{\partial \bar{u}}{\partial y} \right) \right], \quad (2.2)$$

$$\frac{\partial \bar{v}}{\partial t} = - 0.00283 \frac{\bar{v} q}{h} - 0.993 \left( \frac{\partial h}{\partial y} + \frac{\partial b}{\partial y} \right)
- 1.025 \bar{v} \frac{\partial \bar{v}}{\partial y} - 1.017 \bar{u} \frac{\partial \bar{v}}{\partial x}
+ 0.094 \frac{\bar{q}}{h} \left[ \frac{\partial}{\partial x} \left( h^2 \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^2 \frac{\partial \bar{v}}{\partial y} \right) \right]
+ 0.084 \bar{u}^2 - \bar{v}^2 \left[ \frac{\partial}{\partial x} \left( h^2 \frac{\partial \bar{v}}{\partial x} \right) - \frac{\partial}{\partial y} \left( h^2 \frac{\partial \bar{v}}{\partial y} \right) \right], \quad (2.3)$$

where $\bar{q} = \sqrt{\bar{u}^2 + \bar{v}^2}$ is the mean speed of the flow. The momentum equa-
tions (2.2)–(2.3) incorporate inertial terms $\partial q/\partial t$, advection terms $\bar{q}(\partial \bar{q}/\partial x)$, bed drag terms $\bar{q}\bar{q}/h$, gravitational forcing $\tan \theta - \nabla (h + b)$, and other terms related to the turbulent mixing. Note that although equations (2.1)–(2.3) are expressed in terms of depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$, they are derived not by depth-averaging, but instead by systematically accounting for interaction between vertical profiles of the velocity, the stress, bed drag, lateral space variations and bed topography.

Section 2.2 describes the governing continuity and Navier–Stokes equations in the turbulent mean variables of the turbulent fluid. The Smagorinski large eddy closure is used to approach the turbulence. Boundary conditions on the free surface and on the bed are formulated.

Section 2.3 embeds the physical problem into a family of artificial problems to empower the modelling and theoretical support by centre manifold theory. Centre manifold theorems support that there is a slow manifold in the system. Then computer algebra in section 2.4 approximates the slow manifold in terms of depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$, and the fluid depth $h(x,y,t)$. Equations (2.1)–(2.3) are the simplified version of the evolutions on the slow manifold. Section 2.5 analyses the eigenvalue problem and indicates the model (2.1)–(2.3) is stable.

Section 2.6 builds the staggered scheme for numerically solving the proposed model (2.1)–(2.3). Then, section 2.7 predicts the turbulent flow in the straight, meander and river-like channel flows. Numerical results show that the model (2.1)–(2.3) is reasonable to simulate the environmental large scale turbulent flow.

The research of section 2.2, 2.4 and the applications of the straight and meander open channel flows in section 2.7 have been referred and published in the proceeding of the 18th Australasian Fluid Mechanics Conference in 2012 (Cao & Roberts 2012, e.g.).

### 2.2 Detailed equations of the turbulence model

This section details the modified Reynolds-averaged Navier–Stokes equations and the boundary conditions on the free surface and on the mean bed.

#### 2.2.1 The coordinate system

Figure 2.1 depicts turbulent fluid flowing over physically rough bed with rocks, plants and other roughness on the bed. Consider three dimensional incompressible turbulent fluid flowing down a bumpy bed with some small mean slope, shown in Figure 2.1. Define Cartesian coordinates which contains...
Figure 2.1: Schematic diagram of three dimensional turbulent fluid over physically rough ground with rock, plants and other roughness on the bed. Define the coordinate system by \((x, y, z)\) and the gravity \(g\). The fluid has a depth \(h(x, y, t)\) and turbulent mean velocity \(q(x, y, z, t)\). Denote the mean bed \(z = b(x, y)\) and the free surface \(z = h + b\).
the lateral directions \(x_1 = x\) and \(x_2 = y\) and the normal direction \(x_3 = z\). The bumpy bed has a mean slope angle \(\theta\) in the \(x\) direction. Let the turbulent fluid have depth \(h(x, y, t)\) over the mean bed located at \(z = b(x, y)\). Denote the turbulent mean velocity field \(q(x, y, z, t)\) with the components \((u, v, w) = (u_1, u_2, u_3)\) and the turbulent mean pressure field by \(p(x, y, z, t)\).

### 2.2.2 Turbulent mean variables

The velocity field \(q(x, y, z, t)\) and pressure \(p(x, y, z, t)\) are turbulent mean but not depth-averaged. Take the turbulent lateral mean velocity \(u(x, y, z, t)\) in the \(x\) direction for example. Physically, the turbulent fluid has a turbulent lateral velocity \(\tilde{u}(x, y, z, t)\). We separate this velocity \(\tilde{u}(x, y, z, t)\) into a turbulent mean velocity \(u(x, y, z, t)\) and a fluctuation \(u'(x, y, z, t)\). Mathematically, we take the ensemble average for the physical velocity \(\tilde{u}(x, y, z, t)\). Because the ensemble average of the fluctuation \(u'(x, y, z, t)\) is zero, then the ensemble average of the physical velocity \(\tilde{u}(x, y, z, t)\) reduces to the turbulent mean velocity \(u(x, y, z, t)\).

This chapter aims to base upon studying the dynamics of the turbulent mean variables to derive a turbulent model in terms of the depth-averaged lateral velocities \(\bar{u}(x, y, t)\) and \(\bar{v}(x, y, t)\).

The bed topography \(z = b(x, y)\) and the fluid depth \(h(x, y, t)\) are also turbulent mean. Environmental fluids flow over very complex topographic beds. The photograph in Figure 2.2 exhibits the river bed with rocks, plants and other roughness, which affects the dynamics of the fluid. For a pilot study, we assume that the bed topography is independent of the time. Rocky bed breaks up any classic boundary layer. Therefore, we separate the physical rocky bed \(z = b(x, y)\) into a turbulent mean bed \(z = b(x, y)\) and a fluctuation \(z = b'(x, y)\). Mathematically, the ensemble average of the physical bed \(z = \bar{b}(x, y)\) reduces to the turbulent mean bed \(z = b(x, y)\). The physical fluid depth \(\tilde{h}(x, y, t)\) is from the physical bed \(z = b(x, y)\) to the free surface \(z = h(x, y, t) + b(x, y)\). The bed \(z = \bar{b}(x, y)\) is sometimes within the fluid and sometimes not. Thus, the ensemble of the fluid depth \(\tilde{h}(x, y, t)\) reduces to the turbulent mean depth \(h(x, y, t)\).

### 2.2.3 Nondimensionalization of the variables

This subsection nondimensionalizes the turbulent mean variables of the flow. Generally, the vertical variables are nondimensionalized differently to the lateral variables (Pittaluga & Seminara 2003, Constantin & Johnson 2008,
e.g.). However, in our work, centre manifold theorems (Roberts 1997, 1998, e.g.) are most conveniently applied using the same scaling for the vertical and lateral variables. Thus, we adopt a characteristic depth of the turbulent fluid $H$ as the length scale, the velocity scale $\sqrt{gH}$, and the fluid density $\rho$. Then we have the following nondimensionalization of the independent and dependent variables,

$$x^* = \frac{x}{H}, \quad y^* = \frac{y}{H}, \quad z^* = \frac{z}{H}, \quad t^* = \frac{\sqrt{gH}}{H} t,$$

$$u^* = \frac{1}{\sqrt{gH}} u, \quad v^* = \frac{1}{\sqrt{gH}} v, \quad w^* = \frac{1}{\sqrt{gH}} w,$$

$$p^* = \frac{1}{\rho gH} p, \quad g^* = \frac{H \cos \theta}{gH} g.$$

For simplicity, we omit the subscript on the nondimensional variables in the following.

We aim to model flows with very large Reynolds number using the Smagorinski model of turbulence. The Reynolds number $Re = \frac{H \sqrt{gH}}{\nu_f}$, where $\nu_f$ is the fluid kinematic viscosity and is assumed very small in our work. Thus, the Reynolds number $Re$ in our work is effectively infinite. Schultz & Flack (2013) reported the Reynolds number scaling of turbulent channel flow for the Reynolds number ranged from 10,000 to 300,000. Experimental measurements by Schultz & Flack (2013) show that the mean flow and Reynolds
shear stresses are effectively independent of the large Reynolds number. Thus our model is assumed to be independent of the large Reynolds number.

### 2.2.4 The governing Smagorinski equations

The nondimensional governing partial differential equations for the incompressible, three dimensional, turbulent mean fluid fields are the Reynolds-averaged continuity and momentum equations,

\[
\nabla \cdot \mathbf{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]

\[
\frac{\partial q}{\partial t} + q \cdot \nabla q = -\nabla p + \nabla \cdot \tau + g.
\]

The variable \(\tau\) is the turbulent mean deviatoric stress tensor, which is approximated via the eddy viscosity \(\nu\) in section 2.2.5. The vector \(\mathbf{g} = (\tan \theta, 0, -1)\) is the nondimensional gravity, where the component \(\tan \theta\) is the slope of the mean bed.

### 2.2.5 Smagorinski large eddy closure

A closure is required for the nonlinear turbulence stresses in the Reynolds-averaged Navier–Stokes equation (2.8). The turbulence stresses are modelled via an eddy viscosity \(\nu\), such as the Boussinesq eddy viscosity (Nwogu 1993, e.g.), the turbulence mixing theory (Fredsoe 1984, e.g.), and Smagorinski model for the sub-grid scale eddy viscosity (Leveque et al. 2007, e.g.).

We use the Smagorinski eddy closure to approximate the turbulence stresses. In the Smagorinski model the effects of turbulence are modelled via an eddy viscosity \(\nu\), in an assumed mean stress-strain equation of

\[
\tau_{ij} = 2\nu \dot{\varepsilon}_{ij},
\]

with the indexes \(i,j = 1,2,3\) corresponding to the \(x\), \(y\) and \(z\) directions. Define the turbulent mean strain-rate tensor (Leveque et al. 2007, Meyers et al. 2007, e.g.)

\[
\dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

and then the nondimensional turbulent mean stress tensor for the turbulent flow is \(-p\delta_{ij} + 2\nu \dot{\varepsilon}_{ij}\). When the eddy viscosity \(\nu\) is constant, the turbulent mean stress tensor \(-p\delta_{ij} + 2\nu \dot{\varepsilon}_{ij}\) models a Newtonian fluid. In the Smagorinski
model (Ozgokmen et al. 2007, e.g.), the eddy viscosity $\nu$ varies linearly with the magnitude $\dot{\epsilon}$ of the second invariant of the strain-rate tensor,

$$\nu = \ell^2 \dot{\epsilon} \quad \text{where} \quad |\dot{\epsilon}|^2 = \sum_{i,j} \dot{\epsilon}_{ij}^2,$$

where $\ell$ is the turbulent mixing length. The turbulent mixing length $\ell$ is scaled by $\ell = \sqrt{c_t h}$, where $c_t$ is a constant related to the strength of turbulent mixing and $h$ is the fluid depth. The eddy viscosity becomes

$$\nu = c_t h^2 \dot{\epsilon} \quad \text{where} \quad |\dot{\epsilon}|^2 = \sum_{i,j} \dot{\epsilon}_{ij}^2,$$

(2.11)

Roberts et al. (2008) recommended the proportionality constant $c_t \approx 0.02$ for turbulent environmental flows through comparison with established channel flow experiments (Nezu 2005, e.g.).

Thus, equations (2.9)–(2.11) give the turbulent mean deviatoric stress tensor

$$\tau_{ij} = 2c_t h^2 (\dot{\epsilon}) \dot{\epsilon}_{ij} = c_t h^2 \dot{\epsilon} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

(2.12)

### 2.2.6 The boundary conditions

This subsection formulates boundary conditions on the mean bed $z = b(x,y)$ and on the free surface $z = \eta(x,y,t) = h(x,y,t) + b(x,y)$ in terms of the turbulent mean velocity field $q(x,y,z,t)$, the fluid depth $h(x,y,t)$ and the turbulent mean pressure $p(x,y,z,t)$.

On the mean bed, no fluid penetrating the ground requires $q \cdot n = 0$. Since the unit normal vector to the mean bed is

$$n = \frac{1}{1 + b_x^2 + b_y^2} (-b_x, -b_y, 1),$$

(2.13)

this ‘no penetration’ requires

$$w = ub_x + vb_y \quad \text{on} \quad z = b.$$ (2.14)

Strictly, there are turbulent mean closure issues in the boundary conditions. On the physical bed $b$, the exact non-mean ‘no penetration’ boundary condition is $\dot{w} = \dot{u}b_x + \dot{v}b_y$. Decompose the physical variables into the turbulent mean variables and small fluctuations, such as the vertical physical velocity $\dot{w} = w + w'$. Then the exact ‘no penetration’ conditions becomes

$$w + w' = (u + u')(b_x + b_x') + (v + v')(b_y + b_y'), \quad \text{on} \quad z = b + b'.$$ (2.15)
Expand the left-hand of equation (2.15) by Taylor series in $b'$, take the ensemble average and obtain

$$
\langle w(b + b') \rangle \approx \langle w(b) \rangle + \left\langle b' \frac{\partial w(b)}{\partial z} \right\rangle + \left\langle \frac{1}{2} b'^2 \frac{\partial^2 w(b''')}{\partial z^2} \right\rangle,
$$

$$
= w(b) + \frac{1}{2} \left\langle b'^2 \frac{\partial^2 w(b''')}{\partial z^2} \right\rangle,
$$

(2.16)

$$
\langle w'(b + b') \rangle \approx \langle w'(b) \rangle + \left\langle b' \frac{\partial w(b''')}{\partial z} \right\rangle = \left\langle b' \frac{\partial^2 w(b''')}{\partial z^2} \right\rangle,
$$

(2.17)

where the angle brackets represent taking ensemble average, the ensemble average of the small fluctuations are zeros, and $b''$ and $b'''$ are in the range $[b, b + b']$. Similarly, expand the right-hand by Taylor series in $b'$ and take ensemble average of such expansion. Assuming that the remaining ensemble averaged $\langle \cdots \rangle$ terms in equations (2.16) and (2.17) are negligible, obtain the turbulent mean ‘no penetration’ boundary condition (2.14) on the turbulent mean bed $z = b(x, y)$.

We posit a slip law on the mean bed to account for a negligibly thin turbulent boundary layer:

$$
q_{\text{tan}} = c_u h \frac{\partial q_{\text{tan}}}{\partial n} \quad \text{on} \quad z = b,
$$

(2.18)

where $q_{\text{tan}}$ represents the velocity tangential to the mean bed. Roberts et al. (2008) found the constant $c_u \approx 1.85$ matched open channel flow observations. In a wider range of applications, the coefficient $c_u$ would change for different bed roughness. Unit vectors tangential to the mean bed in the $x$ and $y$ directions are

$$
t_x = \frac{1}{\sqrt{1 + b_x^2}} (1, 0, b_x) \quad \text{and} \quad t_y = \frac{1}{\sqrt{1 + b_y^2}} (0, 1, b_y).
$$

The slip boundary condition (2.18) on the mean bed $z = b$ indicates the turbulent mean boundary conditions of

$$
\frac{1}{\sqrt{1 + b_x^2}} (u + wb_x) = \frac{c_u h}{\sqrt{1 + b_x^2 + b_y^2}} \frac{\partial}{\partial n} (u + wb_x),
$$

(2.19)

$$
\frac{1}{\sqrt{1 + b_y^2}} (v + wb_y) = \frac{c_u h}{\sqrt{1 + b_x^2 + b_y^2}} \frac{\partial}{\partial n} (v + wb_y),
$$

(2.20)

where the derivative $\partial/\partial n = -b_x \partial/\partial x - b_y \partial/\partial y + \partial/\partial z$. 25
On the free surface (that is, on its turbulent mean position), the kinematic condition that no fluid crosses the free surface is

\[
\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} = w \quad \text{on } z = \eta = h + b, \tag{2.21}
\]

Assume the air is so light that the only stress it can supply is pressure. Relative to atmospheric pressure, the pressure on the free surface is zero. Thus the turbulent mean stress normal to the free surface is also zero: on \( z = \eta \),

\[
-p + \frac{\tau_{33} - 2\eta_x \tau_{13} - 2\eta_y \tau_{23} + \eta_x^2 \tau_{11} + 2\eta_x \eta_y \tau_{12} + \eta_y^2 \tau_{22}}{1 + \eta_x^2 + \eta_y^2} = 0. \tag{2.22}
\]

There must be no turbulent mean, tangential stress at the free surface,

\[
(1 - \eta_x^2)\tau_{13} + \eta_x(\tau_{33} - \tau_{11}) - \eta_y(\tau_{12} + \eta_x \tau_{23}) = 0 \quad \text{on } z = \eta, (2.23)
\]

\[
(1 - \eta_y^2)\tau_{23} + \eta_y(\tau_{33} - \tau_{22}) - \eta_x(\tau_{12} + \eta_y \tau_{13}) = 0 \quad \text{on } z = \eta. (2.24)
\]

2.3 Centre manifold theory supports the modelling

This section embeds the physical problem into a family of problems by modifying the boundary conditions \((2.23)–(2.24)\) using an artificial parameter \(\gamma\). Such modification empowers modelling and theoretical support by centre manifold theory. Then linear analysis implies the spectrum which supports the existence of the slow manifold.

2.3.1 Embed the physical problem in a family of problems

This subsection embeds the physical problem in a family of problems to empower theory support. Artificially modify the surface condition \((2.23)–(2.24)\) on the tangential stresses to have an artificial forcing proportional to the square of the local, free surface, velocity:

\[
(1 - \eta_x^2)\tau_{13} + \eta_x(\tau_{33} - \tau_{11}) - \eta_y(\tau_{12} + \eta_x \tau_{23})
\]

\[
= \frac{(1 - \gamma) \sqrt{2c_t}}{(1 + c_\text{u})(1 + 2c_\text{u})} \cdot u \sqrt{u^2 + v^2} \quad \text{on } z = \eta = h + b, \tag{2.25}
\]

\[
(1 - \eta_y^2)\tau_{23} + \eta_y(\tau_{33} - \tau_{22}) - \eta_x(\tau_{12} + \eta_y \tau_{13})
\]
\[ (1 - \gamma)\sqrt{\frac{2c_t}{1 + c_u(1 + 2c_u)}} \nu \sqrt{u^2 + v^2} \] on \( z = \eta = h + b \). \hspace{1cm} (2.26)

When evaluating at \( \gamma = 1 \) this artificial right-hand side becomes zero so the artificial surface conditions (2.25)–(2.26) reduce to the physical surface condition (2.23)–(2.24).

However, when the artificial parameter \( \gamma = 0 \), associated with the mean slope and the lateral derivatives are negligible, \( \tan \theta = \partial_x = \partial_y = 0 \), then equations (2.25)–(2.26) reduce to \( \nu \partial u / \partial z = \nu (u/\eta) \) and \( \nu \partial v / \partial z = \nu (v/\eta) \), which imply that two neutral modes of the dynamics are the lateral shear flow \( (u, v) \propto (z/h + c_u) \). \(^1\)

Conservation of fluid provides a third neutral mode in the dynamics. Thus, when \( \gamma = \tan \theta = \partial_x = \partial_y = 0 \), a three parameter subspace of equilibria exists corresponding to some uniform lateral shear, turbulent mean, flow, \( (u, v) \propto (z/h + c_u) \), on a fluid of any constant fluid depth \( h \). For large enough lateral length scales, these equilibria occur independently at each location \( x \) and \( y \) (Roberts 1988, 2008a, e.g.) and hence the subspace of equilibria are in effect parameterised by \( \bar{u}(x,y) \), \( \bar{v}(x,y) \) and \( h(x,y) \).

Provided we can treat lateral derivatives \( \partial_x \) and \( \partial_y \) as modifying influences, that is provided solutions vary slowly enough in \( x \) and \( y \), centre manifold theorems (Roberts 1988, Chicone 2006, e.g.) assure us three vitally important properties:

- this subspace of equilibria are perturbed to a slow manifold, where the evolutions are slow, that exists for a finite range of gradients \( \partial_x \) and \( \partial_y \), and parameters \( \gamma \) and \( \tan \theta \), and which may be parameterised by the depth-averaged lateral velocities \( \bar{u}(x,y,t) \) and \( \bar{v}(x,y,t) \), and the local fluid depth \( h(x,y,t) \);

- the slow manifold attracts solutions from all nearby initial conditions;

- and that a formal power series in the parameters \( \gamma \), \( \tan \theta \) and gradients \( \partial_x \) and \( \partial_y \) approximate the slow manifold to the same order of error as the order of the residuals of the governing differential equations.

That is, the theorems support the existence, accurate relevance and construction of slow manifold models such as (2.1)–(2.3).

\(^1\)The Euler parameter of a toy problem suggests introducing a factor \( 1 - \gamma/6 \) into the left-hand side of the tangential stress boundary conditions (2.23)–(2.24) in order to improve convergence in the parameter \( \gamma \) when evaluated at the physically relevant \( \gamma = 1 \). For the moment, the present work omits such a factor.
2.3.2 Linear dynamics of the system

The linear dynamics of the system support the application of centre manifold theory. For the flat bed of \( b = \text{constant} \), and with \( \tan \theta = \gamma = 0 \), the base problem (2.7)–(2.26) has the equilibrium of a shear flow which in terms of the vertical scale \( Z = (z - b)/h \),

\[
\begin{align*}
  h &= \text{constant}, \\
  u &= 2\bar{u}\frac{Z + c_u}{1 + 2c_u}, \\
  v &= 2\bar{v}\frac{Z + c_u}{1 + 2c_u}, \\
  w &= 0, \\
  p &= h(1 - Z),
\end{align*}
\]

(2.27)

where \( \bar{u} \) and \( \bar{v} \) are the depth-averaged lateral velocities, and \( \bar{q} = \sqrt{\bar{u}^2 + \bar{v}^2} \) is the mean speed. Environmental turbulent flows have significant eddy viscosity. In this linear analysis, we assume the eddy viscosity \( \nu \) is effectively constant. Then we consider the dynamics of the PDEs (2.7)–(2.8) linearised the perturbations \( (h', u', v', w', p') \) about each of these equilibria. Because environmental turbulent fluids have very large lateral scales compared with the depth. As the lateral variations vary slow they do not affect the dominant linear process. We treat the lateral derivatives \( \partial_x \) and \( \partial_y \) as ‘small’ options. Thus, the PDEs (2.7)–(2.8) and the boundary conditions (2.14)–(2.24), considering the effective \( \partial_x = \partial_y = \tan \theta = \gamma = 0 \), result in a linear problem

\[
\begin{align*}
  \frac{\partial w'}{\partial z} &= 0, \\
  \frac{\partial u'}{\partial t} + w'\frac{\partial u'}{\partial z} &= \nu\frac{\partial^2 u'}{\partial z^2}, \\
  \frac{\partial v'}{\partial t} + w'\frac{\partial v'}{\partial z} &= \nu\frac{\partial^2 v'}{\partial z^2}, \\
  \frac{\partial w'}{\partial t} &= -\frac{\partial p'}{\partial z} + \nu\frac{\partial^2 w'}{\partial z^2}, \\
  w' &= 0 \quad \text{on} \quad Z = 0, \\
  u' &= c_uh'\frac{\partial u'}{\partial z} + c_uh\frac{\partial u'}{\partial z} \quad \text{on} \quad Z = 0, \\
  v' &= c_uh'\frac{\partial v'}{\partial z} + c_uh\frac{\partial v'}{\partial z} \quad \text{on} \quad Z = 0, \\
  \frac{\partial h'}{\partial t} &= w' \quad \text{on} \quad Z = 1, \\
  -p' + 2\nu\frac{\partial w'}{\partial z} &= 0 \quad \text{on} \quad Z = 1,
\end{align*}
\]

(2.29a–2.29i)
Equations (2.29a) and (2.29e) indicate $w' = 0$. Equation (2.29h) implies $h' = \text{constant}$, which corresponds to the freedom already in (2.27) so without loss of generality we here set $h' = 0$. Equations (2.29d) and (2.29i) implies $p' = 0$. Equations (2.29b)–(2.29c) and (2.29j)–(2.29k) indicate $u'$ and $v'$ have solutions in the form $[A \sin(kz) + B \cos(kz)] \exp(\lambda t)$, where $k$ is a nondimensional wavenumber. Substitute these solution forms into equations (2.29b)–(2.29c) and obtain $\lambda = -\nu k^2$. Substituting these solution forms into boundary conditions (2.29j)–(2.29k) leads to

$$\tan k = \frac{k}{1 + c_u(1 + c_u)k^2}. \quad \text{(2.30)}$$

Figure 2.3 shows that equation (2.30) has the non-zero wavenumbers $k > \pi$. Therefore, the eigenvalues $\lambda = -\nu k^2 < -\nu \pi^2$. There is eigenvalues $\lambda = 0$ corresponding to the freedom to vary the fluid depth $h$ and depth-averaged velocities $\bar{u}$ and $\bar{v}$. Thus, there is a spectral gap between the eigenvalues $\lambda = 0$ and $\lambda < -\nu \pi^2$. Centre manifold theory (Roberts 1988, Potzsche & Rasmussen
then supports the existence of a slow manifold in the three dimensional turbulent system.

2.4 Reduced model of the fluid dynamics

This section focusses on interpreting the application of centre manifold theory and the resulting low order model of the turbulent flow. Computer algebra in Appendix A.1 readily approximates the slow manifold in terms of the depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \), and the fluid depth \( h(x, y, t) \). Evolution of these terms model the dynamics of the turbulent flow. The dominant terms in the model agree with established modelling (Rodi 1984, Yulistiyanto et al. 1998, Bousmar 2002, e.g.). Our modelling ensures more subtle effects are modelled, effects that can be important in complex flow regimes.

2.4.1 Computer algebra constructs the slow manifold

Instead of depth-averaging equations, we apply centre manifold theory to deal with the turbulent dynamics across the fluid layer. Roberts (2008b), in a freely available report, detailed the computer algebra that constructed the slow manifold model in 2D flow.

I generalise this computer algebra program for 3D turbulent flow, as described in Appendix A.1. This empowers the computer algebra program to derive evolution equations for the water depth \( h(x, y, t) \) and the depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \). In the computer algebra, the residuals of the actual governing equations (2.7)–(2.12) and boundary conditions (2.14)–(2.26) are calculated. The iteration in the algebra is performed until all the residuals are zero to some order of error. Centre manifold theorems (Roberts 1988, Potzsche & Rasmussen 2006, e.g.) demonstrate that the correctness of the residual computation assures the correctness of the resulting governing equations. The key to obtain correct resulting equations is the proper coding of the continuity equation (2.7), the Smagorinski model (2.8) and the boundary conditions (2.14)–(2.26).

In the centre manifold framework we can choose any reasonable measure of the emergent fluid dynamics in order to parameterise the model: we choose the depth-averaged lateral velocities and fluid depth. This contrasts with usual modelling where models for depth-averaged quantities arise from heuristic depth-averaging of equations.
2.4.2 The order of errors in the construction

The order of errors in the construction is phased in terms of the small parameters. The lateral spatial structure of large scale environmental turbulent fluids varies relatively slowly compared with the vertical structure. We consider the lateral spatial derivatives $\partial_x$ and $\partial_y$ are small. Therefore, there are three small physical factors in the system, the lateral spatial derivatives $\partial_x$ and $\partial_y$, and the small mean slope $\tan \theta$ in the $x$ direction. Thus, errors $O(\partial_x^p + \partial_y^p + \tan^p \theta)$ denote the error terms of the slow manifold. That each term in model has less than $p$ factors in total of $\partial_x$, $\partial_y$ and $\tan \theta$. The bigger the exponent number $p$, the higher order of the modelling.

The artificial small parameter $\gamma$ has no physical meaning but is introduced to establish the slow manifold. We compute to high order in the artificial parameter $\gamma$ and then evaluate at $\gamma = 1$. Complexity grows quickly.

2.4.3 The low leading model

Computer algebra in Appendix A.1 derives the slow manifold in the system in terms of the depth $h(x,y,t)$ and the depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$. The computer algebra replaces much tedious derivation by hand. This subsection records the low leading order evolutions of these terms on the slow manifold.

Truncating to errors $O(\partial_x^2 + \partial_y^2 + \tan^2 \theta, \gamma^5)$, the computer derives that the leading order evolution of $h(x,y,t)$, $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$ are described by the fluid conservation equation and by effective lateral momentum equations:

\[
\frac{\partial h}{\partial t} = -\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right),
\]
\[
\frac{\partial \bar{u}}{\partial t} = \left(-0.00293\gamma + 0.000105\gamma^2 + 0.000009\gamma^3 + 0.0000025\gamma^4\right) \frac{\bar{u}q}{h}
+ (0.985 + 0.00799\gamma - 0.000536\gamma^2 + 0.000025\gamma^3 + 0.0000025\gamma^4) \left[\tan \theta - \frac{\partial(h + b)}{\partial x}\right]
+ (-1.045 + 0.204\gamma - 0.000189\gamma^2 + 0.00007\gamma^3 + 0.000006\gamma^4) \bar{u} \frac{\partial \bar{u}}{\partial x}
+ (-1.03 + 0.0136\gamma - 0.000134\gamma^2 + 0.000042\gamma^3 + 0.000039\gamma^4) \bar{v} \frac{\partial \bar{u}}{\partial y}
+ (-0.0152 + 0.0071\gamma - 0.000082\gamma^2 + 0.0000028\gamma^3) \left(\frac{\bar{u}^2}{h} \frac{\partial h}{\partial x} + \frac{\bar{u}\bar{v}}{h} \frac{\partial h}{\partial y}\right)
+ O(\partial_x^2 + \partial_y^2 + \tan^2 \theta, \gamma^5),
\]
\[
\frac{\partial \bar{v}}{\partial t} = \left(-0.00293\gamma + 0.000105\gamma^2 + 0.000009\gamma^3 + 0.0000025\gamma^4\right) \frac{\bar{v}q}{h}
\]
Table 2.1: Partial sums from evaluating coefficients at $\gamma = 1$ of selected terms in equation (2.32) indicates that the power series in $\gamma$ converge quickly.

<table>
<thead>
<tr>
<th>$\gamma^0$</th>
<th>$\tan \theta$</th>
<th>$\bar{u}\bar{q}/h$</th>
<th>$\bar{u}\partial \bar{u}/\partial x$</th>
<th>$\bar{u}^2/h\partial h/\partial x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9849</td>
<td>0.9849</td>
<td>0.002931</td>
<td>-1.045</td>
<td>-0.01519</td>
</tr>
<tr>
<td>0.9929</td>
<td>0.9929</td>
<td>-0.002826</td>
<td>-1.025</td>
<td>-0.008173</td>
</tr>
<tr>
<td>0.9924</td>
<td>0.9924</td>
<td>-0.002817</td>
<td>-1.025</td>
<td>-0.008145</td>
</tr>
<tr>
<td>0.9924</td>
<td>0.9924</td>
<td>-0.002815</td>
<td>-1.025</td>
<td>-0.008122</td>
</tr>
</tbody>
</table>

\[
+ (-0.989 - 0.00366\gamma + 0.0000273\gamma^2 - 0.000025\gamma^3) [\frac{\partial (h + b)}{\partial y}] \\
+ (-1.042 + 0.0166\gamma - 0.000055\gamma^2 + 0.000077\gamma^3 + 0.000053\gamma^4) v \frac{\partial \bar{v}}{\partial y} \\
+ (-1.026 + 0.00975\gamma - 0.0000006\gamma^2 + 0.00005\gamma^3 + 0.000031\gamma^4) \bar{u} \frac{\partial \bar{v}}{\partial x} \\
+ (-0.0152 + 0.0071\gamma - 0.000082\gamma^2 + 0.000028\gamma^3) \left( \frac{\bar{u}\partial h}{h} \frac{\partial\bar{h}}{\partial x} + \frac{\bar{v}^2}{h} \frac{\partial h}{\partial y} \right) \\
+ \mathcal{O}(\partial_x^2 + \partial_y^2 + \tan^2 \theta, \gamma^5),
\] (2.33)

where $\bar{q} = \sqrt{\bar{u}^2 + \bar{v}^2}$ is the mean speed of the flow.

Equations (2.31)–(2.33) are a result of taking into account the relatively slow variations in the lateral directions $x$ and $y$ via small but non-zero lateral derivatives $\partial_x$ and $\partial_y$. They are a form of the slowly varying approximation (Roberts 1997, e.g.). Equations (2.31)–(2.33) model the turbulent fluid flow with the introduced artificial parameter $\gamma$. Upon setting the artificial parameter $\gamma = 1$, equations (2.31)–(2.33) model the physical problem.

To the error $\mathcal{O}(\partial_x^3 + \partial_y^3 + \tan^3 \theta, \gamma^5)$, the momentum equation (2.32) incorporates the inertial term $\partial \bar{u}/\partial t$, self-advection term $\bar{u}\partial \bar{u}/\partial x + \bar{v}\partial \bar{v}/\partial y$, varying fluid thickness $\bar{u}^2/h\partial h/\partial x + \bar{u}\bar{v}/h\partial h/\partial y$, bed drag term $\bar{u}\bar{q}/h$, and gravitational forcing $\tan \theta - \partial (h + b)/\partial x$. The turbulent mixing terms will be included by truncating to errors $\mathcal{O}(\partial_x^4 + \partial_y^4 + \tan^4 \theta, \gamma^5)$ in section 2.4.4.

Every coefficient in equations (2.32)–(2.33) is a series in $\gamma$. Computations indicate that series in $\gamma$ converge quickly. Table 2.1 shows the partial sums for $\gamma = 1$ of four selected terms in equation (2.32). The values in the table indicate that evaluating the expansions to terms $\gamma^3$ for $\gamma = 1$ gives sufficiently accurate coefficients in the evolution of the depth-averaged lateral velocities $\bar{u}$ and $\bar{v}$.

Although equations (2.31)–(2.33) are expressed in terms of depth-averaged
lateral velocities, they are derived not by depth-averaging, but instead by systematically accounting for interaction between vertical profiles of the velocity, the stress, bed drag, lateral space variations and bed topography. The form of coefficients in equations (2.31)–(2.33) are supported by dynamical systems theory: the detail in the equations reflects that a slow manifold is in principle composed of exact solutions of the Smagorinski dynamics (2.8) and hence accounts for all interactions up to a given order of analysis no matter how small the numerical coefficient in the interactions.

2.4.4 Model the turbulent dispersion

The turbulent lateral dispersion in the turbulent flow requires the model to resolve second derivatives in \( x \) and \( y \). Derive the second derivative terms, such as the dissipation terms, by extending the derivation to errors \( O(\partial^3_x + \partial^3_y + \tan^3 \theta, \gamma^3) \). Executing the computer algebra in Appendix A.1, evaluating at \( \gamma = 1 \), leads to

\[
\frac{\partial h}{\partial t} = - \left( \frac{\partial h \bar{u}_x}{\partial x} + \frac{\partial h \bar{v}_y}{\partial y} \right),
\]

\[
\frac{\partial \bar{u}}{\partial t} = -0.00293 \frac{\bar{u} \bar{q}}{h} + 0.993 \left[ \tan \theta - \frac{\partial (h + b)}{\partial x} \right]
\]

\[
- 1.025 \frac{\partial \bar{u}}{\partial x} - 1.017 \frac{\partial \bar{v}}{\partial y} - 0.00817 \left( \frac{\bar{u}^2}{h} \frac{\partial h}{\partial x} - \frac{\bar{u} \bar{v}}{h} \frac{\partial h}{\partial y} \right)
\]

\[
+ \left( 0.0171 q + 0.339 \frac{\bar{u}^2}{q} - 0.171 \frac{\bar{u}^4}{q^3} \right) h \frac{\partial^2 \bar{u}}{\partial x^2}
\]

\[
+ \left( 0.173 q - 0.339 \frac{\bar{u}^2}{q} + 0.171 \frac{\bar{u}^4}{q^3} \right) h \frac{\partial^2 \bar{u}}{\partial y^2}
\]

\[
+ \left( 0.507 \frac{\bar{u} \bar{v}}{q} - 0.342 \frac{\bar{u}^3 \bar{v}}{q^3} \right) h \frac{\partial}{\partial x} \left( \frac{\partial \bar{u}}{\partial y} \right)
\]

\[
+ \left( 0.0116 q - 0.170 \frac{\bar{u}^2}{q} + 0.342 \frac{\bar{u}^4}{q^3} \right) h \frac{\partial}{\partial x} \left( \frac{\partial \bar{v}}{\partial y} \right)
\]

\[
+ \left( 0.0351 q + 0.678 \frac{\bar{u}^2}{q} - 0.351 \frac{\bar{u}^4}{q^3} \right) \frac{\partial h}{\partial x} \frac{\partial \bar{u}}{\partial x}
\]

\[
+ \left( 0.366 q - 0.709 \frac{\bar{u}^2}{q} + 0.351 \frac{\bar{u}^4}{q^3} \right) \frac{\partial h}{\partial y} \frac{\partial \bar{u}}{\partial y}
\]

\[
+ O(\partial^3_x + \partial^3_y + \tan^3 \theta, \gamma^3),
\]

\[
\frac{\partial \bar{v}}{\partial t} = -0.00293 \frac{\bar{v} \bar{q}}{h} - 0.993 \frac{\partial (h + b)}{\partial y}
\]
\[-1.025 \frac{\partial \bar{v}}{\partial y} - 1.017 \bar{u} \frac{\partial \bar{v}}{\partial x} - 0.00809 \left( \frac{\bar{u} \partial h}{h} \frac{\partial h}{\partial x} - \frac{\bar{v}^2}{h} \frac{\partial h}{\partial y} \right) \] 

(2.34l)

\[+ \left( 0.00429 q - 0.00251 \frac{\bar{u}^2}{q} + 0.171 \frac{\bar{u}^4}{q^3} \right) h \frac{\partial^2 \bar{v}}{\partial x^2} \] 

(2.34m)

\[+ \left( 0.184 q + 0.00321 \frac{\bar{u}^2}{q} - 0.171 \frac{\bar{u}^4}{q^3} \right) h \frac{\partial^2 \bar{v}}{\partial y^2} \] 

(2.34n)

\[+ \left( 0.166 \frac{\bar{u} \bar{v}}{q} + 0.342 \frac{\bar{u} \bar{v}^3}{q^3} \right) h \frac{\partial}{\partial x} \left( \frac{\partial \bar{v}}{\partial y} \right) \] 

(2.34o)

\[+ \left( 0.184 q - 0.513 \frac{\bar{u}^2}{q} + 0.342 \frac{\bar{u}^4}{q^3} \right) h \frac{\partial}{\partial x} \left( \frac{\partial \bar{u}}{\partial y} \right) \] 

(2.34p)

\[+ \left( 0.00866 q + 0.00667 \frac{\bar{u}^2}{q} + 0.351 \frac{\bar{u}^4}{q^3} \right) \frac{\partial h}{\partial x} \frac{\partial \bar{v}}{\partial x} \] 

(2.34q)

\[+ \left( 0.358 q + 0.0257 \frac{\bar{u}^2}{q} - 0.351 \frac{\bar{u}^4}{q^3} \right) \frac{\partial h}{\partial y} \frac{\partial \bar{v}}{\partial y} \] 

(2.34r)

\[+ \mathcal{O} (\partial^3_x + \partial^3_y + \tan^3 \theta, \gamma^3). \] 

(2.34s)

The momentum equations in the model (2.34) include the effects of drag and gravity in lines (2.34b) and (2.34k), advection in lines (2.34c) and (2.34l), dissipation in lines (2.34d)–(2.34g) and (2.34m)–(2.34p), and other turbulent mixing effects in lines (2.34h)–(2.34i) and (2.34q)–(2.34r).

The model (2.34) includes all the effective terms in established modelling (Rodi 1984, Yulistiyanto et al. 1998, Bousmar 2002, e.g.). Equation (2.34a) is a direct consequence of conservation of fluid. Hence, I only concentrate on the momentum equation of the depth-averaged lateral velocity \( \bar{u} \), because the momentum equation of the depth-averaged lateral velocity \( \bar{v} \) is similar, by symmetry, to the depth-averaged lateral velocity \( \bar{u} \).

Bousmar (2002) reported the depth-averaged Saint–Venant equation to describe the evolution of the depth-averaged velocity \( \bar{u} \); that is

\[ \frac{\partial \bar{u}}{\partial t} = -0.002 \frac{\bar{u} q}{h} - \frac{\partial (h + b)}{\partial x} - \bar{u} \frac{\partial \bar{u}}{\partial x} - \bar{v} \frac{\partial \bar{u}}{\partial y} + 0.089 \frac{\partial^2 \bar{u}}{\partial x^2} + 0.045 \frac{\partial^2 \bar{u}}{\partial y^2} \]

\[ - \frac{1}{h} \left[ 0.0077 \frac{\partial}{\partial x} (h \bar{u}^2) + \frac{\partial}{\partial y} (0.0014 h \bar{u}^2) \right]. \] 

(2.35)

The momentum equation of the depth-averaged lateral velocity \( \bar{u} \) in the model (2.34) contains all the terms in established equation (2.35). However, the coefficients of these terms are slightly different. For example, the coefficient of the gravity terms \( \tan \theta - \partial (h + b) / \partial x \) is 0.993 in our model, which is 0.7% smaller than the coefficient 1 in equation (2.35). The coefficient of the drag term \( \bar{u} q / h \) is 0.00293 in our model, which is nearly 50% bigger.
than the corresponding coefficient in equation (2.35), due to the calibration with environmental flow (Roberts et al. 2008, e.g.). The coefficients of the advection terms $\bar{u}\partial\bar{u}/\partial x$ and $\bar{v}\partial\bar{u}/\partial y$ are 1% and 1.4% bigger in our model than in the model (2.35). Roberts (1997) commented that the differences of the coefficients for these terms are due to the different vertical shape of the leading order velocity profile $u(Z)$. That these terms are only slightly different confirms the traditional conservation arguments are appropriate for such environmental flow.

However, the model (2.34) also represents more subtle effects, which can be important in complex flow regimes. For example, consider that the depth-averaged lateral velocity $\bar{v}$ is negligible, and then from the terms in lines (2.34d)–(2.34e) we obtain the dispersion terms $0.01\bar{h}\bar{u}(\partial^2\bar{u}/\partial x^2 + \partial^2\bar{u}/\partial y^2) + 0.168\bar{h}\bar{u}\partial^2\bar{u}/\partial x^2$, which includes an enhanced dispersion term $0.168\bar{h}\bar{u}\partial^2\bar{u}/\partial x^2$. This is supported by the similar enhanced shear dispersion terms derived by Mercer & Roberts (1994) and Roberts (2004) in pipe flows, as originally recognised by Taylor (1953).

In practice one might only implement those terms in the model (2.34) which are important in a specific application. For example, in the application of the two dimensional (no $y$ dependence) Smagorinski model on simulating dam breaking flow, Georgiev et al. (2009) estimated and compared the effect of each term in equation (2.34j). Georgiev et al. (2009) reported that apart from the self-advection term $\bar{u}\partial\bar{u}/\partial x$, only the term $\bar{u}\partial\bar{h}/\partial x\partial\bar{u}/\partial x$ related to the turbulent mixing contributes significantly to the water front speed in their dam breaking simulations.

The initially quoted model (2.1)–(2.3) is a simplified version of the model (2.34), where I choose the terms with the important effects of inertia, bed drag, gravity and dissipation to model turbulent environmental turbulent fluids.

### 2.4.5 Vertical distribution of the velocity

This subsection explores the vertical distribution of the lateral velocity $u$. Computer algebra in Appendix A.1 derives the slow manifold of the turbulent flow in out-of-equilibrium dynamics.

The centre manifold approach does not impose a specific cross-sectional velocity distribution as done by other methods, instead our approach empowers the Smagorinski equations to determine the cross-sectional structures. Recall that the locally scaled vertical coordinate $Z = (z - b)/h$. We concentrate on the vertical distribution of the lateral velocity $u(Z)$. Truncate to errors $O(\partial^2 + \partial^2 + \tan^2 \theta, \gamma^3)$. Executing the computer algebra in Appendix A.1 and evaluating at $\gamma = 1$ leads to the following approximation of the lateral
Figure 2.4: The profiles of the equilibrium lateral velocity $u(Z)$ in the vertical at different bed slopes. The line curves are from equation (2.37) and the dash curves are by the approximation (2.38).
velocity on the slow manifold:

\[
\begin{align*}
\mathbf{u}(Z) & \approx \bar{u}(0.816 + 0.445Z - 0.0916Z^2 - 0.0307Z^3 - 0.00383Z^4 - 0.000418Z^5) \\
& \quad + \tan \theta \frac{h}{q} (2.208 + 1.204Z - 14.31Z^2 + 8.069Z^3 - 1.569Z^4 \\
& \quad + 0.954Z^5 + 0.586Z^6 + 0.119Z^7) \\
& \quad + \frac{h \bar{u} \partial \bar{u}}{q} (2.326 + 1.269Z - 13.52Z^2 + 4.585Z^3 + 0.894Z^4 \\
& \quad + 0.783Z^5 + 0.533Z^6 + 0.118Z^7 + 0.0106Z^8) \\
& \quad + \frac{h \bar{v} \partial \bar{u}}{q} (2.352 + 1.283Z - 13.25Z^2 + 3.622Z^3 + 1.53Z^4 \\
& \quad + 0.708Z^5 + 0.543Z^6 + 0.129Z^7 + 0.0127Z^8) \\
& \quad + \frac{h}{q} \left( \frac{\partial h}{\partial x} + \frac{\partial b}{\partial x} \right) (-2.208 - 1.204Z + 14.31Z^2 - 8.069Z^3 \\
& \quad + 1.569Z^4 - 0.954Z^5 - 0.586Z^6 - 0.119Z^7),
\end{align*}
\]

(2.36)

Physically, equation (2.36) describes the vertical details of the lateral velocity \( \mathbf{u}(Z) \) in terms of the vertical position \( Z \), fluid depth \( h \), depth-averaged velocities \( \bar{u} \) and \( \bar{v} \), and the slope \( \tan \theta \) of the mean bed \( b \).

We concentrate on the vertical distribution of the lateral velocity \( \mathbf{u}(Z) \) in steady flow, and compare our prediction of the lateral velocity \( \mathbf{u}(Z) \) with relative published experimental data (Schultz & Flack 2007, 2013, e.g.). Consider turbulent fluid flowing down a flat bed of a slope \( \tan \theta \); that is the mean bed \( b = 0 \). The steady fluid flow has the nondimensional equilibrium depth \( H = 1 \), and then the nondimensional equilibrium depth-averaged lateral velocities are \( \bar{q} = U = 18.7 \tan^{1/2} \theta \) and \( V = 0 \) via the evolution equations (2.2)–(2.3). Schultz & Flack (2013) experimentally studied the smooth-wall turbulent channel flow with the Reynolds number \( \text{Re} \) up to 300,000, and showed that the mean flow is approximately independent of the Reynolds number. In their early work, Schultz & Flack (2007) showed when the relative roughness, the ratio of the roughness height and the boundary-layer thickness, is small, the mean velocity shape for the rough and smooth walls are similar in the outer layer.

However, compared with large roughness in the environmental flows under consideration, the roughness \( \approx 1.5 \times 10^{-3} \), the ratio between the roughness height and the fluid depth, in the experiments of Schultz & Flack (2007) are small. To compare with the experimental data (Schultz & Flack 2007, 2013, e.g.), we derive the equilibria profiles, evaluate the lateral velocity at these
Figure 2.5: Vertical distribution of the lateral velocity from: (blue curve) the approximation (2.37); and (red circle curve) the experimental measurements (Schultz & Flack 2013, Fig. 2). The ratio $u(Z)/u(1)$ is independent of the slope $\tan \theta$, where $u(1)$ is the lateral velocity at the level $Z = 1$.

equilibria, and obtain the profile of the lateral velocity in this steady flow

$$
\frac{u(Z)}{\sqrt{\tan \theta}} \approx 2.18 + 1.19Z - 0.297Z^2 - 0.0533Z^3 - 0.0173Z^4 - 0.00366Z^5 - 0.00115Z^6 - 0.000089Z^7.
$$

(2.37)

Figure 2.4 plots the vertical profile of the lateral velocity $u(Z)$ with the parameter $c_t = 0.02$ (from equation (2.11)) at three different slopes. When the slope of the mean bed increases, the fluid flows faster. The lateral velocity shows a small slip on the mean bed, which corresponds to the turbulent mixing across the fluid on the mean bed. For an indicative comparison, we integrate Smagorinski equation (2.8) in the vertical by assuming $\partial_t = \partial_x = 0$, and obtain an approximation

$$
u(Z) \approx \tan^{1/2} \theta \left(3.56 \log(1.42 - Z) + 37.6 \log(3.42 + Z) - 31.8\right).
$$

(2.38)

The dash curves in Figure 2.4 show the profile of the approximation (2.38) at three different slopes. The agreement between our lateral velocity approximation (2.37) and approximation (2.38) indicates our model is reasonable.

38
Figure 2.6: Shear stress profile of: (blue curve) our approximation $\tau_z$; and (red triangle curve) the experimental measurements of flow over rough bed (Schultz & Flack 2007, Fig. 9). The error bars show the ±9% uncertainty in their experiments.
Figure 2.5 compares the profile of the lateral velocity from the approximation (2.37) (blue line curve) with the experimental measurements (red circle curve) by Schultz & Flack (2013). From the approximation (2.37), the velocity ratio \( u(\mathbf{z}) / u(1) \) is independent of the slope \( \tan \theta \). Our prediction of the lateral velocity in the vertical agrees reasonably the experimental lateral velocity by Schultz & Flack (2013), except near the mean bed. That is because we expect that the large roughness of environmental flows over stones, roots and debris typically breaks up any log boundary layer. Thus, we do not resolve the turbulent log layer, and are interested in the dynamics determined by the relatively large scale of the fluid depth.

Shear stress arises in the turbulent fluid. We explore the shear stress \( \tau_{xz} \) in the steady flow of nondimensional equilibrium depth \( H = 1 \), and depth-averaged lateral velocities \( \bar{q} = U = 18.7 \tan^{1/2} \theta \) and \( V = 0 \). In this steady flow, we predict the shear stress \( \tau_{xz} \) has the near linear profile of

\[
\frac{\tau_{xz}(\mathbf{z})}{\tan \theta} \approx 0.997 - 0.999Z + 0.000284Z^2 - 0.00995Z^3 \\
+ 0.00776Z^4 + 0.000791Z^5 + 0.000072Z^6. 
\]  

(2.39)

Figure 2.6 shows that our approximation of shear stress \( \tau_{xz} \) (blue curve) approaches the experimental measurements (red triangle curve) by Schultz & Flack (2007). The shear stress is approximately straight possibly due to the need to dissipate the near constant forcing of gravity. A difference occurs near the bed, because we do not resolve the log boundary layer.

2.5 Eigenvalue analysis of the model

This section explores the eigenvalue analysis of the nondimensional Smagorinski turbulence model (2.1)–(2.3). The zero and negative real-parts of the eigenvalues indicate the model is stable.

Consider the turbulent flow over the mean bed \( z = 0 \) has an equilibrium of depth \( H = 1 \), and depth-averaged lateral velocities \( \bar{q} = U \neq 0 \) and \( V = 0 \), so the mean speed of the flow is \( \bar{q} = \sqrt{U^2 + V^2} = U \). Recall that the mean slope of the mean bed is \( \tan \theta \). The momentum equation (2.2) implies the equilibrium of depth-averaged lateral velocity \( U \approx 18.7 \tan^{1/2} \theta \).

We impose small perturbations to this equilibrium of the flow, and seek solutions of the form

\[
h = 1 + \hat{h} \exp(\lambda t + ik_x x + ik_y y), \\
\bar{u} = \bar{q} = 18.7 \tan^{1/2} \theta + \hat{u} \exp(\lambda t + ik_x x + ik_y y), \\
\bar{v} = 0 + \hat{v} \exp(\lambda t + ik_x x + ik_y y),
\]  

(2.40)

40
Figure 2.7: The fast branch of the rate $\Re \lambda$ versus the nondimensional wavenumbers $k_x$ and $k_y$ in the x and y directions in equation (2.41). The colour corresponds to the values of $\Im \lambda$. The mean slope is set to $\tan \theta = 0.001$. The negative values of the colour bar indicate that waves propagate in the x direction.
where \( k_x \) and \( k_y \) are nondimensional wavenumbers in \( x \) and \( y \) directions, \( \hat{h}, \hat{u}, \) and \( \hat{v} \) are small amplitudes of the perturbations. Substitute the solution form \((2.40)\) into the governing equations \((2.1)–(2.3)\), and obtain the eigenvalue problem of

\[
\lambda \begin{bmatrix} \hat{h} \\ \hat{\bar{u}} \\ \hat{\bar{v}} \end{bmatrix} = M \begin{bmatrix} \hat{h} \\ \hat{\bar{u}} \\ \hat{\bar{v}} \end{bmatrix},
\]

where the coefficient matrix

\[
M = \begin{bmatrix}
-18.7ik_x \tan^{1/2} \theta & -ik_x & -ik_y \\
-0.993ik_x & -[0.053 + 19.2ik_x + 3.33(k_x^2 + k_y^2)] \tan^{1/2} \theta & 0 \\
-0.993ik_y & 0 & -[0.053 + 19.2ik_y + 3.33(k_x^2 + k_y^2)] \tan^{1/2} \theta
\end{bmatrix}.
\]

The characteristic equation \((2.41)\) has three solution branches of the eigenvalue \( \lambda \), one fast branch and two slow branches. Figure 2.7 exhibits the fast branch of the growth rate \( \Re \lambda \) varying with the nondimensional wavenumbers \( k_x \) and \( k_y \) through equation \((2.41)\). The color scaling is determined by the values of \( \Im \lambda \). The mean bed has a slope \( \tan \theta = 0.001 \), which indicates subcritical flow occurs as \( U = 18.7 \tan^{1/2}(0.001) \approx 0.6 < 1 \). The values represented by the colour bar indicate that decaying waves may exist. Negative values of the growth rates indicate the modes decay in time. The growth rate \( \Re \lambda \) decays more quickly versus the wavenumber \( k_y \), which means the waves in the \( y \) direction decay to steady faster. Figure 2.8 plots the two slow branches of the growth rates. The upper graph indicates the upstream waves decay quickly in \( k_x \), while the lower graph indicates the downstream waves decay quickly in \( k_y \).
Figure 2.8: The two slow branches of the growth rate $\Re \lambda$ versus the nondimensional wavenumbers $k_x$ and $k_y$ in the $x$ and $y$ directions according to the equation (2.41). The colour corresponds to the values of $\Im \lambda$. The upper/lower graph indicates there are upstream/downstream propagating damped waves.
Fluid flowing down a slope generates turbulence that provides the drag to balance the gravitational forcing. The bigger the slope tan \(\theta\), the larger the turbulence, and the stronger the decay. For example, the growth rate \(\Re \lambda \approx -0.884\) at the nondimensional wavenumber \(k_x = 2\) and \(k_y = 2\) when the mean slope tan \(\theta = 0.001\) in the fast branch in Figure 2.7, whereas when the slope tan \(\theta = 0.01\) the corresponding growth rate \(\Re \lambda \approx -2.67\), which is three times as fast in decay.

Figure 2.7–2.8 show that the growth rates \(\Re \lambda\) are not positive, which indicates the model (2.1)–(2.3) is stable.

2.6 Numerical scheme for numerically solving the model

This section describes the staggered grid numerical method to compute solutions of the 3D turbulence model (2.1)–(2.3) in several physical scenarios. The staggered method is used here because of its reasonableness and stability (Armfield & Street 2005, e.g.). Previous work (Cao 2012) shows how the staggered method effectively works for seeking numerical solutions of 2D models for thin film flow and turbulence flow. This section develops the staggered grid scheme for computing solutions of the model (2.1)–(2.3).

2.6.1 Developing the staggered grid scheme

Consider a domain with length \(L_x\) and width \(L_y\). Let there be \(N_x\) intervals in the \(x\) direction and \(N_y\) intervals in the \(y\) direction. Then the spatial steps are \(\delta x = L_x/N_x\) and \(\delta y = L_y/N_y\), respectively. Define the grid points \(x_i = i\delta x\) and \(y_j = j\delta y\) with \(i = 1 : N_x\) and \(j = 1 : N_y\). Figure 2.9 exhibits the diagram of the staggered grid near the point \((x_i, y_j)\). Locate the depth \(h\) at the blue points. Locate the depth-averaged lateral velocity \(\bar{u}\) at the half-points of the \(x\)-grid (red points) and the depth-averaged lateral velocity \(\bar{v}\) at the half points of the \(y\)-grid (green points). Define \(h_{ij}(t) = h(x_i, y_j, t)\), \(\bar{u}_{i+1/2,j}(t) = \bar{u}(x_{i+1/2}, y_j, t)\) and \(\bar{v}_{i,j+1/2}(t) = \bar{v}(x_i, y_{j+1/2}, t)\). Therefore, this staggered grid discretises the physical fields.

2.6.2 Approximation of spatial derivatives

This subsection approximates the spatial derivatives by the developed staggered grid scheme. Compute the depth \(h\) at the half step grids (that is, red
Figure 2.9: The staggered grid scheme at the point \((x_i, y_j)\) and its neighbouring points. Locate the depth \(h\) at the blue points. Locate the mean velocity \(\bar{u}\) at the centre of \(x\)-grids (red points) and the mean velocity \(\bar{v}\) at the centre of \(y\)-grids (green points).
and green points in Figure 2.9) by

\[ h_{i+1/2,j} = \frac{1}{2} (h_{i,j} + h_{i+1,j}) , \quad h_{i,j+1/2} = \frac{1}{2} (h_{i,j} + h_{i,j+1}) , \]

\[ h_{i+1/2,j+1/2} = \frac{1}{2} (h_{i+1/2,j+1} + h_{i+1/2,j}) \]

Similarly, for the mean velocities \( \bar{u} \) and \( \bar{v} \), compute

\[ \bar{u}_{i,j} = \frac{1}{2} (\bar{u}_{i+1/2,j} + \bar{u}_{i-1/2,j}) , \quad \bar{u}_{i,j+1/2} = \frac{1}{2} (\bar{u}_{i,j} + \bar{u}_{i,j+1}) , \]

\[ \bar{u}_{i+1/2,j+1/2} = \frac{1}{2} (\bar{u}_{i+1/2,j+1} + \bar{u}_{i+1/2,j-1}) , \]

\[ \bar{v}_{i,j} = \frac{1}{2} (\bar{v}_{i,j+1/2} + \bar{v}_{i,j-1/2}) , \quad \bar{v}_{i+1/2,j} = \frac{1}{2} (\bar{v}_{i,j} + \bar{v}_{i+1,j}) , \]

\[ \bar{v}_{i+1/2,j+1/2} = \frac{1}{2} (\bar{v}_{i,j+1/2} + \bar{v}_{i+1,j+1/2}) . \]

The local mean flow speed at the grid point \((x_i, y_j)\) is \( \bar{q}_{i,j} = \sqrt{\bar{u}_{i,j}^2 + \bar{v}_{i,j}^2} \).

Then we approximate the right-hand side of the PDEs (2.1)–(2.3) by standard centre difference approximation, and obtain

\[
\frac{\partial h_{i,j}}{\partial t} = -\frac{1}{\delta x} (h_{i-1/2,j} \bar{u}_{i+1/2,j} - h_{i+1/2,j} \bar{u}_{i-1/2,j}) \\
- \frac{1}{\delta y} (h_{i,j+1/2} \bar{v}_{i,j+1/2} - h_{i,j-1/2} \bar{v}_{i,j-1/2}) ,
\]

\[
\frac{\partial \bar{u}_{i+1/2,j}}{\partial t} = -0.00283 \bar{u}_{i+1/2,j} \bar{q}_{i+1/2,j} \\
+ 0.993 \left[ \tan \theta - \frac{1}{\delta x} (h_{i+1,j} - h_{i,j}) - \frac{1}{\delta x} (b_{i+1,j} - b_{i,j}) \right] \\
- 1.017 \frac{1}{\delta y} \bar{v}_{i,j+1/2} (\bar{u}_{i,j+1} - \bar{u}_{i,j}) \\
- 1.025 \frac{1}{\delta x} \bar{u}_{i,j+1/2} (\bar{u}_{i+1,j} - \bar{u}_{i,j}) \\
+ 0.094 h_{i+1/2,j} \bar{q}_{i+1/2,j} \frac{4}{\delta x^2} (\bar{u}_{i+1,j} - 2\bar{u}_{i+1/2,j} + \bar{u}_{i,j}) \\
+ 0.084 \frac{\bar{u}_{i+1/2,j}^2 - \bar{u}_{i+1,j}^2}{q_{i+1/2,j}} \frac{4}{\delta x^2} (\bar{u}_{i+1,j} - 2\bar{u}_{i+1/2,j} + \bar{u}_{i,j}) \\
+ 0.188 q_{i+1/2,j} \frac{1}{\delta x^2} (h_{i+1,j} - h_{i,j}) (\bar{u}_{i+1,j} - \bar{u}_{i,j}) \\
+ 0.168 \frac{\bar{u}_{i+1/2,j}^2 - \bar{v}_{i+1/2,j}^2}{q_{i+1/2,j}} \frac{1}{\delta x^2} (h_{i+1,j} - h_{i,j}) (\bar{u}_{i+1,j} - \bar{u}_{i,j})
\]
\[ + 0.094 h_{i+1/2,j} q_{i+1/2,j} \frac{4}{\delta y^2} \left( \bar{u}_{i+1/2,j+1/2} - 2\bar{u}_{i+1/2,j} + \bar{u}_{i+1/2,j-1/2} \right) \]
\[ + 0.084 \frac{\bar{u}_{i+1/2,j}^2 - \bar{v}_{i+1/2,j}^2}{q_{i+1/2,j}} \frac{4}{\delta y^2} \left( \bar{u}_{i+1/2,j+1/2} - 2\bar{u}_{i+1/2,j} + \bar{u}_{i+1/2,j-1/2} \right) \]
\[ + 0.188 q_{i+1/2,j} \frac{1}{\delta y^2} \left( h_{i+1/2,j+1} - h_{i+1/2,j} \right) \left( \bar{u}_{i+1/2,j+1/2} - \bar{u}_{i+1/2,j} \right) \]
\[ + 0.168 \frac{\bar{u}_{i+1/2,j}^2 - \bar{v}_{i+1/2,j}^2}{\delta y^2 q_{i+1/2,j}} \left( h_{i+1/2,j+1} - h_{i+1/2,j} \right) \left( \bar{u}_{i+1/2,j+1/2} - \bar{u}_{i+1/2,j} \right) , \]
\[ \frac{\partial \bar{v}_{i,j+1/2}}{\partial t} = \ldots , \] (2.43)

where the approximation (2.44) is a similar expression to (2.43). The matlab function in Appendix B.1 are based on these standard centre difference approximations.

Boundary conditions need to be specified. Typically, choose periodic boundary conditions for all the three variables, \( h, \bar{u} \) and \( \bar{v} \), both in the \( x \) and \( y \) direction, that is
\[ h(x, y, t) = h(x + L_x, y, t) , \quad h(x, y, t) = h(x, y + L_y, t) , \]
\[ \bar{u}(x, y, t) = \bar{u}(x + L_x, y, t) , \quad \bar{u}(x, y, t) = \bar{u}(x, y + L_y, t) , \]
\[ \bar{v}(x, y, t) = \bar{v}(x + L_x, y, t) , \quad \bar{v}(x, y, t) = \bar{v}(x, y + L_y, t) . \]

2.6.3 Numerical eigenvalue analysis

This subsection calculates the numerical eigenvalues of the staggered discretisation (2.42)–(2.43). Figure 2.10 plots the growth rates \( \Re \lambda \) and frequency \( \Im \lambda \) of the model (2.42)–(2.43). The mean bed has a mean slope \( \tan \theta = 0.001 \). There are \( N_x = 20 \) and \( N_y = 4 \) intervals in the \( x \) and \( y \) directions. The domain has a length \( L_x = 20\pi \) and width \( L_y = \pi \), so the spatial steps \( \delta x = \pi \) and \( \delta y = \pi/4 \). There are total 117 pairs of eigenvalues and another six real eigenvalues \( 0, -0.0017, -0.0034, -0.157, -0.173 \) and \( -0.328 \). Figure 2.10 shows the features:

- the symmetric non-zero \( \Im \lambda \) indicates waves, where the negative real-part indicates they decay due to the dissipation;
- the ellipses are due to upstream/downstream asymmetry;
- the groups correspond to different wavenumbers in \( y \) direction; and
- the closely spaced eigenvalues have different wavenumber in \( x \) direction.
Figure 2.10: Numerical eigenvalues of the model (2.42)–(2.44) by the staggered numerical scheme with $N_x = 20$ and $N_y = 4$ intervals. The mean slope $\tan \theta = 0.001$ indicates the subcritical flow as $U \approx 0.6 < 1$. The length of the domain $L_x = 20\pi$ and the width $L_y = \pi$, so the steps $\delta x = \pi$ and $\delta y = \pi/4$. There are 117 pairs of eigenvalues and another six real eigenvalues 0, $-0.0017$, $-0.0034$, $-0.157$, $-0.173$ and $-0.328$. 

\[ -0.35 \quad -0.3 \quad -0.25 \quad -0.2 \quad -0.15 \quad -0.1 \quad -0.05 \quad 0 \]
\[ -0.35 \quad -0.3 \quad -0.25 \quad -0.2 \quad -0.15 \quad -0.1 \quad -0.05 \quad 0 \]
Figure 2.11: Fluid along a straight channel (2.45) with the channel width $2\beta = 8$, shallows $B = 0.1$ and the mean slope $\tan \theta = 0.01$. The flow depth of the middle channel is $h = 1$.

2.7 Modelling fluids along open channels

This section focuses on the application of the model (2.1)–(2.3) on open channel fluids. Three types of open channels are considered, including the straight channels in section 2.7.1, curved channels in section 2.7.2 and river-like channels in section 2.7.3. The water covers the entire domain in order to avoid, at this stage, complications of moving contact lines between wet and dry bed: the flow is in the channel and over a surrounding flood plain. Numerical results indicate that the model (2.1)–(2.3) is reasonable to describe the dynamics of the turbulent flows.

2.7.1 Modelling flows along straight channels

The subsection simulates the turbulent flow along straight open channels. This turbulent channel flow is compared with viscous open channel flow by Roberts & Li (2006) and the experiments and analysis of turbulent flow over flood
Figure 2.12: The histories of the mean speed $\bar{q}$ at three observed stations across the channel for: (blue) flow along the straight channel in Figure 2.11, and (red) meandering channel in Figure 2.15. These curves indicate the flow converges to a steady state after the time $t = 300$ for straight channel and $t = 500$ for meandering channel.

Consider a domain with the nondimensional length $L_x = 40$ and width $L_y = 20$. Let $x$ be the down-stream and $y$ be the cross-stream coordinate. Choose a quartic shape for the channel to make smooth transitions to and from the shallows and the channel:

$$z = b(x, y) = -B - (1 - B) \left\{ \max \left[ 0, 1 - \left( \frac{y}{\beta} \right)^2 \right] \right\}^2,$$  \hspace{1cm} (2.45)

where $\beta$ denotes the half-width of the channel, $B$ the depth of water on the shallow ‘flood plain’ on either side of the channel, and the mid-depth of the channel is one (non-dimensionally) as we set the mean water level to be at $z = 0$. Figure 2.11 displays the open channel where the width $2\beta = 8$, the depth of shallows $B = 0.1$ and a mean slope $\tan \theta = 0.01$ in the $x$ direction. For comparison, the channel of Bousmar (2002) was about twice as deep in the channel as in the surrounding flood plain.

Numerical simulations were simply implemented using centred difference approximations to the spatial derivatives as in equations (2.42)–(2.44) on the staggered grids in space described in section 2.6. Time integration was performed by Matlab’s ode15s.

The simulations typically start the fluid with zero velocity and a flat free surface ($z = 0$). The blue curves in Figure 2.12 represent the mean velocity $\bar{q}$ in time along the straight channel. Transients in the simulations decayed on a non-dimensional time of typically $t \approx 300$. Figure 2.13 shows that fast flow developed in the deeper channel and slow flow on the shallow regions. In a viscous flow in a small open channel, Roberts & Li (2006) found that the flow was eight times as fast in the channel as in the shallows when the channel is three times as deep in the middle as the surrounding shallows. Equation (2.2) suggests the equilibrium downstream velocity in the shallows is $\sqrt{0.993 \sin(0.01)} \times 0.1/0.00283 \approx 0.59$, which corresponds to the mean numerical result in Figure 2.13. The equilibrium downstream velocity for a fluid of depth one is $\sqrt{0.993 \sin(0.01)} \times 1/0.00283 \approx 1.87$, but in our channel the peak velocity is only approximate $1.42$ as in Figure 2.13: such simulations show that when the shape of the bed becomes complex, the equilibrium downstream velocity decreases through lateral mixing and dissipation. For example, for the lesser slope $\tan \theta = 0.001$, the equilibrium downstream peak velocity over a flat bed is $0.58$, in mid-channel with width $2\beta = 14$ is $0.436$, in mid-channel with width $2\beta = 8$ the peak velocity is $0.406$, and in a slightly meandering channel with a width $2\beta = 8$ is $0.399$. Without the effect of the mixing and dissipation terms, such as $0.094q/h \partial/\partial x \left(h^2 \partial \bar{u}/\partial x \right)$, in the model (2.1)–(2.3), the equilibrium mean velocity $\bar{u}$ in mid-channel with width
Figure 2.13: The depth-averaged lateral velocity $\bar{u}$ along the straight channel in Figure 2.11 at the time $t = 800$. The channel is nine times as deep as in the surrounding shallows, but the peak downstream velocity is two times faster.

Figure 2.14: The depth-averaged lateral velocity $\bar{v}$ along the straight channel in Figure 2.11 at time $t = 800$. The humps and hollows in the flow direction indicate weak travelling vortices on the shear induced by the interactions between the channel and shallow regions.
2\beta = 8 is 0.453, which is closer to the equilibrium 0.58. Thus, these two mixing and dissipation terms reduce the equilibrium mean velocity $\bar{u}$ for complex bed shapes.

Figure 2.14 displays the depth-averaged lateral transverse velocity $\bar{v}(x, y, t)$ at time $t = 800$, which indicates that weak horizontal vortices grow on the shear in the transition between the channel and shallow regions. These weak mixing vortices travel downstream. Similar vortices were observed by Roberts & Li (2006) in numerical simulations of viscous open channel flow and by Bousmar & Zech (2004) in experiments of turbulent flow along channels in a flume. Computation shows that when the differences of the downstream velocity between the channel and the shallows increase, stronger travelling vortices are generated, which corresponds to the analysis of Sofialidis & Prinos (1999).

### 2.7.2 Flows along meandering channels

This section describes simulations of turbulent flow along a slightly sloping bed with a meandering open channel. The simulations are compared with the numerical results of Liu et al. (2009) and Demuren (1993) who calculated the two and three dimensional turbulence flows in meandering channels by a lattice Boltzmann model and a finite volume numerical model, respectively.
Figure 2.16: The depth-averaged lateral velocity $\bar{u}(x, y, t)$ along the meandering channel in Figure 2.15 at time $t = 800$. Fast flow is developed in the deeper channel and slow flow on the shallow regions. The downstream velocity $\bar{u}$ reaches maximum at the bends.
Figure 2.17: The depth-averaged lateral velocity $\bar{v}(x,y,t)$ along the meandering channel in Figure 2.15 at time $t = 800$. The transverse velocity $\bar{v}$ attains maximum and minimum at the connection of the bends.
Figure 2.18: Plots of the depth-averaged lateral velocity $\bar{u}(x,y,t)$ across the channel in Figure 2.15 for the fluid flowing through a channel bend. The left-hand side is the inner bank of the channel bend and the right-hand side is the outer bank. The blue curve is at $x = 16.4$, the green curve is at $x = 22.6$ and the red curve is at $x = 28.7$.

Here I describe simple meandering open channels by the bed

$$z = b(x,y) = -B - (1 - B) \left\{ \max \left[ 0, 1 - \left( \frac{y - \kappa_1 \cos(\kappa_2 x)}{\beta} \right)^2 \right] \right\}^2,$$  \hspace{1cm} (2.46)

where the parameter $\kappa_2$ determines the wavelength $2\pi/\kappa_2$ of the meandering channel, the parameter $\kappa_1$ is the half-width of the extent of the meanders, and the parameters $\beta$ and $B$ are the half-width and depth of the shallows as before. Figure 2.15 exhibits the meandering channel with the channel width $2\beta = 8$, shallow depth $B = 0.1$, mean slope $\tan \theta = 0.001$, curvatures $\kappa_1 = 1$ and $\kappa_2 = 4\pi/L_x$. I simulate the turbulent flow along such channel by the equations (2.1)–(2.3) with periodic boundary conditions in both $x$ and $y$ directions for both the flow and channel.

The red curves in Figure 2.12 show that the transients decay over times of typically $t = 500$ in the numerical simulations. Figure 2.16–2.17 exhibit the depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$ along the meandering channel in Figure 2.15 at time $t = 800$. The depth-averaged lateral velocity $\bar{u}$
reaches maximum at the bends. The depth-averaged lateral velocity $\bar{v}$ attains maximum and minimum at the connection of the bends. These phenomenon is consistent with the results of Liu et al. (2009) who modelled the water in meandering channels with $60^\circ$ and $90^\circ$ consecutive bends and a width of $0.3 \text{ m}$.

Plots of the water depth do not show any features of much interest: plots are dominated by the variations in the bed. Figure 2.18 plots the the depth-averaged lateral velocity $\bar{u}(x, y, t)$ across the channel in Figure 2.15 for the flow through a channel bend. The left-hand side is the inner bank of the channel bend and the right-hand side is the outer bank. The bend is between the positions of $x = 15$ and $x = 30$. Demuren (1993) calculated the water depth and the depth-averaged longitudinal and transverse velocities of three dimensional flows in meandering channels with a natural bed configuration by a finite volume numerical method. Simulations of Demuren (1993) at fifteen observed stations of the meandering channel indicate that the location of the maximum velocity shifts from the inner bank to the outer bank as the water flows through the bends of the channel. Figure 2.18 shows the maximum depth-averaged lateral velocity $\bar{u}$ shifts from the inner bank (blue curve) to the outer bank (red curve) as the flow through the channel bend. This agrees with the computations by Demuren (1993) and the analysis of secondary flow across the channel by Sofialidis & Prinos (1999).

### 2.7.3 Flows along river-like open channels

Physically, rivers have very rough and curved beds and banks, which result in complicated fluid dynamics. This section reports on the primarily numerical simulations of flow along river-like open channels by the model (2.1)–(2.3).

The river-like compound channels potentially have arbitrary topography both on the bed of the shallows and the main channel. Describe a river-like channel by

$$ z = b(x, y) = -B - (1 - B) f(x, y), \quad (2.47) $$

where $B$ denotes the shallow depth. The function $f(x, y)$ shapes a river-like channel. Figure 2.19 shows the flow along a river-like compound channel. The shallow depth parameter $B = 0.3$, the maximum depth of channel is 1. The river-like channel has a mean slope $\tan \theta = 0.001$. The river-like channel only includes one bend around the position $x = 20$. The bed of the shallows and the channel are curved. The banks of the river-like channel are randomly bent.

For a pilot study, we seek the steady state flow; that is, the left-hand sides of the model (2.1)–(2.3) are zeros. Numerical solutions are found by
Figure 2.19: Flow along a river-like channel with the shallow depth parameter $B = 0.4$ and the mean slope $\tan \theta = 0.001$. 
Figure 2.20: The depth-averaged lateral velocity $\bar{u}$ of the flow along the river-like channel in Figure 2.19. Fast flow is developed in the deeper channel.

Figure 2.21: The depth-averaged lateral velocity $\bar{v}$ of the flow along the river-like channel in Figure 2.19.
the matlab function `fsolve`, which zeros the right-hand side ODEs of the PDEs (2.1)–(2.3).

Figure 2.20–2.21 show the depth-averaged lateral velocities $\bar{u}(x,y)$ and $\bar{v}(x,y)$ of the flow along the river-like channel in Figure 2.19. Similarly to the findings in section 2.7.1 and 2.7.2, horizontal vortices are observed and the maximum of velocity $\bar{u}$ shifts from inner bank to the outer bank. However, when the shallow depth parameter $B$ decreases, the water dips down in the fastest part of the shallows until the water depth approaches zero, which is singular in our modelling. For example, when the shallow depth $B = 0.4$, the shallows have a minimum depth of $0.22$ and the main channel has a maximum flow depth of $1.03$. When the shallow depth $B < 0.4$, the flow depth in the shallows quickly approaches zero and the simulation breaks down.

Numerical results in Figure 2.20–2.21 indicate that the model (2.1)–(2.3) is reasonable to simulate flow in physically complex bed topography.

### 2.8 Conclusion

A 3D Smagorinski model (2.1)–(2.3) is derived to simulate the turbulent flows. Such a model is reduced from the conservation equation and Reynolds-averaged Navier–Stokes equations with boundary conditions on the free surface and on the mean bed. To avoid depth-averaging fluid variables, centre manifold theory is used. The model (2.1)–(2.3) accounts for the interactions between the vertical profiles and lateral spatial variations. It is reassuring that the dominant terms in the model (2.1)–(2.3) agree with the established modelling (Rodi 1984, Yulistiyanto et al. 1998, Bousmar 2002, e.g.). The approach then seeks to ensure more subtle effects are modelled, effects that can be important in complex flow regimes. Examples are modelling dam breaking waves by Georgiev et al. (2009) using a 2D Smagorinski model (no $y$ dependence in equation (2.2)). Vertical lateral velocity and shear stress distributions in steady flow are plotted in Figure 2.4–2.6, which agree with published experimental data (Schultz & Flack 2007, 2013, e.g.). Thus the model is reasonable and effective to simulate turbulent flows. Analytical eigenvalue analysis indicates that the model converges and is stable. Numerical eigenvalues are explored by numerically solving the model in a staggered grid scheme.

Application to simulating channel flows shows that the model (2.1)–(2.3) predicts environmental turbulent fluids reasonably well. Fast flow is developed in the main channel. Weak vortices arise at the interfaces between the main channel and shallows. The maximum depth-averaged lateral velocity $\bar{u}$ shifts from the inner bank to the outer bank as the flow through the channel bend,
as shown in Figure 2.18. These results quantitively agree with the analysis and numerical simulations of published work (Bousmar 2002, Liu et al. 2009, e.g.).

Future work could be to improve the model (2.1)–(2.3), such as extending to spherical coordinates in a rotating frame. Further future work could focus on evaluating the application of the model to complex flows.
Chapter 3

Modelling suspended sediment transport in turbulent floods

3.1 Introduction

Modelling sediment erosion and sediment transport is another important branch of environmental fluid dynamics (van Rijn 1984, Celik & Rodi 1988, e.g.). For example, Figure 1.2 shows the Yellow River in China, which is famous for carrying large amounts of sediment. Modelling the sediment in such river is important for studying the changes of river channel morphology and flood routing processes (He et al. 2008, e.g.).

Typically, there are three important sediment transport loads in the environmental fluids: bed load, suspended load and wash load. Fredsoe & Deigaard (1992), in their book, detailed these three kinds of sediment transports beneath ocean waves. For a pilot study, this chapter only aims to model the suspended sediment in turbulent flow. Most previous work only studied the suspended sediments in uniform flows (Hunt 1954, van Rijn 1984, Celik & Rodi 1988, e.g.) or by depth averaging flow and sediment equations (Wu et al. 2000, Pittaluga & Seminara 2003, e.g.). This chapter, based on centre manifold theory instead of depth averaging governing equations, derives the following non-dimensional suspended sediment model of the horizontal evolution of the depth-averaged concentration $\bar{c}(x,y,t)$:

$$\frac{\partial \bar{c}}{\partial t} \approx - \frac{w_f}{h} \bar{c} \left( 0.938 + 28.87 \frac{w_f}{q} \right) + \frac{w_f}{h} \epsilon_{ae} \left( 0.984 - 51.3 \frac{w_f}{q} \right)$$

$$- 1.007 \exp \left( -3.073 \frac{w_f}{q} \right) \left( u \frac{\partial \bar{c}}{\partial x} + v \frac{\partial \bar{c}}{\partial y} \right)$$

$$+ 0.033q \left[ \frac{\partial}{\partial x} \left( h^2 \frac{\partial \bar{c}}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^2 \frac{\partial \bar{c}}{\partial y} \right) \right]$$
\[ + 0.027 \frac{\overline{u}^2 - \overline{v}^2}{q} \left[ \frac{\partial}{\partial x} \left( h^2 \frac{\partial \overline{c}}{\partial x} \right) - \frac{\partial}{\partial y} \left( h^2 \frac{\partial \overline{c}}{\partial y} \right) \right], \quad (3.1d) \]

where \( h \) is the depth of the fluid, \( \overline{u}(x, y, t) \) and \( \overline{v}(x, y, t) \) are depth-averaged lateral velocities of the fluid, \( \overline{q} = \sqrt{\overline{u}^2 + \overline{v}^2} \) the mean local flow speed, \( b(x, y) \) is the mean bed, \( w_f \) the constant falling velocity, and \( c_{\text{ne}} \) an equilibrium reference concentration on the mean bed \( z = b \). This model includes the sediment erosion and deposition in line (3.1a), the advection in line (3.1b), and the anisotropic dispersion in lines (3.1c)–(3.1d).

Section 3.2 describes the underlying governing advection-diffusion equation and the boundary conditions on the free surface and on the mean bed. Section 3.3 embeds the physical problem in a family of problems by artificially modifying the boundary conditions on the free surface and on the mean bed. Computer algebra in Appendix A.1 constructs the slow manifold of the system. Then on the slow manifold, section 3.4 derives the low dimensional model of the suspended sediment, and section 3.5 describes the low order approximation of the suspended sediment.

Section 3.6 interprets the application of the suspended sediment model (3.1) in modelling the suspended sediment in open channel flows and under large waves. A staggered grid scheme implements the numerical simulation. The numerical results indicate that our model is reasonable to describe the dynamics of the suspended sediment in turbulent flows.

### 3.2 Detailed equations of suspended sediment

This section describes the governing advection-diffusion equation of suspended sediment and the boundary conditions on the free surface and near the mean bed. The section also defines the variables of falling velocity, diffusion factor and a reference equilibrium concentration introduced in the boundary condition on the mean bed.

#### 3.2.1 The governing equations

Consider the turbulent flow flowing down a mean bed carrying sediment. For a pilot study, this section only considers the suspended sediment and neglects the bed load on the mean bed. Figure 3.1 depicts the diagram of the suspended sediment in the turbulent flow. Define a coordinate system with \( x, y \) for the lateral directions along the bed and \( z \) for the direction normal to the mean bed slope. Section 2.2.1 discussed that all the variables of the fluid are turbulent mean. Similarly, here the concentration of the suspended sediment is also a turbulent mean.
The fluid of depth $h(x,y,t)$ flows down the sloped mean bed $z = b(x,y)$ at the turbulent mean velocity $q(x,y,z,t)$; the velocity vector $q = (u,v,w)$ in the $(x,y,z)$ directions, respectively. The suspended sediment has a turbulent mean concentration $c(x,y,z,t)$ (volume fraction). The mean bed $z = b(x,y)$ has a mean slope $\tan \theta$ in the $x$ direction.

We assume the particle size of the suspended sediment is small. The particle size of the suspended sediment affects the falling velocity of the particle, the deposition and entrainment (van Rijn 1984, Fredsoe & Deigaard 1992, e.g.). Walling et al. (2000) concluded that the averaged median particle size of the suspended sediment in rivers ranges from $4.1 \mu m$ to $13.5 \mu m$. For simplicity, assume all the particles of the suspended sediment have the same falling velocity as a sphere of diameter $d$.

The advection-diffusion equation of the suspended sediment in the field of the turbulent mean concentration $c(x,y,z,t)$ is then
\[
\frac{\partial c}{\partial t} + \nabla \cdot (qc) = -\nabla \cdot (w_f c n_g) + \nabla \cdot (\epsilon_s \nabla c),
\] (3.2)
where $w_f$ is the falling velocity of the particles, $\epsilon_s$ the diffusion factor and the unit vector $n_g = (\sin \theta, 0, -\cos \theta)$ is the direction of gravity.

Equation (3.2) generally describes the sediment transport in fluids.
3.2.2 The falling velocity

The falling velocity $w_f$ is related to the mean particle size $d$, the relative density $s$ and the gravity $g$. The relative density $s = \rho_m / \rho$, where $\rho$ is the density of the turbulent flow and $\rho_m$ the density of the particles. Natural suspended particles have variety of shapes, which affect the falling velocity of the particles (Jimenez & Madsen 2003, e.g.). Jimenez & Madsen (2003) used a shape factor of 0.7 and a roundness value of 3.5 to account for the shape effect for the falling velocity, where the shape factor is equal to the shortest diameter of the natural particles divided by the square root of the product by the longest and intermediate diameters. For a spherical particle, the shape factor equals to one.

We consider all the particles are effectively spherical with a mean diameter $d$ and then neglect shape effects (Engelund & Fredsoe 1976, van Rijn 1984, Wu et al. 2000, e.g.). Thus, by balancing the drag force $c_D \rho w_f^2 d^2 \pi/8$ by the sum $(s - 1)d^2 \pi/6$ of the gravity and buoyancy, I set the falling velocity

$$w_f = \sqrt{\frac{4(s - 1)gd}{3c_D}}.$$  

(3.3)

The drag coefficient $c_D$ depends on the grain Reynolds number $w_f d / \nu_f$, where the $\nu_f$ is the fluid viscosity. Fredsoe & Deigaard (1992) reported that the drag coefficient $c_D \approx 1.4$ for the large grain Reynolds number of natural sands, typically $Re > 500$.

Turbulence in the fluid apparently increases the falling velocity of the particles: Wang & Maxey (1993) reported that the falling velocity of heavy particles is approximate 50% bigger in turbulence than in still fluid. Nonetheless the falling velocity in our work is still small. This is because we require small mean size $d$ of the particles which implies a small falling velocity according to equation (3.3).

3.2.3 The diffusion factor

The effective diffusion coefficient $\epsilon_s$ links the suspended sediment to the fluid turbulence. The diffusion $\epsilon_s$ is proportional to the eddy viscosity of the turbulent flow (van Rijn 1984, Celik & Rodi 1988, Lopez & Garcia 1998, Zedler & Street 2001, Yoon & Kang 2005, e.g.). For example, van Rijn (1986) proposed the diffusion $\epsilon_s = \beta \phi \nu$, where $\nu$ is the eddy viscosity, the factor $\beta$ describes the relationship between the sediment diffusion and fluid diffusion, and the factor $\phi$ describes the sediment particles damping the turbulence. However, we require small mean particle size $d$ and thus assume no damping.
effect to the turbulence by the suspended sediments, hence we assume the damping factor $\phi = 1$.

van Rijn (1984) computed the factor $\beta = 1 + 2(\omega_f/u_f)^2$ with the velocity ratio $\omega_f/u_f$ ranging from 0.1 to 1, where $\omega_f$ is the falling velocity of particles and $u_f$ the shear velocity near the bed. We assume the falling velocity $\omega_f$ small enough, so that the term $(\omega_f/u_f)^2$ is negligible. Therefore, the factor $\beta$ is approximately one in our work.

Many research articles (Celik & Rodi 1988, Lopez & Garcia 1998, Zedler & Street 2001, e.g.) proposed the diffusion $\epsilon_s = \nu/\sigma_c$, where $\nu$ is the eddy viscosity of the turbulent flow and $\sigma_c$ the turbulent Schmidt number. Celik & Rodi (1988) reported that the computed suspended sediment concentration agrees better with the experimental measurements for lower values of the turbulent Schmidt number $\sigma_c$ in the range 0.5 to 1 for the open channel flow. The factor $\beta \approx 1$ corresponds to a turbulent Schmidt number $\sigma_c = 1$ according to $\beta = 1/\sigma_c$. This chapter sets the diffusion factor $\epsilon_s$ equal to the turbulent eddy viscosity $\nu$, defined by equation (2.11) in Section 2.2.4.

3.2.4 The boundary conditions

This subsection formulates the boundary conditions of the suspended sediment on the free surface and on the mean bed.

On the free surface, the sediment flux normal to the surface is zero, which requires

$$-(n_\eta \cdot n_g) \omega_f c + \epsilon_s \frac{\partial c}{\partial n_\eta} = 0 \quad \text{on} \quad z = \eta,$$

where $\eta = h + b$ is the free surface, the unit vector

$$n_\eta = \frac{1}{\sqrt{1 + \eta_x^2 + \eta_y^2}} (-\eta_x, -\eta_y, 1),$$

is normal to the free surface and the unit vector $n_g = (\sin \theta, 0, -\cos \theta)$ is the direction of the gravity.

On the mean bed $z = b$, the downward flux across the reference level is proportional to the difference with an equilibrium, that is

$$-(n_b \cdot n_g) \omega_f c + \epsilon_s \frac{\partial c}{\partial n_b} = -(n_b \cdot n_g) \omega_s (c - c_{ae}) \quad \text{on} \quad z = b,$$

where the vector

$$n_b = \frac{1}{\sqrt{1 + b_x^2 + b_y^2}} (-b_x, -b_y, 1),$$
is the unit vector normal to the bed, \( \mathbf{c}_{ae} \) is the equilibrium reference concentration on the mean bed, and the variable \( w_s \) has the unit of velocity, namely the deposition velocity (Lick 1982, e.g.). Lick (1982) argued that the deposition velocity \( w_s \) is approximately equal to the falling velocity \( w_f \) for the mean particle size \( d \) bigger than \( 1.0 \mu m \). We assume the deposition velocity \( w_s = w_f \). Thus, the boundary condition (3.5) on the mean bed becomes

\[
- \epsilon_s \frac{\partial c}{\partial n_b} = - (\mathbf{n}_b \cdot \mathbf{n}_g) w_f c_{ae} \quad \text{on} \quad z = b , 
\]

which means the upward net flux across the mean bed \( z = b \) comes from the entrainment \( w_f c_{ae} \) due to the fluid turbulence.

### 3.2.5 The equilibrium reference concentration

This subsection discusses the equilibrium reference concentration \( c_{ae} \). Entrainment experiments (Lick 1982, Celik & Rodi 1988, e.g.) show that the equilibrium reference concentration \( c_{ae} \) at least depends on the turbulent stress (shear stress, shear velocity, e.g.), fluid contents (relative density, gravity viscosity, e.g.), and the composition of the sediments (mean particle size, e.g.).

Einstein (1950) computed the equilibrium reference concentration by relating the bed load \( q_b \) to the friction velocity \( u_f \),

\[
c_{ae} = \frac{1}{11.6} \frac{q_b}{2d u_f} ,
\]

where \( d \) is the mean size of the sediments. However, the bed load in our work is not modelled and so this equation is not suitable.


van Rijn (1984) approximated the equilibrium reference concentration as

\[
c_{ae} = 0.015 d^{0.7} \frac{u_f^{0.2}}{\alpha} \left( \frac{u_f^2}{u_{fc}^2} - 1 \right)^{1.5} \frac{1}{[g(s-1)]^{0.1}} ,
\]

which is determined by the mean particle size \( d \), the shear velocity \( u_f \), the reference level \( \alpha \), the water viscosity \( \nu_w \), the critical shear velocity \( u_{fc} \), the relative density \( s \) and the gravity \( g \). van Rijn (1984) pointed out that the reference level \( \alpha \) is limited, with a minimum value of 1% of the water depth \( h \).
Our work approximates the equilibrium reference concentration $c_{ae}$ by following van Rijn (1984). This chapter approximates the critical shear velocity $u_{fc} = \sqrt{\theta_c g d (s - 1)}$ with the critical Shield number $\theta_c = 0.06$ for large Reynolds number, generally $Re > 1000$ (Fredsoe & Deigaard 1992, e.g.). Assume the reference level $a$ is 1% of the fluid depth $h$. By substituting the nondimensional water viscosity $\nu_w \approx 1.0 \times 10^{-6} \text{m}^2/\text{s}$, the typical relative density $s = 2.65$ (Fredsoe & Deigaard 1992, e.g.), the critical Shield number $\theta_c = 0.06$ and the gravity $g = 9.8 \text{m/s}^2$, we approximate the equilibrium reference concentration

$$c_{ae} \approx 0.075 \frac{u_f^3}{d^{0.8}}, \quad (3.7)$$

where the coefficient 0.75 has the unit of $s^3/\text{m}^{1.2}$. Thus, the equilibrium reference concentration (3.7) only depends on the mean particle size $d$ and the shear velocity $u_f$.

### 3.2.6 The mixing density of the fluid and sediment

The suspended sediment influences the fluid turbulence. The turbulent eddy viscosity $\nu$ is produced by both the fluid momentum and sediment mass (Smith & McLean 1977, Glenn & Grant 1987, Yoon & Kang 2005, e.g.).

Our work assumes the factor of the mixing density $\rho_{mix}$ of the fluid and suspended sediment satisfying

$$\frac{1}{\rho_{mix}} = \frac{1}{\rho + c(\rho_m - \rho)} = \frac{1}{\rho} \left(1 + c(s - 1)\right) = \frac{1}{\rho} \left[1 - c(s - 1) + c^2(s - 1)^2\right] + O([c(s - 1)]^3), \quad (3.8)$$

where $\rho_m$ is the sediment density, $\rho$ the density of the fluid and $s = \rho_m/\rho$ the relative density. Generally the natural concentration $c$ in rivers is small. Take the Yellow River in Figure 1.2 for example. The Yellow River is notable for carrying large amount of silt. Tregear (1965) in his book (page 219) referenced an estimated density of 35 kg/m$^3$, which implies the sediment fraction $cs \approx 3.5\%$. Typically, a quartz sediment has a relative density $s = 2.65$ \footnote{http://en.wikipedia.org/wiki/Quartz}, which indicates the sediment fraction of the Yellow River $c \approx 1.3\%$. This concentration $c$ acts on the eddy viscosity $\nu$. 

\footnote{http://en.wikipedia.org/wiki/Quartz}
3.3 Centre manifold theory supports the modelling

This section embeds the physical problem in a family of problems by artificially modifying the boundary conditions (3.4)–(3.6). Such modification empowers the modelling and the theoretical support by centre manifold theory.

To distinguish the artificial parameter $\gamma$ in the boundaries of the turbulent flow in Section 2.2.6, I use a different artificial parameter $\gamma_c$ to embed the sediment physics. Artificially modify the physical boundary conditions (3.4) and (3.6) to

\[
\epsilon_s \left( -\frac{\partial (h+b)}{\partial x} \frac{\partial c}{\partial x} - \frac{\partial (h+b)}{\partial y} \frac{\partial c}{\partial y} + \gamma_c \frac{\partial c}{\partial z} \right) + 2(1 - \gamma_c) \epsilon \frac{c}{h} \\
+ \left[ 1 + (1 - \gamma_c) \frac{w_f}{6} \right] w_f c \left[ \cos \theta + \sin \theta \frac{\partial (h+b)}{\partial x} \right] = 0 \quad \text{on} \quad z = h + b , \tag{3.9}
\]

\[
\epsilon_s \left[ \frac{\partial b \partial c}{\partial x \partial x} - \frac{\partial b \partial c}{\partial y \partial y} + (2 - \gamma_c) \frac{\partial c}{\partial z} \right] + 2(1 - \gamma_c) \epsilon \frac{c}{h} \\
+ \left[ 1 + (1 - \gamma_c) \frac{w_f}{6} \right] w_f c ae \left[ \cos \theta + \sin \theta \frac{\partial b}{\partial x} \right] = 0 \quad \text{on} \quad z = b , \tag{3.10}
\]

Upon setting the embedded parameter $\gamma_c = 1$ in the left-hand sides, equations (3.9)–(3.10) recover the original physical boundary conditions (3.4)–(3.6).

When the embedded parameter $\gamma_c = 0$, equations (3.9)–(3.10) form an artificial problem, which is used to find a slow manifold in the system. The extra term $w_f/6$ in equations (3.9)–(3.10) ensures conservation to errors $O(w_f^4)$. Without such term, the model only conserves sediment to errors $O(w_f^2)$. The reason for implementing such high order errors is that the information about different falling velocities comes into the model in $O(w_f^2)$ terms.

Section 2.3 discusses the theoretical support for the existence of a slow manifold in the turbulent flow system. Similarly, when the artificial parameter $\gamma_c = 0$, the lateral gravity and lateral derivatives are negligible, $\tan \theta = \partial_x = \partial_y = 0$, and the falling velocity $w_f = 0$. equation (3.9) requires the concentration $c = 0$ on the free surface, and equation (3.10) indicates

\[
\frac{\partial c}{\partial z} + \frac{c}{h} = 0 \quad \text{on} \quad z = b ,
\]

which implies a neutral mode of the sediment dynamics is $c \propto 1 - z/h$.

Thus, when $\gamma_c = \tan \theta = \partial_x = \partial_y = w_f = 0$, a one parameter subspace of equilibria exists corresponding to some turbulent mean concentration
$c \propto (1 - z/h)$, on a fluid of any constant fluid depth $h$. This one parameter subspace joins with the three parameter subspace of section 2.3.1 to make a four parameter subspace. For large enough lateral length scales, these four parameter subspace occurs independently at each location $x$ and $y$ (Roberts 1988, 2008a, e.g.) and hence the space of equilibria are in effect parameterised by $\tilde{c}(x, y)$, and the turbulent flow fields of $\tilde{u}(x, y)$, $\tilde{v}(x, y)$ and $h(x, y)$. As discussed in Section 2.3, the theorems (Roberts 1988, Chicone 2006, Potzsche & Rasmussen 2006, e.g.) support the existence, accurate relevance and construction of slow manifold models such as (3.1).

### 3.4 Reduced model of the suspended sediment

This section interprets the application of centre manifold theory and the resultant leading model of the suspended sediment. Computer algebra in Appendix A.1 derives the evolution equation of the depth-averaged concentration $\tilde{c}(x, y, t)$. The dominant terms in the model agree with established modelling (Wu 2004, Duan & Nanda 2006, e.g.). The approach ensures more subtle effects are modelled, effects that can be important in complex flow regimes.

Section 3.2–3.3 discussed the dimensional physical variables in the physical problem. Similarly to section 2.2.3, we nondimensionalise the variables with respect to a characteristic depth of the turbulent fluid $H$, and lateral velocity scale $\sqrt{gh}$, and the mixing density $\rho_{\text{mix}}$. Recall that the concentration $c$ is in volume fraction. Hereafter we concentrate on deriving a nondimensional model to model the dynamics of the suspended sediment in turbulence flow, and applying this nondimensional model to physical problems in section 3.6.

#### 3.4.1 The low dimensional model

The order of error in the construction is phased in terms of the small parameters. Here the small parameters are the lateral derivatives $\partial_x$ and $\partial_y$, the small mean slope $\tan \theta$, the falling velocity $w_f$, and the artificial parameters $\gamma$ for fluid and $\gamma_c$ for suspended sediment. Generally work to errors $O(\partial_x^{\gamma/2} + \partial_y^{\gamma/2} + \gamma^{\gamma/2} + \tan^{\gamma/2} \theta + w_f^\gamma)$ which denote the order of the error terms. Thus, each term in model has less than $p$ factors in total of these five small parameters. The bigger the exponent number $p$, the higher the order of the modelling.

The artificial small parameter $\gamma_c$ has no physical meaning but is introduced to establish the slow manifold. However, to evaluate accurately at $\gamma_c = 1$, we need higher orders in the artificial parameter $\gamma_c$. 

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By truncating to errors $O(\partial_{x}^{3/2} + \partial_{y}^{3/2} + \tan^{3/2} \theta + w_{f}^{3} + \gamma^{3}, \gamma^{5})$ and omitting the intricate details of the derivation, executing the computer algebra listed in Appendix A.1 leads to the following governing equation modelling the suspended sediment in terms of depth-averaged concentration:

$$\frac{\partial \bar{c}}{\partial t} = \bar{c} \frac{w_{f}}{h} \left[ (-1.5 + 0.375\gamma c^{2} + 0.186\gamma^{3}) \right]$$

(3.11a)

$$+ \frac{w_{f}}{q} (-16.5 + 24.75\gamma c - 43.31\gamma^{2} - 9.281\gamma^{3} + 15.47\gamma^{4})$$

(3.11b)

$$+ \gamma c \frac{d}{h} \left( 0.00476 - 0.00202\gamma c - 0.00257\gamma^{2} - 0.000779\gamma^{3} \right)$$

(3.11c)

$$+ c_{ae} \frac{w_{f}}{h} \left( 0.75 + 0.375\gamma c - 0.0938\gamma^{2} - 0.0469\gamma^{3} \right)$$

(3.11d)

$$+ \frac{w_{f}}{q} (-33 - 39.19\gamma c + 7.218\gamma^{2} + 13.41\gamma^{3} + 0.258\gamma^{4})$$

(3.11e)

$$+ \left( \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y} \right) \left( -0.893 - 0.0536\gamma c - 0.0536\gamma^{2} - 0.0134\gamma^{3} \right)$$

(3.11f)

$$+ \bar{c} \frac{\partial u}{\partial x} \left( 0.0515 - 0.0238\gamma c - 0.0267\gamma^{2} - 0.00738\gamma^{3} + 0.00298\gamma^{4} \right)$$

(3.11g)

$$+ \bar{c} \frac{\partial v}{\partial y} \left( 0.193 - 0.212\gamma c - 0.0364\gamma^{2} + 0.0358\gamma^{3} + 0.027\gamma^{4} \right)$$

(3.11h)

$$+ \bar{c} \frac{\partial u}{\partial x} \left( 0.0515 - 0.0238\gamma c - 0.0267\gamma^{2} - 0.00738\gamma^{3} + 0.00298\gamma^{4} \right)$$

(3.11i)

$$+ \gamma c \bar{q} \left( \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y} \right) \left( -0.143 + 0.195\gamma c + 0.01\gamma^{2} - 0.0438\gamma^{3} \right)$$

(3.11j)

$$+ \bar{c} \frac{\partial u}{\partial x} \left( 0.144 - 0.195\gamma c - 0.01\gamma^{2} + 0.0438\gamma^{3} + 0.0244\gamma^{4} \right)$$

(3.11k)

$$+ \bar{c} \frac{\partial v}{\partial y} \left( 0.144 - 0.195\gamma c - 0.01\gamma^{2} + 0.0438\gamma^{3} + 0.0244\gamma^{4} \right)$$

(3.11l)

$$+ O(\partial_{x}^{3/2} + \partial_{y}^{3/2} + \tan^{3/2} \theta + w_{f}^{3} + \gamma^{3}, \gamma^{5}) .$$

Equation (3.11) models the suspended sediment with the introduced artificial parameters $\gamma_{c}$ and $\gamma$. To this order, equation (3.11) includes sediment erosion effects in lines (3.11a)–(3.11c), the advection in lines (3.11d)–(3.11h), and
Table 3.1: Partial sums from evaluating coefficients at $\gamma = \gamma_c = 1$ of selected terms in equation (3.11) indicates that the power series in $\gamma_c$ converges.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\bar{u}\partial\bar{c}/\partial x$</th>
<th>$c\partial\bar{u}/\partial x$</th>
<th>$c\partial\bar{v}/\partial y$</th>
<th>$\bar{u}\bar{c}/h(\partial h/\partial x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma^0$</td>
<td>-0.893</td>
<td>0.052</td>
<td>0.193</td>
<td>-0.144</td>
</tr>
<tr>
<td>$\gamma^1$</td>
<td>-0.946</td>
<td>0.028</td>
<td>-0.024</td>
<td>0.052</td>
</tr>
<tr>
<td>$\gamma^2$</td>
<td>-1</td>
<td>0.001</td>
<td>-0.060</td>
<td>0.062</td>
</tr>
<tr>
<td>$\gamma_c^3$</td>
<td>-1.013</td>
<td>-0.006</td>
<td>-0.024</td>
<td>0.018</td>
</tr>
<tr>
<td>$\gamma_c^4$</td>
<td>-1.007</td>
<td>-0.003</td>
<td>0.003</td>
<td>-0.003</td>
</tr>
</tbody>
</table>

the effects by the changes of fluid depth and bed topography in lines (3.11i)–(3.11k). The lateral turbulent mixing effects are incorporated by section 3.4.2.

Recall that Table 2.1 shows the power series in $\gamma$ converges quickly in $\gamma^3$. Table 3.1 shows the partial sums of coefficients of selective terms evaluating at $\gamma = \gamma_c = 1$. The series in $\gamma_c$ in the coefficients converges. The coefficient of the selective term $\bar{u}\partial\bar{c}/\partial x$ approximately converges to $-1$ to errors $O(\gamma_c^5)$. To this order, the coefficients of other selective terms in Table 3.1 approximately converge to zero. Thus, we conclude that truncating to errors $O(\gamma_c^5)$ and upon setting the parameters $\gamma_c = \gamma = 1$, the coefficient in lines (3.11d) approximately converges to $-1$, and coefficients in lines (3.11e)–(3.11h) approximately converge to zero.

### 3.4.2 Model lateral dispersion of the sediment

Lateral dispersion of sediment requires the model to resolve second derivatives in $x$ and $y$ directions. The second derivative terms are derived by extending to high order errors. Computer algebra of Appendix A.1 computes the following to errors $O(\partial^{5/2}_x + \partial^{5/2}_y + \gamma^{5/2} + \tan^{5/2}\theta + w^{5/2}_f, \gamma_c^2)$.

Recall that although our model (3.11), for example, are expressed by the depth-averaged variables, they are not from depth-averaging but from centre manifold theory. To compare with established depth-averaged suspended sediment model (Wu 2004, Duan 2004, Duan & Nanda 2006, e.g.), we record our sediment model in the term of transport rate:

\[
\frac{\partial \bar{c}}{\partial t} \approx -\frac{w_f}{h} \left(0.938 + 28.87 \frac{w_f}{q}\right) \bar{c} + \frac{w_f}{h} \left(0.984 - 51.3 \frac{w_f}{q}\right) c_{ae} - \left(1.007 - 3.094 \frac{w_f}{q}\right) \left(\bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y}\right)
\]
\[ + 0.384(s - 1) \frac{\bar{c}}{q^2} \left( \frac{\partial \bar{c}}{\partial x} + \frac{\partial \bar{c}}{\partial y} \right) \] 

\[ + 0.00606h^2q \frac{\partial^2 \bar{c}}{\partial x^2} + 0.0602h^2q \frac{\partial^2 \bar{c}}{\partial y^2} \] 

\[ + 0.0541h^2 \frac{u^2}{q} \left( \frac{\partial^2 \bar{c}}{\partial x^2} - \frac{\partial^2 \bar{c}}{\partial y^2} \right) \] 

\[ + 0.0121h\bar{q} \frac{\partial h}{\partial x} \frac{\partial \bar{c}}{\partial x} + 0.171h\bar{q} \frac{\partial h}{\partial y} \frac{\partial \bar{c}}{\partial y} \] 

\[ + 0.0121h\bar{q} \frac{\partial h}{\partial x} \frac{\partial \bar{c}}{\partial y} + 0.171h\bar{q} \frac{\partial h}{\partial y} \frac{\partial \bar{c}}{\partial x} \] 

\[ + \left( 0.0037\bar{c}\bar{q} - 0.0186\bar{c}\bar{u} \right) \frac{\partial^2 h}{\partial x^2} \] 

\[ + \left( -0.0149\bar{c}\bar{q} + 0.0186\bar{c}\bar{u} \right) \frac{\partial^2 h}{\partial y^2} . \] 

The model (3.12) includes the vertical sediment distribution effect in lines (3.12a)–(3.12b), the advection in lines (3.12c)–(3.12d), and the turbulent dispersion in lines (3.12e)–(3.12j). In practice one might only implement those terms in equation (3.12) which are important in a specific application. The initially introduced equation (3.1) is a simplified version of the model (3.12). Equation (3.1) proposes the asymptotically equivalent coefficient \( 1.007 \exp(-3.073w_f/\bar{q}) \) for the advection terms, whose Taylor expansion \( 1.007 - 3.094w_f/\bar{q} \) is approximately the same as the original coefficient for small ratio \( w_f/\bar{q} \).

The model (3.12) includes all the dominated terms in established modelling (Wu 2004, Duan 2004, Duan & Nanda 2006, e.g.). For example, Duan (2004) derived the following depth-averaged advection-diffusion equation of suspended sediment:

\[ \frac{\partial \bar{c}}{\partial t} = -\frac{w_f}{h}(\bar{c} - c_{ae}) - \bar{u} \frac{\partial \bar{c}}{\partial x} - \bar{v} \frac{\partial \bar{c}}{\partial y} + 0.13h\bar{q} \left( \frac{\partial^2 \bar{h}\bar{c}}{\partial x^2} + \frac{\partial^2 \bar{h}\bar{c}}{\partial y^2} \right), \] 

which consists of the effects of vertical distribution \( w_f/h(\bar{c} - c_{ae}) \), advection \( \bar{u}\partial\bar{c}/\partial x \) and \( \bar{v}\partial\bar{c}/\partial y \), and dispersion \( h\bar{q}\partial^2\bar{h}\bar{c}/\partial x^2 \) and \( h\bar{q}\partial^2\bar{h}\bar{c}/\partial y^2 \). But the derived model (3.12) also includes more subtle effects, which could be important for suspended sediment in complex flow regimes. The model (3.12) further gives modifications in presence of the ratio \( w_f/\bar{q} \) due to different distributions of sediment in the vertical for the different levels of turbulent mixing, parameterised by the mean flow speed \( \bar{q} \). The coefficients in equation (3.12) are a little different to the established model (3.13). Take the
advection term $\bar{u}\partial \bar{c}/\partial x$ for example, the established model (3.13) has the coefficient 1, whereas in our equation (3.12), the coefficient of such terms is $1.007 - 3.094w_f/\bar{q}$. Physically, a higher falling velocity $w_f$ means sediment concentrates more near bed where the mean advection velocity is lower, and hence net transport will be slower. Our model (3.12) has smaller dispersion terms compared with the established model (3.13).

The suspended sediment affects the flow momentum equations by variations in the mean density. We need to modify the turbulence flow momentum equations (2.2)–(2.3) to account for the dominant effects of the sediment. Recall that in the suspended sediment system the mixing density $\rho_{\text{mix}}$ includes the density of the flow $\rho$ and the density of the sediment $\rho_s$. This mixing density changes the fluid inertia. Thus, the flow momentum equations become

\[
\frac{\partial \bar{u}}{\partial t} \approx \text{RHS of equation (2.2)} + 0.00257(s - 1)\frac{\bar{u}\bar{c}\bar{q}}{h} - 0.298(s - 1)h\frac{\partial \bar{c}}{\partial x},
\]

\[
\frac{\partial \bar{v}}{\partial t} \approx \text{RHS of equation (2.3)} + 0.00257(s - 1)\frac{\bar{v}\bar{c}\bar{q}}{h} - 0.298(s - 1)h\frac{\partial \bar{c}}{\partial y}.
\]

The sediment increases the effective density of the flow. The increased mass makes the drag less effective according to the terms $\bar{u}\bar{c}\bar{q}/h$ and $\bar{v}\bar{c}\bar{q}/h$. The terms $h\partial \bar{c}/\partial x$ and $h\partial \bar{c}/\partial y$ are from modification to the hydrostatic pressure due to the extra density. The terms $h\partial \bar{c}/\partial x$ and $h\partial \bar{c}/\partial y$ physically add a force to encourage flow from the high level concentration $\bar{c}$ to the low level concentration $\bar{c}$, as shown in Figure 3.2. A flow would occur until the water depth adjusted to exactly counter the added density.

Thus, equation (3.12), coupled with the conservation equation (2.1) and modified momentum equations (3.14)–(3.15), is valid to approximate the physically suspended sediment in turbulent flow.

### 3.4.3 The suspended sediment in steady flow

Consider the turbulent flow with suspended sediment flowing on a flat mean bed of constant slope $\tan \theta$; that is, the mean bed $b = 0$. Consider the suspended sediment in steady flow of depth $H = 1$. The equilibrium velocities are $U > 0$ and $V = 0$, so the mean speed $\bar{q} = U$. Recall that we consider the concentration fraction is small, $\bar{c} < 0.01$. Thus, in practice, in (3.14)–(3.15) the terms $\bar{u}\bar{c}\bar{q}/h$ and $\bar{v}\bar{c}\bar{q}/h$ are negligible. Then the evolution (3.14) predicts the equilibrium $\bar{q} = U = 18.7\tan^{1/2} \theta$. For the suspended sediment in steady
Figure 3.2: The extra force from the extra hydrostatic pressure (all other things being equal).

flow, equation (3.11), evaluating at $\gamma = \gamma_c = 1$, predicts the nondimensional depth-averaged concentration:

$$\bar{c} / c_{ae} = \frac{0.984w_f - 51.3w_f^2/\bar{q}}{0.938w_f + 0.0156w_f^2 + 28.87w_f^2/\bar{q} + 0.0647 \tan \theta / \bar{q}}.$$  

by substituting the equilibrium speed $\bar{q} = 18.7 \tan^{1/2} \theta$ and the falling velocity (3.3), this simplifies to

$$\frac{\bar{c}}{c_{ae}} \approx \frac{1.05 \tan^{1/2} \theta d^{1/2} - 3.67d}{\tan^{1/2} \theta d^{1/2} + 0.021 \tan^{1/2} \theta d + 2.06d + 0.0029 \tan \theta}. \quad (3.16)$$

In the falling velocity (3.3), take the relative density $s = 2.65$, the nondimensional drag coefficient $c_D = 1.4$. Equation (3.16) shows that the ratio of the depth-averaged concentration $\bar{c}$ and the equilibrium reference concentration $c_{ae}$ on the mean bed $z = 0$ varies with the mean particle size $d$ for the fixed mean slope $\tan \theta$.

Figure 3.3 plots the ratio $\bar{c} / c_{ae}$ varying with the nondimensional mean particle size $d$. The ratio $\bar{c} / c_{ae}$ has a maximum value of 0.8 for small mean particle size $d$. As the mean slope $\tan \theta$ increases, the nondimensional depth-averaged concentration $\bar{c}$ approximately approaches $0.8c_{ae}$. I suggest the reason for this phenomenon is that the increasing slope drives faster flows with more turbulence that mixes the sediment more in the vertical.
Figure 3.3: The ratio of the depth-averaged concentration $\bar{c}$ and the equilibrium reference concentration $c_{ae}$ on the mean bed $z = 0$ varies with the nondimensional mean particle size $d$ via equation (3.16) in steady flow. The maximum $\bar{c}/c_{ae} \approx 0.8$. As the mean slope $\tan \theta$ increases, the depth-averaged concentration $\bar{c}$ approximately gets close to $0.8c_{ae}$. 
3.5 **Vertical distribution of suspended sediment**

This section describes the vertical distribution of the suspended sediment. Most research (Engelund & Fredsoe 1976, Celik & Rodi 1984, van Rijn 1984, e.g.) has focused on the vertical distribution of the suspended sediment concentration in steady uniform channel flows. Computer algebra in Appendix A.1 derives the slow manifold of the suspended sediment concentration in out-of-equilibrium dynamics.

3.5.1 **The low order approximation of the suspended sediment concentration**

Recall that the locally scaled vertical coordinate \( Z = (z - b)/h \). Truncate to errors \( \mathcal{O}(\delta^{3/2} + \delta^{3/2} \tan^{3/2} \theta + w_f^3 + \gamma_e^{3/2}) \) and omit the intricate details of the derivation. Executing the computer algebra in Appendix A.1 and evaluating at \( \gamma = \gamma_e = 1 \) leads to the following approximation of the suspended sediment concentration on the slow manifold:

\[
\begin{align*}
c(Z) &= \bar{c} (0.985 + 0.0422Z - 0.00756Z^2 - 0.0139Z^3) + \bar{c} \frac{w_f}{\bar{q}} (28.36 - 5.156Z - 77.34Z^2) \quad (3.17a) \\
&+ \bar{c} \frac{w_f^2}{\bar{q}} (-0.430 - 0.430Z + 2.578Z^2 - 0.859Z^3) \quad (3.17b) \\
&+ c_{ae} \frac{w_f}{\bar{q}} (56.72 - 166.3Z + 77.34Z^2 + 2.578Z^3) \quad (3.17c) \\
&+ c_{ae} \frac{w_f^2}{\bar{q}} (-0.430 + 1.074Z - 0.430Z^3) \quad (3.17d) \\
&+ \frac{h}{\bar{q}} \left( \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y} \right) (2.578 + 0.921Z - 17.68Z^2 + 11.42Z^3) \quad (3.17e) \\
&+ h \frac{c_{ae} \delta \bar{u}}{\bar{q}} (-0.17 + 0.449Z - 0.0392Z^2 - 1.322Z^3) \quad (3.17f) \\
&+ h \frac{c_{ae} \delta \bar{v}}{\bar{q}} (-1.774 + 1.244Z + 2.471Z^2 - 1.486Z^3) \quad (3.17g) \\
&+ \frac{\bar{c}}{\bar{q}^3} \left( \bar{u} \frac{\partial \bar{h}}{\partial x} + \bar{v} \frac{\partial \bar{h}}{\partial y} \right) (-0.17 + 0.449Z - 0.0392Z^2 - 1.322Z^3) \quad (3.17h) \\
&+ \frac{\bar{c}}{\bar{q}^3} \left( \bar{u} \frac{\partial \bar{b}}{\partial x} + \bar{v} \frac{\partial \bar{b}}{\partial y} \right) (0.918 - 0.809Z^2 + 549Z^2 + 1.343Z^3) \quad (3.17i)
\end{align*}
\]
\[
+ \frac{\bar{c}}{\bar{q}^3} \left( \bar{u} \frac{\partial b}{\partial x} + \bar{v} \frac{\partial b}{\partial y} \right) \left( 0.918 - 0.809 Z^2 + 1.343 Z^3 \right) \quad (3.17k)
- \tan \theta \frac{h \bar{c}}{\bar{q}^3} \left( 0.918 - 0.809 Z^2 + 1.343 Z^3 \right) \quad (3.17l)
+ O\left( \partial_x^{3/2} + \partial_y^{3/2} + \tan^{3/2} \theta + w_f^3 + \gamma^{3/2}, \gamma_5 \right).
\]

Equation (3.17) describes the low-order shape of the slow manifold in state space with introduced artificial parameters. Physically this equation describes the details of the suspended sediment concentration associated with given patterns of fluid depth \( h \), depth-averaged velocities \( \bar{u} \) and \( \bar{v} \), and the depth-averaged concentration \( \bar{c} \). The terms in equation (3.17) have physical interpretations. For example, the line (3.17a) approximates the mean concentration, together with the lines (3.17b)–(3.17e), which forms the basic distribution of the concentration \( c(Z) \) in the vertical in the presence of the depth-averaged concentration \( \bar{c} \), the equilibrium reference concentration \( c_{ae} \), the falling velocity \( w_f \) and the mean fluid speed \( \bar{q} \). The lines (3.17f)–(3.17i) describe the effect by mixing of the vertical distribution of the concentration. The lines (3.17j)–(3.17k) describe the effect due to the change of the bed topography. The line (3.17l) describes the gravity affecting the vertical distribution of the suspended sediment.

### 3.5.2 Distribution of the suspended sediment in steady flow

This subsection computes the vertical distribution of the suspended sediment in steady flow.

First, compute the nondimensional equilibrium reference concentration \( c_{ae} \) on the mean bed \( z = b \) in steady flow. We consider the steady flow has the nondimensional equilibrium of depth \( H = 1 \) and depth-averaged velocities \( \bar{q} = U = 18.7 \tan^{1/2} \theta \) and \( V = 0 \). We approximate the nondimensional shear velocity \( u_f = \bar{q}/C' \), where \( C' = 18 \log(4/d) \) is the Chezy coefficient (van Rijn 1984, e.g.). Thus, equation (3.7) gives the nondimensional equilibrium reference concentration:

\[
c_{ae} = \frac{3.26 \tan^{1.5} \theta}{d^{0.8}(1.39 - \log(d))^{3}},
\]

which only depends on the nondimensional mean slope \( \tan \theta \) and the nondimensional mean particle size \( d \). Figure 3.4 plots the nondimensional equilibrium reference concentration \( c_{ae} \) varying with the nondimensional mean particle size \( d \) in steady fluid flow down a mean bed. The fluid depth \( H = 1 \)
Figure 3.4: Profile of the nondimensional equilibrium reference concentration \( c_{ae} \) as a function of the nondimensional mean particle size \( d \) in steady flow. The mean slope \( \tan \theta = 0.01 \). When the nondimensional particle mean size \( d > 0.8 \times 10^{-4} \), the nondimensional equilibrium reference concentration \( c_{ae} \approx 0.005 \).
and the mean slope \( \tan \theta = 0.01 \). When the particle mean size \( d \) is bigger than \( 0.8 \times 10^{-4} \), the equilibrium reference concentration \( c_{ae} \) approximately approaches 0.005. The nondimensional equilibrium reference concentration \( c_{ae} \) increases as the mean slope \( \tan \theta \) increases via equation (3.18).

Second, we compute the vertical distribution of the suspended sediment in steady flow. Consider the turbulent flow with suspended sediment on the flat mean bed of constant slope; that is the mean bed \( b = 0 \). For example and for simplicity, we consider the vertical distribution of the suspended sediment concentration in steady fluid flow of nondimensional equilibrium depth \( H = 1 \). Then the nondimensional equilibrium depth-averaged lateral velocities are \( \bar{q} = U = 18.7 \tan^{1/2} \theta \) and \( V = 0 \). Therefore, for the steady flow, the approximate vertical distribution of sediment (3.17) simplifies to

\[
c(Z) \approx \bar{c} \left( 0.985 + 0.0422Z - 0.00756Z^2 - 0.0139Z^3 \right) \\
+ \bar{c} \frac{w_f}{\bar{q}} \left( 28.36 - 5.156Z - 77.34Z^2 \right) \\
+ \bar{c} \frac{w_f^2}{\bar{q}} \left( -0.430 - 0.430Z + 2.578Z^2 - 0.859Z^3 \right) \\
+ c_{ae} \frac{w_f}{\bar{q}} \left( 56.72 - 166.3Z + 77.34Z^2 + 2.578Z^3 \right) \\
+ c_{ae} \frac{w_f^2}{\bar{q}} \left( -0.430 + 1.074Z - 0.430Z^3 \right).
\]  

Equation (3.19) shows that the concentration \( c(Z) \) depends on the vertical coordinate \( Z \), the falling velocity \( w_f \), the depth-averaged concentration \( \bar{c} \), the equilibrium reference concentration \( c_{ae} \) and the mean flow speed \( \bar{q} \). The falling velocity \( w_f \) varies with the mean particle size \( d \) and the mean slope \( \tan \theta \) according to equation (3.3). Equation (3.16) expresses the ratio of the depth-averaged concentration \( \bar{c} \) and the equilibrium reference concentration \( c_{ae} \) on the mean bed \( z = b \) varying with the mean particle size \( d \) and the mean slope \( \tan \theta \). Recall that the mean flow speed \( \bar{q} = U = 18.7 \tan^{1/2} \theta \). Thus, the concentration profile \( c(Z) \) depends on the vertical coordinate, the mean particle size \( d \) and the mean slope \( \tan \theta \).

Figure 3.5 depicts the profiles of the nondimensional suspended sediment concentration \( c(Z) \) in the vertical for four different nondimensional mean particle size \( d \). The bed has a mean slope \( \tan \theta = 0.01 \). The figure assumes the particles have relative density \( s = 2.65 \), nondimensional drag coefficient \( c_D = 1.4 \) and the nondimensional gravity \( g = 1 \). According to Figure 3.4, the nondimensional equilibrium reference concentration \( c_{ae} \approx 0.005 \) for the mean particle size \( d > 6 \times 10^{-5} \). For small nondimensional mean particle size \( d \), the nondimensional concentration \( c(Z) \) is approximately linear in the vertical
Figure 3.5: Profiles (line curves) of the suspended sediment concentration \( c(Z) \) in the vertical for three different nondimensional mean particle size \( d \) according to equation (3.19). The mean slope \( \tan \theta = 0.01 \). The equilibrium reference concentration \( c_{ae} = 0.005 \). The dash curves are the corresponding steady analytical approximation (3.21).
Figure 3.6: Vertical distribution of the suspended sediment: (blue curve) from equation (3.19); (red circle) the numerical prediction by Celik & Rodi (1988); and (green stars) the corresponding experimental data used by Celik & Rodi (1988). The nondimensional mean particle size $d = 1.65 \times 10^{-4}$ and then the falling velocity $w_f = 0.0161$ in our simulation.
coordinate \( Z \). In these steady flow, the approximation of the diffusion is

\[
\epsilon_s(Z) \approx \bar{q}(0.00628 - 0.00269Z - 0.000733Z^2) \\
+ \tan \theta \frac{1}{\bar{q}}(0.00978 - 0.2605Z + 0.247Z^2).
\]

(3.20)

For an indicative comparison, we integrate equation (3.2) from the bottom to the free surface in the steady flow, and obtain an approximation

\[
c(Z) \approx c_{ae} \left[ \frac{5.29 + Z}{3.26(1.62 - Z)} \right]^{-197.45w_f/\bar{q}}.
\]

(3.21)

The dash curves in Figure 3.5 depict the approximation (3.21). When the nondimensional mean particle size \( d \) is small, the computed vertical distribution (3.19) is approximately the same with the approximation (3.21). When the nondimensional mean particle size \( d \) increases, there is a difference between the computed vertical distribution (3.19) and the approximation (3.21) at the upper flow.

Figure 3.6 plots the vertical distribution of the suspended sediment from equation (3.19) (blue curve), the numerical prediction (red circles) by Celik & Rodi (1988), and the corresponding experimental data (green stars) used by Celik & Rodi (1988). Celik & Rodi (1988) calculated the suspended sediment transport in unidirectional channel flow, where they used the nondimensional variables of fluid depth \( H = 1 \), mean velocity \( U \approx 1.8 \), mean particle size \( d = 1.65 \times 10^{-4} \) and falling velocity \( w_f = 0.0165 \). Our simulation (blue curve) agrees with the numerical prediction by Celik & Rodi (1988) except at the bottom. This difference at the bottom is because we have small entrainment at the mean bed. Our simulation (blue curve) is good enough to predict the experimental data (green stars). The trends of the suspended sediment concentration qualitatively agrees with other published experimental measurements (Cellino & Graf 1999, Yoon & Kang 2005, e.g.).

3.6 Numerical simulations of the suspended sediment in turbulent flow

This section implements the numerical simulation of the suspended sediment model (3.1) in open channel flows and large waves. A staggered grid scheme is used to implement the numerical simulation. Numerical results show the model (3.1) is reasonable.
Figure 3.7: The staggered grid scheme at the point \((x_i, y_j)\) and its neighbouring points. Locate the depth \(h\) and the depth-averaged concentration \(\bar{c}\) at the blue points. Locate the depth-averaged lateral velocity \(\bar{u}\) at the half-points of the \(x\)-grid (red points) and the depth-averaged lateral velocity \(\bar{v}\) at the half points of the \(y\)-grid (green points).
3.6.1 The staggered numerical scheme

This subsection describes the staggered grid scheme for simulations of the suspended sediment model (3.1), coupled together with the turbulence model (2.1) and (3.14)–(3.15). All variables and parameters are nondimensional.

Consider a domain with length $L_x$ and width $L_y$. Let there be $N_x$ intervals in the $x$ direction and $N_y$ intervals in the $y$ direction. The spatial steps are $\delta x = L_x/N_x$ and $\delta y = L_y/N_y$. Define the grid points $x_i = i\delta x$ and $y_j = j\delta y$ with $i = 1 : N_x$ and $j = 1 : N_y$. Figure 3.7 exhibits the staggered grid near the point $(x_i, y_j)$. Discretise the depth $h$ and the depth-averaged concentration $\bar{c}$ at the blue points. Discretise the depth-averaged lateral velocity $\bar{u}$ at the half-points of the $x$-grid (red points) and the depth-averaged lateral velocity $\bar{v}$ at the half points of the $y$-grid (green points). Recall that $h_{i,j}(t) = h(x_i, y_j, t)$, $\bar{u}_{i+1/2,j}(t) = \bar{u}(x_{i+1/2}, y_j, t)$ and $\bar{v}_{i,j+1/2}(t) = \bar{v}(x_i, y_{j+1/2}, t)$. Define $\bar{c}_{i,j}(t) = \bar{c}(x_i, y_j, t)$. Thus, this staggered grid discretises the physical fields.

Approximate the suspended sediment PDE (3.1) on this staggered grid. Section 2.6 defines the values of $h_{i+1/2,j}$, $h_{i+1,j}$, $h_{i+1/2,j+1/2}$, and similarly the depth-averaged velocities $\bar{u}$ and $\bar{v}$. Similarly, compute approximations to the depth-averaged concentration $\bar{c}$ at the half step grids,

$$\bar{c}_{i+1/2,j} = \frac{1}{2} (\bar{c}_{i,j} + \bar{c}_{i+1,j}) , \quad \bar{c}_{i,j+1/2} = \frac{1}{2} (\bar{c}_{i,j} + \bar{c}_{i+1,j}) ,$$

$$\bar{c}_{i+1/2,j+1/2} = \frac{1}{2} (\bar{c}_{i+1/2,j+1} + \bar{c}_{i+1,j+1/2}) .$$

The mean flow speed at the grid point $(x_i, y_j)$ is $\bar{q}_{i,j} = \sqrt{\bar{u}_{i,j}^2 + \bar{v}_{i,j}^2}$. Note that compute the mean bed $b(x, y)$ at the same grid positions as the depth $h$. Then, use the centre difference method to approximate the suspended sediment model (3.1)

$$\frac{d\bar{c}}{dt} \approx - \frac{w_f}{h_{i,j}} \left( \frac{0.938 + 28.87}{q_{i,j}} \right) \frac{w_f}{h_{i,j}} \bar{c}_{i,j} + \frac{w_f}{h_{i,j}} \bar{c}_{ae} \left( \frac{0.984 - 51.3}{q_{i,j}} \right)$$

$$- 1.007 \exp \left( -3.073 \frac{w_f}{q_{i,j}} \right) \left( h_{i+1/2,j} \bar{u}_{i+1/2,j} \bar{c}_{i+1/2,j} - h_{i-1/2,j} \bar{u}_{i-1/2,j} \bar{c}_{i-1/2,j} \right) + \frac{h_{i,j+1/2} \bar{v}_{i,j+1/2} \bar{c}_{i,j+1/2} - h_{i,j-1/2} \bar{v}_{i,j-1/2} \bar{c}_{i,j-1/2}}{\delta y}$$

$$+ \bar{v}_{i,j} h_{i,j+1/2} + b_{i,j+1/2} - h_{i,j-1/2} b_{i,j-1/2} \frac{\delta y}{\delta y}$$

$$+ 0.033 h_{i,j} q_{i,j} \left( \frac{\bar{c}_{i+1/2,j} - 2\bar{c}_{i,j} + \bar{c}_{i-1/2,j}}{\delta x^2} + \frac{\bar{c}_{i,j+1/2} - 2\bar{c}_{i,j} + \bar{c}_{i,j-1/2}}{\delta y^2} \right)$$
\[ + 0.027h_{i,j} \frac{\bar{u}_{i,j}^2 - \bar{v}_{i,j}^2}{\bar{q}_{i,j}} \left( \frac{\bar{c}_{i+1/2,j} - 2\bar{c}_{i,j} + \bar{c}_{i-1/2,j}}{\delta x^2} \right) - \frac{\bar{c}_{i,j+1/2} - 2\bar{c}_{i,j} + \bar{c}_{i,j-1/2}}{\delta y^2} \right), \]

(3.22)

where the falling velocity \( w_f \) and the equilibrium reference concentration \( c_{ae} \) are constants determined by the mean particle size \( d \) and the mean slope \( \tan \theta \). The matlab function \texttt{ode15s} is used to integrate these equations in time, listed in Appendix B.2.

Boundary conditions need to be specified. Typically, choose periodic boundary conditions for all fields, now including the depth-averaged concentration \( \bar{c} \), both in the \( x \) and \( y \) direction, that is

\[ \bar{c}(x, y, t) = \bar{c}(x + L_x, y, t), \quad \bar{c}(x, y, t) = \bar{c}(x, y + L_y, t) \]

Thus, the suspended sediment PDE (3.1) is approximated by the spatial discretisation. Numerical solutions of the model (3.1), coupled with the flow equations (2.1) and (3.14)–(3.15), are implemented with some given initial conditions.

### 3.6.2 Simulating suspended sediment in open channels

This section nondimensionally simulates the suspended sediment in straight and meandering channels by the suspended model (3.1), coupled with the flow equations (2.1) and (3.14)–(3.15). The water covers the entire domain in order to avoid, at this stage, complications of moving contact lines between wet and dry bed: the flow is in the channel and over a surrounding flood plain. The simulations agree with published findings that the depth-averaged concentration is higher in the channel than in the shallows, and the maximum of the depth-averaged concentration shifts from the outer bank to the inner bank (Lin & Falconer 1996, Demuren & Rodi 1986, Ye & McCorquodale 1997, Duan 2004, e.g.).

**Simulating suspended sediment in a straight open channel**

Consider fluid with suspended sediment flowing down a straight channel of mid-depth one. As shown in Figure 2.11, the width of the channel is \( 2\beta = 8 \), and the shallow depth \( B = 0.1 \) so the depth parameter of the channel is 0.9. The channel has a mean slope \( \tan \theta = 0.01 \) in the flow direction.

Numerical solutions of the fields of depth \( h(x, y, t) \), depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \), and depth-averaged concentration \( \bar{c}(x, y, t) \) are implemented using centred difference approximations to the spatial derivatives in equations (2.42)–(2.43) and equation (3.22) on a staggered grid in
Figure 3.8: The history of the depth-averaged concentration $\bar{c}$ in the turbulent flow over straight open channel at three stations across the channel. The channel has a length $L_x = 40$ and a width $2\beta = 8$. The shallow depth is $B = 0.1$, so the depth parameter of the channel is 0.9. The channel has a mean slope $\theta = 0.01$ in the flow direction. The sediment has a nondimensional mean particle size $d = 6 \times 10^{-5}$. 
Figure 3.9: The distribution of the depth-averaged suspended sediment concentration \( \bar{c} \) along the straight channel at time \( t = 400 \). The mean particle size \( d = 6 \times 10^{-5} \), so the falling velocity \( w_f = 0.0097 \) and the equilibrium reference concentration \( c_{ae} = 0.0057 \) on the mean bed \( z = b \).

space described in Section 2.6 and Section 3.6.1. Time integration is implemented by the Matlab function \texttt{ode15s}.

Simulations typically start the fluid with a flat free surface \((z = 0)\), zero depth-averaged lateral velocity \( \bar{v} \) and depth-averaged concentration \( \bar{c} \), and equilibrium depth-averaged lateral velocity \( \bar{u} = 1.87 \). The numerical results of the flow fields of depth-averaged velocities \( \bar{u} \) and \( \bar{v} \) are the same as in section 2.7.1. Thus, we concentrate on the depth-averaged concentration \( \bar{c} \) in the straight channel flow.

Figure 3.8 shows the history of the depth-averaged concentration \( \bar{c} \) in the turbulent flow over straight open channel, which demonstrates the solution reaches a steady state after the time \( t \approx 250 \). Figure 3.9 shows the distribution of the depth-averaged suspended sediment concentration \( \bar{c} \) along the straight channel at time \( t = 400 \). The channel has a width \( 2\beta = 8 \) and a mean slope \( \theta = 0.01 \) in the flow direction. The shallow depth is \( B = 0.1 \), so the depth parameter of the channel is then 0.9. The mean particle size \( d = 6 \times 10^{-5} \), so the falling velocity \( w_f = 0.0097 \) and the equilibrium reference concentration \( c_{ae} = 0.0057 \) on the mean bed \( z = b \). Lin & Falconer (1996) and Lin & Falconer (1997) solved the diffusion equation (3.2) using an operator splitting algorithm, which is to split the equation (3.2) into several smaller
Figure 3.10: The distribution of the depth-averaged concentration $\bar{c}$ across the straight channel at time $t = 400$. The channel is nine times as deep as in the shallow regions. The flow in the channel transports suspended sediment 70% more than in the shallows.

and simple sub-equations and to solve these equations by the ULTIMATE QUICKEST scheme. Numerically predicted depth-averaged concentration $\bar{c}$ by Lin & Falconer (1996) was independent of the flow direction $x$ in the steady flow. Figure 3.9 shows that the depth-averaged concentration $\bar{c}$ is independent of coordinate $x$.

We are interested in the depth-averaged concentration $\bar{c}$ in the cross section of the channel. Figure 3.10 plots the depth-averaged concentration $\bar{c}$ across the straight channel at time $t = 400$. The flow in the channel mixes more suspended sediment than in the shallows, because the flow in the channel has faster mean speed and hence stronger turbulent mixing. Pizzuto (1987) pointed out that the suspended sediment will be transported away from the channel centre, and will be deposited by the turbulent eddies at the interfaces of the channel and shallows. Thus, in Figure 3.10 the suspended sediment will be transported from the channel (high concentration) to the shallows (low concentration) by turbulent dispersion. The vortices found at the interfaces between the channel and shallows in Figure 2.14 also aid the suspended sediment being deposited away from the channel.
Figure 3.11: The meandering channel is covered by zero level fluid with suspended sediment. The channel has a length of 40 and a width of $2\beta = 8$. The depth of the shallow regions is $B = 0.5$, so the depth parameter of the channel is 0.5. The meandering channel has the curvatures $\kappa_1 = 1.5$ and $\kappa_2 = 2\pi/L_x$, and has a mean slope $\tan \theta = 0.01$ in the flow direction.

Simulating suspended sediment in a meandering channel

Consider the fluid with suspended sediment flowing down a meandering channel. Here I describe simple meandering open channels by the nondimensional bed

$$z = b(x, y) = -B - (1 - B) \left\{ \max \left[ 0, 1 - \left( \frac{y - \kappa_1 \cos(\kappa_2 x)}{\beta} \right)^2 \right] \right\}^2,$$  (3.23)

where the parameter $\kappa_2$ determine the wavelength $2\pi/\kappa_2$ of the meandering channel, the parameter $\kappa_1$ is the half-width of the extent of the meanders, and the parameters $\beta$ and $B$ are the half-width and depth of the shallows. Figure 3.11 shows the shape of the slightly meandering channel, with a length of 40 and a width of $2\beta = 8$. The meandering channel has the curvatures $\kappa_1 = 1.5$ and $\kappa_2 = 2\pi/L_x$, and has a mean slope $\tan \theta = 0.01$ in the flow direction. The depth of the shallow regions is $B = 0.5$, so the depth parameter of the channel is 0.5. Compared with the straight channel, the reason for enlarging the shallow depth $B$ is that the complex topography makes the water depth dip down close zero, which is singular in our modelling.
Figure 3.12: The distribution of the depth-averaged suspended sediment concentration $\bar{c}$ along the meandering channel. The depth-averaged concentration $\bar{c}$ is higher in the channel than in the shallows.
Figure 3.13: Plots of the depth-averaged concentration $\bar{c}$ across channel bend in region of $15 < x < 30$. The left-hand side is the outer bank and the right-hand side is the inner bank. The maximum of the depth-averaged concentration $\bar{c}$ shifts from the outer bank to the inner bank at the bend.

I simulate the suspended sediment in the turbulent flow along such channel by the suspended model (3.1), together with the flow equations (2.1) and (3.14)–(3.15) with periodic boundary conditions in both $x$ and $y$ directions for both the flow and channel. The Matlab function $\text{fsolve}$ is used to find the equilibrium of the PDEs (3.1) and (2.1)–(2.3).

Figure 3.12 shows the depth-averaged suspended sediment concentration $\bar{c}$ along the meandering channel. The depth-averaged concentration $\bar{c}$ is higher in the channel than in the shallows, and reaches maximum at the channel bends. Compared with over the straight open channel, the depth-averaged concentration $\bar{c}$ varies along the channel. Demuren & Rodi (1986) pointed out that the mixing of pollutants is stronger in the meandering channel than in the corresponding straight channel. However, in our simulation, the maximum depth-averaged concentration in the meandering mid-channel is a little smaller than that in the corresponding straight mid-channel ($B = 0.5$), shown in Figure 3.14. That is because the meandering channel generates
Figure 3.14: Plots of the depth-averaged concentration $\bar{c}$ across the channel at $x = 19$ in the straight (blue) and meandering (red) channels. The flood plain has shallow depth 0.5.

strong turbulent mixing at the interfaces between the main channel and the shallows, shown in Figure 2.17. The strong turbulent mixing makes the suspended sediment be deposited away from the channel.

Figure 3.13 plots the depth-averaged concentration $\bar{c}$ in the cross section at the bend of the channel in region of $15 < x < 30$. At this bend, the left-hand side is the outer bank of the meandering channel and the right-hand side is the inner bank. Demuren & Rodi (1986), Ye & McCorquodale (1997) and Duan (2004) reviewed the experiments of pollutant dispersion in meandering channels (Chang 1971, e.g.), and acquired good agreements between the reviewed experimental data and their numerical calculations by depth-averaging the momentum, continuity equations and the convection-diffusion equation for mass transport. Both their numerical calculations and the reviewed experiments show that the pollutants spread from the outer bank to the inner bank of the meandering channels. In our simulation, when the fluid flows through the bend, the largest depth-averaged suspended sediment concentration $\bar{c}$ shifts from the outer bank to the inner bank, which agrees with the published findings (Demuren & Rodi 1986, Ye & McCorquodale 1997, Duan 2004, e.g.).
3.6.3 Simulating suspended sediment in large waves

This section explores the depth-averaged concentration in waves on an inclined flat and rippled bed by the suspended sediment model (3.1), coupled with the modified momentum equations (3.14)–(3.15). Numerical results qualitatively agree with the experimental measurements and observations of the suspended sediment in physical waves (Ribberink & Al-Salem 1995, Black 1994, Lavelle et al. 1984, e.g.).

Simulating suspended sediment in large waves over an inclined flat bed

Consider the fluid of depth \( h(x, y, t) \) with suspended sediment of depth-averaged concentration \( \bar{c}(x, y, t) \) flowing down an inclined flat bed at the mean slope \( \tan \theta \). The fluid has the depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \). Let the mean bed have nondimensional length \( L_x = 100 \) and width \( L_y = 10 \).

In simulations, all the variables are nondimensional. Let the mean slope of the bed be \( \tan \theta = 0.01 \). Assume the sediment has the mean particle size \( d = 6 \times 10^{-5} \), so the falling velocity \( w_f = 0.0097 \) according to equation (3.3) using the relative density \( s = 2.65 \) and the drag coefficient \( c_D = 1.4 \). According to equation (3.18), the equilibrium reference concentration \( c_{ae} = 0.0057 \). The flow has the equilibrium of depth \( H = 1 \), and depth-averaged lateral velocities \( \bar{U} = 1.87 \) and \( \bar{V} = 0 \) from momentum equations (3.14)–(3.15), and depth-averaged concentration \( \bar{c} \approx 0.0036 \) from equation (3.1). The simulations initially impose a small perturbation \( 0.2 \sin(2\pi/L_x x) \) to the equilibrium.

For a pilot study, we consider the depth-averaged velocity \( \bar{v} = 0 \) throughout. Roll waves will be generated on the free surface in the flow direction (Anders & Ekvall 1993, Balmforth & Mandre 2004, e.g.). Figure 3.15 shows the time series of the depth-averaged lateral velocity \( \bar{u} \) (blue curves) at the position \( x = 50 \). We are not interested in the sinusoidally oscillated waves before the time \( t \approx 50 \), which is dominated by the initial conditions. After time \( t \approx 50 \), periodic roll waves are established.

Figure 3.15 also plots the time series of the depth-averaged concentration \( \bar{c} \) (red curves). The depth-averaged concentration \( \bar{c} \) rises slightly fast and falls slowly in a period; for example, see the period during time \( t \approx 220 \) to \( t \approx 250 \). This phenomenon is because the turbulent mixing picks up and mixes the sediment fast into suspension and then the falling velocity factor makes the sediment fall when the depth-averaged velocity \( \bar{u} \) slows. This mixing qualitatively agrees with the theoretical prediction of the suspended sediment in the combined wave-current motion by Fredsoe et al. (1985),
Figure 3.15: Time series of the depth-averaged lateral velocity $\bar{u}$ and depth-averaged suspended sediment concentration $\bar{c}$ of the fluid flowing down an inclined flat bed at the position $(x, y) = (50, 0)$. The bed has a mean slope $\tan \theta = 0.01$ in the flow direction. The mean particle size $d = 6 \times 10^{-5}$, so the falling velocity $w_f = 0.0097$ and the equilibrium reference concentration $c_{ae} = 0.0057$. 

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Figure 3.16: Plot the depth $h(x, y, t)$ (black curve) and depth-averaged concentration $\bar{c}(x, y, t)$ (red curve) in the $x$ direction at time $t = 220$ from the simulation in Figure 3.15.
and experimental measurements in oscillatory flow (Ribberink & Al-Salem 1995, e.g.) and physical observations in ocean currents (Lavelle et al. 1984, Soulsby et al. 1994, e.g.). Figure 3.15 also shows that the time series of the depth-averaged concentration $\bar{c}$ and the depth-averaged velocity $\bar{u}$ are approximately $\pi/2$ out of phase, which means the maximum concentration occurs as the velocity decreases. This is due to the time needed to mix the sediment (Fredsoe et al. 1985, Staub et al. 1996, e.g.).

Figure 3.16 plots the depth $h(x, y, t)$ and depth-averaged concentration $\bar{c}(x, y, t)$ in the $x$ direction at time $t = 220$ in Figure 3.15. A turbulent bore is located near $x = 75$. The depth-averaged concentration $\bar{c}$ reaches peak near $x = 40$. This corresponds to the time series at $x = 50$ at time $t \approx 220$ in Figure 3.15. Figure 3.15–3.16 show one peak in a period of the depth-averaged concentration $\bar{c}$, which qualitatively agrees with the predictions by Davies et al. (1997) of the suspended sediment in waves and currents over plane beds. However, at least two peaks in a period were observed in experimental measurements and observations (Ribberink & Al-Salem 1995, Staub et al. 1996, e.g.). Black (1994) compared numerical simulation of suspended sediment in an asymmetric wave cycle over a plane bed with established experimental data to examine the peaks appearing in a period. Black (1994), as a pilot study, showed that the peaks are relative to the eddy viscosity, entrainment and flow reversal. In our simulation, the only peak in a period is caused by the strong turbulent mixing in the turbulent flow.

Simulating suspended sediment in large waves over a rippled bed

Consider the fluid of depth $h(x, y, t)$ with suspended sediment of depth-averaged concentration $\bar{c}(x, y, t)$ flowing down a rippled bed. The fluid has the depth-averaged lateral velocities $\bar{u}(x, y, t)$ and $\bar{v}(x, y, t)$ along the bed. Let the bed have nondimensional length $L_x = 100$ and width $L_y = 10$. The mean slope of the bed is $\tan \theta = 0.01$. The green curve in Figure 3.18 shows the ripples on the bed. The ripples have the maximum nondimensional height 0.4 and length 20. The bed has zero mean.

For this pilot study, we consider the depth-averaged velocity $\bar{v} = 0$ throughout. The flow has mean equilibrium depth $H = 1$, then the mean equilibrium depth-averaged lateral velocity $\bar{U} \approx 1.86$ and mean equilibrium depth-averaged concentration $\bar{c} \approx 0.0035$. Recall the Froude number $\bar{U}/\sqrt{gH} = 1.86 > 1$. Thus, we predict supercritical flow arises in the simulation.

Simulate the suspended sediment in the fluid flowing over the rippled bed of Figure 3.18 by the suspended model (3.1), coupled with the modified turbulence model (3.14)–(3.15) with periodic boundary conditions in
Figure 3.17: Time series of the depth-averaged lateral velocity $\bar{u}$ (blue curves) and the depth-averaged concentration $\bar{c}$ (red curves) at the trough $x = 50$ (line curves) and at the crest $x = 60$ (dash curves) in Figure 3.18. The mean particle size $d = 6 \times 10^{-5}$, so the falling velocity $w_f = 0.0097$ and the equilibrium reference concentration $c_{ae} = 0.0057$. 
both \( x \) and \( y \) directions for both the flow and bed. All the variables in the simulations are nondimensional. Figure 3.17 plots the time series of the depth-averaged lateral velocity \( \bar{u} \) (blue curves) and the depth-averaged concentration \( \bar{c} \) (red curves) at a trough \( x = 50 \) (line curves) and at a crest \( x = 60 \) (dash curves). The periodic depth-averaged velocity \( \bar{u} \) indicates that large roll waves are generated on the free surface (Balmforth & Mandre 2004, e.g.). The depth-averaged velocity \( \bar{u} \) is bigger at the trough \( x = 50 \) (blue line curve) than at the crest \( x = 60 \) (blue dash curve), that indicates strong turbulent mixing arises at the trough (Zedler & Street 2001, e.g.). Then the turbulent mixing produces slightly bigger depth-averaged concentration \( \bar{c} \) at the trough \( x = 50 \) (red line curve) than at the crest \( x = 60 \) (red dash curve). Zedler & Street (2006), in their calculation of suspended sediment over rippled beds, found far from the bed, the concentration at the crest and trough are approximate \( \pi \) out of phase. Zedler & Street (2006) commented that this phase lag is due to the vortex produced by the ripple near the trough. However, in our simulation, there no significant lag happens, which is due to our ripple not producing a vortex near the trough.

Figure 3.18 plots the depth \( h \) (black) and depth-averaged velocity \( \bar{u} \) (blue)
Figure 3.19: Plots of the depth $h$ (black) and the depth-averaged concentration $\bar{c}$ (red) in the $x$ direction at time $t = 180$ in Figure 3.17. The depth-averaged concentration $\bar{c}$ is ahead to reach maximum over a ripple.
of the fluid flowing over the rippled bed (green) in the x direction at time $t = 180$ in Figure 3.17. The dash green line represents the zero mean bed level. Figure 3.18 exhibits the supercritical flow as the fluid flowing over each ripple on the bed. The depth $h$ rises at the crest and the depth-averaged velocity $\bar{u}$ declines at the crest, which corresponds to the depth-averaged velocity $\bar{u}$ reaches minimum at the crest in Figure 3.17. Figure 3.19 plots the depth $h$ and the depth-averaged concentration $\bar{c}$ in x-direction at time $t = 180$ in Figure 3.17. The depth-averaged concentration $\bar{c}$ is approximate $\pi/2$ phase ahead the depth $h$. That is because the strong turbulent mixing at the troughs makes the concentration peak quickly.

Similar to Figure 3.16, there is still only one significant peak in one period in Figure 3.18–3.19. Zedler & Street (2006), who reported numerical results of large eddy simulation of the flow and suspended sediment over sinusoidal ripples, found three peaks on the time series of concentration at the crest and trough. Their ripples have a height to wavelength ratio of 0.1, which is five times steeper than our ripples. Zedler & Street (2006) commented that these peaks are mainly due to the vortex, shear stress and advection near the ripple. In our simulation, the only significant peak in a period is due to the turbulent mixing.

Figure 3.20 compares at the time $t = 180$ the depth $h$ (black) and depth-averaged concentration $\bar{c}$ (red) of the flow over ripples with different heights. The dash curves represent the depth $h$ and depth-averaged concentration $\bar{c}$ for the ripple height 0.4, while the line curves are for the ripple height 0.6. The fluid depth $h$ is usually bigger at the crest of the steeper ripple, but the depth-averaged concentration $\bar{c}$ becomes smaller for the steeper ripples. However, the laboratory experiment by Osborne & Vincent (1996), whose ripples have an approximate height to wavelength ratio of 0.2, verifies that steep asymmetric ripples under shoaling waves produce greater concentrations higher in the water column than low steepness ripples. Such difference is possibly because the ripples with small height in our simulation do not produce strong vortices that enhance the pick up of sediment into suspension. The phenomenon of smaller depth-averaged concentration for steeper ripples is because the increased fluid depth for steeper ripples produces small depth-averaged concentration according to the erosion and deposition $w_1\bar{c}/h$ in the governing equation (3.1).

### 3.7 Conclusion

This chapter derives a suspended sediment model (3.1) to simulate the suspended sediment in turbulent flows. The model consists of the effects of
Figure 3.20: Plots of the depth $h$ (black curves) and depth-averaged concentration $\bar{c}$ (red curves) of the fluid flowing over the rippled bed with ripple height 0.4 (dash curves) and 0.6 (line curves) in the x direction at time $t = 180$. 
sediment erosion, advection, and dispersion. Section 3.3 embedded the physical boundary conditions on the free surface and on the mean bed in a family of artificial problems to access a slow manifold in the system. When the parameter $\gamma_c = 1$ recovers the original physical problem and $\gamma_c = 0$ introduces an arbitrary problem. Based on the small variations $\gamma_c = \tan \theta = \partial_x = \partial_y = 0$, a two parameter family of equilibria exists to support the existence of a slow manifold in the system. Computer algebra detailed in Appendix A.1 leads to the evolution equation in the field of depth-averaged concentration $\overline{c}(x, y, t)$. It is reassuring that the dominant terms in our model agree with the established modelling (Wu 2004, Duan 2004, Duan & Nanda 2006, e.g.). Then our model includes more subtle effects, that could be important for suspended sediment in complex flow regimes. The low order approximation of the suspended sediment governs the vertical distribution. The trends of the suspended sediment concentration corresponds to the published experimental measurements (Cellino & Graf 1999, Yoon & Kang 2005, e.g.).

Section 3.6 implemented numerical simulations of the suspended sediment in open channel flows and under large waves by the suspended sediment model (3.1), coupled with the modified momentum equations (3.14)–(3.15). A staggered grid scheme is used to implement the numerical simulation. Section 3.6.2 explores the suspended sediment in straight and meandering channels. The simulations agree qualitatively with published findings that the depth-averaged concentration is higher in the channel than in the shallows, and the maximum of the depth-averaged concentration shifts from outer bank to the inner bank (Lin & Falconer 1996, Demuren & Rodi 1986, Ye & McCorquodale 1997, Duan 2004, e.g.). Section 3.6.3 explores the distribution of suspended sediment in large waves over inclined flat and rippled beds. The time series of the depth-averaged suspended sediment concentration rises fast and falls slowly. Supercritical flow arises when the fluid flowing over the rippled bed. The plots of the depth-averaged concentration $\overline{c}$ in space show that high concentration arises at the troughs and low concentration arises at the crests. These results qualitatively agree with the experimental measurements and observations of the suspended sediment in physical waves (Ribberink & Al-Salem 1995, Black 1994, Davies et al. 1997, e.g.).

Further research based on this chapter could improve the suspended model by invoking bed loads. One might apply the proposed suspended model to simulate the suspended sediment in complex physical waves, such as in dam-breaking waves, river floods and beach waves.
Chapter 4

Multiscale modelling couples patches of wave-like simulations

4.1 Introduction

Some multiscale models are presented for dissipative systems (E & Engquist 2003, Kevrekidis et al. 2003, Roberts & Kevrekidis 2005, Hou et al. 2008, e.g.). The aim of our macroscopic modelling is to describe the evolution system over large time and lateral space scales through a known microscopic simulator which only describes the system over small times and spatial domains. The microscopic simulator provides the necessary data for the macroscopic model, so when the microscopic simulator improves, the overall results will improve. The weakly or non-dissipative wave dynamics over large spatial domains, such as tsunamis, floods and rivers, are complicated, as shown as Figure 1.1. A macroscopic model for such large scale waves is expensive. This chapter aims to adapt a gap-tooth scheme to the wave-like system.

For dissipative systems, the gap-tooth scheme was introduced and reviewed by Kevrekidis et al. (2003) and Samaey et al. (2005, 2009). The scheme uses microscale simulations on small patches of space, coupling the simulations over the intervening space, to simulate the system over a macroscale. The gap-tooth method adapts to whatever microscale simulator is provided. Roberts & Kevrekidis (2005, 2007) developed patch coupling conditions to ensure high order accuracy in the gap-tooth scheme for a class of dissipative systems. Roberts et al. (2011) extended such gap-tooth scheme to two dimensional nonlinear reaction-diffusion equations. One aim of this chapter is to show that analogous coupling conditions also work well for weakly dissipative or non-dissipative wave dynamics.

Section 4.2 establishes a gap-tooth multiscale simulation of linear wave
Figure 4.1: Gap-tooth simulation of the depth $h$ (circles) and velocity $u$ (stars) in a progressive wave of equation (4.1) on $[0,2\pi]$ at three times. There are $m = 10$ patches and $n = 9$ microscale grid points on each patch (excluding patch edges). The patches are coupled by simple linear interpolation. The microscale modes oscillates more rapidly than the macroscale modes.
dynamics on a staggered grid. The microscale simulation is a pared down version of the nonlinear equations of the shallow water dynamics because our aim is to adapt the approach to problems such as turbulent floods and tsunamis.

Our first challenge is to develop a gap-tooth scheme for the canonical linear wave PDEs
\[
\frac{\partial h}{\partial t} = -\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial t} = -\frac{\partial h}{\partial x},
\]
where in a water wave application \( h \) and \( u \) would represent the fields of ‘water depth’ and ‘fluid velocity’. We need this gap-tooth scheme to preserve the wave dynamics in this system.

The aim is to simulate the evolution of the fields \( h(x, t) \) and \( u(x, t) \), periodic in the \( x \) direction, on the macroscale length \( L = 2\pi \). Figure 4.1 exhibits a patch simulation of equation (4.1) through microscale simulation on \( m = 10 \) patches, each of width \( l = \pi/15 \), with macroscale spacing \( D = \pi/5 \). No computations were performed on the empty space between the shown patches. Each patch appearing in Figure 4.1 integrates a spatial discretisation of the PDEs (4.1) on a microscale spatial grid of \( n = 9 \) points (not counting the two edge points of each patch), with microscale grid spacing \( d = l/n = \pi/150 \).

The microscale spatial discretisation of equation (4.1) represents a finely detailed model or particle simulation which is assumed far too expensive to use over a large domain (Kevrekidis et al. 2003, e.g.). Section 4.3 introduces one scheme to couple such patches of microscale simulators by interpolation in order to simulate the macroscale wave-like dynamics to varying degrees of accuracy.

Figure 4.1 shows that microscale waves oscillate rapidly in comparison to the macroscale waves. Nonetheless the macroscale waves appear to propagate unscathed by the bath of microscale vibrations. Section 4.4 explores the eigenvalues of the implemented gap-tooth scheme to support the veracity of the simulations in Figure 4.1. An earlier version of the research of section 4.2–4.4 was published in the refereed Proceedings of the 16th Biennial Computational Techniques and Applications Conference (Cao & Roberts 2013).

In further research, Section 4.5 explores the nonlinear slow manifold of the gap-tooth simulation with nonlinear microscale simulators. Then section 4.6 applies the gap-tooth simulation to dam-breaking waves.

### 4.2 The linear microscale simulator

This section describes the microscale simulation of canonical, linear, weakly damped, wave-like dynamics. Staggered microscale grids are used to represent
Figure 4.2: A gap-tooth solution field at one time: the circles for the depth $h$; and the stars for the fluid velocity $u$. The rectangle depicts on the $j$th microscale patch which Figure 4.3 shows in detailed zoom.

the fields of depth $h$ and velocity $u$. Section 4.3 then couples these microscale schemes on small patches in order to simulate macroscale dynamics.

This section considers the wave-like dynamics of two fields analogous to the water depth $h(x, t)$ and water velocity $u(x, t)$. The one dimensional linear governing equations analogous to the mass and momentum equations are

$$\frac{\partial h}{\partial t} = - \frac{\partial u}{\partial x},$$

$$\frac{\partial u}{\partial t} = - \frac{\partial h}{\partial x} - \nu_0 u + \nu_2 \frac{\partial^2 u}{\partial x^2},$$

where the parameters $\nu_0$ and $\nu_2$ are constant coefficients of analogues to bed friction and fluid viscosity respectively.

The first task is to code a microscale simulator of the linear equations (4.2) and (4.3) via a spatial discretisation on the microscale in each patch. Figure 4.2 shows we envisage patches centred on a macroscale grid of point $x = X_j$ with equal spacing step $D$. We now focus on the simulation within the $j$th patch shown schematically in Figure 4.2. Within each patch use a staggered grid of $n$ interior grid points and two boundary points. The microscale grid spacing is $d = l/(n + 1)$, where $l$ is the width of each patch. We use two different types of patch, one for even $j$ and one for odd $j$, in order to later create a macroscale grid that is staggered. Figure 4.3 shows the microscale staggered grid for the depth $h_{j,i}$ and velocity $u_{j,i}$ at the $i$th point of the micro-grid of the $j$th patch: using two different patches depending upon whether the patch index $j$ is odd or even. The microscale simulator discretises equations (4.2)
\[
\begin{align*}
\frac{dh_{j,i}}{dt} &= -\frac{u_{j,i+1} - u_{j,i-1}}{2d}, \\
\frac{du_{j,i}}{dt} &= -\frac{h_{j,i+1} - h_{j,i-1}}{2d} - \nu_0 u_{j,i} + \nu_2 \frac{u_{j,i+2} - 2u_{j,i} + u_{j,i-2}}{4d^2}.
\end{align*}
\]

Figure 4.3: Scheme of the staggered grid points of the depth \(h_{j,i}\) (blue points) and velocity \(u_{j,i}\) (magenta points) at the \(i\)th micro-grid point on the odd \(j\)th patch (top) and the even \(j\)th patch (bottom). This diagram shows the cases for \(n = 5\) interior grid points in each patch.

and (4.3) on the interior of the \(j\)th patch via centred differences in space as

\[
\begin{align*}
\frac{dh_{j,i}}{dt} &= -\frac{u_{j,i+1} - u_{j,i-1}}{2d}, \\
\frac{du_{j,i}}{dt} &= -\frac{h_{j,i+1} - h_{j,i-1}}{2d} - \nu_0 u_{j,i} + \nu_2 \frac{u_{j,i+2} - 2u_{j,i} + u_{j,i-2}}{4d^2}.
\end{align*}
\]

Figure 4.1 was generated by such microscale simulations with the damping parameters \(\nu_0 = \nu_2 = 0\), and with patches being coupled together as described in the section 4.3. Time integration is done by Matlab \texttt{ode15s}.

### 4.3 Couple microscale patches across gaps

This section aims to describe the wave-like dynamics via a macroscale staggered grid of patches. The patches are coupled by interpolating information from neighbouring patches into boundary values for each of the microscale patch simulators of Section 4.2.
Figure 4.4: Scheme of the staggered macroscale grid of patches. Let each of \( m \) patches be centred on the macroscale grid points \( x = X_j = jD \), where \( D = L/m \) is the macroscale spacing and \( L \) is the length of the whole domain. Let \( h_j(x, t) \) and \( u_j(x, t) \) represent the microscale grids of the depth and fluid velocity.

### 4.3.1 Develop the coupling conditions

As shown schematically in Figure 4.4, let each of \( m \) patches be centred on equi-spaced macroscale grid points \( x = X_j = jD \), where \( D = L/m \) is the macroscale spacing and \( L \) is the length of the whole macroscale domain. Each patch has relatively small width \( l \). Note that we generally use lowercase for microscale quantities and uppercase for macroscale quantities. Let each patch around a macroscale grid point \( X_j \) execute microscale simulation (4.4)–(4.5) of the wave-like dynamics. Section 4.2 introduced that two different microscale simulations are defined for odd and even \( j \), and Figure 4.4 illustrates these simulations alternating to form a macroscale staggered grid. The edge of each patch is a distance \( l/2 \) from its macroscale grid point. Define the scale ratio \( r = l/(2D) \) to characterise the size of each patch relative to the distance between neighbouring patches: when \( r = 1/2 \) the neighbouring patches meet as in holistic discretisation by Roberts (2001); and when \( r = 1 \) the patches overlap which is an interesting case as it empowers a slow manifold view of nonlinear wave-like dynamics. When the ratio \( r \) is small, the patches form a relatively small part of the physical domain to provide a computationally efficient scheme for multiscale simulation.

As also shown in Figure 4.5, let macroscale grid values \( H_j = h_j(X_j, t) \) for even \( j \) and \( U_j = u_j(X_j, t) \) for odd \( j \). Interpolate these macroscale grid values to provide boundary values for each microscale patch. Computationally, all microscale simulators could execute in parallel with the only necessary communication between patches being these macroscale grid values.\(^1\) Finite

\(^1\)Such computational communication is synchronous as analysed herein. However,
Figure 4.5: Scheme shows interpolating macroscale grid values $H_j$ and $U_j$ to provide edge values on each patch. The green arrows provide edge values of odd $j$ patches by interpolating macroscale grid values $U_j$. The cyan arrows provide edge values of even $j$ patches by interpolating macroscale grid values $H_j$.

difference operators interpolate the macroscale grid values. Define a shift operator $\mathcal{E}$ over two patches, for example, $\mathcal{E}h(x,t) = h(x + 2D, t)$ and $\mathcal{E}U_j = U_{j+2}$. Note the identities for discrete operators (Roberts & Kevrekidis 2005, 2007, e.g.),

\[
\begin{align*}
\text{centred mean} & \quad \mu = \frac{1}{2}(\mathcal{E}^{1/2} + \mathcal{E}^{-1/2}), \\
\text{centred difference} & \quad \delta = \mathcal{E}^{1/2} - \mathcal{E}^{-1/2}, \\
\text{shift} & \quad \mathcal{E} = 1 + \mu \delta + \frac{1}{2} \delta^2. 
\end{align*}
\]

Now, the edges of the $j$th patch are at $x = X_j \pm (r/2)2D$ which is a fraction $r/2$ of a shift from the macroscale grid point $X_j$. The corresponding shift

\[
\mathcal{E}^{\pm r/2} = (1 \pm \mu \delta + \frac{1}{2} \delta^2)^{r/2} = \frac{\mu}{\sqrt{1 + \delta^2/4}}(1 \pm \mu \delta + \frac{1}{2} \delta^2)^{r/2},
\]

as $\mu^2 = 1 + \frac{1}{4} \delta^2$. Expanding the right-hand side of (4.9) in the Taylor series in small difference $\delta$, and replacing $\mu^2$ by $1 + \frac{1}{4} \delta^2$, the shifts to the edges of a patch are

\[
\mathcal{E}^{\pm r/2} = [\mu \pm \frac{1}{2} r \delta + \frac{1}{8}(-1 + r^2)\mu \delta^2 \pm \frac{1}{48}(-r + r^3)\delta^3
\]

asynchronous computation and communication could be modelled by including a couple of intermediary variables into the dynamics between the actual macroscale value and its use as a patch boundary value. For example, for each macroscale variable $H_j$ one could define two intermediaries, $H'_j$ and $H''_j$ say, with dynamics $\tau dH'_j/dt = -H'_j + H_j$ and $\tau dH''_j/dt = -H''_j + H'_j$ for some mean delay $2\tau$ caused by asynchronous computation. Then the model to analyse is one where variables $H''_j$, instead of $H_j$, were interpolated to provide patch boundary values.

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This expansion then empowers us to express values on the edges of the \( j \)th patch, at \( x = X_j \pm rD \), in terms of the macroscale grid values \( H_j \) and \( U_j \). Thus for odd/even \( j \) set the microscale, patch boundary, ‘depth’/‘velocity’ as:

\[
(h_j, u_j)|_{x = X_j \pm rD} \approx \left[ \gamma \mu \pm \frac{1}{2} \gamma r \delta + \frac{1}{6} \gamma^2 (-1 + r^2) \mu \delta^2 \pm \frac{1}{18} \gamma^3 (-r + r^3) \delta^3 \right. \\
+ \frac{1}{384} \gamma^3 (9 - 10r^2 + r^4) \mu \delta^4 \pm \frac{1}{3840} \gamma^3 (9r - 10r^3 + r^5) \delta^5 \\
+ \frac{1}{4608} \gamma^4 (-225 + 259r^2 - 35r^4 + r^6) \mu \delta^6 \\
\left. \pm \frac{1}{645120} \gamma^4 (-225r + 259r^3 - 35r^5 + r^7) \delta^7 \right] (H_j, U_j),
\]

(4.11)

where the parameter \( \gamma \) conveniently labels the spatial extent of the interpolation. The interpolation (4.11) couples the patches together.

One obtains various accuracies by truncating the coupling (4.11) to various orders in the label \( \gamma \). For example, truncating to errors \( O(\gamma^2) \) gives linear interpolation, \( h_j = (\mu \pm \frac{1}{2} r \delta) H_j \), from the nearest neighbour patches as Figure 4.5 illustrates. Whereas truncating to errors \( O(\gamma^3) \) gives cubic interpolation from nearest and next nearest patches. The details recorded in the coupling (4.11), that is to errors \( O(\gamma^5) \), are equivalent to interpolating a seventh order polynomial through the eight neighbouring macroscale values \( H_{j \pm 1}, H_{j \pm 3}, H_{j \pm 5} \) and \( H_{j \pm 7} \). Expect that such high order interpolation achieves high order accuracy as it does for dissipative systems (Roberts & Kevrekidis 2005, e.g.).

**4.3.2 Numerical simulations verify the coupling conditions**

Numerical simulations of the wave-like microscale discrete system (4.4)–(4.5) with coupling condition (4.11) verifies that the proposed coupling works. Figure 4.1 shows the gap-tooth simulation of the depth \( h \) (circles) and velocity \( u \) (stars) in the governing equations (4.4)–(4.5) with the parameters \( \nu_0 = \nu_2 = 0 \) on \([0, 2\pi]\) at three times. There are \( m = 10 \) patches and \( n = 9 \) microscale grid points on each patch (excluding patch edges). The patches are coupled by the linear coupling condition (4.11) to errors \( O(\gamma^2) \). A sinusoidal wave \( 0.1 \sin(kx) \) is initially imposed to the constant depth \( h = 0.2 \) and \( u = 0 \) at time \( t = 0 \), shown as the \( t = 0 \) graph. The \( t = 1 \) and \( t = 3 \) graphs show that the microscale modes oscillate fast on each patch, whereas the macroscale modes propagate slowly unscathed by the bath of microscale vibrations.
Figure 4.6: Gap-tooth simulations of the depth $h$ (circles) and velocity $u$ (stars) governing by equations (4.4) and (4.5) on $[0, 2\pi]$ via $m = 10$ patches and $n = 9$ microscale grid points on each patch at three times. A quintic polynomial is used from the coupling (4.11) to error $O(\gamma^4)$. The scale ratio $r = 1/6$, and the damping coefficients $\nu_0 = 0.01$ and $\nu_2 = 0.03$. The microscale modes smooth quickly and the macroscale progressive wave decay slowly.
Figure 4.6 plots the gap-tooth simulations of the depth $h$ and velocity $u$ governed by a microscale discretisation of (4.4)–(4.5) with $m = 10$ patches and $n = 9$ microscale grid points on each patch. A fifth order polynomial is used from the coupling (4.11): that is, to error $O(\gamma^4)$ in the label $\gamma$. The scale ratio $r = 1/6$, and the coefficients $\nu_0 = 0.01$ and $\nu_2 = 0.03$. The initial condition ($t = 0$ graph), with some random noise within each patch, smooths rapidly by microscale ‘diffusion’ $\gamma u_{xx}$ to a quasi-equilibrium ($t = 1$ graph). The macroscale wave decays slowly due to the ‘bed friction’ $\nu_0 u$, shown in $t = 30$ graph.

The multiscale modelling reduces the numerical cost of the simulations for large enough scale separation between the microscale computation and the resolved macroscale structures. For an indicative example, when the microscale grid spacing is $d = 0.0026$, our Matlab multiscale code with ten patches on the domain takes a computation time of 0.38 seconds in one period, whereas a corresponding Matlab code that resolves the microscale dynamics over the whole domain takes 21 seconds. In this example, the complete microscale simulation is 56 times slower to compute than our multiscale patch simulation. In more spatial dimensions expect the computational time savings to be much larger as the fraction of space occupied by patches is expected to be much smaller in higher dimensions.

4.4 Linear analysis of the coupled dynamics

This section linearly analyses the proposed gap-tooth multiscale modelling. The focus is on the performance of the coupling conditions so this section analyses both the wave-like PDEs (4.2)–(4.3), and the numerical eigenvalues of its microscale discretisation (4.4)–(4.5), and both on patches coupled by (4.11).

4.4.1 Coupled wave-like equations (4.2)–(4.3)

On the $j$th patch assume the fields of the depth $h_j$ and velocity $u_j$ have solutions in the forms of

$$h_j(x, t) = C_h e^{\lambda t + i(\ell + \ell \xi)} \quad \text{and} \quad u_j(x, t) = C_u e^{\lambda t + i(\ell + \ell \xi)}, \quad (4.12)$$

for microscale space variable $\xi = (x - X_j)/D$, the complex growth rate $\lambda$, macroscale wavenumber $k$, and microscale wavenumber $\ell$. These wavenumbers are both defined relative to the macroscale grid spacing $D$. Substitute the solution form (4.12) into the equations (4.2)–(4.3), factor $e^{\lambda t + i(\ell + \ell \xi)}$ and
obtain

\[ \lambda C_h = -i \ell D C_u \quad \text{and} \quad \lambda C_u = -i \ell D C_h - \nu_0 C_u - \nu_2 \frac{\ell^2}{D^2} C_u. \quad (4.13) \]

For non-trivial solutions for \( C_h \) and \( C_u \) the complex growth-rate must satisfy

\[ \lambda^2 + \left( \nu_0 + \nu_2 \frac{\ell^2}{D^2} \right) \lambda + \frac{\ell^2}{D^2} = 0, \quad (4.14) \]

which determines the dispersion relationship between the growth rate \( \lambda \) and microscale wavenumber \( \ell \).

The microscale wavenumber \( \ell \) is then determined by the coupling conditions as a function of macroscale wavenumber \( k \). On the \( j \)th patch, assume the fluid velocity \( u_j \) have solutions in the forms of

\[ u_{j0} = a_0 e^{\lambda t + i(kj + \ell \xi)} + b_0 e^{\lambda t - i(kj - \ell \xi)}, \]
\[ u_{j1} = a_1 e^{\lambda t + i(kj + \ell \xi)} + b_1 e^{\lambda t - i(kj - \ell \xi)}, \]

such that together with equation (4.13), the depth \( h_j \) have the solutions in the forms of

\[ h_{j0} = -(\lambda - i\nu) \frac{D}{\ell} a_0 e^{\lambda t + i(kj + \ell \xi)} + (\lambda - i\nu) \frac{D}{\ell} b_0 e^{\lambda t - i(kj - \ell \xi)}, \]
\[ h_{j1} = -(\lambda - i\nu) \frac{D}{\ell} a_1 e^{\lambda t + i(kj + \ell \xi)} + (\lambda - i\nu) \frac{D}{\ell} b_1 e^{\lambda t - i(kj - \ell \xi)}, \]

where \( \nu = \nu_0 + \nu_2 \frac{\ell^2}{D^2} \), and \( a_i \) and \( b_i \) are coefficients with \( i = 0, 1 \). The subscripts 0 and 1 represent the solutions of the depth \( h_j \) and fluid velocity \( u_j \) on the odd \( j \)th patch and even \( j \)th patch respectively. One can straightforwardly check that the solutions (4.15)–(4.18) satisfy the governing equations (4.2)–(4.3). Substitute such solutions (4.15)–(4.18) into the coupling conditions (4.11) to obtain the relationship of the microscale wavenumber \( \ell \) and the macroscale wavenumber \( k \). Appendix A.2 lists a computer algebra program to use the coupling conditions (4.11) to derive the high order characteristic equation

\[ \pm \sin \ell r + r \sin k + \frac{1}{6} (r - r^3) \sin^3 k + \frac{1}{120} (9r - 10r^3 + r^5) \sin^5 k = O(k^7). \]

Expanding a small microscale wavenumber \( \ell \) in equation (4.19) in a Taylor series in the macroscale wavenumber \( k \), obtain from the explicitly given terms that

\[ \ell = \pm k + \frac{1}{5040} (-225 + 259r^2 - 35r^4 + r^6) k^7 + O(k^9). \]
That is, to the order of analysis used to obtain (4.19) the coupling conditions (4.11) ensure that there exist microscale wavenumbers which agree precisely with the macroscale wavenumber $\ell = \pm k + O(k^7)$. This agreement, via the dispersion relation (4.14), ensures the high order accuracy of the gap-tooth scheme with these coupling conditions.

The characteristic equation (4.19) also has solutions with non-small $\ell$. For macroscale wavenumber $k = 0$, these are $\ell = n\pi/r$ for integer $n$. For such large microscale wavenumber $\ell$, from (4.14) the growth-rate $\lambda$ has large imaginary part indicating rapid microscale oscillations/waves within patches. These rapid microscale waves are not of interest to the macroscale dynamics—often microscale dissipation will damp them as seen in Figure 4.6. Importantly though, and even without dissipation, for non-zero macroscale wavenumber $k$ one can see that the solutions $\ell$ of the characteristic equation (4.19) will remain real as seen in Figure 4.1. Consequently, the coupling conditions (4.11) maintain the microscale waves as waves; the coupling conditions do not turn the microscale waves into unstable modes that would wreck the macroscale simulation.

4.4.2 Coupled microscale discretisations (4.4)–(4.5)

Numerical eigenvalues of the dynamics confirms the stability of the gap-tooth scheme with coupling conditions (4.11). Consider the microscale simulator (4.4)–(4.5) with the quintic coupling conditions (4.11) to error $O(\gamma^4)$. The fields $h_{j,i} = 1$ and $u_{j,i} = 0$, say, is a steady state. Characterise the dynamics in the neighbourhood of this the steady state via the spectrum of the Jacobian. Numerical differentiation of the simulation function gives a sufficiently good approximation to the Jacobian, then standard routines compute the complete spectrum of eigenvalues. Figure 4.7 plots the real (growth rate) and imaginary part (frequency) of the numerical eigenvalues about this steady state for one set of parameters. Figure 4.7 shows several eigenvalues with small $\Re \lambda$—these are the macroscale waves which decay slowly through ‘bottom friction’ as they propagate.

Figure 4.7 also shows many eigenvalues with large imaginary and negative real parts: these represent microscale waves within the patches that decay rapidly through microscale dissipation. Such rapid microscale waves, and their decay, is seen in the initial transients shown in Figure 4.6.

The gap in the growth rate $\Re \lambda$, shown in Figure 4.7 as the gap in $\Re \lambda$ between 0 and nearly $-2$, is interesting. This gap indicates that the set of macroscopic waves form a slow subspace, $\Re \lambda \approx 0$, among the multiscale dynamics of the waves on the coupled patches.

Table 4.1 lists the macroscale numerical eigenvalues in Figure 4.7 and the
Figure 4.7: Distribution (stretched) of the real and imaginary parts of the numerical eigenvalues for equations (4.4)–(4.5) with $m = 10$ patches and $n = 9$ microscale grid points on each patch. The length scale ratio $r = 1/6$ and the coefficients $\nu_0 = 0.01$ and $\nu_2 = 0.03$. There are 44 complex conjugate pairs of values, and another two real values 0 and $-0.1$.

Table 4.1: Numerical eigenvalues for the macroscale wave modes from Figure 4.7 for $m = 10$ and $m = 14$ patches; compare with exact eigenvalues from equation (4.14) for the linear dynamics about the steady state.

<table>
<thead>
<tr>
<th>$\ell/D$</th>
<th>multiplicity</th>
<th>$\lambda$, eqn (4.14)</th>
<th>$\lambda$, $m = 10$</th>
<th>$\lambda$, $m = 14$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>one</td>
<td>$-0.000, -0.010$</td>
<td>$0, -0.010$</td>
<td>$0, -0.010$</td>
</tr>
<tr>
<td>$\pm 1$</td>
<td>two</td>
<td>$-0.020 \pm 0.998i$</td>
<td>$-0.020 \pm 0.998i$</td>
<td>$-0.020 \pm 0.999i$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>two</td>
<td>$-0.065 \pm 1.999i$</td>
<td>$-0.056 \pm 1.836i$</td>
<td>$-0.063 \pm 1.968i$</td>
</tr>
</tbody>
</table>
exact eigenvalues from equation (4.14) about the steady state. The numerical eigenvalues are close to the exact eigenvalues. Table 4.1 shows the conservation of fluid (eigenvalue zero) and the decay of bulk velocity due to bed friction (eigenvalue −0.010) are correct to at least three decimal places. The longest wave mode on the domain, ℓ/D = ±1, which in the absence of friction would have frequency one, also has its eigenvalue correct to three decimal places. Numerical tests show that when the number of patches increases, the numerical eigenvalues become closer to the analytical eigenvalues, as shown in Table 4.1. For example, for the wavenumber ℓ/D = ±2 mode, upon increasing the number of patches from m = 10 to m = 14 finds the eigenvalues λ change from −0.056±1.836i to −0.063±1.968i, which are significantly more accurate when compared to the exact eigenvalues −0.065±1.999i.

4.5 Nonlinear analysis of the gap-tooth simulation

This section explores the gap-tooth simulation with a nonlinear microscale simulator on a patch. Numerical eigenvalue analysis suggests that there is an appropriate slow manifold in such an application of the gap-tooth scheme. Numerical gap-tooth simulations show that the coupling condition (4.11) works well for the multiscale model with the nonlinear microscale simulator.

4.5.1 The nonlinear microscale simulator

This subsection invokes a nonlinear Smagorinski model of turbulent floods (Roberts 2008b, Cao & Roberts 2012, e.g.). The microscale simulator on a patch is a finite difference code of the Smagorinski model. For simplicity, consider the Smagorinski model in one spatial dimension.

Consider fluid flowing down an inclined flat bed, so the bed z = 0. Define a coordinate system with x representing horizontal position and z for position normal to the bed. The dynamics of the turbulent flow are straightforwardly modelled by the nondimensional Smagorinski model in terms of fluid depth h(x, t) and depth-averaged lateral velocity $\bar{u}(x, t)$ (Roberts 2008b, e.g.):

$$\frac{\partial h}{\partial t} = -\frac{\partial (h\bar{u})}{\partial x}, \quad (4.21)$$

$$\frac{\partial \bar{u}}{\partial t} = 0.985 \left(\tan \theta - \frac{\partial h}{\partial x}\right) - 0.003 \frac{\bar{u}|\bar{u}|}{h} - 1.045 \frac{\partial \bar{u}}{\partial x} + 0.26h|\bar{u}| \frac{\partial^2 \bar{u}}{\partial x^2}, \quad (4.22)$$
where $\tan \theta$ is the slope of the mean bed. Equation (4.21) is the conservation of the fluid. The momentum equation (4.22) represents nonlinear, and consists of the effects of turbulent bed drag $\bar{u} \partial \bar{u}/\partial x$, turbulent dissipation $h|\bar{u}|^2 \partial^2 \bar{u}/\partial x^2$ and gravitational hydrostatic forcing $\tan \theta - \partial h/\partial x$.

Code the microscale simulator of the nonlinear equations (4.21)–(4.22) via a spatial staggered discretisation on the microscale in each patch. Similarly to Figure 4.3, on the $j$th patch, use the microscale staggered grids of spacing $d$ for the depth $h_{j,i}$ and turbulent mean velocity $\bar{u}_{j,i}$ at the $i$th point of the microgrid of the $j$th patch. Thus, approximate Smagorinski model (4.21)–(4.22) on the $j$th patch with centred differences in space as

$$\frac{\partial h_{j,i}}{\partial t} = - \left( \frac{h_{j,i+1} + h_{j,i}}{4d} + \frac{h_{j,i-1} + h_{j,i}}{4d} \right) \bar{u}_{j,i} + \frac{0.003}{h_{j,i}} \frac{\bar{u}_{j,i} | \bar{u}_{j,i} |}{h_{j,i} - 1} \frac{\partial^2 \bar{u}_{j,i}}{\partial x^2} - \frac{1.545}{4d} \frac{\bar{u}_{j,i} \bar{u}_{j,i+1} - \bar{u}_{j,i} \bar{u}_{j,i-2}}{4d} + 0.26h_{j,i} | \bar{u}_{j,i} | \frac{\bar{u}_{j,i+1} + \bar{u}_{j,i-2}}{4d^2}. \quad (4.23)$$

Equations (4.23)–(4.24), together with the coupling conditions (4.11), generates the gap-tooth simulation of the turbulent flow over the flat bed.

### 4.5.2 Numerical eigenvalue analysis of the nonlinear problem

Numerical eigenvalues of the dynamics supports the gap-tooth scheme with nonlinear microscale simulator (4.23)–(4.24) and the coupling conditions (4.11). Omitting terms $O(\gamma^3)$, the boundary conditions (4.11) become a cubic interpolation of the neighbouring macroscale values to approximate the values on a patch edges. Consider the equilibrium of depth $h_{j,i} = 1$ and depth-averaged lateral velocity $\bar{u}_{j,i} \approx 18.1 \tan^{1/2} \theta$. Characterise the dynamics in the neighbourhood of this equilibrium via the spectrum of the Jacobian. Numerical differentiation of the simulation function gives a sufficiently good approximation to the Jacobian, then standard routines compute the complete spectrum of eigenvalues.

Figure 4.8 plots the growth rate ($\Re \lambda$) and frequency ($\Im \lambda$) for $m = 10$ patches and $n = 9$ microscale grids on a patch. There are 40 pairs of eigenvalues with large negative real parts which represent the microscale modes within the patches. Most of these negative real parts are between $-2$ and $-150$. Thus, these microscale waves within the patches decay very rapidly through the microscale turbulent dissipation, dominantly $h|\bar{u}|^2 \partial^2 \bar{u}/\partial x^2$. The large imaginary parts of these eigenvalues indicate the fast oscillation of the microscale waves within the patches.
Figure 4.8: Distribution (stretched) of the real and imaginary parts of the numerical eigenvalues for equations (4.23)–(4.24) with $m = 10$ patches and $n = 9$ microscale grid points on each patch. The length of the domain is $L = 2\pi$ and the length scale ratio $r = 1/6$. The slope is $\tan \theta = 0.001$.

Figure 4.9: Zoom into the values in the red box in Figure 4.8. There are four complex conjugate pairs of values, and another two real values 0.0002 and $-0.0042$. 
The eigenvalues with small real parts in the red box are for the macroscale modes. Figure 4.9 zooms in these small values and shows that there are totally four pairs of small negative growth rates, together with the two real values \(0.0002\) and \(-0.0042\). These small values imply that the macroscale waves decay slowly, primarily through the small nonlinear bed drag \(\overline{u} |\overline{u}| / h\). Such small decay rates indicate that the system needs a long time to become steady as the ‘eddy diffusivity’ only weakly damps waves. The non-zero imaginary parts of these four pairs of eigenvalues indicate that macroscale waves are supported on the free surface of the fluid. The positive growth rate \(0.002\) need be specified, which is due to the numerical errors.

The gap between the growth rate \(\Re \lambda \approx 0\) and \(\Re \lambda \approx -2\), as shown in Figure 4.8, indicates there is a nonlinear centre manifold of the gap-tooth dynamics (Roberts 1988, Chicone 2006, Potzsche & Rasmussen 2006, e.g.).

### 4.5.3 Numerical gap-tooth simulation of the nonlinear problem

This subsection numerically explores the macroscale turbulent fluid flow on a slightly inclined flat bed by the gap-tooth scheme with the nonlinear microscale turbulent model (4.23)–(4.24) and the coupling conditions (4.11). The coupling conditions (4.11) are truncated to the error \(O(\gamma^3)\) to give a cubic interpolation of the neighbouring macroscale values to approximate the edge values of a patch.

Numerical simulations are straightforwardly implemented in equations (4.23)–(4.24) on staggered grids in space as described by section 2.6. Time integration is performed by Matlab’s `ode15s`.

Figure 4.10 plots the numerical gap-tooth results for the depth \(h\) and depth-averaged lateral velocity \(\overline{u}\) at three times. At the initial time \(t = 0\) graph, a perturbation of \(0.2 \sin(2\pi/L_x)\) with small random noise is imposed to the equilibrium of depth \(h = 1\) and depth-averaged lateral velocity \(\overline{u} \approx 0.57\) from equation (4.22). The \(t = 2\) graph shows that the microscale structures within a patch smooths quickly by the dissipation \(h |\overline{u}| \partial^2 \overline{u} / \partial x^2\). This corresponds to the large decay rates of the microscale modes in Figure 4.8. In addition, macroscale waves propagate on the free surface. The macroscale waves over the whole domain decay slowly, shown in the \(t = 4\) graph. This slow decay of macroscale waves corresponds to the eigenvalues of small real part but non-zero frequencies in Figure 4.8. A large length scale simulation is discussed in the following section 4.6, using the gap-tooth scheme with the nonlinear microscale turbulent model (4.23)–(4.24) and the coupling conditions (4.11).
Figure 4.10: Gap-tooth simulation of the depth $h$ (circles) and depth-average lateral velocity $\bar{u}$ (stars) by the nonlinear microscale simulator (4.23)–(4.24) and the coupling conditions (4.11) on $[0, 2\pi]$ via $m = 10$ patches and $n = 9$ microscale grid points on each patch at three times. The scale ratio is $r = 1/6$ and the mean slope of the bed is $\tan \theta = 0.001$. 

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Figure 4.11: The initial conditions of the dam breaking. Consider the domain has a nondimensional length $L$. The dam located at $x = L/2$ holds water of nondimensional depth $h = 1$ in the upstream and a nondimensional shallow depth $h = 0.1$ in the downstream. Case 1 shows the dam lies in between the patches and Case 2 shows the dam is in the middle of a patch.

### 4.6 Gap-tooth simulation of dam breaking

This section applies the gap-tooth scheme to simulate dam-breaking. Figure 4.11 shows a dam standing in the middle of the domain that holds back water. The ground is horizontal and let $x$ denote the horizontal position. The domain has a length $L$. Initially the dam holds water upstream of nondimensional depth $h = 1$. To avoid poor conditioning in the numerical calculation, downstream of the dam let the water have a shallow depth, for example 0.1. At time $t = 0$ the dam breaks and the upstream water rushes downstream.

Simulate the dam-breaking waves by the gap-tooth scheme with the microscale turbulent model (4.21)–(4.22) and the coupling conditions (4.11). Comparison among the gap-tooth simulation, the microscale simulation over the whole domain and the experimental data (Stansby et al. 1998, e.g.) shows that the gap-tooth scheme reasonably simulates the dam-breaking waves.

#### 4.6.1 Distribute patches of the dam breaking

There are typically two ways to distribute patches in the macroscale domain: a patch includes the dam, and the dam lies in between two patches.

First, let the dam stand in between the patches. Case 1 in Figure 4.11 distributes six patches in the macroscale domain. The dam stands in the
middle of the gap between the third and fourth patches. An advantage of such a choice is to avoid the sharp discontinuity at the dam, because it arises in the gap which is not represented in the gap-tooth simulation.

Second, let the dam be on a patch. Case 2 in Figure 4.11 distributes five patches in the whole domain. The dam is included on the centre of the third patch. This choice would show the dynamics at the dam when the dam breaks.

4.6.2 Numerical gap-tooth simulation of dam breaking

This section explores numerical gap-tooth simulations of the dam-breaking waves. Georgiev et al. (2009) used an earlier version of model (4.21)–(4.22) simulating the dam breaking waves over all space in the domain, not by gap-tooth scheme. The comparison of the calculations and experimental data by Georgiev et al. (2009) shows that the model (4.21)–(4.22) is a reasonable way to model dam breaking waves. This section compares the gap-tooth simulation with the microscale simulation over the whole space domain (not gap-tooth) and with experimental data (Stansby et al. 1998, e.g.), and indicates that putting the dam within a patch appears better.

I computed both the gap-tooth simulation and the microscale simulation over the whole domain for the dam-breaking waves. Experiments of dam-break water flow in a flume by Stansby et al. (1998) show some of the phenomena of the dam breaking waves. In the experiment of Stansby et al. (1998), the initial water depth behind the dam is 10 cm and a horizontal length of 200 cm. The dam stands at the centre of the considered length. For comparison, nondimensionalise the depth 10 cm to one, such that the nondimensional length is \( L = \frac{200 \text{ cm}}{10 \text{ cm}} = 20 \). In this pilot study of gap-tooth simulation, distribute both \( m = 10 \) and \( m = 22 \) patches on the whole macroscale domain and \( n = 9 \) microscale grid points on a patch. For comparison and for consistency, let the spatial step in the microscale simulation over the whole macroscale space domain (not by gap-tooth) have the same microscale spatial step \( \delta x \). For simplicity, use the linear coupling conditions (4.11), truncate to error \( O(\gamma^2) \). That is,

\[
\bar{u}(X_j \pm rd, t) = \frac{\gamma}{2} \left[ (U_{j+1} + U_{j-1}) \pm r(U_{j+1} - U_{j-1}) \right],
\]

which means interpolating the closest neighbouring macroscale values to give edge values of a patch.

The case of the dam being within a patch. Figure 4.12 plots the gap-tooth simulation (red for \( m = 10 \) patches and black for \( m = 22 \) patches), the
Figure 4.12: Comparison among the gap-tooth simulation: (red) for $m = 10$ patches; (black) $m = 22$ patches; (green) the microscale simulation over the whole domain; and (blue) the experimental data (Stansby et al. 1998, Fig. 8c) of the dam breaking. The dam is on a patch. The nondimensional shallow depth is $h = 0.45$ in front of the dam. The scale ratio $r = 1/6$, such that the microscale step $\delta x = 1/15$. 

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Figure 4.13: The water area in Figure 4.12 of the gap-tooth simulation with: (black) $m = 22$ patches; (red) $m = 10$ patches; (green) the microscale simulations over the whole space domain; and (blue) the Stansby’s experiments varies in time.
Figure 4.14: Comparison among the gap-tooth simulation: (red) $m = 10$ patches; (black) $m = 22$ patches; (green) the microscale simulation over the whole domain; and (blue) the experimental data (Stansby et al. 1998, Fig. 8c) of the dam breaking. The dam is on a patch. The nondimensional shallow depth is $h = 0.1$ in the front of the dam. The scale ratio $r = 1/8$, such that the microscale step $\delta x = 1/20$. 

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Figure 4.15: Comparison among the gap-tooth simulation: (red) $m = 10$ patches; (black) $m = 22$ patches; (green) the microscale simulation over the whole domain; and (blue) the experimental data (Stansby et al. 1998, Fig. 8c) of the dam breaking. The dam lies in between patches. The nondimensional shallow depth is $h = 0.45$ in the front of the dam. The scale ratio $r = 1/6$, such that the microscale step $\delta x = 1/15$. 

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microscale simulation over the whole domain (green), and the experimental data (blue) (Stansby et al. 1998, Fig. 8) at four times. The $t = 0$ graph shows that the water has depth one upstream and depth 0.45 downstream, which corresponds to the depth ratio of 0.45 in the experiments (Stansby et al. 1998, Fig. 8c). Recall the nondimensional length scale $H$, velocity $\sqrt{gH}$, and time $H/\sqrt{gH} = \sqrt{H/g}$. Nondimensionalise the time in the experiment of Stansby et al. (1998) by $\sqrt{0.1 \text{ m} / (10 \text{ m} / \text{s}^2)} = 0.1 \text{ s}$. Therefore Stansby’s plots are at nondimensional times $t = 0$, $t = 2$, $t = 5.2$ and $t = 7.6$.

The $t = 2$ graph shows that a turbulent bore arises in the experiment and microscale simulation over the whole domain. The turbulent bore in the gap-tooth simulation is smoothed by the relating large spacing between patches. At later times $t = 5.2$ and $t = 7.6$ graphs, the gap-tooth simulation agrees better with the experimental data for the waves behind the water bore. The height of the bore in the gap-tooth simulation is smaller than both in the experimental and in the microscale simulation over the whole domain. The bore in the gap-tooth simulation lags that in the experiment, while the bore in the microscale simulation over the whole domain is ahead of that in the experiment. The possible reason is that the gap-tooth simulation near the bore involves the error $O(D^2) \sim 0.8$ for the macroscale step $D = L/m = 20/22 \approx 0.9$, while the microscale simulation over the whole macroscale domain involves the error $O(d^2) \sim 0.001$ for the microscale step $d = 2rD/(n + 1) = 0.03$. When the number of patches increases, the error $O(D^2)$ declines, then the gap-tooth simulation performs better, as shown by the black curves in comparison to the coarser red curves.

However, the gap-tooth simulation saves computer time. The gap-tooth scheme only takes a compute time of $0.78 \text{ s}$ for $m = 22$ patches to simulate to the $t = 7.6$ graph in Figure 4.12. Whereas the microscale simulation over the whole domain with the same microscale step $\delta x = 0.03$ needs a compute time of $74.3 \text{ s}$ to simulate over the same time. The whole domain simulation is nearly 100 times slower than the gap-tooth simulation.

Figure 4.13 shows the water area in Figure 4.12 varying in time: the red and black curves are for $m = 10$ and $m = 22$ patches in the gap-tooth simulation respectively; the green curve is for the microscale simulation over the whole domain; and blue curve is for the experimental data (Stansby et al. 1998, Fig. 8c). Initially, they have the same water area. The green curve remains unchanged and indicates that the microscale simulation over the whole domain conserves water. Stansby’s experimental (blue curve) shows a bigger water area possibly due to the entrainment of air, whereas the gap-tooth simulation (red and black curves) leads to smaller water area due to the relatively coarse separation of the patches.
Consider a shallower water in front of the dam. To avoid singular points at the dam, the dam is smoothed by the function \( \tanh(x) \). Figure 4.14 plots the gap-tooth simulation (red and black curves for \( m = 10 \) and \( m = 22 \) patches respectively), the microscale simulation over the whole macroscale domain (green curve), and the experimental data (blue curve) (Stansby et al. 1998, Fig. 8b). The initial shallow depth in front of the dam is 0.1. The numerical simulations need a smaller microscale step to avoid singular points. Thus, reduce the scale ratio \( r = 1/8 \), so the microscale step \( \delta x = 1/20 \). Compared with Figure 4.12, the height of the turbulent bore in the gap-tooth simulations are significantly smaller than that in the experiment and microscale simulation over the whole domain. Because, near the bore, the involved error ratio \( \mathcal{O}(D^2)/\mathcal{O}(d^2) \) between the gap-tooth simulation and microscale simulation over the whole domain becomes big. Thus, we predict that the gap-tooth simulation would better approximate the dam-breaking waves with more patches and deeper water in front of the dam.

Consider the dam lying in between the patches. Figure 4.15 plots the gap-tooth simulation (red and black curves for \( m = 10 \) and \( m = 22 \) patches respectively), the microscale simulation over the whole macroscale domain (green curve), and the experimental data (blue curve) by Stansby et al. (1998) at four times. The \( t = 0 \) graph shows the initial depth, corresponding to the depth ratio of 0.45 in the experiments of Stansby et al. (1998). The dam is not simulated in the gap-tooth simulation. The \( t = 2 \) graph shows this gap-tooth simulation is not as good as the corresponding results in Figure 4.12. Then the \( t = 5.2 \) and \( t = 7.6 \) graphs shows the same results with that in Figure 4.12. Therefore, we find putting the dam on a patch appears better.

4.6.3 Boundary conditions of the dam breaking

This subsection discusses the invoked boundary conditions at the upstream and downstream in the gap-tooth simulation of the dam-breaking waves. The gap-tooth simulation of the dam-breaking waves requires boundary conditions at the upstream \( x = 0 \) and downstream \( x = L \). Typically, no-flow boundary conditions are usually considered in dam breaking (Abdolmaleki et al. 2004, Ozgokmen et al. 2007, e.g.). We consider three types of boundary condition at the upstream \( x = 0 \) and downstream \( x = L \):

- constant depth \( h = 1 \) at the upstream or the depth \( h = 0.45 \), for example, at the downstream;
- no flux, \( \partial \bar{u}/\partial x = 0 \), at the upstream or at the downstream;
Figure 4.16: There are at least two ways to implement boundary conditions at the upstream \( x = 0 \) and downstream \( x = L \) in the gap-tooth simulation of dam breaking: (a) invokes the boundary conditions for the macroscale values of the leftmost and rightmost patches; (b) invokes the boundary conditions for the microscale values on the leftmost and rightmost patches.

- and zero turbulent mean velocity, \( \bar{u} = 0 \), at the upstream or at the downstream, which is equivalent to no fluid flowing through the upstream or downstream, \( \partial h / \partial x = 0 \) according to the momentum equation (4.22).

These boundary conditions indicate at least \( \left( \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right)^2 = 49 \) possible combinations of boundary conditions in the gap-tooth simulation of dam-breaking.

In the gap-tooth simulation, boundary conditions are invoked to either the macroscale modes or the microscale modes on the leftmost and rightmost patches. Figure 4.16 shows the diagram of invoking boundary conditions: (a) invokes boundary conditions to the macroscale modes; (b) invokes boundary conditions to the microscale modes. Figure 4.12–4.15 implement the boundary conditions \( \bar{u}_{1,1} = 0 \) and \( h_{m,1} = h_{m,n+2} = H_{m-1} \), where \( \bar{u}_{1,1} \) is the left edge of the first patch, \( h_{m,1} \) and \( h_{m,n+2} \) are the edges of the \( m \)th patch, and \( H_{m-1} \) is the macroscale value on the \((m - 1)\)th patch.

Further work could explore the gap-tooth simulation of dam-breaking with different boundary conditions.
4.7 Conclusion

We invoked a staggered grid to discretely model the microscale and another staggered grid for the macroscale simulators of wave-like dynamics. Classic polynomial interpolation underlies the coupling of patches proposed in section 4.3 to form a gap-tooth scheme. The resultant numerical simulations indicated that this could be a useful scheme for wave-like dynamics. Numerical simulations in section 4.5 and 4.6 indicate that the scheme also works for nonlinear wave-like dynamics. Section 4.4 reported algebraic analysis and numerical determination of eigenvalues that both confirm the accuracy of the proposed gap-tooth scheme for wave-like dynamics. In particular, Figure 4.7 and 4.8 shows the clear separation between the dynamics of the macroscale waves of interest, and the microscale waves within each patch. Section 4.6 applies the gap-tooth scheme on the dam-breaking waves. In Figure 4.12–4.15, although the turbulent bore lags and the height of this bore is a bit smaller, we predict the gap-tooth simulation better approximate the dam-breaking waves with more patches and bigger shallow depth in front of the dam. We also find putting the dam on a patch appears better.

Future work could apply such gap-tooth scheme to other wave dynamics with more inertia or extend such gap-tooth scheme to two dimensions (Roberts et al. 2011, e.g.).
Chapter 5

Multiscale modelling couples patches of two-layer of fluid film flow

Chapter 4 developed a gap-tooth scheme for wave-like phenomenon. However, in most applications, a microscale simulator will have many internal modes. In the gap-tooth simulation, an outstanding issue is that at each time step we need to ‘lift’ macroscale data to an appropriate microscale configuration. As a first attempt to address this issue for wave dynamics, this chapter aims to use the gap-tooth scheme to explore viscous flow of a layer of fluid.

The flow of rainwater on the road, windscreen or other draining problems (Chang 1987, 1994, e.g.), and paint and coating flows (Weinstein & Ruschak 2004, e.g.) are a few examples of fluid film flows. Dynamics of such thin film flows have been studied extensively (Benjamin 1957, Roberts 1997, 1998, Roy et al. 2002, e.g.). A first aim of this chapter is to develop a multilayer model for such thin film flow. The reason of developing a multilayer model is that it will have more microscale modes than classic one-layer models, but without the full complexity of fully resolved vertical structures.

Consider a thin fluid flow of depth \( h(x, t) \) on an inclined plate with the slope \( \tan \theta \). This chapter artificially assumes two layers in the thin fluid flow, which have no distinguishing physical feature, as shown in Figure 5.1. Based on centre manifold theory, this chapter derives a two-layer model in the flow fields of depth \( h(x, t) \) and layer mean velocities \( \bar{u}_1(x, t) \) in the lower layer and \( \bar{u}_2(x, t) \) in the upper layer:

\[
\frac{\partial}{\partial t} \begin{pmatrix} h \\ \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = G(h, \bar{u}_1, \bar{u}_2, \tan \theta, Re),
\]  

(5.1)
where the vector function $G$ in the right-hand side represents PDEs in lateral space. These PDEs depend on the depth $h(x,t)$, the layer mean velocities $\bar{u}_1(x,t)$ and $\bar{u}_2(x,t)$, the Reynolds number $Re$, and the plate slope $\tan \theta$. The functions $G$ include the effects of gravity, drag, advection and dispersion.

Section 5.1 describes the governing Navier–Stokes equation and the boundary conditions on the free surface and on the plate. In Section 5.2, centre manifold theory underpins that there is a slow manifold in the two layer problem. To be able to form the slow manifold, we embed the physical problem into a family of artificial problems using an artificial parameter $\gamma$, where $\gamma = 1$ recovers the original physical problem. The spectrum of eigenvalues implies the existence of the slow manifold.

Computer algebra in Appendix A.3 constructs the slow manifold. Section 5.3 then reports the leading order two-layer model on the slow manifold. This two-layer model ensures more subtle effects and resolves more internal modes, which are necessary to ‘lift’ macroscale data to an appropriate microscale configuration in the gap-tooth scheme.

Stability analysis in Section 5.4 shows that instability happens at high wavenumber. Section 5.4 introduces a regularising operator to stabilise the model. Numerical eigenvalues and numerical simulations of fluid film flow on the inclined plate support that the two-layer model is feasible.

Then section 5.5 applies the gap-tooth scheme to simulate the fluid film flow with the two-layer model being the microscale simulator. Section 5.5.3 establishes the slow manifold of the two-layer model. The coupling conditions are developed by lifting the one-layer mean velocity $\bar{u}(x,t)$ to the two-layer mean velocities $\bar{u}_1(x,t)$ and $\bar{u}_2(x,t)$. The successful numerical gap-tooth simulations indicate that the gap-tooth scheme works for the thin film flow.

5.1 Detailed equations for two layer thin film flow

This section describes the 2D continuity and modified Navier–Stokes equations of the layer thin fluid flow and the boundary conditions on the free surface and on the plate.

5.1.1 The governing equations

Consider a thin fluid flow of depth $h(x,t)$ flowing down an inclined plate with slope $\tan \theta$. Artificially assume the fluid film flow has two artificial layers, which have no distinguishing physical feature. Figure 5.1 shows the two layers of a thin film flow. Denote the coordinate system by $x$ and $z$
Figure 5.1: Diagram of the two-layer modelling of thin film flow. Assume the fluid film flow has two artificial layers, which have no distinguishing physical feature. Each layer has a thickness \( h(x,t)/2 \) and a mean layer velocity \( u_j(x,z,t) \) with \( j = 1 \) for lower layer and \( j = 2 \) for upper layer. The plate has a slope \( \tan \theta \). Then \( x \) and \( z \) are the coordinates and \( g \) is gravity.

along and normal to the plate. Each layer has a thickness \( h(x,t)/2 \) and velocity field \( q_j(x,z,t) = (u_j,w_j) \) in each layer, where \( j = 1 \) is for lower layer and \( j = 2 \) is for upper layer. Describe the pressure field by \( p_j(x,z,t) \) in each layer.

The nondimensional continuity and Navier–Stokes equations for the dynamics of fluid film flow are

\[
\nabla \cdot q_j = \frac{\partial u_j}{\partial x} + \frac{\partial w_j}{\partial z} = 0, \tag{5.2}
\]

\[
\frac{\partial q_j}{\partial t} + q_j \cdot \nabla q_j = -\nabla p_j + \frac{1}{Re} \nabla^2 q_j - g, \tag{5.3}
\]

where \( Re \) is the Reynolds number characterising the importance of the inertial terms compared to viscous dissipation and the vector \( g = (\tan \theta, 1) \) is the nondimensional gravity.

In PDEs (5.2)–(5.3), we have scaled quantities so that fluid density \( \rho = 1 \). In terms of a typical fluid velocity \( U \), a typical film thickness \( H \) and the slope angle of the plate \( \theta \), the Reynolds number \( Re = HU/\nu_f \) with \( \nu_f \) being the viscosity of the fluid; the component of the gravity along the plate is \( (gH \sin \theta)/(U^2 \cos \theta) = \tan \theta \) via the choice of the inviscid shallow water wave velocity \( U = \sqrt{gH} \) for small slope angle \( \theta \); and then similarly the component of gravity normal to the plate is one.
5.1.2 Boundary conditions

This section formulates the boundary conditions on the plate and on the free surface.

On the plate, prescribing no-slip requires

\[ u_1 = w_1 = 0 \quad \text{on} \quad z = 0. \tag{5.4} \]

On the free surface, the pressure is assumed to be zero (relative to atmosphere pressure). The stress normal to the surface is affected by surface tension. At this stage, we assume the surface tension is zero on the free surface. Thus the viscous mean stress normal to the free surface is zero, that is

\[ (\mathbf{n} \cdot \tau) \cdot \mathbf{n} - p_2 = 0, \tag{5.5} \]

where the unit vector \( \mathbf{n} = (-h_x, 1)/\sqrt{1 + h_x^2} \) normal to the free surface, \( \tau \) is the mean deviatoric stress tensor on free surface, similarly defined in equation (2.9), and \( p_2 \) is the pressure on the free surface. The boundary condition (5.5) leads to

\[ 2 \left[ \frac{\partial w_2}{\partial z} + \left( \frac{\partial h}{\partial x} \right)^2 \frac{\partial u_2}{\partial x} - \frac{\partial h}{\partial x} \left( \frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x} \right) \right] \]

\[ - \text{Re} \left[ 1 + \left( \frac{\partial h}{\partial x} \right)^2 \right] p_2 = 0 \quad \text{on} \quad z = h, \tag{5.6} \]

which agrees with the normal stress boundary on the free surface of viscous fluid film flow on flat and curved plates by Roberts (1997, 1998) with zero surface tension.

The free surface has zero tangential stress, \( \mathbf{n}_t \cdot \tau \cdot \mathbf{n} = 0 \), where \( \mathbf{n}_t \) is any vector tangential to the surface. This boundary condition results in

\[ \left[ 1 - \left( \frac{\partial h}{\partial x} \right)^2 \right] \left( \frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x} \right) + 2 \left( \frac{\partial h}{\partial x} \right) \left( \frac{\partial w_2}{\partial z} - \frac{\partial u_2}{\partial x} \right) = 0 \quad \text{on} \quad z = h. \tag{5.7} \]

On the free surface, the kinematic condition is

\[ \frac{\partial h}{\partial t} = w_2 - u_2 \frac{\partial h}{\partial x} \quad \text{on} \quad z = h. \tag{5.8} \]

On the artificial interface of the two artificial layers, continuity of the physical fields requires

\[ p_1 = p_2, \quad \text{on} \quad z = h/2, \tag{5.9} \]
Thus, the equations (5.2)–(5.3), together with the boundary conditions (5.4)–(5.12) describe the dynamics of the fluid film flow on an inclined plate.

5.2 Centre manifold theory supports the modelling

This section finds that centre manifold theory supports a slow manifold in the system. Embed the physical problem in a family of problems to empower the theoretical and dynamical support using an artificial parameter $\gamma$. Linear analysis determines the spectrum, which then supports the existence of a slow manifold.

5.2.1 Embed physical problem in a family of artificial problem

Modify the surface condition (5.7) on the tangential stresses to have an artificial forcing proportional to the upper layer averaged velocity:

$$
\left[1 - \left(\frac{\partial h}{\partial x}\right)^2\right] \left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) + 2\left(\frac{\partial h}{\partial z}\right)^2 \left(\frac{\partial w_2}{\partial z} - \frac{\partial u_2}{\partial z}\right)
= (1 - \gamma)\frac{2}{h} \left[u_2(h) - u_2\left(\frac{h}{2}\right)\right] \quad \text{on} \quad z = h.
$$

(5.13)

When evaluating at $\gamma = 1$, the artificial boundary condition (5.13) reduces to the physical boundary conditions (5.7).

Similarly, artificially modify the derivative condition (5.12) on the artificial interface of the two artificial layers

$$
\left(1 - \frac{\gamma}{2}\right) \frac{\partial u_1}{\partial z} = \frac{\gamma}{2} \frac{\partial u_2}{\partial z} + 2(1 - \gamma)\frac{u_1}{h} \quad \text{on} \quad z = h/2.
$$

(5.14)

When evaluated at $\gamma = 1$, this artificial boundary condition (5.14) recovers the originally physical boundary condition (5.12).

In the absence of lateral variations, evaluating at $\gamma = 0$, equations (5.13) and (5.14) indicate that two neutral modes of the dynamics are the layer
shear flows \((\bar{u}_1, \bar{u}_2) \propto (z, h/2)\) and \((\bar{u}_1, \bar{u}_2) \propto (0, z)\). Conservation of fluid provides a third neutral mode in the dynamics. Thus, under the conditions \(\gamma = \partial_x = \tan \theta = 0\), there are three neutral modes corresponding to some uniform shear flows \((\bar{u}_1, \bar{u}_2) \propto (z, h/2)\) and \((\bar{u}_1, \bar{u}_2) \propto (0, z)\) on a fluid of any thickness \(h\). Similarly with discussion in section 2.3.1, centre manifold theorems (Roberts 1988, Potzsche & Rasmussen 2006, e.g.) support the existence, accurate relevance and construction of slow manifold models.

Centre manifold theory (Roberts 1988, Chicone 2006, Potzsche & Rasmussen 2006, e.g.) then assures us that there exists a slow manifold, that we can construct, of the nonlinear dynamics and under changes in the parameters. Evaluating the resulting slow manifold model at the physical case of parameter \(\gamma = 1\) then provides a model for the physical flow dynamics.

### 5.2.2 The spectrum

In large lateral length scales, the lateral variations vary slowly and do not significantly affect the viscous decay of lateral shear which is the dominant linear process. Thus, at the equilibrium of constant flow thickness and zero layer velocities, together with the small parameters \(\partial_x = \tan \theta = 0\) and \(\gamma = 1\), the PDEs (5.2)–(5.3) and the boundary conditions (5.4)–(5.14) lead to a linear problem of

\[ \frac{\partial w_j}{\partial z} = 0, \quad \text{(5.15a)} \]
\[ \frac{\partial u_j}{\partial t} - \frac{1}{Re} \frac{\partial^2 u_j}{\partial z^2} = 0, \quad \text{(5.15b)} \]
\[ \frac{\partial w_j}{\partial t} - \frac{1}{Re} \frac{\partial^2 w_j}{\partial z^2} + \frac{\partial p}{\partial z} - 1 = 0, \quad \text{(5.15c)} \]
\[ u_1 = w_1 = 0 \quad \text{on} \quad z = 0, \quad \text{(5.15d)} \]
\[ -p_2 + \frac{2}{Re} \frac{\partial w_2}{\partial z} = 0 \quad \text{on} \quad z = h, \quad \text{(5.15e)} \]
\[ \frac{\partial u_2}{\partial z} = 0 \quad \text{on} \quad z = h, \quad \text{(5.15f)} \]
\[ \frac{\partial h}{\partial t} - w_2 = 0 \quad \text{on} \quad z = h, \quad \text{(5.15g)} \]
\[ p_1 - p_2 = 0 \quad \text{on} \quad z = h/2, \quad \text{(5.15h)} \]
\[ w_1 - w_2 = 0 \quad \text{on} \quad z = h/2, \quad \text{(5.15i)} \]
\[ u_1 - u_2 = 0 \quad \text{on} \quad z = h/2, \quad \text{(5.15j)} \]
\[ \frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial z} = 0 \quad \text{on} \quad z = h/2. \quad \text{(5.15k)} \]
Equations (5.15a) and (5.15i) indicate \( w_1 = w_2 = 0 \). Equations (5.15c), (5.15e) and (5.15h) indicate \( p_1 = p_2 = 0 \). Equation (5.15f) then indicates \( h = \text{constant} \). Equations (5.15b), together with equations (5.15d), (5.15e) and (5.15j)–(5.15k), indicate \( u_j \propto \sin[ \pi (2n - 1)z/2h] \exp(\lambda_n t) \) for the subscript \( n = 1, 2, \ldots \). Substituting these solutions into the linear problem leads to

\[
\lambda_n = -\frac{1}{\text{Re}} \left[ \frac{(2n - 1)\pi}{2h} \right]^2 \quad \text{where} \quad n = 1, 2, \ldots ,
\]

which gives the eigenvalues of \( \lambda_1 = -\pi^2/(4h^2 \text{Re}) \), \( \lambda_2 = -9\pi^2/(4h^2 \text{Re}) \), and other eigenvalues headed by \( \lambda_3 = -25\pi^2/(4h^2 \text{Re}) \).

There is also eigenvalue \( \lambda_0 = 0 \) corresponding to the freedom to vary the fluid film thickness \( h \). Thus, there is a spectral gap between the eigenvalues \( \lambda_2 \) and \( \lambda_3 \). This gap indicates that there should be a three-mode model based on the dynamics of the fluid thickness \( h(x,t) \) and layer mean velocities \( \bar{u}_1(x,t) \) and \( \bar{u}_2(x,t) \).

This gap is for \( \gamma = 1 \) which accesses for \( \gamma = 0 \). When the parameter \( \gamma = 0 \), there are two neutral modes of the dynamics are the layer shear flows \( (\bar{u}_1, \bar{u}_2) \propto (z,h/2) \) and \( (\bar{u}_1, \bar{u}_2) \propto (0,z) \). Thus, there are three zero eigenvalues and other negative eigenvalues in the system when \( \gamma = 0 \). Centre manifold theory (Chicone 2006, Potzsche & Rasmussen 2006, e.g.) then supports the existence of a slow manifold in the system of two layer fluid film flow. This manifold is called semi-slow manifold because of the small gap between the non-zero eigenvalues \( \lambda_2 \) and \( \lambda_3 \).

### 5.3 Model the two layer thin fluid flow

This section focusses on constructing the semi-slow model of the two-layer equations of section 5.1, and interpreting the model. Computer algebra in Appendix A.3 constructs the semi-slow models.

#### 5.3.1 Computer algebra constructs the slow manifold

Computer algebra constructs the slow manifold of the two layer thin fluid flow, as described in Appendix A.3. The computer algebra program derives the semi-slow model in the flow fields of depth \( h(x,t) \), lower layer mean velocity \( \bar{u}_1(x,t) \) and upper layer mean velocity \( \bar{u}_2(x,t) \). In the computer algebra program, the residuals of the governing equations (5.2)–(5.3) and boundary conditions (5.6)–(5.14) are calculated. The iteration is performed until all the residuals are zero to some order of error. Centre manifold theorems (Roberts 1988, Potzsche & Rasmussen 2006, e.g.) demonstrate that
the correctness of the residual computation assures the correctness of the resulting governing equations. The key to obtain correct resulting equations is the proper coding of the continuity equation (5.2), the Navier–Stokes equation (5.3) and the boundary conditions (5.6)–(5.14).

5.3.2 The order of errors in the construction

The order of errors in the construction is phrased in terms of the small parameters. The lateral spatial structure varies relatively slowly compared with the vertical structure. Thus, consider the lateral spatial derivative $\partial_x$ is small in the lateral direction. Then two small factors, the lateral spatial derivative $\partial_x$ and the slope of the plate $\tan \theta$, exist in the two layer thin fluid flow system. In some application, we need discard high order terms in $\bar{u}_i$ (Roberts 1998, e.g.). Thus $O(\bar{u}_1^p + \bar{u}_2^p + \partial_x^p + \tan^p \theta)$ denote the error terms for some exponent $p$, which means each term in the model has less than $p$ factors in total of these four parameters. The bigger the exponent number $p$, the higher the order of the modelling.

The artificial small parameter $\gamma$ has no physical meaning but is introduced to establish the slow manifold. However, relatively high orders of the artificial parameter $\gamma$ are required so that evaluating at $\gamma = 1$ is accurate.

5.3.3 The low leading order model of the two layer flow

Computer algebra in Appendix A.3 derives the physical flow fields of pressures $p_1$ and $p_2$, and layer velocities $u_1$ and $u_2$ in terms of the film thickness $h$, layer mean velocities $\bar{u}_1$ and $\bar{u}_2$, and scaled local vertical coordinate $Z = z/h$. This section records the dominant terms in such physical flow fields.

By truncating to errors $O(\bar{u}_1^3 + \bar{u}_2^3 + \partial_x^3 + \tan^3 \theta, \gamma^2)$ and omitting the intricate details of the derivation, executing the computer algebra in Appendix A.3 leads to

$$p_1 = (1 - Z)h + h^2 \frac{\partial^2 h}{\partial x^2} \left[ -0.422 + 0.104Z - 0.5Z^2 \right] + \gamma (-0.0148 - 0.0107)$$

$$+ h \tan \theta \frac{\partial h}{\partial x} \left[ (0.0469 - 0.104Z) + \gamma (0.0375 + 0.0107Z) \right]$$

$$+ \frac{2}{Re} \left[ \left( \frac{\partial \bar{u}_1}{\partial x} - \frac{\partial \bar{u}_2}{\partial x} - 2 \frac{\partial \bar{u}_1}{\partial x} Z \right) + \frac{1}{h} \frac{\partial h}{\partial x} (\bar{u}_1 + \bar{u}_2 - 2 u_1 Z) \right]$$

$$+ \gamma \frac{1}{Re} \left[ (4.875 \frac{\partial \bar{u}_1}{\partial x} - 0.125 \frac{\partial \bar{u}_2}{\partial x} - 1.5 \frac{\partial \bar{u}_1}{\partial x} Z + 0.5 \frac{\partial \bar{u}_2}{\partial x} Z) \right]$$
\[ \begin{align*}
+ \frac{1}{h} \frac{\partial h}{\partial x} (-7.625 \bar{u}_1 + 2.375 \bar{u}_2 + 1.5 \bar{u}_1 Z - 0.5 \bar{u}_2 Z) \\
+ \mathcal{O}(\bar{u}_1^3 + \bar{u}_2^3 + \bar{d}_1^3 + \tan^3 \theta, \gamma^2),
\end{align*} \tag{5.16} \]

\[ \begin{align*}
p_2 &= (1 - Z) h \\
+ h^2 \frac{\partial^2 h}{\partial x^2} \left[ (-0.417 + 0.115 Z - 0.5 Z^2) + \gamma (-0.0178 - 0.0049 Z) \right] \\
+ h \tan \theta \frac{\partial h}{\partial x} \left[ (0.0521 - 0.115 Z) + \gamma (0.0404 + 0.0049 Z) \right] \\
+ \frac{4}{\text{Re}} \left[ \left( -\frac{\partial \bar{u}_1}{\partial x} - \frac{\partial \bar{u}_2}{\partial x} + 2 \frac{\partial \bar{u}_1}{\partial x} Z \right) + \frac{1}{h} \frac{\partial h}{\partial x} (2 \bar{u}_1 - \bar{u}_2 - 2 \bar{u}_1 Z + \bar{u}_2 Z) \right] \\
+ \gamma \frac{1}{\text{Re}} \left[ (11 \frac{\partial \bar{u}_1}{\partial x} - 3 \frac{\partial \bar{u}_2}{\partial x} - 13.75 \frac{\partial \bar{u}_1}{\partial x} Z + 4.25 \frac{\partial \bar{u}_2}{\partial x} Z) \right] \\
+ \frac{1}{h} \frac{\partial h}{\partial x} (-13.75 \bar{u}_1 + 4.25 \bar{u}_2 + 13.75 \bar{u}_1 Z - 4.25 \bar{u}_2 Z) \\
+ \mathcal{O}(\bar{u}_1^3 + \bar{u}_2^3 + \bar{d}_1^3 + \tan^3 \theta, \gamma^2),
\end{align*} \tag{5.17} \]

\[ \begin{align*}
u_1(Z) &= \left[ (4.0 + 1.5 \gamma) \bar{u}_1 Z - 0.5 \gamma \bar{u}_2 Z + (4.0 \bar{u}_2 - 12.0 \bar{u}_1) \gamma Z^3 \right] \\
+ \text{Re} \, h \frac{\partial^2 h}{\partial x^2} \left[ (-0.104 + 0.0107 \gamma) Z + 0.5 Z^2 \right] \\
- (0.5 + 0.123 \gamma) Z^3 + 0.225 \gamma Z^3] \\
+ \text{Re} \, h \tan \theta \left[ (0.104 - 0.0107 \gamma) Z - 0.5 Z^2 \right] \\
+ (0.5 + 0.123 \gamma) Z^3 - 0.225 \gamma Z^3] \\
+ \mathcal{O}(\bar{u}_1^3 + \bar{u}_2^3 + \bar{d}_1^3 + \tan^3 \theta, \gamma^2),
\end{align*} \tag{5.18} \]

\[ \begin{align*}
u_2(Z) &= \left[ (6.0 - 11.9 \gamma) \bar{u}_1 + (-2.0 + 4.13 \gamma) \bar{u}_2 + (4.0 - 17.8 \gamma) \bar{u}_2 Z \right] \\
+ (-8.0 + 48.3 \gamma) \bar{u}_1 Z + (27.0 \bar{u}_2 - 69.0 \bar{u}_1) \gamma Z^2 \\
+ (34.0 \bar{u}_1 - 14.0 \bar{u}_2) \gamma Z^3] \\
+ \text{Re} \, h \frac{\partial^2 h}{\partial x^2} \left[ (-0.089 + 0.10 \gamma) + (0.45 - 0.68 \gamma) Z \right] \\
+ (-0.63 + 1.8 \gamma) Z^2 + (0.25 - 2.5 \gamma) Z^3 + 1.8 \gamma Z^4 - 0.49 \gamma Z^5] \\
+ \text{Re} \, h \tan \theta \left[ (0.089 - 0.10 \gamma) + (-0.45 + 0.68 \gamma) Z \right] \\
+ (0.63 - 1.8 \gamma) Z^2 + (-0.25 + 2.5 \gamma) Z^3 - 1.8 \gamma Z^4 + 0.49 \gamma Z^5] \\
+ \mathcal{O}(\bar{u}_1^3 + \bar{u}_2^3 + \bar{d}_1^3 + \tan^3 \theta, \gamma^2),
\end{align*} \tag{5.19} \]

where \( \text{Re} \) is the Reynolds number and \( \tan \theta \) is the slope of the plate in the flow direction. Equations (5.16)–(5.19) describe the low order shape of the manifold in the state space. Physically, upon setting the parameter \( \gamma = 1 \), these four equations (5.16) and (5.19) describe the vertical structures of the
pressures and layer velocities associated with the terms of depth \( h(x,t) \) and layer mean velocities \( \bar{u}_1(x,t) \) and \( \bar{u}_2(x,t) \).

Computer algebra in Appendix A.3 also determines the evolutions of the depth \( h(x,y) \) and layer mean velocities \( \bar{u}_1(x,t) \) and \( \bar{u}_2(x,t) \) on this semi-slow manifold in high order \( \gamma \):

\[
\frac{\partial h}{\partial t} = -0.5 \frac{\partial}{\partial x} (h \bar{u}_1 + h \bar{u}_2), \quad (5.20)
\]

\[
\frac{\partial \bar{u}_1}{\partial t} = (0.75 + 0.0438\gamma + 0.0365\gamma^2 - 0.00439\gamma^3 + 0.0000522\gamma^4
-0.000305\gamma^5 - 0.0000393\gamma^6) \left( \tan \theta - \frac{\partial h}{\partial x} \right)
- \frac{1}{\text{Re}} \left[ (18\gamma + 1.35\gamma^2 + 0.0723\gamma^3 - 0.0869\gamma^4 - 0.0112\gamma^5 - 0.00637\gamma^6) \frac{\bar{u}_1}{h^2}
+ (6.0\gamma + 1.05\gamma^2 - 0.038\gamma^3 - 0.022\gamma^4 - 0.008\gamma^5 - 0.00218\gamma^6) \frac{\bar{u}_2}{h^2} \right] + \mathcal{O}(\bar{u}_3^3 + \bar{u}_3^3 + \partial^3 x + \tan^3 \theta, \gamma^7), \quad (5.21)
\]

\[
\frac{\partial \bar{u}_2}{\partial t} = (1.25 - 0.195\gamma + 0.00126\gamma^2 - 0.00126\gamma^3 - 0.0003062\gamma^4
-0.000185\gamma^5 - 0.0000231\gamma^6) \left( \tan \theta - \frac{\partial h}{\partial x} \right)
- \frac{1}{\text{Re}} \left[ (15\gamma + 8.29\gamma^2 - 0.22\gamma^3 - 0.0278\gamma^4 - 0.0135\gamma^5 - 0.00351\gamma^6) \frac{\bar{u}_1}{h^2}
- (9.0\gamma - 3.17\gamma^2 + 0.0456\gamma^3 + 0.0173\gamma^4 + 0.00559\gamma^5 + 0.00146\gamma^6) \frac{\bar{u}_2}{h^2} \right] + \mathcal{O}(\bar{u}_3^3 + \bar{u}_3^3 + \partial^3 x + \tan^3 \theta, \gamma^7), \quad (5.22)
\]

To this order, equation (5.20) is a direct consequence of conservation of fluid. The momentum equations (5.21)–(5.22) include the effects of viscous drag \((\bar{u}_1 + \bar{u}_2)/h^2\), and the gravity forcing \(\tan \theta + \partial h/\partial x\). The advection and dispersion effects are included by section 5.3.4 in the momentum equations by extending to errors \(\mathcal{O}(\bar{u}_4^4 + \bar{u}_4^4 + \partial^4 x + \tan^4 \theta, \gamma^7)\).

Equations (5.16)–(5.22) are dynamical equations with the introduced artificial parameter \(\gamma\). Every coefficient in these equations is a series in \(\gamma\). Computations indicate that the coefficients depending upon \(\gamma\) converge quickly. The partial sums in Table 5.1 indicate that the series in \(\gamma\) converges quickly for \(\gamma = 1\). Thus, to errors \(\mathcal{O}(\gamma^6)\), it is apparent that all the showed digits are accurate. Roberts (1997, 1998) and Roberts et al. (2002) reported similar convergence in other physical problems. Hereafter we calculate every coefficient in the model up to errors \(\mathcal{O}(\gamma^6)\) evaluating at the artificial parameter \(\gamma = 1\).
Table 5.1: Partial sums from evaluating coefficients at $\gamma = 1$ of selected terms in equation (5.21)–(5.22) indicates that the power series in $\gamma$ converges quickly in $\gamma^5$.

\[
\begin{array}{cccccccc}
\gamma^0 & -0.75 & -1.125 & 0 & 0 & 0 & 0 \\
\gamma^1 & -0.7938 & -0.9297 & 18.0 & 15.0 & 6.0 & -9.0 \\
\gamma^2 & -0.8302 & -1.004 & -19.35 & 6.712 & 7.05 & -5.288 \\
\gamma^3 & -0.8258 & -1.002 & -19.42 & 6.933 & 7.012 & -5.333 \\
\gamma^4 & -0.8259 & -1.002 & -19.34 & 6.960 & 6.712 & -5.350 \\
\gamma^5 & -0.8256 & -1.002 & -19.32 & 6.974 & 6.982 & -5.356 \\
\gamma^6 & -0.8255 & -1.002 & -19.32 & 6.977 & 6.98 & -5.357 \\
\end{array}
\]

5.3.4 Derive dispersion terms of the two-layer model

This subsection derives the dispersion terms in the two-layer model by resolving the second derivatives in lateral coordinate $z$.

Truncating to errors $O(\bar{u}_1^4 + \bar{u}_2^4 + \theta^4 + \tan^4 \theta, \gamma^6)$, omitting the intricate details of the derivation and upon setting the artificial parameter $\gamma = 1$, the low leading order evolution of the depth $h(x,t)$, the lower layer mean velocity $\bar{u}_1(x,t)$ and the upper layer mean velocity $\bar{u}_2(x,t)$ on the semi-slow manifold are described by the flow conservation equation and by effective lateral momentum equations:

\[
\begin{align*}
\frac{\partial h}{\partial t} & = -0.5 \frac{\partial}{\partial x} (h\bar{u}_1 + h\bar{u}_2), \\
\frac{\partial \bar{u}_1}{\partial t} & = 0.826 \left( \tan \theta - \frac{\partial h}{\partial x} \right) - 0.0164h \tan \theta \frac{\partial^2 h}{\partial x^2} \\
& + \frac{1}{Re} \left( -19.3 \frac{\bar{u}_1}{h^2} + 6.98 \frac{\bar{u}_2}{h^2} \right) \\
& - 1.48\bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} - 0.225\bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + 0.142\bar{u}_2 \frac{\partial \bar{u}_1}{\partial x} + 0.0728\bar{u}_1 \frac{\partial \bar{u}_2}{\partial x} \\
& + \frac{1}{Re} \left( -3.84 \frac{\partial^2 \bar{u}_1}{\partial x^2} + 2.52 \frac{\partial^2 \bar{u}_2}{\partial x^2} \right) \\
& + Re h^2 \frac{\partial h}{\partial x} \left( -0.00256 \frac{\partial \bar{u}_1}{\partial x} + 0.00182 \frac{\partial \bar{u}_2}{\partial x} \right) \\
& - Re h^2 \frac{\partial^2 h}{\partial x^2} (0.00221\bar{u}_1 + 0.0009\bar{u}_2)
\end{align*}
\]
\[
+ \text{Re} \ h^3 \left( 0.000879 \frac{\partial^2 \bar{u}_1}{\partial x^2} + 0.000656 \frac{\partial^2 \bar{u}_2}{\partial x^2} \right)
+ \frac{1}{h} \frac{\partial h}{\partial x} \left( -0.251 \bar{u}_1^2 + 0.540 \bar{u}_1 \bar{u}_2 - 0.335 \bar{u}_2^2 \right)
+ 0.0263 \left( \frac{\partial h}{\partial x} \right)^3 - 0.706h \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} - 0.338h^2 \frac{\partial^3 h}{\partial x^3}
+ \frac{1}{\text{Re} \ h} \frac{\partial h}{\partial x} \left( 1.82 \frac{\partial \bar{u}_1}{\partial x} - 2.43 \frac{\partial \bar{u}_2}{\partial x} \right) + \frac{1}{\text{Re} \ h} \frac{\partial^2 h}{\partial x^2} (2.03 \bar{u}_1 - 0.0168 \bar{u}_2)
+ \frac{1}{\text{Re} \ h^2} \left( \frac{\partial h}{\partial x} \right)^2 ( -4.82 \bar{u}_1 + 2.18 \bar{u}_2 )
+ \mathcal{O}(\bar{u}_1^4 + \bar{u}_2^4 + \partial^4 x + \tan^4 \theta, \gamma^6),
\]
\[
\frac{\partial \bar{u}_2}{\partial t} = 1.002 \left( \tan \theta - \frac{\partial h}{\partial x} \right) - 0.0286h \tan \theta \frac{\partial^2 h}{\partial x^2}
+ \frac{1}{\text{Re} \ h} \left( 6.98 \bar{u}_1 - 5.36 \bar{u}_2 \right)
- 1.25u_1 \frac{\partial \bar{u}_1}{\partial x} - 1.57u_2 \frac{\partial \bar{u}_2}{\partial x} + 0.768u_2 \frac{\partial \bar{u}_1}{\partial x} + 0.930u_1 \frac{\partial \bar{u}_2}{\partial x}
+ \frac{1}{\text{Re} \ h} \left( -1.98 \frac{\partial^2 \bar{u}_1}{\partial x^2} + 5.23 \frac{\partial^2 \bar{u}_2}{\partial x^2} \right)
+ \text{Re} \ h^2 \frac{\partial h}{\partial x} \left( -0.00549 \frac{\partial \bar{u}_1}{\partial x} + 0.00359 \frac{\partial \bar{u}_2}{\partial x} \right)
- \text{Re} \ h^2 \frac{\partial^2 h}{\partial x^2} (0.00474 \bar{u}_1 + 0.00303 \bar{u}_2 )
+ \text{Re} \ h^3 \left( 0.000183 \frac{\partial^2 \bar{u}_1}{\partial x^2} + 0.000113 \frac{\partial^2 \bar{u}_2}{\partial x^2} \right)
+ \frac{1}{\text{Re} \ h} \left( -0.789 \bar{u}_1^2 + 1.09u_1 \bar{u}_2 - 0.388 \bar{u}_2^2 \right)
+ 0.0964 \left( \frac{\partial h}{\partial x} \right)^3 - 0.870h \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} - 0.191h^2 \frac{\partial^3 h}{\partial x^3}
+ \frac{1}{\text{Re} \ h} \frac{\partial h}{\partial x} \left( 0.316 \frac{\partial \bar{u}_1}{\partial x} + 6.41 \frac{\partial \bar{u}_2}{\partial x} \right) + \frac{1}{\text{Re} \ h} \frac{\partial^2 h}{\partial x^2} (7.21 \bar{u}_1 - 3.92 \bar{u}_2)
+ \frac{1}{\text{Re} \ h^2} \left( \frac{\partial h}{\partial x} \right)^2 (7.16 \bar{u}_1 - 2.79 \bar{u}_2)
+ \mathcal{O}(\bar{u}_1^4 + \bar{u}_2^4 + \partial^4 x + \tan^4 \theta, \gamma^6),
\]

Equation (5.23) is a direct consequence of conservation of fluid. The momentum equations (5.24)–(5.25) include the effects of gravity $\tan \theta - \partial h/\partial x$,
drag $\bar{u}_i/h^2$, advection $\bar{u}_i \partial \bar{u}_i/\partial x$, dispersion $\partial^2 \bar{u}_i/\partial x^2$, tension $\partial^3 h/\partial x^3$ and other viscous terms, such as $\partial h/\partial x \bar{u}_i/\partial x$, where the subscript $i,j = 1,2$ for lower and upper layer. Compared with the one-layer models (Prokopiou et al. 1991, Roberts 1997, Ruyer-Quil & Manneville 1998, e.g.), the two-layer model (5.24)–(5.25) ensures more subtle effects and resolve more internal modes, which are necessary for a more generic microscale configuration in the gap-tooth scheme.

At this stage we simplify the model in order to ease the application of the gap-tooth scheme to model fluid film flow. Thus we discard terms with small coefficients and then equations (5.24)–(5.25) are simplified to

$$\frac{\partial h}{\partial t} = -0.5 \left( \frac{\partial h \bar{u}_1}{\partial x} + \frac{\partial h \bar{u}_2}{\partial x} \right),$$  \hspace{1cm} (5.26)

$$\frac{\partial \bar{u}_1}{\partial t} \approx 0.826 \left( \tan \theta - \frac{\partial h}{\partial x} \right) + \frac{1}{Re} \left( -19.3 \frac{\bar{u}_1}{h^2} + 6.98 \frac{\bar{u}_2}{h^2} \right) - 1.48 \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} - 0.225 \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + 0.142 \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + 0.0728 \bar{u}_1 \frac{\partial \bar{u}_2}{\partial x} + \frac{(\bar{u}_1 - \bar{u}_2)}{h} \left( -0.25 \bar{u}_1 + 0.34 \bar{u}_2 \right) \frac{\partial h}{\partial x} + \frac{1}{Re} \left( -3.84 \frac{\partial^2 \bar{u}_1}{\partial x^2} + 2.52 \frac{\partial^2 \bar{u}_2}{\partial x^2} \right),$$  \hspace{1cm} (5.27)

$$L \frac{\partial \bar{u}_2}{\partial t} \approx 1.002 \left( \tan \theta - \frac{\partial h}{\partial x} \right) + \frac{1}{Re} \left( 6.98 \frac{\bar{u}_1}{h^2} - 5.36 \frac{\bar{u}_2}{h^2} \right) - 1.25 \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} - 1.57 \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + 0.768 \bar{u}_2 \frac{\partial \bar{u}_1}{\partial x} + 0.930 \bar{u}_1 \frac{\partial \bar{u}_2}{\partial x} + \frac{(\bar{u}_1 - \bar{u}_2)}{h} \left( -0.78 \bar{u}_1 + 0.38 \bar{u}_2 \right) + \frac{1}{Re} \left( -1.98 \frac{\partial^2 \bar{u}_1}{\partial x^2} + 5.23 \frac{\partial^2 \bar{u}_2}{\partial x^2} \right).$$  \hspace{1cm} (5.28)

We use equations (5.26)–(5.28) to model the dynamics of the thin fluid flow of depth $h(x,t)$ by the two artificial layers with the layer mean velocities $\bar{u}_1(x,t)$ and $\bar{u}_2(x,t)$.

### 5.4 Eigenvalue analysis of the two-layer model

This section linearly analyses the two-layer model (5.26)–(5.28). Linear analysis indicates that an unphysical instability appears for high wavenumber. This section then proposes three methods to avoid such instability, includ-
Figure 5.2: The growth rate $\Re \lambda$ varies with the nondimensional wavenumber $k$ from the characteristic equation (5.32) of the model (5.26)–(5.28). The slope of the plate is $\tan \theta = 0$. The Reynolds number is $Re = 1$. An unphysical instability appears for high nondimensional wavenumber $k > 2.5$. The little interval near $k = 2.5$ is due to the numerical error.

5.4.1 Linear analysis of the two-layer model

Consider the fluid film flow with two artificial layers on a flat plate with slope $\tan \theta$. The fluid has an equilibrium of thickness $h = 1$. The model (5.26)–(5.28), assuming $\partial_t = \partial_x = 0$, simplifies to

$$0.826 \tan \theta + \frac{1}{Re} (-19.3 \bar{u}_1 + 6.98 \bar{u}_2) = 0,$$
$$1.002 \tan \theta + \frac{1}{Re} (6.98 \bar{u}_1 - 5.36 \bar{u}_2) = 0,$$

which indicates an equilibrium of layer mean velocities

$$\bar{u}_1 = 0.209 \Re \tan \theta \quad \text{and} \quad \bar{u}_2 = 0.459 \Re \tan \theta. \quad (5.29)$$
Impose small perturbations to this equilibrium and seek solutions in the form
\[ h = 1 + \hat{h} \exp(\lambda t + ikx), \]
\[ \hat{u}_1 = 0.209 \Re \tan \theta + \hat{u}_1 \exp(\lambda t + ikx), \]
\[ \hat{u}_2 = 0.459 \Re \tan \theta + \hat{u}_2 \exp(\lambda t + ikx), \]

for growth rate \( \lambda \) (possibly complex) and nondimensional wavenumber \( k \).

Substitute the form (5.30) into the model (5.26)–(5.28), equate coefficients and derive a linear problem of
\[ \lambda \begin{bmatrix} \hat{h} \\ \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = M \begin{bmatrix} \hat{h} \\ \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \]
where the coefficient matrix \( M \) is
\[
\begin{bmatrix}
-0.334 \tan \theta \Re ik & -0.5ik & -0.5ik \\
-(0.826 + 0.0259 \tan \theta \Re)ik & \frac{1}{\Re}(-19.3 + 3.84k^2) - \frac{1}{\Re}(6.98 - 2.52k^2) - 0.088 \tan \theta \Re ik \\
-(1.002 + 0.0029 \tan \theta \Re)ik & \frac{1}{\Re}(6.98 + 1.98k^2) + \frac{1}{\Re}(-5.36 - 5.23k^2) - 0.526 \tan \theta \Re ik \\
\end{bmatrix}.
\]

The coefficient matrix \( M \) has characteristic equation
\[
\lambda^3 + \lambda^2 \left[ 1.114 \tan \theta \Re ik + \frac{1}{\Re} (1.39k^2 + 24.66) \right] \\
- \lambda \left[ 0.476 \tan^2 \theta \Re k^2 + \tan \theta (-2.265ik^2 - 14.03i - 0.0144 \Re k) + 15.09 \frac{1}{\Re} k^4 - 0.914k^2 - 84.13 \frac{1}{\Re} k^2 - 54.73 \frac{1}{\Re^2} \right] \\
- \left[ 0.0725 \tan^3 \theta \Re k^3 + \tan^2 \theta \Re k^2 (-0.0191 \Re ik + 0.615k^2 + 1.908) + \tan \theta \frac{1}{\Re} k(5.19ik^4 - 0.682 \Re^2 ik^2 - 28.94ik^2 - 18.83i - 0.841 \Re k^3 - 0.198 \Re k) + \frac{1}{\Re} k^2 (0.209k^2 - 18.26) \right] = 0
\]

Figure 5.2 plots the growth rates \( \Re \lambda \) versus the nondimensional wavenumber \( k \) from equation (5.32). All the values represented by the blue curve are negative which nicely reflects viscous decay of lateral shear modes. When the nondimensional wavenumber \( k < 2.5 \), the values represented by the red curve
are negative (viscous decay), and by the green curve are zeros (conservation of fluid). But when the nondimensional wavenumber $k > 2.5$, the green curve increases to positive which implies that instability arises in the system. This instability is nothing to do with physical instabilities, for example, found by Chen (1993), who found the instability of the two liquid films down an inclined plate due to the different viscosity at the interface of the two layer flow and on the free surface.

The instability in Figure 5.2 is not physical. Section 5.4.2 stabilises the instability.

### 5.4.2 Methods of avoiding the instability

This section proposes three ways to stabilise the instability of the two layer thin film flow, and recommends the last regularisation.

First, resolve only low wavenumbers in the numerical solutions. Figure 5.2 shows that the instability only arises at high wavenumber. The critical nondimensional wavenumber $k_{cr} = 2.5$, or the critical wave length $2\pi/(k_{cr})$. In the spatial discretisation of the numerical simulations, the highest nondimensional wavenumber resolved is $k = \pi/\delta x$. Thus choosing the spatial step $\delta x \geq \pi/k_{cr} = 0.4\pi$ would eliminate the unstable modes. Such a large spatial step in the numerical scheme would reduce the resolution of the numerical simulations. Thus, this method of stabilising the instability is not flexible.

Second, we could add a high order dissipation term to the evolution of velocity $\bar{u}_1$ to stabilise the instability. The model (5.26)–(5.28) is truncated to errors $O(\bar{u}_1^4 + \bar{u}_2^4 + \partial_x^4 + \tan^4 \theta, \gamma^6)$, so to the same order of error we can add a higher order term. To stabilise the instability, try adding a high order dissipation term $-Ch^2 \partial_x^4 \bar{u}_1/\partial x^4$ to the equation (5.27), where the positive constant coefficient $C$ is to be determined to counteract the positive growth rates. Thus, the coefficient matrix $M$ in (5.38) becomes

$$
\begin{bmatrix}
-0.334 \tan \theta \text{Re } ik & -0.5ik & -0.5ik \\
-(0.826 + 0.0259 \tan \theta \text{Re } ik) & \frac{1}{\text{Re}}(-19.3 + 3.84k^2 + \text{C}k^4) - 0.244 \tan \theta \text{Re } ik & \frac{1}{\text{Re}}(6.98 - 2.52k^2) - 0.088 \tan \theta \text{Re } ik \\
-(1.002 + 0.0029 \tan \theta \text{Re } ik) & \frac{1}{\text{Re}}(6.98 + 1.98k^2) + 0.935 \tan \theta \text{Re } ik & \frac{1}{\text{Re}}(-5.36 - 5.23k^2) - 0.526 \tan \theta \text{Re } ik
\end{bmatrix}.
$$

(5.33)
Figure 5.3: Plot of the growth rate $\Re \lambda$ varying with the nondimensional wavenumber $k$ from the characteristic equation (5.34). The coefficient $C = 2.4$ in the high order dissipation term. The slope of the plate is $\tan \theta = 0$. The Reynolds number is $Re = 1$. 
Then the coefficient matrix $M$ indicates a characteristic equation

$$
\lambda^3 + \lambda^2 \left[ 1.114 \tan \theta \Re ik + \frac{1}{\Re} (1.39k^2 + 24.66) + \frac{1}{\Re} Ck^4 \right] \\
- \lambda \left[ 0.476 \tan^2 \theta \Re^2 k^2 + \tan \theta k(-2.265ik^2 - 14.03i - 0.0144 \Re k) \right] \\
+ 15.09 \frac{1}{\Re^2} k^4 - 0.914k^2 - 84.13 \frac{1}{\Re^2} k^2 - 54.73 \frac{1}{\Re^2} \\
- 0.87 \tan \theta ik^5 - \frac{1}{\Re^2} (5.23Ck^6 + 5.36Ck^4) \\
- \left[ 0.0725 \tan^3 \theta \Re^3 ik^3 + \tan^2 \theta \Re k^2(-0.0191 \Re ik + 0.615k^2 + 1.908) \right] \\
+ \tan \theta \frac{1}{\Re} k(5.19ik^4 - 0.682 \Re^2 ik^2 - 28.94ik^2 - 18.83i - 0.841 \Re k^3) \\
+ -0.198 \Re k \frac{1}{\Re} k^2(0.209k^2 - 18.26) + \frac{1}{\Re} (0.181C \tan^2 \theta \Re^2 k^6 \\
- 1.80C \tan \theta ik^7 - 1.84C \tan \theta ik^4 \\
- 0.00145C \tan \theta \Re k^5 - 0.501Ck^5) \right] = 0.
$$

(5.34)

Figure 5.3 shows the growth rate $\Re \lambda$ versus nondimensional wavenumber $k$ by the characteristic equation (5.34). The positive coefficient $C = 2.4$ in the high order dissipation term. The slope of the plate is $\tan \theta = 0$. The Reynolds number is $\Re = 1$. The lack of positive growth rate $\Re \lambda$ demonstrates no instability arises. Numerical checks show that when the coefficient $C > 0.12$, no positive growth rate occurs, but the imaginary parts of $\lambda$ are non-zeros, which are physically not realistic for viscous decay of shear modes on the horizontal plate. Numerical checks indicate that when the coefficient $C > 2.3$, no instability appears and all modes decay to steady state. A disadvantage of this method of stabilising the instability is that the high order dissipation term would increase the cost of numerical calculations of the model.

Third, we introduce a regularising operator to stabilise the instability. For the momentum equations (5.24)–(5.25), consider applying the regularising operator $\mathcal{L} = 1 - C \partial_x (h^2 \partial_x)$ to both sides of both equations. The coefficient $C$ is positive. The reason for using $h^2$ in the regularising operator $\mathcal{L}$ is to be dimensionally consistent which means we can cancel the $h^{-2}$ in the drag terms in equations (5.24)–(5.25) and also ensure the regularising operator $\mathcal{L}$ is self-adjoint. This regularising operator would generate dissipation terms to counteract the positive growth rates. I implement the regularising operator for the equations (5.24)–(5.25) in the computer algebra in Appendix A.3, which gives equations, in term of the regularising operator,

$$
\frac{\partial h}{\partial t} = -0.5 \left( \frac{\partial h u_1}{\partial x} + \frac{\partial h u_2}{\partial x} \right),
$$

(5.35)
Figure 5.4: Plot of the growth rate $\Re \lambda$ varying with the nondimensional wavenumber $k$ from the characteristic equation (5.34). The coefficient $C = 0.5$ in the regularising operator $\mathcal{L}$. The slope of the plate is $\tan \theta = 0$. The Reynolds number is $Re = 1$. 

\[ \Re (\lambda/Re) \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.4}
\caption{Plot of the growth rate $\Re \lambda$ varying with the nondimensional wavenumber $k$ from the characteristic equation (5.34). The coefficient $C = 0.5$ in the regularising operator $\mathcal{L}$. The slope of the plate is $\tan \theta = 0$. The Reynolds number is $Re = 1$.}
\end{figure}
\[
L \frac{\partial \bar{u}_1}{\partial t} \approx 0.826 \left( \tan \theta - \frac{\partial h}{\partial x} \right) + \frac{1}{Re} \left( -19.3 \frac{\bar{u}_1}{h^2} + 6.98 \frac{\bar{u}_2}{h^2} \right) - 1.48 \frac{\partial \bar{u}_1}{\partial x} - 0.225 \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + 0.142 \bar{u}_2 \frac{\partial \bar{u}_1}{\partial x} + 0.0728 \bar{u}_1 \frac{\partial \bar{u}_2}{\partial x} + \frac{(\bar{u}_1 - \bar{u}_2)}{h} \left( -0.25 \bar{u}_1 + 0.34 \bar{u}_2 \right) \frac{\partial h}{\partial x} + \frac{1}{Re} \left( -3.84 + 19.3 C \right) \frac{\partial^2 \bar{u}_1}{\partial x^2} + \frac{1}{Re} \left( 2.52 - 6.98 C \right) \frac{\partial^2 \bar{u}_2}{\partial x^2}, \quad (5.36)
\]

\[
L \frac{\partial \bar{u}_2}{\partial t} \approx 1.002 \left( \tan \theta - \frac{\partial h}{\partial x} \right) + \frac{1}{Re} \left( 6.98 \frac{\bar{u}_1}{h^2} - 5.36 \frac{\bar{u}_2}{h^2} \right) - 1.25 \frac{\partial \bar{u}_1}{\partial x} - 1.57 \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + 0.768 \bar{u}_2 \frac{\partial \bar{u}_1}{\partial x} + 0.930 \bar{u}_1 \frac{\partial \bar{u}_2}{\partial x} + \frac{(\bar{u}_1 - \bar{u}_2)}{h} \left( -0.78 \bar{u}_1 + 0.38 \bar{u}_2 \right) \frac{\partial h}{\partial x} + \frac{1}{Re} \left( -1.98 - 6.98 C \right) \frac{\partial^2 \bar{u}_1}{\partial x^2} + \frac{1}{Re} \left( 5.23 + 5.36 C \right) \frac{\partial^2 \bar{u}_2}{\partial x^2}. \quad (5.37)
\]

By substituting the solution form (5.30), equations (5.35)–(5.37) indicate a linear problem

\[
\lambda \begin{bmatrix} \frac{\hat{h}}{(1 + Ck^2)\bar{u}_1} \\ \frac{\hat{h}}{(1 + Ck^2)\bar{u}_2} \end{bmatrix} = M \begin{bmatrix} \frac{\hat{h}}{\bar{u}_1} \\ \frac{\hat{h}}{\bar{u}_2} \end{bmatrix},
\]

where the coefficient matrix \( M \) is

\[
\begin{bmatrix}
-0.334 \tan \theta \Re i k & -0.5 i k \\
-(0.826 + 0.0259 \tan \theta \Re i k) & \frac{1}{Re} (-19.3 + 3.84 k^2) - \frac{1}{Re} (6.98 - 2.52 k^2) + 19.3 C k^2) - 6.98 C k^2) - 0.244 \tan \theta \Re i k & 0.088 \tan \theta \Re i k \\
-(1.002 + 0.0029 \tan \theta \Re i k) & \frac{1}{Re} (6.98 + 1.98 k^2) + 6.98 C k^2) + 5.36 C k^2) - 0.935 \tan \theta \Re i k & 0.526 \tan \theta \Re i k \\
\end{bmatrix} .
\]

(5.38)

This generalised eigenproblem has characteristic equation

\[
\lambda^3 + \lambda^2 \left[ 1.114 \tan \theta \Re i k + \frac{1}{Re} (1.39 k^2 + 24.66) + 24.66 \frac{1}{Re} C k^2 \right] = 0.
\]

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\[- \lambda \left[ 0.476 \tan^2 \theta \Re^2 k^2 + \tan \theta k(-2.265\Re k^2 - 14.03i - 0.0144 \Re k) \\
+ 15.09 \frac{1}{\Re} k^4 - 0.914 k^2 - 84.13 \frac{1}{\Re^2} k^2 - 54.73 \frac{1}{\Re^2} \\
- 14.03 \Re \tan \theta k^2 - \frac{1}{\Re} (54.73 C^2 k^4 + 84.13 \Re C k^2 + 109.46 C k^2) \right] \\
- \left[ 0.0725 \tan^3 \theta \Re^3 \Re k^3 + \tan^2 \theta \Re k^2 (-0.0191 \Re \Re k + 0.615 k^2 + 1.908) \\
+ \tan \theta \Re k (5.19 \Re k^4 - 0.682 \Re^2 \Re k^2 - 28.94 \Re k^2 - 18.83 i - 0.841 \Re k^3 \\
- 0.198 \Re k) + \frac{1}{\Re} k^2 (0.209 k^2 - 18.26) + \frac{1}{\Re} (-18.83 C^2 \tan \theta k^4 \\
- 28.94 C \tan \theta k^4 - 37.65 C \tan \theta k^2 - 18.26 C k^3) + 1.908 C \tan^2 \theta \Re k^3 \\
- 0.198 C \tan \theta k^3 \right] = 0. \tag{5.39} \]

Figure 5.4 plots the growth rate $\Re \lambda$ varying with the nondimensional wavenumber $k$ from the characteristic equation (5.39). The $C = 0.5$ in the regularising operator $L$. The slope of the plate is $\tan \theta = 0$. The Reynolds number is $\Re = 1$. That there is no positive growth rate demonstrates that no instability occurs, even for the high wavenumber. The decay rates represented by the blue and red curves grow quickly for large wavenumber. Numerical checks indicate that the coefficient $C > 0.17$ to eliminate the instabilities. This method is flexible through the good range of choice of the positive coefficient $C$ in the regularising operator $L$. No high order derivatives, such as $h^2 \partial^4 \bar{u}_1 / \partial x^4$, appear in this method. Thus, introducing the regularising operator $L$ is useful to stabilise the model, and hereafter I implement the numerical simulations of the model (5.35) and (5.37) with the coefficient $C = 0.5$.

### 5.4.3 Numerical eigenvalue analysis of the two layer model

This subsection explores numerical eigenvalues and numerical simulations of the thin fluid flow by the two-layer model (5.35) and (5.37).

Consider the viscous thin film fluid flowing on a horizontal plate; that is, the mean slope of the plate is $\tan \theta = 0$. The plate has a length $L = m \pi$, where $m$ is the number of patches that will be discussed in section 5.5. Approximate the spatial derivatives in the two-layer model (5.35)–(5.37) by the staggered grid scheme. Time integration is implemented using the matlab ode15s function.

Numerical eigenvalues are calculated about the steady state of the fluid thickness $h = 1$ and layer mean velocities $\bar{u}_1 = \bar{u}_2 = 0$. Figure 5.5 shows the growth rate $\Re \lambda$ and the frequency $\Im \lambda$ of the thin film flow by the two-layer
Figure 5.5: Plots of the growth rate $\Re \lambda$ and the frequency $\Im \lambda$ of the thin fluid flow by the two layer model (5.35)–(5.37). The domain has a length $L = 10\pi = 31.4$. The Reynolds number $Re = 15$ and the coefficient in the regularising operator $L$ is $C = 0.5$. 
Figure 5.6: Plots of the fluid thickness in the thin fluid flow by the two-layer model (5.35)–(5.37) on the domain $[0 \ 10\pi]$. The Reynolds number $\text{Re} = 15$. The plate is horizontal, and the regularising coefficient $C = 0.5$. Figure 5.7 shows the fluid velocities.
Figure 5.7: Plots of the layer mean velocities $\bar{u}_1$ (blue stars) and $\bar{u}_2$ (red triangles) in the fluid film flow by the two-layer model (5.35)–(5.37) on the domain $[0 \ 10\pi]$. The Reynolds number $Re = 15$.  

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Table 5.2: Numerical frequencies of the two-layer model (5.35)–(5.37) from Figure 5.5; compare with exact frequencies from equation (5.43) for the linear dynamics about the steady state.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Frequency, eqn (5.43)</th>
<th>Frequency, Fig. 5.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0, 0</td>
<td>0, −0.1644</td>
</tr>
<tr>
<td>±0.2</td>
<td>±0.1912</td>
<td>±0.1490</td>
</tr>
<tr>
<td>±0.4</td>
<td>±0.3824</td>
<td>±0.2957</td>
</tr>
</tbody>
</table>

model (5.35)–(5.37). The domain has a length $L = 10\pi = 31.4$. The Reynolds number $Re = 15$ and the coefficient in the regularising operator $L$ is $C = 0.5$. No positive real eigenvalues demonstrate that no instability appears, which supports that the regularising operator $L$ eliminates the instability. Symmetric non-zero imaginary parts of the eigenvalues imply that waves are supported on the free surface. These waves decay, because $\Re \lambda < 0$, through viscous effects. The spectral gap between $\Re \lambda \approx −1.2$ and $\Re \lambda \approx −0.2$ indicates the existence of a slow manifold in the system composed of the relatively slowly decaying waves.

For an indicative comparison, we calculate the eigenvalues of the linear version of the two-layer model (5.35)–(5.37) based upon the equilibrium of depth one,

\[
\frac{\partial h}{\partial t} = -0.5 \left( \frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{u}_2}{\partial x} \right), \tag{5.40}
\]

\[
\frac{\partial \bar{u}_1}{\partial t} = -0.826 \frac{\partial h}{\partial x}, \tag{5.41}
\]

\[
\frac{\partial \bar{u}_2}{\partial t} = -1.002 \frac{\partial h}{\partial x}. \tag{5.42}
\]

Substituting the solution forms (5.30) with the slope $\tan \theta = 0$ leads to a characteristic equation

\[\lambda^2 + 0.914k^2 = 0. \tag{5.43}\]

This characteristic equation implies that equations (5.40)–(5.42) support waves on the free surface of the thin film flow with no viscous dissipation. Table 5.2 shows that the frequencies from Figure 5.5 are close enough to the exact frequencies from equation (5.43). This indicates that our numerical simulation is reasonable.

The two-layer model (5.40)–(5.42) describes the dynamics of the thin fluid flow in terms of thickness $h(x, t)$ and layer mean velocities $\bar{u}_1(x, t)$.
and $\bar{u}_2(x,t)$. Thus, we numerically compute the thickness $h(x,t)$ and layer mean velocities $\bar{u}_1(x,t)$ and $\bar{u}_2(x,t)$ in the fluid flow. Figure 5.6 plots the fluid thickness $h$ and Figure 5.7 plots the layer mean velocities $\bar{u}_1$ (stars) and $\bar{u}_2$ (triangles) in the fluid flow by the two-layer model (5.35)–(5.37) on the domain $[0, 10\pi]$. The $t = 0$ graphs in Figure 5.6–5.7 show an initially imposed a perturbation $0.1 \sin(2\pi/Lx)$ with small random noise to the fluid thickness $h = 1$, and layer mean velocities $\bar{u}_1 = 0$ and $\bar{u}_2 = 0.2$. The modes of the layer mean velocities in Figure 5.7 smooth quickly and then decay quickly due to the viscous decay, as shown from the $t = 2$ to $t = 12$ graphs. These modes of layer mean velocities correspond to the growth rates representing by the red and blue curves in Figure 5.4. However, the modes of the thickness in Figure 5.6 slowly smooth, as shown from $t = 0$ to $t = 6$ graphs, and then decay, shown in $t = 12$ graph. The modes of the thickness correspond to the growth rates representing by the green curve in Figure 5.4.

5.5 Gap-tooth simulation of the two layer thin fluid flow

This section focuses on implementing the gap-tooth simulation of the thin fluid flow. This section uses the two-layer model (5.35)–(5.37) as the microscale simulator within patches. Coupling conditions (5.53) and (5.47)–(5.50) are developed to couple patches together. Appendix B.4 lists the numerical code of this gap-tooth simulation.

Let us focus on one patch, the $j$th patch. Figure 5.8 shows the staggered grid for the depth $h_{j,i}$ (blue points) and the layer mean velocities $\bar{u}_{1,i}$ and $\bar{u}_{2,i}$ (magenta points) at the $i$th micro-grid point on the $j$th patch. The superscripts identify the two mean velocities $\bar{u}_1$ and $\bar{u}_2$ on a patch. As shown schematically in Figure 4.5, let each of $m$ patches be centred on equi-spaced macroscale grid points $x = X_j = jD$, where $D = L/m$ is the macroscale spacing and $L$ is the length of the whole domain. Each patch has relatively small width $l$. Assume each patch has a total of $n$ microscale grid points excluding the two edge grids, so the microscale spatial step $d = l/(n+1)$. Let each patch around a macroscale grid point $X_j$ execute a microscale simulator. Define the mid patch point $x_{j,0} = X_j$. Let the macroscale value $H_j = h_j(X_j, t)$ for the odd $j$ and $U_j = (\bar{u}_{1,i}(X_j, t) + \bar{u}_{2,i}(X_j, t))/2$ for even $j$.

As a basic pilot study of the gap-tooth simulation of two layer fluid flow on a horizontal plate, we first consider the linear version of the two layer model (5.35)–(5.37). Figure 5.9 plots the fluid thickness $h$ and Figure 5.10 plots the layer mean velocities $\bar{u}_1$ (stars) and $\bar{u}_2$ (triangles) in the fluid film flow by the...
Figure 5.8: Scheme of the staggered grid points of the depth \( h_{j,i} \) (blue points) and the mean velocities \( \bar{u}_{1,j,i} \) and \( \bar{u}_{2,j,i} \) (magenta points) at the \( i \)th micro-grid point on the odd \( j \)th patch (top) and the even \( j \)th patch (bottom). This diagram show the cases for \( n = 5 \) interior grid points in each patch.

5.5.1 Coupling conditions on the odd patches

This subsection develops the coupling conditions on the odd patches. The values \( \bar{u}_{1,j,\pm 3} \) and \( \bar{u}_{2,j,\pm 3} \) at the edges of the odd patches are approximated by interpolation of \( U_{j,\pm 1}, U_{j,\pm 3}, \ldots \) of neighbouring patches, and by requiring that the microscale dynamics lie on the slow manifold.

The dynamics in the interior of each patch is given by the microscale simulator \((5.35)–(5.37)\). The regularising operator \( \mathcal{L} = 1 - C \delta_x(h^2 \delta_x) \) of the two layer mean velocities \( \bar{u}_1(x, t) \) and \( \bar{u}_2(x, t) \) on the odd \( j \)th patch requires more details. From Figure 5.8, for odd \( j \), the momentum equations \((5.36)–\)
Figure 5.9: Plots of the fluid thickness in the fluid film flow by the exampled linear two-layer model (5.40)–(5.42) on the domain $[0 \ 10\pi]$. There are $m = 10$ patches and $n = 9$ microscale grids on a patch. Figure 5.10 shows the fluid velocities.
Figure 5.10: Plots of the layer mean velocities $\bar{u}_1$ (blue stars) and $\bar{u}_2$ (red triangles) in the fluid film flow by the exampled linear two-layer model (5.40)–(5.42) on the domain $[0 10\pi]$. There are $m = 10$ patches and $n = 9$ microscale grids on a patch.
(5.37) are of the form

\[
\mathcal{L} \begin{bmatrix}
\partial \bar{u}_{\ell,1}^t / \partial t \\
\partial \bar{u}_{\ell,1}^t / \partial t \\
\partial \bar{u}_{\ell,1}^t / \partial t
\end{bmatrix} = \text{RHS},
\]

where the RHS refers to a discretisation of the right-hand sides of the governing equations (5.35)–(5.37), and the superscript \( \ell = 1, 2 \) for the lower and upper layer. The regularising operator \( \mathcal{L} \) has the matrix form of

\[
\mathcal{L} = \frac{1}{4d^2} \begin{bmatrix}
-Ch_{j,-2}^2 & 1 + C(h_{j,-2}^2 + h_{j,0}^2) & -Ch_{j,0}^2 \\
0 & 1 + C(h_{j,0}^2 + h_{j,2}^2) & -Ch_{j,2}^2
\end{bmatrix},
\]

where \( d \) is the microscale spatial step on a patch. The values of \( \bar{u}_{\ell,\pm 3}^t \) are at the edges of the \( j \)th patch. Recall that the macroscale mean velocity \( \bar{U}_j(t) = (\bar{u}_j^t(X_j, t) + \bar{u}_j^t(X_j, t))/2 \) for the even \( j \) are known by interpolation. Thus, this section completes the set of equations by finding the unknown microscale values \( \partial \bar{u}_{\ell,\pm 3}^t / \partial t \) and \( \bar{u}_{\ell,\pm 3}^t \) from the known macroscale values \( \bar{U}_j \).

The challenge is to deduce microscale values appropriate to the macroscale values. E & Engquist (2003), E et al. (2007) and Malecha et al. (2013) studied a heterogeneous multiscale method (HMM). The HMM contains two main components: an overall macroscale scheme for the macroscale variables and estimating the missing macroscale data by the microscale model. A compression operator \( Q \) and a reconstruction operator \( R \) are defined to satisfy \( Q\bar{u}_j = \bar{U}_j, RU_j = \bar{u}_j \) and \( QRU_j = \bar{U}_j \), where \( \bar{u}_j \) is the microscale variable and \( \bar{U}_j \) the macroscale variable. Such compression and reconstruction operators combine the microscale and macroscale variables. Kevrekidis et al. (2003), Samaey et al. (2005) and Samaey et al. (2009) defined a coarse time-stepper by introducing a lifting operator and a corresponding restriction operator which transform between the microscale and macroscale variables. These works provide the methods to relate the microscale and macroscale variables.

This section analogously constructs a lifting operator to give the values of \( \bar{u}_{\ell,\pm 3}^t \) and \( \partial \bar{u}_{\ell,\pm 3}^t / \partial t \) by the macroscale mean velocity \( \bar{U}_j \). Recall that the one layer mean velocity \( \bar{u}(X_j, t) = \bar{U}_j = (\bar{u}_{j,0}^1 + \bar{u}_{j,0}^2)/2 \). One constraint on the lifting is the coupling conditions (4.11) that gives a linear, cubic or quintic approximation for the one layer mean velocity, such as the quintic

\[
\bar{u}(X_j \pm rD, t) = \frac{1}{2} (\bar{U}_{j+1} + \bar{U}_{j-1}) \pm \frac{r}{2} (\bar{U}_{j+1} - \bar{U}_{j-1})
\]
\[ + \frac{1}{16} (-1 + r^2) (U_{j+2} - U_{j+1} - U_{j-1} + U_{j-2}) \]
\[ \pm \frac{1}{48} (-r + r^3) (U_{j+2} - 3U_{j+1} + 3U_{j-1} - U_{j-2}) . \]  
\[ (5.46) \]

The other requirement is that the patch be on the slow manifold of macroscale waves. The following section 5.5.3 details the slow manifold of the two-layer model (5.36)–(5.37) in term of the mean velocity \( \bar{u} \). Equations (5.56)–(5.60) in section 5.5.3 then give the two layer mean velocities \( \bar{u}_l \) in term of the mean velocity \( \bar{u} \). Thus, the macroscale values of \( \bar{u}_{j,+,3} \) and \( \partial \bar{u}_{j,+,3}/\partial t \) are approximated by the macroscale values \( U_j \) in the coupling condition (5.46). For example and for simplicity, truncate the slow manifold description (5.56)–(5.57) and (5.59)–(5.60) to errors \( O(\epsilon^2) \) and obtain the values of \( \bar{u}_{j,+,3} \) and \( \partial \bar{u}_{j,+,3}/\partial t \) on the odd \( j \)th patch as

\[
\bar{u}_{1,+,3} = 0.587 \bar{u}(x_j \pm rD, t) + 0.0129 \Re \tan \theta , \quad (5.47)
\]
\[
\bar{u}_{2,+,3} = 1.413 \bar{u}(x_j \pm rD, t) - 0.0129 \Re \tan \theta , \quad (5.48)
\]
\[
\frac{\partial \bar{u}_{1,+,3}}{\partial t} = -1.482 \frac{1}{\Re h_{j,+2}} \bar{u}(x_j \pm rD, t) + 0.489 \tan \theta , \quad (5.49)
\]
\[
\frac{\partial \bar{u}_{2,+,3}}{\partial t} = -3.526 \frac{1}{\Re h_{j,+2}} \bar{u}(x_j \pm rD, t) + 1.168 \tan \theta , \quad (5.50)
\]

where \( d \) is the microscale spatial step and we approximate the derivative \( \partial h/\partial x = 0 \) in equations (5.56)–(5.60) on the slow manifold. Thus, equations (5.47)–(5.48) are the coupling conditions on the odd patches.

### 5.5.2 Coupling conditions on the even patches

This subsection develops the coupling conditions on the patches with even \( j \). The values \( h_{j,+,3} \) at the edges of the even patches are approximated to give coupling conditions.

Simulate on each patch by the microscale simulator (5.35)–(5.37). The regularising operator \( L = 1 - C \partial_x (h^2 \partial_x) \) of the two layer mean velocities \( \bar{u}_1(x, t) \) and \( \bar{u}_2(x, t) \) on the even \( j \)th patch requires more details. For example, we record here details for \( n = 0, \pm 1, \pm 2, \pm 3 \) as in Figure 5.8; exactly same for larger \( n \). We need approximate the second spatial derivatives \( \partial_x^2 \) in the left-hand and right-hand sides of the (5.36)–(5.37) at the positions \( x_{j,+,2} \), which requires two virtual grid values \( \bar{u}_{1,+,4} \). We set the values \( \bar{u}_{1,+,4} = \bar{u}_{1,+,0} \) on the slow manifold, Thus, in Figure 5.8, for even \( j \), the momentum equations (5.36)–
(5.37) are of the form

$$\begin{bmatrix}
\frac{\partial \bar{u}_{\ell,-4}}{\partial t} \\
\frac{\partial \bar{u}_{\ell,-2}}{\partial t} \\
\frac{\partial \bar{u}_{\ell,0}}{\partial t} \\
\frac{\partial \bar{u}_{\ell,2}}{\partial t} \\
\frac{\partial \bar{u}_{\ell,4}}{\partial t}
\end{bmatrix} = \text{RHS},$$

(5.51)

where the regularising operator $\mathcal{L}$ is discretised in the matrix form

$$\begin{bmatrix}
-\text{Ch}_{j,-3}^2 & 1 + C(h_{j,-3}^2 + h_{j,-1}^2) & -\text{Ch}_{j,-1}^2 & 0 & 0 \\
0 & -\text{Ch}_{j,-1}^2 & 1 + C(h_{j,-1}^2 + h_{j,1}^2) & -\text{Ch}_{j,1}^2 & 0 \\
0 & 0 & -\text{Ch}_{j,1}^2 & 1 + C(h_{j,1}^2 + h_{j,3}^2) & -\text{Ch}_{j,3}^2
\end{bmatrix},$$

(5.52)

and where the RHS refers to a discretisation of the right-hand sides of the momentum equations (5.36)–(5.37), the superscript $\ell = 1, 2$ for the lower and upper layer, and $d$ is the microscale spatial step.

Hence, the regularising operator $\mathcal{L}$ needs the values of $h_{j,\pm3}$. The coupling conditions (4.11) in chapter 4 approximate the values of $h_{j,\pm3}$ by interpolating the neighbouring macroscale values of $H_{j,\pm1}, H_{j,\pm2}, \ldots$. The coupling conditions (4.11) gives a linear, cubic or quintic interpolation, such as the quintic

$$h_{j,\pm3} = h(X_j \pm rD, t) = \frac{1}{2}(H_{j+1} + H_{j-1}) \pm \frac{r}{2}(H_{j+1} - H_{j-1})$$

$$+ \frac{1}{16}(-1 + r^2)(H_{j+2} - H_{j+1} - H_{j-1} + H_{j-2})$$

$$\pm \frac{1}{48}(-r + r^3)(H_{j+2} - 2H_{j+1} + 3H_{j-1} - H_{j-2}),$$

(5.53)

where $r$ is the ratio of between the macroscale step and half of the width of a patch. Thus, equations (5.53), together with equations (5.47)–(5.48), couples the patches together in the macroscale domain. The patches in Figure 5.9–5.10 are coupled by these quintic approximations of the coupling conditions.
5.5.3 The low leading order model of one layer flow

This section derives a slow manifold of the two-layer model (5.36)–(5.37) in terms of the mean velocity \( \bar{u} \). This slow manifold is used by section 5.5.1–5.5.2 in order to lift the macroscale information to the microscale simulation on patches.

The slow manifold of the two layer model is exactly the same, to the order of analysis, as the one layer model (Roberts 1997, e.g.), that is, to the slow manifold of the original fluid dynamics. For simplicity, we base the analysis upon a symmetric linear operator with slow eigenspace where the two layer velocities are in the ratio 1:2. To derive the slow manifold of the two-layer model (5.36)–(5.37), we embed the physical model (5.36)–(5.37) into a family of artificial problems by using the artificial parameter \( \gamma' \).

\[
\begin{align*}
- \frac{\partial \bar{u}_1}{\partial t} + \text{RHS}1 + (1 - \gamma') \frac{E_u}{\Re \h^2} (-4\bar{u}_1 + 2\bar{u}_2) &= 0, \\
- \frac{\partial \bar{u}_2}{\partial t} + \text{RHS}2 + (1 - \gamma') \frac{E_u}{\Re \h^2} (2\bar{u}_1 - \bar{u}_2) &= 0,
\end{align*}
\]

where RHS1 and RHS2 are the right-hand sides of the evolutions (5.24)–(5.25). The variable \( E_u \) denotes the Euler parameter and changes the strength of the linear basis, which is \( E_u = 9/2 \) by experiments in order to get good convergence in \( \gamma' \). When the parameter \( \gamma' = 1 \), equations (5.54)–(5.55) recover the original equations (5.24)–(5.25). When the artificial parameter \( \gamma' = 0 \), the linear operator is

\[
\frac{E_u}{\h^2 \Re} \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}
\]

for updates in the system.

For a specific model I choose to truncate to the error \( \mathcal{O}(\bar{u}^4 + \partial^4 \bar{u} + \tan^4 \theta, \gamma^7) \). Then executing computer algebra in Appendix A.3 and evaluating at the artificial parameter \( \gamma' = 1 \), leads to the slow manifold

\[
\begin{align*}
\bar{u}_1 &\approx 0.587\bar{u} + 0.0129 \Re \h^2 \left( \tan \theta - \frac{\partial h}{\partial x} \right) \\
&- 0.0468\h^2 \frac{\partial^2 \bar{u}}{\partial x^2} - 0.205h \frac{\partial h \partial \bar{u}}{\partial x} + 0.0700h\bar{u} \frac{\partial^2 h}{\partial x^2} \\
&+ \Re \left( 0.00465h^2 \frac{\partial \bar{u}}{\partial x} - 0.0115h\bar{u} \frac{\partial h}{\partial x} - 0.0105h^3 \frac{\partial h \partial^2 h}{\partial x \partial x^2} \right), \\
\bar{u}_2 &\approx 1.413\bar{u} - 0.0129 \Re \h^2 \left( \tan \theta - \frac{\partial h}{\partial x} \right) \\
&+ 0.0468\h^2 \frac{\partial^2 \bar{u}}{\partial x^2} + 0.205h \frac{\partial h \partial \bar{u}}{\partial x} - 0.0700h\bar{u} \frac{\partial^2 h}{\partial x^2}
\end{align*}
\]
\[- \text{Re} \left( 0.00465 h^2 \frac{\partial \bar{u}}{\partial x} - 0.0115 h \bar{u}^2 \frac{\partial h}{\partial x} - 0.0105 h^3 \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} \right) \right] . \quad (5.57)\]

Equations (5.56)–(5.58) are expressed in terms of the depth-averaged velocity \( \bar{u}(x, t) \).

On the slow manifold, the conservation of mass equation is
\[
\frac{\partial h}{\partial t} = - \frac{\partial h \bar{u}}{\partial x} . \quad (5.58)
\]

The computer algebra in Appendix A.3 differentiates the equations (5.56) and (5.57) with respect to the time \( t \), and gives the rate of change of the two layer velocities in term of the mean velocity \( \bar{u} \) and fluid depth \( h \)
\[
\frac{\partial \bar{u}_1}{\partial t} \approx 0.489 \left( \tan \theta - \frac{\partial h}{\partial x} \right) - 1.482 \frac{1}{\text{Re} h^2} \bar{u} - 0.904 \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{1}{\text{Re}} \left( 0.0167 h^3 \frac{\partial \bar{u}}{\partial x^2} + 0.0438 h^2 \frac{\partial h^2}{\partial x \partial x^2} + 0.0359 h \bar{u} \left( \frac{\partial h}{\partial x} \right)^2 \right)
- \text{Re} \tan \theta \left( 0.0298 h \bar{u} \frac{\partial h}{\partial x} + 0.0184 h^2 \frac{\partial \bar{u}}{\partial x} \right) , \quad (5.59)
\]
\[
\frac{\partial \bar{u}_2}{\partial t} \approx 1.168 \left( \tan \theta - \frac{\partial h}{\partial x} \right) - 3.526 \frac{1}{\text{Re} h^2} \bar{u} - 2.107 \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{1}{\text{Re}} \left( 0.00819 h^3 \frac{\partial \bar{u}}{\partial x^2} + 0.0482 h^2 \frac{\partial h^2}{\partial x \partial x^2} + 0.0875 h \bar{u} \left( \frac{\partial h}{\partial x} \right)^2 \right)
+ \text{Re} \tan \theta \left( 0.0847 h \bar{u} \frac{\partial h}{\partial x} + 0.0355 h^2 \frac{\partial \bar{u}}{\partial x} \right) . \quad (5.60)
\]

Equations (5.56)–(5.60) provide microscale information from the macroscale fields \( \bar{u} \) and \( h \).

The computer algebra in Appendix A.3 also leads to the evolution on the slow manifold of the one-layer model; the momentum equation is
\[
\frac{\partial \bar{u}}{\partial t} \approx 0.829 \left( \tan \theta - \frac{\partial h}{\partial x} \right) - 2.504 \frac{1}{\text{Re} h^2} \bar{u} - 1.505 \bar{u} \frac{\partial \bar{u}}{\partial x} - 0.151 \bar{u}^2 \frac{\partial h}{\partial x} . \quad (5.61)
\]
Figure 5.11: Plots of the growth rate $\Re \lambda$ versus the frequency $\Im \lambda$ in the gap-tooth simulation of the thin film flow. There are $m = 10$ patches and $n = 9$ microscale grids on a patch. The Reynolds number $Re = 15$ and the coefficient in the regularising operator $L$ is $C = 0.5$.

Equation (5.61) has the same terms with the low order equation (11) of the thin film flow model by Roberts (1997). The coefficients of these terms are the same to a relative error of less than 2.2%. This agreement partially verifies that the two-layer model is a reasonable model of the fluid film flow.

5.5.4 Numerical patches simulations of the two layer thin film flow

This section explores the numerical gap-tooth simulation of the thin fluid flow. Figure 5.9–5.10 show the example of the gap-tooth simulation with the straightforward linear version of the two-layer model (5.35)–(5.37). This section explores the numerical gap-tooth simulation with the microscale simulator (5.35)–(5.37) and the coupling conditions (5.53) and (5.46)–(5.50). Numerical eigenvalues and simulations show that the gap-tooth scheme works for the fluid film flow with the two-layer model (5.35)–(5.37) being the microscale simulator.

Consider the thin fluid flow on a horizontal plate: that is, the mean slope $\tan \theta = 0$. Distribute $m$ patches in the macroscale domain of length $L = m\pi$, so the distance between the neighbouring patches is $D = L/m = \pi$. On each
patch, divide the patch into \( n + 1 \) equal microscale intervals by \( n + 2 \) grid points, then the distance between neighbouring microscale points is \( d = 2rD/(n + 1) \), where \( r \) is the ratio between half of the width of each patch and the macroscale inter-patch distance \( D \).

Approximate the spatial derivatives in the right-hand side of the two-layer model (5.35)–(5.37) on the staggered microscale grids. Time integration invokes the Matlab \texttt{ode15s} function.

Figure 5.11 plots the growth rate \( \Re \lambda \) versus the frequency \( \Im \lambda \) in the gap-tooth simulation of the thin fluid flow. There are \( m = 10 \) patches and \( n = 9 \) microscale grids on a patch. The plate has a length \( L = m\pi \approx 31.4 \), so the distance between the neighbouring patches is \( D = L/m = 3.14 \). The ratio \( r = 1/6 \), so the microscale step on a patch is \( \delta x = 2rD/(n + 1) \approx 0.17 \). The Reynolds number \( \text{Re} = 15 \) and the coefficient in the regularising operator \( L \) is \( C = 0.5 \). The negative growth rates imply the waves decay in time.

There are 90 pairs of eigenvalues for the fluid depth \( h \) (twice) and the layer mean velocities \( \bar{u}_1 \) and \( \bar{u}_2 \):

- the values with large decay rates \( \Re \lambda < -1 \) and zero imaginary parts are for the shear modes;
- the values with large decay rates \( \Re \lambda < -1 \) but with large imaginary
Figure 5.13: Plots of the fluid thickness in a gap-tooth simulation of the thin fluid flow on the domain $[0 \ 10\pi]$. There are $m = 10$ patches and $n = 9$ microscale grids on a patch. The Reynolds number $Re = 15$ and the regularising coefficient $C = 0.5$. Figure 5.14 shows the corresponding fluid velocities.
parts ($|\lambda| > 2$) are for microscale modes, which indicate fast oscillations are generated on a patch; and

- the values with small growth rates ($|\Re \lambda| < 0.2$) in the red rectangle, zoomed in by Figure 5.12, are for macroscale waves, which include four group values with non-zero imaginary parts and another two real values $\Re \lambda = 0$ and $\Re \lambda \approx -0.16$.

Figure 5.13 plots the free surface of the thin film flow in a gap-tooth simulation. There are $m = 10$ patches and $n = 9$ microscale grids on a patch. The $t = 0$ graph shows that initially impose a perturbation $0.2 \sin(2\pi/Lx)$ with small random noises to the equilibrium of fluid thickness one and initial layer mean velocities $\bar{u}_1 = 0$ and $\bar{u}_2 = 0.2$, so the mean layer velocity $\bar{u} = (\bar{u}_1 + \bar{u}_2)/2 = 0.1$. The $t = 2$ to $t = 10$ graphs show that the microscale modes on a patch smooth quickly and the macroscale waves over the whole domain slowly decay, which agrees with the large decay rates of the microscale values and small decay rates of the macroscale modes in Figure 5.11. Further numerical simulation shows that the macroscale waves decay to the equilibrium of thickness one after time $t \approx 30$ for the Reynolds number Re = 15. When the Reynolds number Re becomes smaller, the waves over the whole domain decay faster to the equilibrium due to the stronger viscous effects.

Figure 5.14 plots the corresponded layer mean velocities $\bar{u}_1$ (blue stars) and $\bar{u}_2$ (red triangles) in the gap-tooth simulation. There are $m = 10$ patches and $n = 9$ microscale grids on a patch. The four graphs show that the two layer mean velocities decay slowly, and will decay to the equilibrium of layer mean velocities $\bar{u}_1 = \bar{u}_2 = 0$ after a long time, $t \approx 30$ for Reynolds number Re = 15 for example.

5.6 Conclusion

This chapter applies the gap-tooth scheme to the thin fluid flow with a derived two-layer model (5.35)–(5.37) being the microscale simulator. The two-layer model (5.35)–(5.37) provides more microscale modes, but without the full complexity of fully resolved vertical structures.

Based on centre manifold theory, section 5.1–5.3 derive a two-layer model (5.35)–(5.37) from the continuity and modified Navier–Stokes equations. The model (5.35)–(5.37) includes the effects of gravity, drag, advection and dispersion. The slow manifold of the two-layer model agrees with the one-layer model of thin film flow by Roberts (1997). Eigenvalues analysis in section 5.4 indicates unphysical instability appearing. We introduce a regularising operator $L$ to stabilise the model. Numerical eigenvalues and
Figure 5.14: Plots of the layer mean velocities $\bar{u}_1$ (blue stars) and $\bar{u}_2$ (red triangles) in the gap-tooth simulation of the fluid film flow on the domain $[0, 10\pi]$. There are $m = 10$ patches and $n = 9$ microscale grids on a patch.
simulations in Figure 5.5–5.7 of thin film flow over horizontal plate indicate the two-layer model (5.35)–(5.37) is reliable.

Then, we simulate the thin film flow by the gap-tooth scheme with the two-layer model (5.35)–(5.37) being the microscale simulator. Section 5.5.3 algebraically derives the slow manifold of the two-layer model (5.35)–(5.37) in order to lift the macorscale information to the microscale simulation on patches. Then coupling conditions (5.53) and (5.47)–(5.48) are developed to couple the patches. Numerical eigenvalues in Figure 5.11–5.12 show that the macroscale modes decay slowly over the whole domain, while the microscale modes oscillate fast and decay quickly to quasi-equilibrium quickly. Non-zero frequencies $\Im \lambda$ in Figure 5.12 indicate waves are supported on the free surface over the whole domain. Numerical simulations in Figure 5.13–5.14 show that the gap-tooth scheme with the two-layer model (5.35)–(5.37) being microscale simulator work well enough for the thin film flow.

Future work could extend the two-layer model (5.35)–(5.37) to multi-layer models by assuming there are artificial multi-layers in the fluid. Then the underlying system becomes a direct numerical simulation of the fluid flow. Another future work would increase the Reynolds number to simulate turbulence in the patches.
Chapter 6

Conclusion

This dissertation achieves two aims: first, chapter 2–3 using centre manifold theory derives models for large scale simulations of turbulent environmental flows; and second chapter 4–5 develops a gap-tooth scheme for wave-like system to reduce the expense of numerical simulations. Section 6.1 summarises the results of the first aim and section 6.2 concludes the results of the second aim. Then section 6.3 suggests possible further directions based on the work in this dissertation.

6.1 Summary of the turbulent flows and sediment transport

The first aim of the dissertation is to derive new models to more appropriately approach environmental turbulent fluids and sediment transports.

Chapter 2 explores the implications of changing the theoretical base from depth-averaging to a slow manifold of the turbulent Smagorinski large eddy closure. Based on centre manifold theory, chapter 2 derives the new turbulence model (2.1)–(2.3) to describe the dynamics of large scale turbulent flows. The model (2.1)–(2.3) includes all the effective terms in established modelling (Rodi 1984, Yulistiyanto et al. 1998, Bousmar 2002, e.g.). For example, Bousmar (2002) reported the depth-averaged Saint–Venant equation (2.35). The model (2.1)–(2.3) include all the terms in equation (2.35), but with different coefficients. These different coefficients are due to the different vertical shape of the leading order velocity profile. However, our model (2.1)–(2.3) ensures more subtle effects, such as the enhanced turbulent dissipation term $0.084(\bar{u}^2 - \bar{v}^2)/h\bar{q}\delta^2\bar{u}/\delta x^2$ that has not previously been resolved. Section 2.4.5 explores the vertical distribution of the lateral velocity and shear stress. Our simulation corresponds to experimental data by Schultz
Eigenvalues analysis in section 2.5 shows that our model is stable. Section 2.7 applies the model (2.1)–(2.3) to channel flows. We find that fast flow developed in the main channel and slow flow in shallow regions, as shown in Figure 2.13 and 2.20. Weak travelling vortices appear at the interfaces of the main channel and shallows in Figure 2.14 and 2.21. The depth-averaged lateral velocity \( \bar{u} \) reaches maximum at the bends (Figure 2.16), while the depth-averaged lateral velocity \( \bar{v} \) attains maximum and minimum at the connection of the bends (Figure 2.17). The maximum velocity shifts from the inner bank to the outer bank when the flow through the bend (Figure 2.18). These findings agree with some published findings (Bousmar 2002, Roberts & Li 2006, Demuren 1993, Liu et al. 2009, e.g.).

Chapter 3 adapts the turbulent modelling and dynamics to include the transport and interaction with suspended sediment. Based on centre manifold theory, chapter 3 derives a new suspended sediment model (3.1). This model includes the effects of sediment erosion, advection and dispersion. Compared with established model (3.13) by Duan (2004), our model includes all the effective terms and then ensures more subtle effects. For example, the modifications in presence of the ratio \( w_f/\bar{q} \) in advection terms enhanced the advection effect. Section 3.5 describes the vertical distribution of the suspended sediment. The vertical distribution of suspended sediment in steady flow approximately agrees with the analytical analysis, shown in Figure 3.5.

Section 3.6 applies the new model (3.1), coupled with the modified momentum equations (3.14)–(3.15), to simulate sediment in the channel flows in section 3.6.2 and in large waves in section 3.6.3. In section 3.6.2, we find that the depth-averaged horizontal concentration is independent of the \( x \) coordinates, and fast flow in the channel generates high concentration, as shown in Figure 3.9 and 3.12. The maximum of the depth-averaged concentration shifts from the outer bank to the inner bank when the flow through the channel bend, shown in Figure 3.13. The findings of sediment in channel flows agree with some published finding (Lin & Falconer 1996, Demuren & Rodi 1986, Ye & McCorquodale 1997, Duan 2004, e.g.). In section 3.6.3, we explore the suspended sediment in roll waves. We find that the depth-averaged concentration \( \bar{c}(x,y,t) \) are approximately in phase with the depth-averaged lateral velocity \( \bar{u}(x,y,t) \), but slightly lags to reach peaks. The two peaks of the depth-averaged concentration in one period are possibly due to the turbulent mixing and flow reversal. In one period of waves, the depth-averaged concentration \( \bar{c} \) is smaller over the rippled plane than over the flat plane, which is different to the findings Os-
borne & Vincent (1996), who found steep asymmetric ripples under shoaling waves produce greater concentrations higher in the water column than low steepness ripples. The possible reason is that we only involve simulations over a small spacial domain.

### 6.2 Summary of the gap-tooth simulations

The second aim of the dissertation is to develop the gap-tooth scheme to reduce the expense of large scale simulations of wave-like systems.

Chapter 4 invokes a staggered grid to discretely model the microscale and another staggered grid for the macroscale simulators of wave-like dynamics. The coupling conditions (4.11) are developed in the gap-tooth scheme in section 4.3. Algebraic analysis and numerical determination of eigenvalues both confirm the accuracy of the proposed gap-tooth scheme for wave-like dynamics in section 4.4. In particular, Figure 4.7 shows a clear separation between the dynamics of the macroscale waves of interest, and the microscale waves within each patch. As a pilot study of nonlinear problem, section 4.6 applies the gap-tooth scheme to explore dam-breaking waves. Figure 4.12–4.15 show the comparison among the gap-tooth simulation, the microscale simulation over the whole domain and the published experimental data (Stansby et al. 1998, e.g.). The bore predicted by the gap-tooth simulation is smaller than both the experimental data and the microscale simulation over the whole domain. The possible reason is that on a patch in the gap-tooth simulation near the bore involves the error $O(D^2) \sim 0.8$ for the macroscale step $D = L/m = 20/22 \approx 0.9$, while the microscale simulation over the whole macroscale domain involves the error $O(d^2) \sim 0.001$ for the microscale step $d = 2rD/(n+1) = 0.03$. Overall, the gap-tooth scheme is feasible to compute large scale wave-like dynamics.

Chapter 5 then attempts to use the gap-tooth scheme to explore viscous flow of a layer of fluid. This chapter develops a new two-layer model (5.35)–(5.37) for a viscous layer of fluid, which has more microscale modes than classic one-layer models, but without the full complexity of fully resolved vertical structures. Eigenvalue analysis (Figure 5.2) indicates that an unphysical instability appears for high wavenumber. In the two-layer model (5.35)–(5.37), we introduce a regularising operator to stabilise the model (Figure 5.4).

The gap-tooth scheme is used to model the viscous layer of fluid with the microscale simulator of the developed two-layer model (5.35)–(5.37). To create microscale details for each patch appropriate to the macroscale information, computer algebra leads to the classic one-layer model (5.56)–(5.60) from the developed two-layer model (5.35)–(5.37). Thus, the coupling conditions (5.53) and (5.47)–(5.50) couple the small patches. Numerical simulations of the
viscous layer of fluid are successfully implemented by such gap-tooth scheme. Results (Figure 5.11–5.14) show that the developed gap-tooth scheme is feasible to model the viscous layer of fluid.

6.3 Future directions

This section mentions the possible future directions based on the work in this dissertation.

Chapter 2 derived the model (2.1)–(2.3) in a Cartesian coordinate system. One future work could improve the model (2.1)–(2.3) to general curvilinear coordinates as special cases, including spherical coordinates, and rotating frame of reference to study fluid flowing over specific topographies on a rotating planet. For example, Roberts & Li (2006), based on centre manifold theory, derived a model for viscous flow in a curvilinear coordinate system.

The model (2.1)–(2.3) includes subtle physical terms. Georgiev et al. (2009) simulated dam-breaking waves by the 1D version of the model (2.1)–(2.3), and numerically specified these subtle terms. Thus, future work is to explore the influence of these subtle terms in the application to complex flows.

The sediment model (3.1) in chapter 3 should be improved. One direction is to include bed load on the mean bed (Wu et al. 2000, e.g.), and then explore the total sediment transport in the flow. In the model (3.1) we consider that the mean particle size $d$ is small and then the falling velocity $w_f$ is small. One future direction is to explore the suspended sediment with larger mean particle size and falling velocity (Yang & Shy 2003, e.g.). Future work could be to apply the proposed suspended model to simulate suspended sediment in complex physical waves, such as in dam-breaking waves, river floods and beach waves. Another practiced issue is how to incorporate boundary conditions on the flow and sediment at the waters edge on a beach or river.

In chapter 4–5, we develop the gap-tooth scheme for large scale wave-like systems. The application of the gap-scheme on dam-breaking waves indicates the validity of this scheme. Therefore, future work could apply this scheme to complex large scale physical waves, including river floods, tides and tsunamis. The new two-layer model (5.26)–(5.28) could be extended to multi-layer models by assuming there are artificial multi-layers in the fluid flow. Then the underlying system becomes a direct numerical simulation of the fluid flow. Further research would increase the Reynolds number to simulate turbulence in the patches.
Appendix A

Reduce programs

This appendix lists the computer algebra programs that constructed the slow manifold models. I used the free Reduce package\(^1\). Section A.1 describes code that derived the turbulence model in chapter 2 and the suspended sediment model in chapter 3. The code of section A.2 derived the high order characteristic equation (4.19) for the coupling conditions in chapter 4. Section A.3 lists the code for the two-layer model of the viscous layer of fluid used in chapter 5.

A.1 Computer algebra models the turbulent flow and sediment transport

This section lists the computer algebra to construct the slow manifold model of the turbulent flow system in chapter 2 and of the sediment transport model in chapter 3.

A.1.1 Explanation of symbols

Denote the fluid depth \( h(x, y, t) \) by \( h \), depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \) by \( uu \) and \( vv \), depth-averaged suspended sediment concentration \( \bar{c}(x, y, t) \) by \( cc \), and their time derivatives \( \eta_t = gh, \bar{u}_t = gu, \bar{v}_t = gv \) and \( \bar{c}_t = gc \). Denote the mean bed \( b(x, y) \) by \( b \). The coefficients of lateral and normal gravitational forcing are represented by \( grx, gry \) and \( grz := 1 \). Use \( qq \) represent the mean flow speed \( \bar{q} = \sqrt{\bar{u}^2 + \bar{v}^2} \) and \( rqq \) for the reciprocal of this mean speed.

\(^1\)http://www.reduce-algebra.com
A.1.2 Definition of operators

Use the operator $h(m,n)$ to denote lateral derivatives of the fluid depth, $\partial_x^m \partial_y^n h$. Similarly use the operators $uu(m,n)$, $vv(m,n)$, $cc(m,n)$ to denote lateral derivatives of the depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$, and the depth-averaged concentration $\bar{c}(x,y,t)$. These operators depend upon time and lateral space. Then the lateral derivative $\partial_x h(m,n) = \partial_x^m \partial_y^n \bar{c}$, and the time derivative $\partial_t h(m,n) = \partial_x^m \partial_y^n \bar{c}$, for example. Define readable abbreviations for $h(x,y,t)$ and its first spatial derivatives. We use $d$ to count the number of lateral derivatives so we can easily truncate the asymptotic expansion.

1 % Computer algebra constructs the slow manifold in the
turbulent fluid system and the suspended sediment system.
linelength 60;
on div; off allfac; on revpri;
5 % define operators
factor vv,uu,cc,qq,rqq,h,ct,gx,gz,gam,r2,b;
operator h; operator b; operator uu;
operator vv; operator cc;
9 % these operators depend upon time and space
10 hx:=h(1,0)*d$ hy:=h(0,1)*d$
11 depend h,xx,yy,tt;
12 depend uu,xx,yy,tt;
13 depend vv,xx,yy,tt;
14 depend cc,xx,yy,tt;
15 depend b,xx,yy;
16 let {  
  df(h,xx)=>h(1,0), df(h(~m,~n),xx)=>h(m+1,n)  
  , df(h,yy)=>h(0,1), df(h(~m,~n),yy)=>h(m,n+1)  
  , df(h,tt)=>gh, df(h(~m,~n),tt)=>df(gh,xx,m,yy,n)  
  , df(uu(~m,~n),xx)=>uu(m+1,n)  
  , df(uu(~m,~n),yy)=>uu(m,n+1)  
  , df(uu(~m,~n),tt)=>df(gu,xx,m,yy,n)  
  , df(vv(~m,~n),xx)=>vv(m+1,n)  
  , df(vv(~m,~n),yy)=>vv(m,n+1)  
  , df(vv(~m,~n),tt)=>df(gv,xx,m,yy,n)  
  , df(cc(~m,~n),xx)=>cc(m+1,n)  
  , df(cc(~m,~n),tt)=>df(gc,xx,m,yy,n)  
  , df(b,xx)=>b(1,0), df(b(~m,~n),xx)=>b(m+1,n)  
  , df(b,yy)=>b(0,1), df(b(~m,~n),yy)=>b(m,n+1)  
};
Use stretched coordinates $zz$, $xx$, $yy$ and $tt$ to denote $Z = (z - b)/h$, $X = x$, $Y = y$ and $T = t$ in (2.4). The free surface is then simply $Z = 1$. These structured coordinates also enable easy counting of lateral derivatives.

31 \% stretch the coordinates with the fluid depth
32 depend $xx,x,y,z,t$;
33 depend $yy,x,y,z,t$;
34 depend $zz,x,y,z,t$;
35 depend $tt,x,y,z,t$;
36 let{ $df(\sim a,x) = df(a,xx)*d-zz*hx/h*df(a,zz)$
37     -d*df(b,xx)/h*df(a,zz)$
38     , $df(\sim a,y) = df(a,yy)*d-zz*hy/h*df(a,zz)$
39     -d*df(b,yy)/h*df(a,zz)$
40     , $df(\sim a,t) = df(a,tt) - zz*gh/h*df(a,zz)$
41     , $df(\sim a,z) = df(a,zz)/h$};

Define the operators for the mean flow speed $\bar{q}$ and its reciprocal. The last
simplification rule for $rqq$ breaks the symbolic symmetric between the two
lateral directions. However the benefit of canonical representation outweighs
the cost of loss of symbolic symmetry.

43 \% define the mean speed
44 depend $qq,uu(0,0),vv(0,0)$;
45 let{ $qq^2 = uu(0,0)^2+vv(0,0)^2$
46     , $df(qq,\sim aa) = (uu(0,0)*df(uu(0,0),aa)$
47     +vv(0,0)*df(vv(0,0),aa))*rqq$}
48 $qq^2 = uu(0,0)^2+vv(0,0)^2$
49 $qq^2 = uu(0,0)^2+vv(0,0)^2$
50 $qq^2 = uu(0,0)^2+vv(0,0)^2+
51 \% operators quickly solve the integrations
52 \% When wf is SMALL,
53 \% define operator csolv to solve $d^c/dz^2=rhs$.
54 operator csolv; linear csolv;
let { csolv(zz^~~n,zz) => (zz^(n+2)-1
+ (1-zz)*2*(n+2)/(n+3))/(n+2)/(n+1)
, csolv(1,zz) => (zz^2-1 +(1-zz)*4/3)/2 }

operator wsolv; linear wsolv;
let { wsolv(zz^~~n,zz) => zz^(n+1)/(n+1)
, wsolv(1,zz) => zz }
operator psolv; linear psolv;
let { psolv(zz^~~n,zz) => (1-zz^(n+1))/(n+1)
, psolv(1,zz) => (1-zz) }
operator mean; linear mean;
let { mean(zz^~~n,zz) => 1/(n+1)
, mean(1,zz) => 1 }
operator usolv; linear usolv;
let { usolv(zz^~~n,zz) => (zz^(n+2)
-(cu+zz)/(n+3)/(cu+1/2))/(n+2)/(n+1)
, usolv(1,zz) => (zz^2 -(cu+zz)/3/(cu+1/2))/2 }

For each equation, write out the number of terms in its residual throughout
iteration.

procedure mylength(res);
begin
return if res=0 then 0 else length(res);
end;

A.1.3 Initial approximation

Start the iteration from the linear solution (2.28) that the turbulent mean
velocities are the linear \( u = \bar{u}(c_u + Z)/(c_u + \frac{1}{2}) \) and
\( v = \bar{v}(c_u + Z)/(c_u + \frac{1}{2}) \),
the turbulent mean concentration \( c = 2\bar{c}(1 - Z) \), the turbulent pressure
\( p = h(1 - Z) \), and all other fields are zero, \( w = 0 \). The parameter \( c_u \)
determines the turbulent slip on the bed, and \( c_u \approx 1.85 \) matches open
channel flow observations (Roberts et al. 2008, e.g.). The initial evolutions
of these variables are also zero, \( \bar{u}_t = gu = 0, \bar{v}_t = gv = 0, \bar{c}_t = gc = 0 \) and
\( \eta_t = gh = 0 \).

% initial approximation
let r2^2=>2; \% r2=sqrt2
u:=uu(0,0)*(cu+zz)/(cu+1/2);
v:=vv(0,0)*(cu+zz)/(cu+1/2);
p:=grz*(1-zz)*h;
c:=cc(0,0)*2*(1-zz);
w:=gh:=gu:=gv:=gc:=0;
Set the initial strains from the linear solution (2.28), initially approximate the magnitude of the strain-rate tensor in equation (2.10), and then set the initial stresses.

\[ \varepsilon_{xx} := \text{df}(u, x); \]
\[ \varepsilon_{yy} := \text{df}(v, y); \]
\[ \varepsilon_{zz} := \text{df}(w, z); \]
\[ \varepsilon_{xz} := (\text{df}(u, z) + \text{df}(w, x))/2; \]
\[ \varepsilon_{xy} := (\text{df}(u, y) + \text{df}(v, x))/2; \]
\[ \varepsilon_{yz} := (\text{df}(v, z) + \text{df}(w, y))/2; \]
\[ \rho_s := \frac{qq*r2/h}{(1+2*cu)}; \]
\[ \tau_{xx} := 2*ct*h^2*\rho_s*\varepsilon_{xx}; \]
\[ \tau_{yy} := 2*ct*h^2*\rho_s*\varepsilon_{yy}; \]
\[ \tau_{zz} := 2*ct*h^2*\rho_s*\varepsilon_{zz}; \]
\[ \tau_{xz} := 2*ct*h^2*\rho_s*\varepsilon_{xz}; \]
\[ \tau_{xy} := 2*ct*h^2*\rho_s*\varepsilon_{xy}; \]
\[ \tau_{yz} := 2*ct*h^2*\rho_s*\varepsilon_{yz}; \]

### A.1.4 Truncation of the asymptotic expansion

The small parameters include the parameter \( d \) counting the number of small lateral derivatives \( \partial_x \) and \( \partial_y \), the lateral gravity \( \tan \theta \), the artificial parameters \( \gamma \) for the turbulent fluid and \( \gamma_c \) for the suspended sediment, and the falling velocity \( w_f \) of the suspended sediment. We use a dummy parameter \( \epsilon \) count the small parameters in a term.

\[ \% \text{truncate orders} \]
\[ \text{ct} := 1/50; \]
\[ \text{cu} := 11/6; \]
\[ \text{d} := \epsilon^2; \]
\[ \text{grz} := 1; \]
\[ \text{grx} := \epsilon^2*gx; \]
\[ \text{gry} := 0; \]
\[ \text{wf, wff}; \]
\[ \text{wf} := \epsilon*wff; \]
\[ \text{factor} \epsilon; \]

This algebra is for deriving slow manifold model of the turbulent flows and sediment transport. The nondimensional effective density are different in the turbulent flow system and in the sediment system. Thus we have the following two cases:
• in the turbulent fluid system, we nondimensionalize the fluid density $\rho = 1$, and then truncate to a relative low error $O(\partial_x^{3/2} + \partial_y^{3/2} + gx^{3/2}, \gamma^3)$;

• in the sediment system, we nondimensionalize the mixing density of the fluid and sediment $\rho = 1 - \epsilon_c (s - 1)$, where $s$ is the relative density, and then truncate to a relative low error $O(\partial_x^{3/2} + \partial_y^{3/2} + gx^{3/2} + \omega^3_f + \gamma^3, \gamma^6)$.

% if 1 for the turbulent flow system;
% if 0 for the sediment system.
% the density of the fluid
if 1 then begin
exc:=1;
let { eps^3=>0};
end else begin
exc:=1-eps*c*(s-1);
let { eps^3=>0, gamc^6=>0};
end;

A.1.5 The iterative loop

Invoke the iterative loop.

for iter:=1:19 do begin ok:=1;
write "ITERATION ",iter;

Update the vertical turbulent mean velocity $w$ with continuity (2.7) and no flow through bed (2.14).

% solve continuity
resc:=df(u,x)+df(v,y)+df(w,z);
resa:=sub(zz=0,w-u*df(b,x)-v*df(b,y));
write length_resc:=mylength(resc);
write length_resa:=mylength(resa);
ok:=if {resc,resa}={0,0} then ok else 0;
w:=w+(dw:==-h*wsolv(resc,zz))-resa;
ezz:=ezz+df(dw,zz)/h;
tzz:=tzz+2*r2*ct/(1+2*cu)*qq*df(dw,zz);

The kinematic condition (2.21) on the free surface gives the evolution of the fluid depth $h(x,y,t)$.

% update thickness evolution
gh:=sub(zz=1, w-u*hx-v*hy-u*df(b,x)-v*df(b,y));
Update pressure from vertical momentum (2.8) and surface normal stresses (2.22).

\[ \text{resw} := df(w, t) + u \cdot df(w, x) + v \cdot df(w, y) + w \cdot df(w, z) + exc \cdot df(p, z) \]
\[ + grz + exc \cdot (-df(txz, x) - df(txy, y) - df(tzz, z)) \]

\[ \text{restn} := \text{sub}(zz=1, -p \cdot (1 + (hx + df(b, x))^2 + (hy + df(b, y))^2) \]
\[ + tzz - 2 \cdot (hx + df(b, x)) \cdot txz - 2 \cdot (hy + df(b, y)) \cdot tyz \]
\[ + (hx + df(b, y))^2 \cdot txx + 2 \cdot (hx + df(b, x)) \]
\[ \times (hy + df(b, y))^2 \times tyy) \];

\[ \text{length\_resw} := \text{mylength}(\text{resw}); \]
\[ \text{length\_restn} := \text{mylength}(\text{restn}); \]
\[ \text{ok} := \text{if} \{\text{resw, restn}\} = \{0, 0\} \text{ then ok else 0}; \]

% update the pressure
\[ p := p + h \cdot \text{psolv}(\text{resw}, zz) + \text{restn}; \]

Update the Smagorinski large eddy stress-shear closure (2.12).

% Smagorinski large eddy closure
\[ \text{exx} := df(u, x); \]
\[ \text{eyy} := df(v, y); \]
\[ \text{ezz} := df(w, z); \]
\[ \text{exz} := (df(u, z) + df(w, x))/2; \]
\[ \text{exy} := (df(u, y) + df(v, x))/2; \]
\[ \text{eyz} := (df(v, z) + df(w, y))/2; \]
\[ \text{rese} := \text{exx}^2 + \text{ezz}^2 + \text{eyy}^2 + 2 \cdot \text{exz}^2 + 2 \cdot \text{exy}^2 + 2 \cdot \text{eyz}^2 - \text{ros}^2; \]
\[ \text{length\_rese} := \text{mylength}(\text{rese}); \]
\[ \text{ok} := \text{if} \ \text{rese} = 0 \ \text{then ok else 0}; \]
\[ \text{ros} := \text{ros} + \text{rese} \cdot h \cdot (cu+1/2)/r2*rqq; \]
\[ \text{txx} := 2 \cdot ctt \cdot h^2 \cdot \text{ros} \cdot \text{exx}; \]
\[ \text{tyy} := 2 \cdot ctt \cdot h^2 \cdot \text{ros} \cdot \text{eyy}; \]
\[ \text{tzz} := 2 \cdot ctt \cdot h^2 \cdot \text{ros} \cdot \text{ezz}; \]
\[ \text{txz} := 2 \cdot ctt \cdot h^2 \cdot \text{ros} \cdot \text{exz}; \]
\[ \text{txy} := 2 \cdot ctt \cdot h^2 \cdot \text{ros} \cdot \text{exy}; \]
\[ \text{tyz} := 2 \cdot ctt \cdot h^2 \cdot \text{ros} \cdot \text{eyz}; \]

Update the turbulent mean velocities \( u \) and \( v \), and their evolutions from the lateral momentum equation (2.8), the slip boundary conditions (2.19)–(2.20) on the mean bed, and the surface tangential stresses (2.22)–(2.24).

% solve lateral momentum
\[ \text{resu} := df(u, t) + u \cdot df(u, x) + v \cdot df(u, y) + w \cdot df(u, z) + exc \cdot df(p, x) \]
\[ - grx + exc \cdot (-df(txx, x) - df(txy, y) - df(txz, z)); \]
\[ \text{resv} := df(v, t) + u \cdot df(v, x) + v \cdot df(v, y) + w \cdot df(v, z) + exc \cdot df(p, y) \]
\[ - gry + exc \cdot (-df(tyy, y) - df(txy, x) - df(tyz, z)); \]
resbu:=sub(zz=0,(-u-w*df(b,x))*(1-df(b,x)^2/2)
+cu*h*(1-df(b,x)^2/2-df(b,y)^2/2)*(
    -df(u+w*df(b,x),x)*df(b,x)
    -df(u+w*df(b,x),y)*df(b,y)
    +df(u+w*df(b,x),z)));
resbv:=sub(zz=0,(-v-w*df(b,y))*(1-df(b,y)^2/2)
+cu*h*(1-df(b,x)^2/2-df(b,y)^2/2)*(
    -df(v+w*df(b,y),x)*df(b,x)
    -df(v+w*df(b,y),y)*df(b,y)
    +df(v+w*df(b,y),z)));
write length_resu:=mylength(resu);
write length_resv:=mylength(resv);
write length_resbu:=mylength(resbu);
write length_resbv:=mylength(resbv);
ok:=if {resu,resv,resbu,resbv}={0,0,0,0} then ok else 0;
% check mean velocities
resuamp:=mean(u,zz)-uu(0,0);
resvamp:=mean(v,zz)-vv(0,0);
write length_resuamp:=mylength(resuamp);
write length_resvamp:=mylength(resvamp);
ok:=if {resuamp,resvamp}={0,0} then ok else 0;
% update lateral mean velocities
u:=u+(du:=resbu*(1-2*zz)/(1+2*cu)
+h*(1+2*cu)*r2/(4*ct)*rqq^3*usolv(
    +qq^2+vv(0,0)^2)*resu
    -uu(0,0)*qq)*zz);
v:=v+(dv:=resbv*(1-2*zz)/(1+2*cu)
+h*(1+2*cu)*r2/(4*ct)*rqq^3*usolv(
    +qq^2+uu(0,0)^2)*resv
    -uu(0,0)*qq)*zz);
ros:=ros+(uu(0,0)*df(du,zz)+vv(0,0)*df(dv,zz))/(r2*h)*rqq;
% compute tangential stresses
exz:=(df(u,z)+df(w,x))/2;
eyz:=(df(v,z)+df(w,y))/2;
txz:=2*ct*h^2*ros*exz;
tyz:=2*ct*h^2*ros*eyz;
resttu:=(-sub(zz=1,
    (1-0*gamm)*((1-(hx+df(b,x))^2)*txz
    +hx+df(b,x))*(tzz-txx)
    -(hy+df(b,y))*(txy+(hx+df(b,x))*tyz))
    -(1-gamm)*r2*ct/(cu+1)/(2*cu+1)*u*qq) );
write length_resttu:=mylength(resttu);
resttv:=-sub(zz=1,
(1-0*gamm)*(((1-(hy+df(b,y))^2)*tyz
+(hy+df(b,y))*(tzz-tyy)
-(hx+df(b,x))*(txy+(hy+df(b,y))*txz))
-(-1-gamm)*r2*ct/(cu+1)/(2*cu+1)*v*qq ));
write length_resttv:=mylength(resttv);
ok:=if {resttu,resttv}={0,0} then ok else 0;
%
% update lateral evolutions
% solve the suspended sediment concentration.
% let the diffusion coefficient es=ct*h^2*ros,
es:=ct*h^2*ros;
sed:=df(c,t)+df(c*u,x)+df(c*v,y)+df(w*c,z)
-df(wf*c,z)+grx*df(wf*c,x)
-df(es*df(c,x),x)-df(es*df(c,y),y)-df(es*df(c,z),z);
% boundary conditions on the free surface and bed.
sedf:=-sub(zz=1,
(1+(1-gamc)*wf/6)*wf*c*(1-grx*df(h+b,x))
+es*(-(df(h,x)+df(b,x))*df(c,x)
-(df(h,y)+df(b,y))*df(c,y)
+df(c,z)*gamc) +2*(1-gamc)*es*c/h
-(df(h+b,x)*u*c+df(h+b,x)*u*c-w*c));
sedb:=-sub(zz=0,
(1+(1-gamc)*wf/6)*wf*(cae)*(1-grx*df(b,x))
+es*(-df(b,x)*df(c,x)-df(b,y)*df(c,y)+(2-gamc)*df(c,z))

Update the turbulent mean concentration of the suspended sediment and its evolution by the advection-diffusion equation (3.2) and the flux boundary condition (3.9) on the free surface and the flux boundary condition (3.10) on the mean bed. Consider the diffusion coefficient $\epsilon_s$ equaling to the eddy viscosity $\nu$ in the turbulent flow system.
\[ +2(1-\text{gamc})\text{es}c/h \]

write rescamp:=mean(c,zz)-cc(0,0);
write length_sed:=mylength(sed);
write length_sedf:=mylength(sedf);
write length_sedb:=mylength(sedb);
ok:=if \{sed,sedf,sedb,rescamp\}\{0,0,0,0\} then ok else 0;
% update the concentration.
gc:=gc+(dc:=-mean(3/2*(1-\text{zz})*sed,zz)
+3/4*(sedb-sedf/ct/r2*(1+2*cu)*rqq)/h);

c:=c+csolv(sed+2*(1-\text{zz})*dc,zz)*h/ct/r2*(1+2*cu)*rqq
-sedf*(\text{zz}-1/2)/ct/r2*(1+2*cu)*rqq;

End the iterative loop when all residuals are zero to the specified order.

showtime;
if ok then write iter:=100000+iter;
end;

Write out the final evolution on the slow manifold.

\%write results
eps:=1$
on rounded; print\_precision 4;
write dhdt:=length(gh);
write dudt:=length(gu);
write dvdt:=length(gv);
write dcdt:=length(gc);
end;

A.2 Computer algebra derives the characteristic equation in gap-tooth simulation

This section lists the computer algebra to derive the high order characteristic equation (4.19) in the gap-tooth simulation in chapter 4. The code substitutes the solutions forms (4.15)–(4.18) of the linear wave-like PDEs (4.2)–(4.3) into the proposed coupling conditions (4.11) to obtain a relationship between the microscale wavenumber \( \ell \) and the macroscale wavenumber \( k \).

Stretch the coordinate \( x_i \) to denote \( X = x/D \), where \( D \) is the macroscale step in the gap-tooth system. Use \( dd \) to represent \( D \).
Define the operator \( \text{cis} \) for the exponential function \( \exp(ix) \). Then define the exponential forms \( \exp[\pm i(\omega t + kj)] \) by \( \text{ee} \) and \( \text{ff} \), respectively, where \( \omega \) is the frequency.

\[
\text{ee} := \text{cis}(\omega t + kj);
\]
\[
\text{ff} := -\text{cis}(-\omega t - kj);
\]

Define the solutions of the ‘velocity’ \( u \) and ‘depth’ \( h \). Check these solutions satisfy the linear wave-like PDEs (4.2)–(4.3).

\[
u := nu0 + nu2*ll^2/dd^2;
\]
\[
u0 := (a0*cis(ll*xi)+b0*cis(-ll*xi))*\text{ee};
\]
\[
h0 := -(omega-i*nu)*dd/ll*a0*cis(ll*xi) + (omega-i*nu)*dd/ll*b0*cis(-ll*xi))*\text{ee};
\]
\[
u1 := (a1*cis(ll*xi)+b1*cis(-ll*xi))*\text{ee};
\]
\[
h1 := -(omega-i*nu)*dd/ll*a1*cis(ll*xi) + (omega-i*nu)*dd/ll*b1*cis(-ll*xi))*\text{ee};
\]

The parameter \( \gamma \) controls the order of the coupling conditions (4.11). In this algebra, we truncate the coupling condition (4.11) to errors \( O(\gamma^4) \). Denote the \( \gamma \) by \( \text{gam} \). This algebra aims to find a relationship between the microscale wavenumber \( \ell \) and the macroscale wavenumber \( k \). We use a dummy parameter \( \text{eps} \) to control the output orders. Then substitute the solution forms into the coupling conditions (4.11).

\[
\text{use eps to control the output orders}
\]
\[
\text{factor i;}
\]
gam := eps;
k := eps^2;
let eps^14 => 0;
% the coupling conditions with the errors O(gam^4)
eehp1 := sub({j = j+1, xi = 0}, h1) $
eehp3 := sub({j = j+3, xi = 0}, h1) $
eehp5 := sub({j = j+5, xi = 0}, h1) $
eehm1 := sub({j = j-1, xi = 0}, h1) $
eehm3 := sub({j = j-3, xi = 0}, h1) $
eehm5 := sub({j = j-5, xi = 0}, h1) $
eeup1 := sub({j = j+1, xi = 0}, u0) $
eeup3 := sub({j = j+3, xi = 0}, u0) $
eeup5 := sub({j = j+5, xi = 0}, u0) $
eeum1 := sub({j = j-1, xi = 0}, u0) $
eeum3 := sub({j = j-3, xi = 0}, u0) $
eeum5 := sub({j = j-5, xi = 0}, u0) $
% the following are errors O(gam^4)
chl := (-sub(xi = +r, h0) + gam*(eehp1 + eehm1)/2 + gam*r/2*(eehp1 - eehm1) + gam^2*(-1+r^2)/16*(eehp3 - eehp1 - eehm1 + eehm3) + gam^2*(-r+r^3)/48*(eehp3 - 3*eehp1 + 3*eehm1 - eehm3) + gam^3*(9-10*r^2+r^4)/768*(eehp5 + eehp3 - 2*eehp1 - 2*eehm1 + eehm3 + eehm5) + gam^3*(9*r-10*r^3+r^5)/3840*(eehp5 - 5*eehp3 + 10*eehp1 - 10*eehm1 + 5*eehm3 - eehm5) ) * ff$
chr := (-sub(xi = -r, h0) + gam*(eehp1 + eehm1)/2 - gam*r/2*(eehp1 - eehm1) + gam^2*(-1+r^2)/16*(eehp3 - eehp1 - eehm1 + eehm3) - gam^2*(-r+r^3)/48*(eehp3 - 3*eehp1 + 3*eehm1 - eehm3) + gam^3*(9-10*r^2+r^4)/768*(eehp5 + eehp3 - 2*eehp1 - 2*eehm1 + eehm3 + eehm5) - gam^3*(9*r-10*r^3+r^5)/3840*(eehp5 - 5*eehp3 + 10*eehp1 - 10*eehm1 + 5*eehm3 - eehm5) ) * ff$
cul := (-sub(xi = +r, u0) + gam*(eeup1 + eeu1)/2 + gam*r/2*(eeup1 - eeu1) + gam^2*(-1+r^2)/16*(eeup3 - eeu1 - eeu1 + eeu1) + gam^2*(-r+r^3)/48*(eeup3 - 3*eeup1 + 3*eeiu1 - eei1) + gam^3*(3-10*r^2+r^4)/768*(eeup5 + eeu1 - 3*eeup1 - 3*eeiu1 - eei1) + gam^3*(3-10*r^2+r^4)/768*(eeup5 - 3*eeup1 - 3*eeiu1 - eei1) ) * ff$
-2*eeum1+eeum3+eeum5) \\
+gam^3*(9*r-10*r^3+r^5)/3840*(eeup5-5*eeup3+10*eeup1 \\
-10*eeum1+5*eeum3-eeum5) )*ff$
\end{verbatim}

Write the resultant characteristic equation and truncate to the error $O(k^7)$ to obtain equation (4.19).

\begin{verbatim}
%%% solutions of characteristic equations
a:=mat((df(chl,a0),df(chl,b0),df(chl,a1),df(chl,b1)) \\
,(df(chr,a0),df(chr,b0),df(chr,a1),df(chr,b1)) \\
,(df(cul,a0),df(cul,b0),df(cul,a1),df(cul,b1)) \\
,(df(cur,a0),df(cur,b0),df(cur,a1),df(cur,b1)) )$  \\
chareqn:=(det(a) where cis(~q)=>cos(q)+i*sin(q))$  \\
chareqn:=trigsimp(chareqn,expand)$  \\
chareqn:=factorize(den(chareqn));
\end{verbatim}

\subsection{A.3 Computer algebra derives the two-layer model}

This section lists the computer algebra to derive a two-layer model of the viscous layer of fluid in chapter 5. Consider the viscous fluid film flow on a flat plane. We partition the spatial domain to have two artificial layers. The flow has the thickness $h(x,t)$, and two layer velocities $u_1(x,z,t)$ of lower layer and $u_2(x,z,t)$ of the upper layer. The two artificial layers have the same thickness, $h/2$. The computer algebra derives the two-layer model
in terms of the flow thickness $h(x,t)$ and the two layer mean velocities $\bar{u}_1(x,t)$ and $\bar{u}_2(x,t)$.

### A.3.1 Explanation of symbols

Denote fluid thickness $h(x,t)$ by $h$, layer mean velocities $\bar{u}_j(x,t)$ by $uu_j$ for the lower layer $j = 1$ and upper layer $j = 2$, and their evolution $\partial h/\partial t = gh$ and $\partial \bar{u}_j/\partial t = gu_j$. The Reynolds number is $re$, and the coefficients of lateral and normal gravitational forcing are $gr_x$ and $gr_z:=1$.

### A.3.2 Definition of operators in the algebra

Use the operator $h(m)$ to denote $m$ lateral derivatives of the fluid thickness $h$, $\partial^m h$, and similarly $uu_j(m)$ denotes $m$ lateral derivatives of the layer mean velocity $\bar{u}_j$, $\partial^m \bar{u}_j$. Define readable abbreviations for $h_x$. Use $d$ to count the number of lateral $x$ derivatives so we can easily truncate the asymptotic expansion. These operators depend upon time and lateral space. Then the spatial derivative $\partial x h(m) = h(m+1)$, and the time derivative $\partial t h(m) = \partial^m gh$, for example.

```plaintext
357 % Computer algebra for exploring two layer
358 % modelling of thin fluid film.
359 on div; off allfac; on revpri;
360 linelength 70$
361 factor gam,d,small,we,re,nu,gx,gy;
362 operator h; operator uu1; operator uu2;
363 hx:=h(1)*d;
364 depend h,xx,tt;
365 depend uu1,xx,tt;
366 depend uu2,xx,tt;
367 let { df(h,xx)=>h(1), df(h(~m),xx)=>h(m+1)
368     , df(h,tt)=>gh , df(h(~m),tt)=>df(gh,xx,m)
369     , df(uu1,xx)=>uu1(1), df(uu1(~m),xx)=>uu1(m+1)
370     , df(uu1,tt)=>gu1 , df(uu1(~m),tt)=>df(gu1,xx,m)
371     , df(uu2,xx)=>uu2(1), df(uu2(~m),xx)=>uu2(m+1)
372     , df(uu2,tt)=>gu2 , df(uu2(~m),tt)=>df(gu2,xx,m)
373 };
```

Use stretched coordinates $zz$, $xx$ and $tt$ to denote $Z = z/h(x,t)$, $X = x$ and $T = t$, and to count lateral derivatives. The free surface of the fluid film is then simply $Z = 1$.  

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dependence

\[ \text{depend } xx,x,z,t; \]
\[ \text{depend } zz,x,z,t; \]
\[ \text{depend } tt,x,z,t; \]
\[ \text{let} \{ \]
\[ \text{df}^{(~a,x)} \Rightarrow \text{df}(a,xx) + d_2 \cdot \text{zz} \cdot h_x / h \cdot \text{df}(a,zz) \]
\[ \text{df}^{(~a,t)} \Rightarrow \text{df}(a,tt) - \text{zz} \cdot gh / h \cdot \text{df}(a,zz) \]
\[ \text{df}^{(~a,z)} \Rightarrow \text{df}(a,zz) / h \]
\[ \} \]

We also define linear operators, which are quicker than native integrations.

operators for updating;

\[ \text{operator } \text{uinv}; \text{linear } \text{uinv}; \]
\[ \text{let} \{ \]
\[ \text{uinv}(1,zz,1) \Rightarrow -zz^2 / 2 + zz / 6 \]
\[ \text{uinv}(1,zz,2) \Rightarrow -zz^2 / 2 + 1 / 8 + 2 / 3 \cdot (zz - 1 / 2) \]
\[ \text{uinv}(zz^{~p},zz,1) \Rightarrow (zz^{~p+2} - zz / (p+3) / 2^p) / (p+1) / (p+2) \]
\[ \text{uinv}(zz^{~p},zz,2) \Rightarrow (zz^{~p+2} + 1 / 2^{p+2}) / (p+1) / (p+2) \]
\[ \} \]

operator mean; linear mean;

\[ \text{let} \{ \]
\[ \text{mean}(1,zz,~a) \Rightarrow 1 \]
\[ \text{mean}(zz^{~p},zz,1) \Rightarrow 1 / (p+1) / 2^p \]
\[ \text{mean}(zz^{~p},zz,2) \Rightarrow 2 - 1 / 2^p / (p+1) \]
\[ \} \]

operator wsolv; linear wsolv;

\[ \text{let} \{ \]
\[ \text{wsolv}(zz^{~n},zz,1) \Rightarrow zz^{n+1} / (n+1) \]
\[ \text{wsolv}(1,zz,1) \Rightarrow zz \]
\[ \text{wsolv}(zz^{~n},zz,2) \Rightarrow (zz^{n+1} - 1 / 2^{n+1}) / (n+1) \]
\[ \} \]

operator psolv; linear psolv;

\[ \text{let} \{ \]
\[ \text{psolv}(zz^{~n},zz,2) \Rightarrow (1 - zz^{n+1}) / (n+1) \]
\[ \text{psolv}(1,zz,2) \Rightarrow 1 - zz \]
\[ \text{psolv}(zz^{~n},zz,1) \Rightarrow (1/2^{n+1} - zz^{n+1}) / (n+1) \]
\[ \} \]

A.3.3 Initial approximation

Start the iteration from the linear solution of shear flow: the lateral velocity \( u_j \) is piecewise linear, the pressure \( p_j \) is linear (as gravity is not ‘small’ we find hydrostatic pressure in the iteration), and all other fields are zero, \( w_j = 0 \).
The evolution of the slow manifold parameters is also zero: \( \partial_t \bar{u}_j = g u_j = 0 \) and \( \partial_t h = g h = 0 \). We use \texttt{small} to count the number of \( \bar{u}_j \) factors in a term.

\% linear initial approximation;
\% use \texttt{small} to count the number of \( U_j \) factors;
\begin{verbatim}
407  u1:=small*uu1*zz*4;
408  u2:=small*(uu1*(6-8*zz)+uu2*(4*zz-2));
409  p1:=small*h(1-zz);
410  p2:=small*h(1-zz);
411  w1:=w2:=gh:=gu1:=gu2:=0;
\end{verbatim}

A.3.4 Truncation of the asymptotic expansion

The small parameters in the system are the parameter \( d \) counting the number of small lateral derivatives \( \partial_x \), the lateral gravity \( \tan \theta \), and the artificial parameters \( \gamma \). We use a parameter \texttt{small} to count the small parameters in a term.

\% truncate the asymptotic expansion;
\begin{verbatim}
414  d:=small;
415  grz:=1;
416  grx:=d*gx;
\end{verbatim}

Then we truncate to the error \( O(\partial_x^3 + \tan^3 \theta, \gamma^7) \).

\% let \( \{\texttt{small}^3=>0, \texttt{gam}^7=>0\} \); 

A.3.5 The iterative loop

\begin{verbatim}
419  for it:=1:10 do begin
420    ok:=1;
421    \% update \( w \) with continuity and no flow through bed;
422    resc1:=df(u1,x)+df(w1,z);
423    w1:=w1+(w1d:=-h*wsolv(resc1,zz,1));
424    resc2:=df(u2,x)+df(w2,z);
425    w2:=w2-h*wsolv(resc2,zz,2)+sub(zz=1/2,w1d);
426    bcw1:=sub(zz=0,w1);
427    bcw2:=sub(zz=1/2,w2-w1);
428    write clengths:={length(resc1),length(resc1)
429      ,length(bcw1),length(bcw2)};
430    ok:=if \{resc1,resc2,bcw1,bcw2\}={0,0,0,0} then ok else 0;
\end{verbatim}
Update the pressure $p$ from vertical momentum (5.3) and surface normal stresses (5.6).

431  % update p from vertical momentum and surface normal stress;
432  resw2:=re*( df(w2,t)+u2*df(w2,x)+w2*df(w2,z) )
433    +df(p2,z) +grz -nu*(df(w2,x,2)+df(w2,z,2));
434  resw1:=re*( df(w1,t)+u1*df(w1,x)+w1*df(w1,z) )
435    +df(p1,z) +grz -nu*(df(w1,x,2)+df(w1,z,2));
436  restn2:= sub(zz=1,-p2*(1+hx^2) +2*nu*(df(w2,z)
437                      +hx^2*df(u2,x)-hx*(df(u2,z)+df(w2,x))) -we*curv );
438  p2:=p2+(p2d:=h*psolv(resw2,zz,2)+restn2);
439  p1:=p1+h*psolv(resw1,zz,1)+sub(zz=1/2,p2d);
440  bcp1:=sub(zz=1/2,p1-p2);
441  write wlengths:={length(resw2),length(resw1)
442                         ,length(restn2),length(bcp1)};
443  ok:=if {resw1,resw2,restn2,bcp1}={0,0,0,0} then ok else 0;

Update the layer velocities $u_j$ and their evolutions from horizontal momentum equations (5.3), no-slip condition (5.4) on the bed, surface tangential stresses (5.13), and boundary condition (5.14) at the artificial interfaces.

444  % update u from horizontal momentum,
445  % bed and surface tangential stress
446  resu1:=-re*( df(u1,t)+u1*df(u1,x)+w1*df(u1,z) )
447      -df(p1,x) +grx +nu*(df(u1,x,2)+df(u1,z,2));
448  resu2:=-re*( df(u2,t)+u2*df(u2,x)+w2*df(u2,z) )
449      -df(p2,x) +grx +nu*(df(u2,x,2)+df(u2,z,2));
450  bctt:=-sub(zz=1, (1-hx^2)*(df(u2,z)+df(w2,x))
451             +2*hx*(df(u2,z)-df(u2,x)) )
452    +(1-gam)*(sub(zz=1/2,df(u2,z)))
453    +(1-gam)*sub(zz=1/2,df(u1,z))
454    +(gam/2)*sub(zz=1/2,df(u2,z));
455  bcbed:=sub(zz=0,u1);
456  ccc:=-sub(zz=1/2,u1)+sub(zz=1/2,u2);
457  cc1:=(1-gam/2)*sub(zz=1/2,df(u1,z))
458      +(gam/2)*sub(zz=1/2,df(u2,z))
459      +(1-gam)*sub(zz=1/2,u1)*2/h;
460  write ulengths:={length(resu1),length(resu2),length(bctt)
461                         ,length(bcbed),length(ccc),length(cc1)};
462  ok:=if {resu1,resu2,bctt,bcbed,cc1,ccc}={0,0,0,0,0,0} then ok else 0;
463  % update the evolution and the lateral velocity;
464  gu1:=gu1+(gd1:=3*nu*cc1/h
465      +mean(3*zz*resu1,zz,1))/re/small;
gu2 := gu2 + (gd2 := 3/2*nu*cc1/h + 3*nu*bctt/h
+ mean(3*zz/2*resu1,zz,1)
+ mean(3*(zz-1/2)*resu2,zz,2)/re/small;

u1 := u1 + (u1d := uinv(resu1-gd1*zz*4,zz,1)*h^2/nu);
u2 := u2 + uinv(resu2-gd1*(6-8*zz)-gd2*(4*zz-2),zz,2)*h^2/nu
+ sub(zz=1/2,u1d)*(3-4*zz);

The kinematic condition (5.8) on the free surface gives the evolution of the fluid thickness $h(x,t)$.

% update the free surface evolution;
gh := sub(zz=1,w2-u2*hx);

End the loop.

showtime;
if ok then write it := it+100000;
end;

Check correct amplitudes for the lateral shear momentum.

% check correct amplitudes for the lateral shear momentum
amp1 := -uu1*small+mean(u1,zz,1);
amp2 := -uu2*small+mean(u2,zz,2);

Stabilise high wavenumber modes using the regularising operator.

% stabilise high wavenumber modes
lgu1 := gu1-a*df(h^2*df(gu1*small,x),x)/small;
lgu2 := gu2-a*df(h^2*df(gu2*small,x),x)/small$;

Write out the final evolution on the slow manifold.

% write out the leading equation;
dhdt := gh;
on rounded; print_precision 4;
lredu1dt := re*lgu1;
lredu2dt := re*lgu2;
end;
A.3.6 The slow manifold of the two-layer model

This slow manifold of the two-layer model should be exactly the same, to the order of analysis, as the one-layer, slow manifold of the original fluid dynamics (Roberts 1997, e.g.).

Use the operator $uu(m)$ to denote $m$ spatial derivatives of the depth-averaged lateral velocity $\bar{u}$, $\partial_x^m \bar{u}$. Similarly, this operator depends upon time and lateral space.

\begin{verbatim}
489 operator uu;
490 depend uu,xx,tt;
491 let { df(uu,xx)=>uu(1), df(uu(~m),xx)=>uu(m+1)
492 , df(uu,tt)=>gu , df(uu(~m),tt)=>df(gu,xx,m)
493 };

Translate spatial derivatives of the two-layer velocities into spatial derivatives of the slow manifold field.

494 let { uu1(~p)=>df(uu1,xx,p)
495 , uu2(~p)=>df(uu2,xx,p)
496 };

Start the iteration from the linear solution. The two-layer velocities are in the ratio $1:2$.

\begin{verbatim}
497 % initial approximation of the two layer velocities
498 uu1:=uu*2/3;
499 uu2:=uu*4/3;
500 gu:=0;

Start the iterative loop. First compute the residuals of the modified PDEs (5.27)–(5.28) for the two layers. Denote the Euler parameter $E_u$ by fac which changes the strength of the linear basis; experiments suggest fac = $9/2$ in order to get good convergence in $\gamma'$. Use gamd to denote the parameter $\gamma'$.

\begin{verbatim}
501 let gamd^7=>0;
502 fac:=9/2;
503 for it:=1:19 do begin
504 res1:=-df(uu1,t)+gu1+(1-gamd)*fac*nu/re/h^2*(-4*uu1+2*uu2);
505 res2:=-df(uu2,t)+gu2+(1-gamd)*fac*nu/re/h^2*(+2*uu1-1*uu2);
506 write ulengths:=map(length(~a),{res1,res2});
507 gu:=gu+(gud:=(res1+2*res2)*3/10);
508 uu1:=uu1+h^2*re/nu/6/fac*(res1-gud*2/3);
509 uu2:=uu2+h^2*re/nu/3/fac*(res2-gud*4/3);
\end{verbatim}

\end{verbatim}
End the iterative loop.

510  if \{res1, res2\} = \{0, 0\} then write it := it + 10000;
511  end;

Check their average is still the mean lateral velocity, and write out the slow manifold equation.

512  \% check the average
513  resamp := (uu1 + uu2) / 2 - uu;
514  \% write out results
515  fullprob := \{gamd => 1, small => 1\};
516  on rounded; print_precision 4$
517  dhdt := sub(fullprob, gh);
518  redundt := sub(fullprob, re*gu);
519  off rounded;
520
521  end;
Appendix B

Matlab codes

This appendix lists the Matlab functions implementing the numerical simulations by the staggered grid schemes in chapter 2–5. These functions empower numerical analysis and simulations for the derived models in the dissertation. These functions implement different physical situations with different boundary conditions. Section B.1 lists the Matlab function of the turbulence model (2.1)–(2.3). Section B.2 lists the Matlab function for the suspended sediment model (3.1), coupled with the modified flow equations (3.14)–(3.15). Section B.3 lists the Matlab function of the gap-tooth simulation of the wave-like systems, including the linear wave-like PDEs and nonlinear PDEs modelling dam breaking-waves. Section B.4 then lists the Matlab function of the gap-tooth simulation with the two-layer model (5.26)–(5.28).

B.1 Matlab code for the turbulent model

This section lists the Matlab codes for numerically implementing channel flows by the model (2.1)–(2.3). We firstly define the index system in the Matlab functions. This index system is used in the Matlab functions both in section B.1 and section B.2.

```
nx=20; % grids in x-direction
ny=40; % grids in y-direction
Lx=40; % considered length in x-direction
Ly=20; % considered length in y-direction
dx=Lx/nx; % spatial step in x-direction
dy=Ly/ny; % spatial step in y-direction
% define the coordinate and indexes.
i=1:nx;
j=1:ny;
```
Then we define a Matlab function \texttt{smag} for the turbulence model (2.1)–
(2.3) by the staggered grid scheme in section 2.6. Figure 2.9 in section 2.6
shows the positions storing the values the physical fields of depth \( h(x, y, t) \),
and depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \). In the function,
we denote the depth \( h(x, y, t) \) by \( h \) at these positions, and similarly denote
the depth-averaged lateral velocities \( \bar{u}(x, y, t) \) and \( \bar{v}(x, y, t) \) at these positions
by \( \bar{u}h0 \) and \( \bar{v}h0 \). Then compute the values of the fields in others positions.
For example, \( \bar{u}0h \) denotes the computed values of the depth-averaged veloc-
ity \( \bar{u}(x, y, t) \) at the positions \((x_i, y_{j+1/2})\). We use \( qh0 \) and \( q0h \) to denote
the local mean flow speed.
\[ hh0 = \frac{(h(k) + h(kp0))}{2}; \]
\[ h0h = \frac{(h(k) + h(k0p))}{2}; \]

% water conservation equation
\[ dhdt = -h(k) \cdot (uh0(k) - uh0(km0))/dx \]
\[ -h(k) \cdot (v0h(k) - v0h(k0m))/dy \]
\[ -u(k) \cdot (hh0(k) - hh0(km0))/dx \]
\[ -v(k) \cdot (h0h(k) - h0h(k0m))/dy \];

% lateral momentum equation
% dudt is staggered on the h0 position
\[ qqh0 = \sqrt{u0h^2 + v0h^2}; \]
\[ dudt = -0.00283 \cdot \gamma \cdot uh0(k) \cdot qvh0/h0h(k) \]
\[ + 0.993 \cdot g \cdot \sin(\theta) \]
\[ -0.993 \cdot (h(kp0) - h(k))/dx \]
\[ -0.993 \cdot (b(kp0) - b(k))/dx \]
\[ -1.017 \cdot vh0(k) \cdot (uh0(k0p) - uh0(k0m))/dy/2 \]
\[ -1.025 \cdot uh0(k) \cdot (u(kp0) - u(k))/dx \]
\[ + hh0(k) \cdot (0.094 \cdot qvh0 - 0.084 \cdot (uh0(k) \cdot 2 - vh0(k) \cdot 2))/qqh0 \]
\[ \cdot (uh0(k0p) - 2 \cdot uh0(k) + uh0(k0m))/dy^2 \cdot 4 \]
\[ + hh0(k) \cdot (0.094 \cdot qvh0 + 0.084 \cdot (uh0(k) \cdot 2 - vh0(k) \cdot 2))/qqh0 \]
\[ \cdot (uh0(k0p) - 2 \cdot uh0(k) + uh0(k0m))/dy^2 \cdot 4 \]
\[ + (0.188 \cdot qvh0 - 0.168 \cdot (uh0(k) \cdot 2 - vh0(k) \cdot 2))/qqh0 \]
\[ \cdot (hh0(k0p) - hh0(k0m))/dy/2 \cdot (uh0(k0p) - uh0(k0m))/dy/2 \]
\[ + (0.188 \cdot qvh0 + 0.168 \cdot (uh0(k) \cdot 2 - vh0(k) \cdot 2))/qqh0 \]
\[ \cdot (h(kp0) - h(k))/dx \cdot (u(kp0) - u(k))/dx; \]

% lateral momentum equation
% dvdt is staggered on the 0h position
\[ qq0h = \sqrt{u0h^2 + v0h^2}; \]
\[ dvdt = -0.00283 \cdot \gamma \cdot v0h(k) \cdot qvh0/h0h(k) \]
\[ - 0.993 \cdot (h(kp0) - h(k))/dy \]
\[ - 0.993 \cdot (b(kp0) - b(k))/dy \]
\[ - 1.017 \cdot v0h(k) \cdot (v0h(k0p) - v0h(k0m))/dx/2 \]
\[ + h0h(k) \cdot (0.094 \cdot qvh0 - 0.084 \cdot (uh0(h) \cdot 2 - vh0(h) \cdot 2))/qqh0 \]
\[ \cdot (v0h(kp0) - 2 \cdot v0h(k) + v0h(k0m))/dx^2 \cdot 4 \]
\[ + h0h(k) \cdot (0.094 \cdot qvh0 + 0.084 \cdot (uh0(h) \cdot 2 - vh0(h) \cdot 2))/qqh0 \]
\[ \cdot (v0h(kp0) - 2 \cdot v0h(k) + v0h(k0m))/dx^2 \cdot 4 \]
\[ + (0.188 \cdot qvh0 - 0.168 \cdot (uh0(h) \cdot 2 - vh0(h) \cdot 2))/qqh0 \]
\[ \cdot (h(kp0) - h(k))/dy \cdot (v(kp0) - v(k))/dy \]
\[ + (0.188 \cdot qvh0 + 0.168 \cdot (uh0(h) \cdot 2 - vh0(h) \cdot 2))/qqh0 \]
\[ \cdot (h0h(kp0) - h0h(k0m))/dx/2 \cdot (v0h(kp0) - v0h(k0m))/dx/2; \]
% form time derivative into a vector
dhuv=[dhdt';dudt';dvdt'];
dhuv=dhuv(:);

B.2 Matlab code for the suspended sediment model

This section lists the Matlab function of simulating the suspended sediment in channel flows and under large waves. The index system is listed in section B.1.

We define a Matlab function sed for the suspended sediment model (3.1), coupled with the modified momentum equations (3.14)–(3.15) by the staggered grid scheme in section 3.6.1. Figure 3.7 in section 3.6.1 shows the positions storing the values the physical fields of depth $h(x,y,t)$, depth-averaged lateral velocities $\bar{u}(x,y,t)$, and depth-averaged concentration $\bar{c}(x,y,t)$. In the function, we denote the depth $h(x,y,t)$ by $h$ at these positions, and similarly denote the depth-averaged lateral velocities $\bar{u}(x,y,t)$ and $\bar{v}(x,y,t)$ at these positions by $uh0$ and $v0h$, and the depth-averaged concentration $\bar{c}(x,y,t)$ by $c$. Then compute the values of the fields in others positions. For example, $u0h$ denotes the computed values of the depth-averaged velocity $\bar{u}(x,y,t)$ at the positions $(x_i,y_{j+1/2})$. We use $qqh0$, $qq0h$ and $qq$ to denote the local mean flow speed.

% time derivatives of suspended sediment model, coupled with modified momentum equations.
function dhuv=sed(t,huvc)
global k kp0 km0 k0p k0m dx dy theta g b wf cae gam s
% create physical field vectors on staggered grid
h=huvc(1:4:end);
uh0=huvc(2:4:end);
v0h=huvc(3:4:end);
c=huvc(4:4:end);
% interpolate velocity fields on the h-grid
u=(uh0(k)+uh0(km0))/2;
v=(v0h(k)+v0h(k0m))/2;
% interpolate these to other staggered grid
u0h=(u(k)+u(k0p))/2;
v0h=(v(k)+v(kp0))/2;
% interpolate onto mid-sq grid
vhh=(v0h(k)+v0h(kp0))/2;
uhh=(uh0(k)+uh0(k0p))/2;
% interpolate h to the staggered grid points
hh0=(h(k)+h(kp0))/2;

% interpolate b to the staggered grid points
bh0=(b(k)+b(kp0))/2;

% interpolate c to the staggered grid points
ch0=(c(k)+c(kp0))/2;

% water conservation equation
dhdt=-h(k).*(uh0(k)-uh0(km0))/dx - h(k).*(v0h(k)-v0h(k0m))/dy - u(k).*(hh0(k)-hh0(km0))/dx - v(k).*(h0h(k)-h0h(k0m))/dy;

% lateral momentum equation
dudt=-0.00283*gam*uh0(k).*qqh0./hh0(k) + 0.993*g*tan(theta) - 0.993*(h(kp0)-h(k))/dx - 0.993*(b(kp0)-b(k))/dx - 1.017*vh0(k).*(uh0(k0p)-uh0(k0m))/dy/2 - 1.025*uh0(k).*(u(kp0)-u(k))/dx - hh0(k).*(0.094*qqh0-0.084*(uh0(k).^2-vh0(k).^2)./qqh0) .* (uh0(k0p)-2*uh0(k)+uh0(k0m))/dy^2/4 - hh0(k).*(0.094*qqh0+0.084*(uh0(k).^2-vh0(k).^2)./qqh0) .* (uh0(kp0)-2*uh0(k)+uh0(km0))/dx^2/4 - (0.188*qqh0-0.168*(uh0(k).^2-vh0(k).^2)./qqh0) .* (hh0(k0p)-hh0(k0m))/dy/2.*(uh0(k0p)-uh0(k0m))/dy/2 - (0.188*qqh0+0.168*(uh0(k).^2-vh0(k).^2)./qqh0) .* (h(kp0)-h(k))/dx.* (u(kp0)-u(k))/dx - 0.298*(s-1)*hh0(k).*(c(kp0)-c(k))/dx;

dvdt=-0.00283*gam*v0h(k).*qq0h./h0h(k) + 0.993*(h(k0p)-h(k))/dy - 0.993*(b(k0p)-b(k))/dy - 1.017*v0h(k).* (v(k0p)-v(k))/dy - 1.025*v0h(k).* (u(kp0)-v0h(km0))/dx/2 - 0.298*(s-1)*hh0(k).*(c(kp0)-c(k))/dx;
\[ +h_0 h(k) \cdot (0.094 \cdot q_0 h - 0.084 \cdot (u_0 h(k)^2 - v_0 h(k)^2) / q_0 h) \ldots \]
\[ \cdot (v_0 (k_0 p) - 2 \cdot v_0 h(k) + v_0 h(k_0 m)) / dy^2 \cdot 4 \ldots \]
\[ +h_0 h(k) \cdot (0.094 \cdot q_0 h + 0.084 \cdot (u_0 h(k)^2 - v_0 h(k)^2) / q_0 h) \ldots \]
\[ \cdot (v_0 (k_0 p) - 2 \cdot v_0 h(k) + v_0 h(k_0 m)) / dx^2 \cdot 4 \ldots \]
\[ +(0.188 \cdot q_0 h - 0.168 \cdot (u_0 h(k)^2 - v_0 h(k)^2) / q_0 h) \ldots \]
\[ \cdot (h_0 h(k_0 p) - h_0 h(k_0 m)) / dx / 2 \cdot (v_0 h(k_0 p) - v_0 h(k_0 m)) / dx / 2 \ldots \]
\[ -0.298 \cdot (s - 1) \cdot h_0 h(k) \cdot (c_0 k_0 p - c_0 k_0 m) / dy ; \]

\% suspended diffusion equation

\% dcdt is staggered on the same position with h

\[ q = \sqrt{u(k)^2 + v(k)^2} ; \]

\[ dcdt = -w_f c(k) / h(k) \cdot (0.938 + 28.87 w_f / q) \ldots \]
\[ + w_f c_{ae} / h(k) \cdot (0.984 - 51.3 w_f / q) \ldots \]
\[ -1.007 \cdot \exp(-3.073 w_f / q(k)) \ldots \]
\[ \cdot (u(k) \cdot (c(k_0 p) - c(k_0 m)) / dx / 2 \ldots \]
\[ + v(k) \cdot (c(k_0 p) - c(k_0 m)) / dy / 2 \ldots \]
\[ +0.033 \cdot h(k) \cdot q(k) \cdot ((c(k_0 p) - 2 \cdot c(k) + c(k_0 m)) / dx^2 \ldots \]
\[ + (c(k_0 p) - 2 \cdot c(k) + c(k_0 m)) / dy^2 \ldots \]
\[ +0.027 \cdot h(k) \cdot u(k) \cdot (c(k_0 p) - 2 \cdot v(k) - v(k_0 m)) / q(k) \ldots \]
\[ \cdot ((c(k_0 p) - 2 \cdot c(k) + c(k_0 m)) / dx^2 - (c(k_0 p) - 2 \cdot c(k) + c(k_0 m)) / dy^2 \ldots \]
\[ -2 \cdot c(k) / c(k_0 m) / dy^2 \ldots \]
\[ +(0.0662 \cdot q_0 - 0.0542 \cdot (u(k)^2 - v(k)^2) / q) \ldots \]
\[ \cdot (h(k_0 p) - h(k_0 m)) / dy / 2 \cdot (c(k_0 p) - c(k_0 m)) / dy / 2 \ldots \]
\[ + (0.0662 \cdot q_0 h_0 + 0.0542 \cdot (u(k)^2 - v(k)^2) / q) \ldots \]
\[ \cdot (h(k_0 p) - h(k_0 m)) / dx / 2 \cdot (c(k_0 p) - c(k_0 m)) / dx / 2 ; \]

\% form time derivative into a vector

\[ dhuv = [dhd t'; dud t'; dvd t'; dcd t'] ; \]

\[ dhuv = dhuv(:) ; \]

### B.3 Matlab code for the gap-tooth simulation

This section lists the Matlab function for the gap-tooth simulation of the wave-like systems, including the linear wave-like PDEs and nonlinear wave-like PDEs.

We define the Matlab function `patchgap`. A matrix `uu` stores the microscale values of the field of ‘depth’ `h` and ‘velocity’ `u`, so the middle values of the columns are for macroscale variables. The `inter0rd` determines the
orders of the coupling conditions: \texttt{interOrd} = 1 is for linear interpolation; 
\texttt{interOrd} = 2 is for cubic interpolation; and \texttt{interOrd} = 3 is for quintic 
interpolation. We use periodic boundary condition over the whole domain. 
The \texttt{nu0}, \texttt{nu2}, and \texttt{c1} are constant coefficients.

% Patches simulation for linear 
% and nonlinear dynamics systems. 
% define a function to contain the nonlinear problem. 
function dhudt=patchgap(t,hu)
    global n m r dx nu0 nu2 gam c1 theta g interOrd Ln
    uu=nan(n+2,m);
i=2:n+1; j=1:m;
    uu(i,j)=reshape(hu,n,m);
    % periodic boundary condition from wrapping patch index 
    jp=[2:m, 1]; jm=[m, 1:m-1];

    % sometimes it is more convenient to generate these as 
    jpp=jp(jp(jp)); jppp=jp(jp(jpp)); and so on
    jpp=[4:m, 1:3]; jmm=[m-2:m, 1:m-3];
    jppp=[6:m,1:5]; jmmm=[m-4:m,1:m-5];
    % coupling edge conditions on each tooth.
    % linear interpolation when \( \gamma^2 = 0 \)
    % quadratic interpolation when \( \gamma^3 = 0 \)
    imid=(n+3)/2;
    uubv=[gam*(uu(imid,jp)+uu(imid,jm))/2 
        +(-1+r^2)*gam^2*(uu(imid,jpp)+uu(imid,jm)-uu(imid,jp) ... 
        -uu(imid,jm)+uu(imid,jmm))/16 
        +(-r+r^3)*gam^2*(uu(imid,jpp)-3*uu(imid,jp) ... 
        +3*uu(imid,jm)-uu(imid,jmm))/48 
        +(9-10*r^2+2*r^4)*gam^3*(uu(imid,jpp)+uu(imid,jp) ... 
        -2*uu(imid,jp)-2*uu(imid,jm)+uu(imid,jmm)+uu(imid,jmmm))/768 
        +(9*r-10*r^3+r^5)*gam^3*(uu(imid,jpp)-5*uu(imid,jp) ... 
        +10*uu(imid,jp)-10*uu(imid,jm) ... 
        +5*uu(imid,jm)-uu(imid,jmm))/3840
    ];
    uu(n+2,j)=sum(uubv(1:2*interOrd,:));
    uubv=[gam*(uu(imid,jp)+uu(imid,jm))/2 
        -gam*r*(uu(imid,jp)-uu(imid,jm))/2 
        +(-1+r^2)*gam^2*(uu(imid,jpp)-uu(imid,jp) ... 
        -uu(imid,jm)+uu(imid,jmm))/16 
        +(-r+r^3)*gam^2*(uu(imid,jpp)-3*uu(imid,jp) ... 
        +3*uu(imid,jm)-uu(imid,jmm))/48 
        +(9*r-10*r^3+r^5)*gam^3*(uu(imid,jpp)-5*uu(imid,jp) ... 
        +10*uu(imid,jp)-10*uu(imid,jm) ... 
        +5*uu(imid,jm)-uu(imid,jmm))/3840
    ];
% first some bed drag
dhudt(i1,j0)=dhudt(i1,j0)-nu0*uu(i1,j0).*abs(uu(i1,j0)) ./((uu(i1+1,j0)+uu(i1-1,j0)))*2;

dhudt(i0,j1)=dhudt(i0,j1)-nu0*uu(i0,j1).*abs(uu(i0,j1)) ./((uu(i0+1,j1)+uu(i0-1,j1)))*2;

% second some gravitational forcing

dhudt(i1,j0)=dhudt(i1,j0)+0.985*g*sin(theta);
dhudt(i0,j1)=dhudt(i0,j1)+0.985*g*sin(theta);

% now viscous drag, perhaps only works for n>=5 ??

nu2/(4*dx^2)*(uu(i1-2,j0)-2*uu(i1,j0)+uu(i1+2,j0)).*abs(uu(i1,j0)).*(uu(i1+1,j0)+uu(i1-1,j0))/2;

uuxx=(uu(i0(1:end-2),j1)-2*uu(i0(2:end-1),j1))/(4*dx^2);

dhudt(i0,j1)=dhudt(i0,j1)+nu2*[0*uuxx(1,:);uuxx;0*uuxx(end,:)].*abs(uu(i0,j1)) ./((uu(i0+1,j1)+uu(i0-1,j1)))/2;

% nonlinear advection terms

-c1*uu(i1,j0).*(uu(i1+2,j0)-uu(i1-2,j0))/4/dx;

uu=-uu(i0(1:end-2),j1)+uu(i0(3:end),j1))/4/dx;

% first derivative on the two ends of each patch

uu1l=(-3*uu(i0(1),j1)+4*uu(i0(2),j1)-uu(i0(3),j1))/4/dx;
uuxr=(uu(i0(end-2),j1)-4*uu(i0(end-1),j1))/4/dx;

+3*uu(i0(end),j1))/4/dx;
dhudt(i0,j1)=dhudt(i0,j1)-c1*uu(i0,j1).*[uu1l;uux;uu1r];

end

% form time derivative into a vector

dhudt=reshape(dhudt(i,j),n*m,1);

**B.4 Matlab code for the fluid film flow by gap-tooth scheme**

This section lists the Matlab functions for numerically simulating the fluid film flow by gap-tooth scheme.

Define a function TwolayerP for the gap-tooth simulation with the microscale simulator of the two-layer model (5.26)–(5.28). The matrix uu1 stores the values of the microscale thickness h and layer mean velocity \( \bar{u}_1 \) on staggered scheme. Similarly the matrix uu2 stores the values of the microscale
thickness $h$ and layer mean velocity $\bar{u}_1$. Note that the thickness $h$ in $uu1$ and $uu2$ are same. Approximate the regularising operator $\mathcal{L}$ in the two-layer model (5.26)-(5.28) by $ccop$. Denote the coefficient $C$ in the regularising operator $\mathcal{L}$ by $cc$.

799 % Explore two layer modelling of thin film
800 % flow by patches method
801 % define a function for the two layer thin film flow model.
802 function dhudt=TwolayerP(t,hu)
803 global n m r dx re gam theta g cc interOrd Ln
804 % forces the two h data to be identical.
805 hu(2:4:end)=hu(1:4:end);
806 % uu1(i,j)=ith microgrid value in jth macropatch of kth field.
807 % with h in both for no good reason
808 % uu1(1,j) and uu1(n+2,j) are the boundaries on every patch.
809 uu1=nan(n+2,m); uu2=uu1;
810 i=2:n+1; j=1:m;
811 uu1(i,j)=reshape(hu(1:2:end),n,m);
812 uu2(i,j)=reshape(hu(2:2:end),n,m);
813
814 % periodic boundary condition from wrapping patch index
815 jp=[2:m, 1]; jm=[m, 1:m-1];
816 % it is more convenient to generate these as
817 jpp=jp(jp(jp)); jppp=jp(jp(jpp));
818 jmm=jm(jm(jm)); jmmm=jm(jm(jmm));
819 % h is when both same; u is when both different;
820 i0=2:2:n+1; i1=3:2:n+1; % even and odd within a patch
821 j0=1:2:m; j1=2:2:m; % corresponding even and odd patches
822
823
824 % coupling edge conditions on each tooth,
825 % h and U are the same interp.
826 % Only couple the average information
827 % (the h data is replicated in U)
828 % linear interpolation when $\text{gam}^2=0$
829 % quadratic interpolation when $\text{gam}^3=0$
830 U=(uu1+uu2)/2;
831 imid=(n+3)/2;
832 Ubv=[gam*(U(imid,jp)+U(imid,jm))/2
833     +gam*r*(U(imid,jp)-U(imid,jm))/2
834     +(-1+r^2)*gam^2*(U(imid,jpp)-U(imid,jp) ....

205
(-r+r^3)*gam^2*(U(imid,jpp)-3*U(imid,jp) ...}
839 +3*U(imid,jm)-U(imid,jmm))/48
840 +U(imid,jmmm))/768
841 +9*r-10*r^3+r^5)*gam^3*(U(imid,jppp)-U(imid,jmm)) ...
842 +10*U(imid,jp)-10*U(imid,jm)+5*U(imid,jmm) ...
843 -U(imid,jmmm))/3840
844];
845 U(n+2,j)=sum(Ubv(1:2*interOrd,:));
846 Ubv=[gam*(U(imid,jp)+U(imid,jm))/2
847 -gam*r*(U(imid,jp)-U(imid,jm))/2
848 +(1+r^2)*gam^2*(U(imid,jpp)-3*U(imid,jm)-U(imid,jmm))/48
849 -(9-10*r^2+r^4)*gam^3*(U(imid,jppp)-5*U(imid,jpp) ...}
850 +10*U(imid,jp)-10*U(imid,jm)+5*U(imid,jmm) ...
851 -U(imid,jmmm))/3840
852 ];
853 U(1,j)=sum(Ubv(1:2*interOrd,:));
854 uu1(1,j1)=U(1,j1); uu1(n+2,j1)=U(n+2,j1);
855 uu2(1,j1)=U(1,j1); uu2(n+2,j1)=U(n+2,j1);
856 \% lift the velocity onto the slow manifold of the two layers
857 uu1(n+2,j0)=0.587*gam*U(n+2,j0) ...
858 +0.0129*re*(sin(theta)-U(n+2,j0).^3/8/dx*0);
859 uu2(n+2,j0)=1.413*U(n+2,j0) ...
860 -0.0129*re*(sin(theta)-U(n+2,j0).^3/8/dx*0);
861 uu1(1,j0)=0.587*U(1,j0) ...
862 +0.0129*re*(sin(theta)-U(1,j0).^3/8/dx*0);
863 uu2(1,j0)=1.413*U(1,j0) ...
864 -0.0129*re*(sin(theta)-U(1,j0).^3/8/dx*0);
865 if Ln==1
866 \%% linear case of the model
867 \% approximate dh/dx=-d(u1+u2)/dx
868 du1dt(i1,j1)=-(U(i1+1,j1)-U(i1-1,j1))/dx/2;
869 }
\begin{verbatim}
876   dhu1dt(i0,j0)=-(U(i0+1,j0)-U(i0-1,j0))/dx/2;
877   dhu2dt=dhu1dt;
878   \%
879   \%
880   \%
881   \%
882   \%
883   \%
884   \%
885   \%
886   \%
887   \%
888   \%
889   \%
890   \%
891   \%
892   \%
893   \%
894   \%
895   \%
896   \%
897   \%
898   \%
899   \%
900   \%
901   \%
902   \%
903   \%
904   \%
905   \%
906   \%
907   \%
908   \%
909   \%
910   \%

\end{verbatim}
u2xx=(uu2(i0(1:end-2),j1)-2*uu2(i0(2:end-1),j1) ... 
+uu2(i0(3:end),j1))/dx^2*4;

dhu1dt(i0,j1)=dhu1dt(i0,j1)+1/re*(-3.84+19.3*cc) ... 
*[uu1(i0(1),j1)*0;u1xx;uu1(i0(end),j1)*0] ... 
+1/re*(2.52-6.98*cc) ... 
*[uu2(i0(1),j1)*0;u2xx;uu2(i0(end),j1)*0];

dhu2dt(i1,j0)=dhu2dt(i1,j0)+1/re*(-1.98-6.98*cc) ... 
(*(uu1(i1-2,j0)-2*uu1(i1,j0)+uu1(i1+2,j0))/dx^2*4 ... 
+1/re*(5.23+5.36*cc) ... 
*(uu2(i1-2,j0)-2*uu2(i1,j0)+uu2(i1+2,j0))/dx^2*4;

%% approximate the advection terms

dhu1dt(i1,j0)=dhu1dt(i1,j0)-1.48*uu1(i1,j0) ... 
.*(uu1(i1+2,j0)-uu1(i1-2,j0))/dx/4 ... 
-0.225*uu1(i1,j0).*uu1(i1+2,j0)-uu2(i1-2,j0))/dx/4 ... 
+0.142*uu2(i1,j0).*uu1(i1+2,j0)-uu1(i1-2,j0))/dx/4 ... 
+0.0728*uu1(i1,j0).*uu2(i1+2,j0)-uu2(i1-2,j0))/dx/4;

u1x=(-uu1(i0(1:end-2),j1)+uu1(i0(3:end),j1))/dx/4;

u2x=(-uu2(i0(1:end-2),j1)+uu2(i0(3:end),j1))/dx/4;

dhu1dt(i0,j1)=dhu1dt(i0,j1)-1.48*uu1(i0,j1) ... 
.*[uu1(i0(1),j1)*0;u1xx;uu1(i0(end),j1)*0] ... 
-0.225*uu2(i0,j1) ... 
.*[uu2(i0(1),j1)*0;u2xx;uu2(i0(end),j1)*0] ... 
+0.142*uu2(i0,j1) ... 
.*[uu1(i0(1),j1)*0;u1xx;uu1(i0(end),j1)*0] ... 
+0.0728*uu1(i0,j1) ... 
.*[uu2(i0(1),j1)*0;u2xx;uu2(i0(end),j1)*0];

dhu2dt(i1,j0)=dhu2dt(i1,j0)-1.25*uu1(i1,j0) ... 
.*(uu1(i1+2,j0)-uu1(i1-2,j0))/dx/4 ... 
-1.57*uu2(i1,j0).*uu2(i1+2,j0)-uu2(i1-2,j0))/dx/4 ... 
+0.768*uu2(i1,j0).*uu1(i1+2,j0)-uu1(i1-2,j0))/dx/4 ... 
+0.930*uu1(i1,j0).*uu2(i1+2,j0)-uu2(i1-2,j0))/dx/4;

dhu2dt(i0,j1)=dhu2dt(i0,j1)-1.25*uu1(i0,j1) ... 
.*[uu1(i0(1),j1)*0;u1xx;uu1(i0(end),j1)*0] ... 
-1.57*uu2(i0,j1) ... 
.*[uu2(i0(1),j1)*0;u2xx;uu2(i0(end),j1)*0] ... 
+0.768*uu2(i0,j1) ... 
.*[uu1(i0(1),j1)*0;u1xx;uu1(i0(end),j1)*0] ...
+0.930*uu1(i0,j1) ...  
.\ast[uu2(i0(1),j1)*u2x;uu2(i0(end),j1)*0];

% approximate the evolution of edges of odd patch.

% needed for the smoothing operator L

dhu1dt(1,j0)=-1.482/re./uu1(2,j0).^2.*U(1,j0) ...
   +0.4904*(g*sin(theta)+uu1(1,j0)/dx*0);

dhu1dt(n+2,j0)=-1.482/re./uu1(n+1,j0).^2.*U(n+2,j0) ...
   +0.4904*(g*sin(theta)+uu1(1,j0)/dx*0);

dhu2dt(1,j0)=-3.526/re./uu2(2,j0).^2.*U(1,j0) ...
   +1.16804*(g*sin(theta)+uu2(2,j0)/dx*0);

dhu2dt(n+2,j0)=-3.526/re./uu2(n+1,j0).^2.*U(n+2,j0) ...
   +1.16804*(g*sin(theta)+uu2(2,j0)/dx*0);

% define the smooth operator L.
% define the index

nn=length(dhu1dt(:))/2;

il=[1:nn];
ilp=[2:nn,1]; ilm=[nn,1:nn-1];

hh=U(1:2:end);

% symmetric operator in front of du/dt of 1-Cd/dx(h^2d/dx)
ccop=sparse(il,il,1+cc/dx^2*(hh(il).^2+hh(ilp).^2),nn,nn) ...
   +sparse(il,ilm,-cc/dx^2*hh(il).^2,nn,nn) ...
   +sparse(il,ilp,-cc/dx^2*hh(ilp).^2,nn,nn);

%ccop=full(ccop),abort
% form final time derivative into a vector

dhudt=hu(:); % make vector the correct size

dhudt(1:2:end)=dhu1dt(2:end-1,:);
dhudt(2:2:end)=dhu2dt(2:end-1,:);
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