

University of Adelaide
School of Chemistry and Physics

Doctor of Philosophy

Regularization and Renormalization of Non-Perturbative
Quantum Electrodynamics via the Dyson-Schwinger Equations

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Abstract

We investigate the effect of using different regularization and renormalization schemes in studying numerical solutions of quenched quantum electrodynamics (QED) using the Dyson-Schwinger equations for the fermion propagator. A primary motivation is to explore dynamical chiral symmetry breaking (DCSB) and mass generation in the theory - typically, QED undergoes a phase transition to a DCSB state above a certain critical value of the interaction strength.

Previous studies have used a cut-off regulator to render the equations finite before applying off-shell renormalization. A major outcome of these studies has been to show the equivalence of dimensional regularization and the cut-off regulator incorporating the gauge covariance improvement which maintains translational invariance of the theory. This was tested numerically using both a minimal vertex and a more realistic Ansatz based on leading log expansion (the Curtis-Pennington vertex). A recasting of the equations for the cut-off regulator in favour of using renormalized quantities exclusively (the regularization-independent method) allowed for much higher precision in extrapolating the ultraviolet end of the fermion propagator, provided one knows its functional form. These are well known by now: for example, the mass function is real power-law behaved below criticality and complex power-law behaved above criticality. The power-law exponents and location of the critical interaction strength, which depend on the choice of vertex, are theoretically calculable, and should take into account the gauge covariance improvement.

Nevertheless, undesirable features remain: for example, the presence of oscillations in the mass function above criticality, and a lack in understanding how to take the chiral limit in a well-defined manner. Whether these defects are fundamental or merely artifacts of the quenched approximation must await a fuller exploration of the unquenched theory which solves the photon equation as well.

Statement of Originality

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree. I give consent to this copy of my thesis when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968. The author acknowledges that copyright of published works contained within this thesis (as listed below) resides with the copyright holder(s) of those works. I also give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library catalogue and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

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