The Wave Function of the Nucleon and its Excited States

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To Mark

Acknowledgements

I sit here staring at a blank page, struggling to find the words to condense the incredible amount of support I've received from friends and family over the last 5 years into a reasonable amount of space. To say 'I couldn't have done it without you all' is an understatement, I wouldn't have thought getting this far was even possible.

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So, Belgian at 5, anyone?

Abstract

In this thesis, we detail the techniques required to perform general lattice QCD calculations. Specifically, we introduce the method by which the continuum theory of quarks and gluons, Quantum Chromodynamics is discretised in order to be solved numerically. We describe the distinct methods by which the discrete actions for the gauge and fermion fields given by naively applying a finite-difference approximation to the continuum theory can be improved, going some way to remove the systematic errors of discretisation. The background field method for placing electromagnetic fields onto a discrete lattice is also introduced.

Techniques required for the calculation of wave functions are then introduced, beginning with the two-point function, which is fundamental in extracting properties of hadrons from the lattice. The variational method, which allows access to the excited states of particles is then introduced. The wave function is then constructed from the two-point function, which forms the basis of the most significant results of this thesis. We also introduce gauge fixing, made necessary by the gauge dependent nature of wave function operators.

The smeared operators used in the construction of these two-point functions are evaluated, by way of two measures designed to measure the coupling strength of these operators to states with a variety of momenta. Of particular interest is the extent to which strong overlap can be obtained with individual high-momentum states. This is vital to exploring hadronic structure at high momentum transfers on the lattice and addressing interesting phenomena observed experimentally. We consider a novel idea of altering the shape of the smeared operator to match the Lorentz contraction of the probability distribution of the high-momentum state, and show a reduction in the relative error of

the two-point function by employing this technique. Our most important finding is that the overlap of the states becomes very sharp in the smearing parameters at high momenta and fine tuning is required to ensure strong overlap with these states.

Making use of the background field methods and the wave functions constructed from the two-point functions, we calculate the probability distributions of quarks in the ground state of the proton, and how they are affected in the presence of a constant background magnetic field. We focus on wave functions in the Landau and Coulomb gauges using the quenched approximation of QCD. We observe the formation of a scalar $u - d$ diquark clustering. The overall distortion of the quark probability distribution under a very large magnetic field, as demanded by the quantisation conditions on the field, is quite small. The effect is to elongate the distributions along the external field axis while localizing the remainder of the distribution. Using optimised smearing parameters calculated from the methods detailed in this thesis, we construct wave functions of high-momentum states, and are able to qualitatively observe high momentum states. We find that, at very high momenta, artefacts are present caused by the poor overlap of these states to the interpolating operators. Careful tuning of the smearing parameters is shown to reduce these artefacts, reinforcing results presented earlier in the thesis.

The culmination of the techniques introduced and the results obtained in this thesis is the application of the eigenvectors from a variational analysis to successfully extract the wave functions of even-parity excited states of the nucleon, including the Roper, in full QCD. We explore the first four states in the spectrum excited by the standard nucleon interpolating field. We find that the states exhibit a structure qualitatively consistent with a constituent quark model, where the ground, first-, second- and third-excited states have 0, 1, 2, and 3 nodes in the radial wave function of the d-quark about two u quarks at the origin. Moreover the radial amplitude of the probability distribution is similar to that predicted by constituent quark models. We present a detailed examination of the quark-mass dependence of the probability distributions for these states, searching for a nontrivial role for the multi-particle components mixed in the finite-volume QCD eigenstates. Finally we examine the dependence of the d-quark probability distribution on the positions of the two u quarks. The results are fascinating, with the underlying S-wave orbitals governing the distributions even at rather large u-quark separations.

Statement of Originality

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution to Dale Roberts and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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