



**Behaviour of Quantum Chromodynamics
Near an Infrared Fixed Point**

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Abstract

In perturbation theory, the QCD running coupling depends on the renormalization scheme or is parameterised by a physical process. The problem is that artefacts of this ambiguity may upset physical conclusions outside the asymptotically free region, in particular near an infrared fixed point. Thus, a non-perturbative definition for the QCD running coupling is required that should be a monotonic and analytic function of the space-like energy scale Q^2 . The most physical coupling is Grunberg's definition for the running coupling as an effective charge $\alpha_G(Q^2)$. However, we find that it works only for sufficiently high energy scales. At some finite values of the energy scale, near the top of the resonance region, the β -function associated with $\alpha_G(Q^2)$ has a false zero below which $\alpha_G(Q^2)$ decreases.

We test this conclusion further by applying chiral perturbation theory to the running coupling based on the method of effective charges. We consider the Drell-Yan ratio $R(q^2)$ and Adler-function $D(Q^2)$ in the time-like and space-like domains, respectively. These quantities tend to a finite value in the both infrared and ultraviolet limits: $R(0) = D(0) = 0.5$ for π^\pm , K^\pm and $R(\infty) = D(\infty) = 2$ for the light quarks. This means that the running coupling becomes negative in the infrared limit. Therefore, neither the time-like nor the space-like effective charges is a monotonic function over the whole energy scale.

We also try to cancel the space-like pole of the proposed exact β -function, using the property of renormalization scheme dependence. Our modified exact β -function is non-singular for all finite values of the running coupling α and has an infrared fixed point even for a small number of quark flavours.

Statement of Originality

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List of workshop proceeding and future publication based on this thesis:

- G. O. Souadi, *When is an Infrared Fixed Point of QCD Physical?*, poster presented at workshop "Determination of the Fundamental Parameters of QCD", Nanyang Technological University, Singapore, March 18-21, 2013.
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