Noncommutative Geometry
Methods in Number Theory

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Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

SIGNED: .....................  DATE: .....................
I undertook a Master of Philosophy program at the University of Adelaide, predominantly for one reason: to be able to study under Elder Professor Mathai Varghese. It has been an incredible experience not hampered at all by the fact that this area of research was new to the both of us. I would like to strongly accentuate how rewarding this experience has been for me and how much I have learnt about the process of research and the life of a mathematician.

Very importantly, I must add that I appreciate Professor Varghese’s infinite patience. Beset by health issues throughout my Masters and other personal tragedies, the timeframe of progress has been rather non-linear and would no doubt try the patience of any other supervisor. I appreciate Professor Varghese sticking by me and believing that I would get past my issues. That is not to say that he did not take me to task on several occasions, but I am much stronger and better off for it.

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Additionally, my studies have been supported by an Australian Postgraduate Award provided by the federal government, which has given me the ability to support myself while studying. I am very grateful for this generosity.

I would like to add a sincere thank you to the examiners of this thesis, who gave me significant words of encouragement as well as advice for improvement. Moreover, I appreciate
them taking time out of their busy schedules to read through this thesis and provide me with a list of errata and recommended fixes that have been fixed and implemented, respectively, for this final edition.
Abstract

Noncommutative geometry deals with many natural spaces for which the classical set-theoretic tools of analysis, such as measure theory, topology, calculus, and metric ideas lose their pertinence, but which correspond very naturally to a noncommutative algebra. While looking for new ways to get information about the Riemann zeta function (as an essential ingredient of the Riemann Hypothesis), Alain Connes and Benoît Bost published a paper [5] in 1995 discussing the use of certain quantum statistical mechanical systems (noncommutative geometric objects) called Bost-Connes systems to recover the class field theory for number fields. The Bost-Connes system surprisingly relates to Galois theory and the maximal abelian extensions of number fields. This has led to speculation that it may have use in trying to get a deeper understanding of abelian extensions of number fields and thus has been seen as one approach to solving Hilbert’s 12th problem, which aims to generalise an incredible result called the Kronecker-Weber Theorem, which asserts that the maximal abelian extension of the rationals, \( \mathbb{Q}^{ab} \), is generated by the roots of unity. Hilbert’s 12th problem asks, for any given number field \( K \), to describe the generators of the maximal abelian extension \( K^{ab} \) of \( K \) in terms of algebraic values of a transcendental function of \( K \). While there has been progress in this area using modern number theoretic methods, it remains unsolved.

This thesis is predominantly an exposition of the research that has been done in this very young and exciting area. In particular, the Bost-Connes system will be reformulated as an enveloping C*-algebra of commensurability classes of \( \mathbb{Q} \)-lattices, as a groupoid C*-dynamical system and as a semigroup crossed product system.

In the thesis a new Bost-Connes type system is constructed, called the partial Connes-Marcolli system and it generalises the systems that are currently studied. It also satisfies a generalised Bost-Connes Problem statement. At the end of the thesis there is also a discussion on how these systems could be used to try to deduce the Kronecker-Weber Theorem from purely noncommutative geometry objects.
Part I

Introduction to Bost-Connes Systems
Chapter 1

Introduction

The French mathematician Alain Connes, world renowned for his work in noncommutative geometry, describes in his seminal work [7], that the reasons for the existence of this branch of mathematics are in particular the existence of many natural spaces for which the classical set-theoretic tools of analysis, such as measure theory, topology, calculus, and metric ideas lose their pertinence, but which correspond very naturally to a noncommutative algebra, and also the fact that there are extensions of the classical tools, such as measure theory, topology, differential calculus and Riemannian geometry, to the noncommutative situation.

Examples of spaces that correspond naturally to a non-commutative algebra include the space of Penrose tilings, the space of leaves of a foliation, the space of irreducible unitary representations of a discrete group, the phase space in quantum mechanics, the Brillouin zone in the quantum Hall effect and space-time. In particular, noncommutative differential geometry studies spaces $X/\sim$, where $X$ is a space and $\sim$ is an equivalence relation on it. This theory considers $X/\sim$ as the noncommutative algebra of coordinates on this space rather than the classical commutative ring of functions of the space and tries to extend the tools of topology and analysis to study it. The graph of the commensurability classes of $\mathbb{Q}$-lattices that is considered as part of this work is treated in this way.

In more recent times, Alain Connes and others have developed this theory to be a useful and relevant tool for tackling problems well outside the usual setting for non-commutative geometry. The motivation behind this is understandable. Many old, perpetually unsolved problems seem unmovable by standard techniques and new and novel ideas must be at the forefront of work into such research paths. Precisely for this reason, while looking for new ways to get information about
the Riemann zeta function (as an essential ingredient of the Riemann Hypothesis), Alain Connes and Benoît Bost published a paper [5] in 1995 discussing the use of certain quantum statistical mechanical systems called Bost-Connes systems to recover the class field theory for number fields. Their intention was to create quantum statistical mechanical systems whose partition function is the Riemann zeta function while also exhibiting non-trivial interaction.

The results were surprising yielding properties of the Bost-Connes system that related to Galois theory and maximal abelian extensions of number fields. This has led to speculation that it may have use in trying to get a deeper understanding of abelian extensions of number fields and thus has been seen as one approach to solving Hilbert’s 12th problem. Hilbert’s 12th problem aims to generalise an incredible result called the Kronecker-Weber Theorem, which asserts that the maximal abelian extension of the rationals, $\mathbb{Q}^{ab}$, is generated by the roots of unity, which are algebraic numbers in $\mathbb{Q}$ and specific values of the exponential function. Since it looks for a generalised analogue of this, Hilbert’s 12th problem asks, for any given number field $K$, to describe the generators of the maximal abelian extension $K^{ab}$ of $K$ in terms of algebraic values of a transcendental function of $K$. While there has been progress in this area using modern number theoretic methods, it remains unsolved.

In particular, the classical Bost-Connes system for $\mathbb{Q}$ uses the Kronecker-Weber Theorem to get an intertwining of Galois action and $KMS_\infty$ states of the system. In this thesis, I will attempt to do the reverse and use the Bost-Connes system and from its properties deduce the Kronecker-Weber Theorem. A few different possible methods for achieving this have been theorized including some discussion by Matilde Marcolli in [34]. This thesis takes this in a slightly different direction by constructing Bost-Connes type systems that can be inducted to the original Bost-Connes system. It is hopeful that this result can be generalised for arbitrary number fields, developing more knowledge of Hilbert’s 12th problem.

The classical Bost-Connes system (with base $\mathbb{Q}$) was described in [5] as the C*-enveloping algebra of a Hecke pair $(P_\mathbb{Q}, P_{\mathbb{Z}})$, which they denoted as $C(P_\mathbb{Q}, P_{\mathbb{Z}})$. Here $P_R$ is the functor that takes a ring $R$ into a group:

$$P_R = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : a, b \in R; a \text{ invertible} \right\}.$$

Benoît Bost and Alain Connes investigated the equilibrium states of this system. For a quantum statistical mechanical system, like the Bost-Connes system, equilibrium states can be characterized by the KMS$_\beta$ condition (at inverse temperature $\beta$). These states form a convex set, so there is a notion of extremal states. Moreover, they studied how these extremal states behaved
on a subalgebra of the system, which they denoted $Q(P_Q, P_Z)$. What they found can be summed up by the following theorem.

**Theorem.** ([5] Theorem 5, [10] Theorem 3.32). The equilibrium states of the above system at inverse temperature $1/\beta$, denoted by $KMS_\beta$ have the following properties:

1. For high temperatures ($0 < \beta \leq 1$) there exists a unique $KMS_\beta$ state $\phi_\beta$ on $C(P_Q, P_Z)$.
2. For low temperatures ($1 < \beta < \infty$) the extremal $KMS_\beta$ states are labeled by the group $Gal(Q^{ab}/Q)$.
3. At zero temperature, the extremal $KMS_\infty$ states $\phi$ have the property that $\phi(Q(P_Q, P_Z)) \subset Q^{ab}$.

Additionally, the group $\hat{\mathbb{Z}}^*$ acts by automorphism on the system $C(P_Q, P_Z)$, and if $\theta : \hat{\mathbb{Z}}^* \to Gal(Q^{ab}/Q)$ is the global class field theory isomorphism for the idèle class group of $Q$, we also have:

$$\gamma(\phi(f)) = \phi(\theta^{-1}(\gamma)f)$$

for all $\gamma \in Gal(Q^{ab}/Q)$ and for all $f \in Q(P_Q, P_Z)$.

One can see that at low temperatures ($1 < \beta$) in the classical Bost-Connes system, the Galois group of the maximal abelian extension of $Q$ labels the extremal equilibrium states, and moreover the values of these states on a certain subalgebra generate $Q^{ab}$. As this subalgebra, $Q(P_Q, P_Z)$, is an arithmetic subalgebra, this situation recovers the Kronecker-Weber Theorem for $Q$. If one generalises the Bost-Connes system to an arbitrary number field $K$, one hopes that it will yield an analogous result. This is the reason behind the application to Hilbert’s 12th problem.

This is a very recent and popular area of research. For instance, there has been progress in the direction of generalising the system to an arbitrary number field and adapting the system to a function field setting. Moreover, the same system has been rewritten in many forms including as an enveloping C*-algebra of a commensurability groupoid of $Q$-lattices, as a semigroup crossed product and most recently in terms of endomotives. Each of these reformulations and subsequent analysis has yielded an abundance of interesting results in a large array of areas, sometimes only tangentially related to the Bost-Connes system. In particular, one notes that the concept of a semigroup crossed product with a C*-algebra has undergone an extensive program of study building on and refining decades of partial results and extending this research to other concepts such as K-theory. More detail about this is included in Chapter 2.
1.1 Thesis summary

In Chapter 2, I begin by recalling the foundations of quantum statistical mechanical systems as well as C*-algebras. With this background, the motivation and construction of the original Bost-Connes system are introduced. A few key statements about the properties of Bost-Connes systems are made including an explanation of directions and progress that mathematicians have taken since Benoît Bost and Alain Connes’ original paper [5].

Although the thesis is mostly concerned with the noncommutative aspect of Bost-Connes systems, there is a real need to have a grasp of at least basic field theory and class field theory to have an understanding of the kind of objects being described and how they act. Chapter 3 brings together the prerequisite literature on field theory, Galois theory, ramification theory, class field theory with particular emphasis on p-adics and the Artin map.

The background provided in Chapter 3 is used in Chapter 4 to prove the Kronecker-Weber theorem in a couple of ways. The aim of the author’s work in the later chapters is aimed towards proving the Kronecker-Weber theorem and hence, a few proofs that are relevant are provided that also give an insight into some of the deeper ideas involved in the constructions made in Chapter 8. There is a proof via class field theory by Eknath Gathe [19] and one via ramification theory by Greenberg [20].

The next part of the thesis is focused on the original Bost-Connes system and is an expository work on the important work done to generalise the system and the directions it has taken mathematicians. In particular, Alain Connes, Matilde Marcolli and Niranjan Ramachandran in [11] found a fascinating reformulation of the Bost-Connes system in terms of commensurability classes of geometric objects called $\mathbb{Q}$-lattices. For this reason, Chapter 5 introduces the theory of groupoids, groupoid C*-algebras and $\mathbb{Q}$-lattices. Moreover, it is shown that the two formulations are indeed isomorphic. It is discussed that this new reformulation has an obvious path to trying to generalise the system to be based on an arbitrary number field.

This generalisation to arbitrary number fields was only partially successful as the system displayed many of the desired properties but was more closely aligned to the maximal cyclotomic extension of a number field $K^{cycl}$, which is deeply understood, as opposed to the maximal abelian extension $K^{ab}$, which is mostly a complete mystery. A new reformulation this time in terms of a semigroup crossed product gives more promising results. So, Chapter 6 elucidates the theory behind semigroups and crossed products of C*-algebras by semigroups. Again, there is a lot of work done to show that the new system is isomorphic to the original Bost-Connes system and has
convenient ways of being generalised to arbitrary number fields. This brings about the definition of the Connes-Marcolli systems, which satisfy the desired generalisation conditions enunciated in the Bost-Connes Problem. The Connes-Marcolli system is the C*-dynamical system
\[(A_K, \sigma_t)\]
where \(A_K\) is the semigroup crossed product C*-algebra \(A_K = C(Y_K) \rtimes J_K^+\), together with the time evolution \(\sigma_t\). \(Y_K\) here is defined as
\[Y_K := \text{Gal}(K^{ab}/K) \times \hat{O}_K\]
defined as the quotient space of the direct product \(\text{Gal}(K^{ab}/K) \times \hat{O}_K\) under the action of \(\hat{O}_K^*\) given by
\[s \cdot (\alpha, \rho) = (r_K(s)^{-1} \alpha, \rho s)\]
where \(\hat{O}_K\) is the ring of finite integral ideals of \(K\), \(J_K^+\) is the monoid of nonzero integral ideals of \(K\) and \(r_K\) is the Artin map for \(K\). The time evolution is given by \(\sigma_t( fu_s) = N_K(s)^i fu_s\), where \(f \in C(Y_K)\) and \(u_s\) the isometry encoding the action of \(s \in J_K^+\).

Chapter 7 aims to describe the theory behind the different properties of a C*-dynamical system. In particular, this chapter explains the concepts of partition function, symmetry groups, Hamiltonian operator as well as KMS\(_\beta\) states. These properties are calculated for the original Bost-Connes system here.

This marks the end of the exposition of material on Bost-Connes systems and Connes-Marcolli systems. In Chapter 8, the author motivates and constructs a Bost-Connes type system that generalises even the Connes-Marcolli system. Trying to prove the Kronecker-Weber theorem by inductively building up the original Bost-Connes system from smaller systems requires the construction of what I have called partial Connes-Marcolli systems. The systems are based on an arbitrary number field but encapsulate the structure of the Galois group of an arbitrary abelian extension rather than the full maximal abelian extension. Moreover, the system has been constructed modulo a finite set of primes of the base number field.

Chapter 9 shows that the partial Connes-Marcolli systems satisfy a modified Bost-Connes problem. This is done by fitting the new construction into a framework first analysed by Marcelo Laca, Nadia Larsen and Sergey Neshveyev in [26]. Using this, all of the properties of the system are calculated.

The last chapter describes how one would use the partial Connes-Marcolli to try and prove the Kronecker-Weber theorem. In particular, a brief analysis describes the partial Connes-Marcolli
systems for base number field \( \mathbb{Q} \). There is a description of how one would take the inductive limit of partial Connes-Marcolli systems. The last ingredient is the use of a theorem by Gunther Cornelissen and Matilde Marcolli in [13] that the isomorphism of number fields is equivalent to isomorphism of their respective generalised Bost-Connes (Connes-Marcolli) systems. There is also a discussion of further work that can be attempted in this direction.
Chapter 2

Original Bost-Connes System

The original Bost-Connes system was published by Alain Connes and Jean-Benoît Bost in their 1995 article ‘Hecke Algebras, Type III factors and Phase Transitions with Spontaneous Symmetry Breaking in Number Theory’ [5]. According to the authors, this work was inspired by the earlier results of Bernard Julia [23], whose research drew upon the desire to present a dictionary between on the one hand, the additive and multiplicative aspects of number theory, and on the other, quantum statistical mechanics (or "solved" quantum systems). Given that there had already been significant work in the understanding of how the generating functions of additive number theory relate to certain aspects of statistical mechanics, Julia focussed on presenting something similar for multiplicative number theory.

Multiplicative number theory is most conveniently encoded by Dirichlet series, which are series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $s$ is complex and $(a_n)_{n \in \mathbb{N}}$ is a complex sequence. One of the most famous examples of a Dirichlet series is the Riemann zeta function. Bernard Julia suggests that this dictionary can be encoded by the equivalence of certain Dirichlet series and partition functions of quantum statistical mechanical systems, which are normalising functions and can be used to calculate equilibrium states of the system. Quantum statistical mechanical systems and partition functions are explained in Section 2.1. It is this idea that led Bost and Connes to consider their work. It is worth mentioning that while this program was of course novel and contained the fascinating aspects of bringing together two rather distinct branches of mathematics, there was also a very clear inspiration in mind; namely the Riemann Hypothesis deals with zeros of the Riemann zeta function. A thermodynamic system that encodes the information of the Riemann zeta function.
can hopefully shed light on this holy grail of mathematics.

They first consider a dynamical system $C^*(\mathbb{N}^*)$ (this will be introduced in Section 2.2), which had properties they were looking for: the partition function was the Riemann zeta function and this system was closely related to the distribution of the prime numbers, but it is not all that interesting, by their own admission, as it is a system without interaction. By this we mean that the mathematics does not take into account interaction between the observables of the system. The lack of interaction means the disorder of the system dominates giving rise to a unique equilibrium state at all temperatures. They rework this system into a system with interaction which turns out to be $C^*(\Gamma, \Gamma_0)$ (to be described in Section 2.3). For high temperatures, we still expect disorder and hence a unique equilibrium state, however, for lower temperatures, order should set in giving rise to the possibility of various thermodynamical phases, or essentially multiple equilibrium states. In the work they did they found that these equilibrium states at low temperatures formed a set whose extreme values generate the maximal abelian Galois group of the rationals. This incredible result is what has prompted this fruitful area of research as is discussed in Section 2.5.

2.1 Dynamical Systems

Before any details of the Bost-Connes system are introduced, it would be remiss of me to omit any explanation of the type of object a Bost-Connes system is. While not strictly required to understand the mathematics, knowledge of dynamical systems and in particular, quantum statistical mechanical systems will shed light on the motivation behind the construction and related properties of Bost-Connes and Bost-Connes-like systems. Many of these facts presented in this section will be familiar to physicists, and I apologise for the brevity of this exposition as I am personally more interested in the mathematical application of this theory to my thesis.

In this thesis, there will be frequent reference to objects called dynamical systems. This is a very general mathematical object, whose geometric interpretation is rather simple; a dynamical system is a triple $(\mathcal{M}, f, \mathcal{T})$, where $\mathcal{M}$ is a manifold, $\mathcal{T}$ is a time domain and $f$ is a time evolution ($t \rightarrow f^t$, with $t \in \mathcal{T}$) such that $f^t$ is a diffeomorphism on $\mathcal{M}$. Examples of what is meant by a manifold include bounded operators on a Hilbert space or on a $C^*$-algebra. The time domains include the set of integers or reals. A diffeomorphism is a smooth invertible function between manifolds whose inverse is also smooth. It acts as an isomorphism between smooth manifolds.

This leads to many different types of systems, including quantum mechanical systems.
Quantum mechanical systems in practice describe physical systems at the microscopic level, where interestingly certain quantities such as electron spin angles are quantised rather than continuous. In the abstract mathematical formulation, a quantum mechanical system contains three fundamental ingredients: observables, states and dynamics. Given a Hilbert space $\mathcal{H}$ (defined in Definition 2.2.4), one views the states as vectors on $\mathcal{H}$, the observables are self-adjoint operators on $\mathcal{H}$ and the dynamics is given by a one parameter group of unitary transformations, which is called a time evolution for convenience. The expected value of an observable $A$ on a state $\phi$ can be calculated using the inner product $\langle A\phi, \phi \rangle$. Also, the time evolution $\sigma_t$ is encoded by the use of a self-adjoint operator Hamiltonian through the formula

$$\sigma_t(A) = e^{itH} A e^{-itH}.$$  

One can also study the dynamics of the system by looking at symmetry groups of the system, which are given by the unitary transformations of the system. So far, only a conceptual image of quantum statistical mechanical systems has been presented. The definitions of the objects herein described, such as Hilbert spaces, self-adjoint operators, unitary operators and C*-algebras, follow in Section 2.2.

When one considers ensembles of $N$ such quantum mechanical systems, as $N$ increases it becomes more convenient to treat this statistically and for large (or infinite) $N$, such an ensemble is called a quantum statistical mechanical system (or C*-dynamical system). Throughout this work, I follow the same structure for a quantum statistical mechanical system as is outlined in Matilde Marcolli’s paper [34]. The construction works by setting up a Gibbs canonical ensemble and then taking the limit $N \to \infty$.

This is done by following these steps. First we need to take a Hilbert space of wave functions $\mathcal{H}_\Gamma$ on a finite region of space $\Gamma$. Now, a state $\rho$ is a positive operator of trace 1 on $\mathcal{H}_\Gamma$. The expected value of an observable $A \in \mathcal{B}(\mathcal{H}_\Gamma)$ on a state $\rho$ is hence given as $\text{trace}(\rho A)$.

The system needs to be analysed at different temperatures $T$. Usually, this is accomplished by considering different values of the inverse temperature $\beta = \frac{1}{kT}$ (where $k$ is the Boltzmann constant, which we shall henceforth set to equal 1). As mentioned earlier in this chapter systems that contain interaction, such as large ensembles, tend to have disorder for small $\beta$ and this disorder dominates and we get a single equilibrium state for each value, but for large $\beta$, the dynamics is ordered and allows for multiple equilibrium states.
We set an equilibrium state to

$$\rho^\beta = \frac{1}{\text{trace}(e^{-\beta H})} e^{-\beta H},$$

where the expression $Z(\beta) = \text{trace}(e^{-\beta H})$ is called the partition function.

At this point, we turn our ensemble of quantum mechanical systems into a quantum statistical mechanical system by turning to the limit $N \to \infty$ (maintaining a finite ratio between $N$ and the volume of our region $\Gamma$). First, note that to do this, it is important to generalise $\mathcal{B}(\mathcal{H}_\Gamma)$ (bounded linear operators on $\mathcal{H}_\Gamma$) to a $C^*$-algebra of observables. Moreover, the equilibrium states now satisfy an equilibrium condition called the Kubo-Martin-Schwinger, or KMS $\beta$, condition that is defined in Definition 7.1.1 and the focus of Chapter 7 and will be explained in significant detail there.

Putting all this together, gives the following definition:

**Definition 2.1.1.** A quantum statistical mechanical system is a $C^*$-algebra of observables $\mathcal{A}$ with a one parameter time evolution $\sigma_t : \mathbb{R} \to \text{Aut}(\mathcal{A})$ given by $t \mapsto e^{itH} A e^{-itH}$ for $A \in \mathcal{A}$ and $H$ a self-adjoint operator called the Hamiltonian. States are positive linear functionals of norm 1 and equilibrium states at each inverse temperature $\beta$ are characterised by the KMS $\beta$ condition.

### 2.2 C*-algebras

In this section we recall the definition of a $C^*$-algebra. This theory can be found in any text on $C^*$-algebras including [15, 16]. First, recall the definition of a Banach algebra.

**Definition 2.2.1.** An algebra $A$ is a vector space (in our case over $\mathbb{C}$) that has an associative multiplication that is distributive over vector addition compatible with scalar multiplication. $A$ is a normed algebra if it has a norm $\| \cdot \|$ that satisfies $\|ab\| \leq \|a\| \cdot \|b\|$ for $a, b \in A$ and the norm of the identity (if it exists) is 1. $A$ is a Banach algebra if it is a normed algebra complete in its norm.

One can equip some algebras with a map called an involution, which can be thought of as a generalisation of complex conjugation to algebras.

**Definition 2.2.2.** An involution on an algebra $A$ is a map $^* : A \to A$ ($a \mapsto a^*$) such that for all $a, b \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$
• $a^{**} = a$,
• $(\lambda_1 a + \lambda_2 b)^* = \overline{\lambda_1}a^* + \overline{\lambda_2}b^*$,
• $(ab)^* = b^*a^*$.

An algebra with an involution is called a $^*$-algebra.

This provides the ingredients to define a C*-algebra.

**Definition 2.2.3.** A Banach $^*$-algebra $A$ is a $C^*$-algebra if it satisfies

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$. This is known as the C*-algebra property.

Recall the definition of a Hilbert space as this is an important ingredient in many examples of C*-algebras.

**Definition 2.2.4.** Also, a Hilbert space is an inner product space which is complete with respect to the distance function induced by the inner product.

It is worth noting some important examples of C*-algebras:

**Examples.**
1. Bounded linear operators $B(H)$ on a Hilbert space $H$. Involution is given by the adjoint operator.
2. Compact linear operators $K(H)$ on a Hilbert space $K$. This is a C*-subalgebra of $B(H)$.
3. Both of the above are examples of concrete C*-algebras which are defined as self-adjoint, norm-closed subalgebras of $B(H)$.
4. Banach space $C(X)$ of all complex-valued functions on compact space $X$ with pointwise multiplication and addition, the operator norm ($\|f\| = \sup_{x \in X} |f(x)|$) and involution given by complex conjugation.
5. Banach space $C_0(X)$ of complex-valued functions on locally compact space $X$ that vanish at infinity (ie. outside of a compact set). The operations are as in the above example.
6. One can construct C*-algebras based on groups, semigroups and groupoids as well as other algebraic structures that preserve the algebraic properties, but also satisfying the C*-algebra conditions. This will be seen again in Chapters 5 and 6 in particular.

Examples 2 and 5 are examples of unital C*-algebras (C*-algebras with identity). However, not all C*-algebras are unital, but it turns out that one can very easily construct a unital C*-algebra that contains our original and preserves its structure. This is done through the following proposition.
Proposition 2.2.5. Let $\mathcal{A}$ be a $C^*$-algebra without identity and let $\mathcal{A}'$ denote the algebra of pairs

$\{(\alpha, A) : \alpha \in \mathbb{C}, A \in \mathcal{A}\}$

with operations

$$(\alpha, A) + (\beta, B) = (\alpha + \beta, A + B),$$

$$(\alpha, A)(\beta, B) = (\alpha \beta, \alpha B + \beta A + AB),$$

$$(\alpha, A)^* = (\overline{\alpha}, A^*).$$

It follows that the definition

$$|(\alpha, A)| = \sup\{ |\alpha B + AB| : B \in \mathcal{A}, |B| = 1 \}$$

yields a norm on $\mathcal{A}'$ with respect to which $\mathcal{A}'$ is a $C^*$-algebra.

The algebra $\mathcal{A}$ is identifiable as the $C^*$-subalgebra of $\mathcal{A}'$ formed by the pairs $(0, A)$ and is a maximal ideal of $\mathcal{A}'$.

In order to further discuss $C^*$-algebras, I introduce some terminology inherited from operator theory that can be found in standard texts such as Rudin [40] and Davidson [15].

Definition 2.2.6. Given an element $a$ in a $C^*$-algebra $\mathcal{A}$, we define

- the spectrum of $a$ as the set

$$\sigma_\mathcal{A}(a) = \{ \lambda : \lambda 1_\mathcal{A} - a \text{ is singular}, \lambda \in \mathbb{C} \}.$$  

If the algebra in question is clear, we can simply write $\sigma(a)$;

- $a$ is self-adjoint if $a^* = a$;

- $a$ is normal if $a^* a = aa^*$;

- $a$ is unitary if $a^* a = aa^* = 1$ where $1$ is the identity of $\mathcal{A}$;

- $a$ is positive if it is self-adjoint and its spectrum is contained in $[0, \infty) \subset \mathbb{R}$.

There are a few useful consequences of these definitions that will come in handy later in my work. Assume $\mathcal{A}$ is a $C^*$-algebra with element $a$ unless otherwise stated.

1. In a Banach algebra $\mathcal{A}$, multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (a, b) \mapsto a \cdot b$ is continuous.
2. In $\mathcal{A}$, the adjoint map $^*$ is isometric, ie. $|a| = |a^*|$.
3. In $\mathcal{A}$, $|a| = \sup_{|x| \leq 1} |ax| = \sup_{|x| \leq 1} |xa|$.
4. If $\mathcal{A}$ has identity, $e$, then $e = e^*$ and $|e| = 1$.
5. The spectrum $\sigma(a)$ of any element $a$ of $\mathcal{A}$ is a non-empty compact set; and the resolvent function $R_a : \mathbb{C} \setminus \sigma(a) \to \mathcal{A}, \lambda \mapsto (\lambda I - a)^{-1}$ is analytic on $\mathbb{C} \setminus \sigma(a)$.
6. $\sigma(a) = \overline{\sigma(a^*)}$.
7. If \( a \) is unitary then \( |\lambda| = 1 \) for all \( \lambda \in \sigma(a) \).

8. If \( a \) is self-adjoint then \( \sigma(a) \) is real.

It is necessary to be able to relate C*-algebras. Recall the definition of algebra homomorphisms.

**Definition 2.2.7.** A homomorphism \( \pi : A_1 \rightarrow A_2 \) between algebras \( A_1 \) and \( A_2 \) is a map preserving all algebraic structure. If the homomorphism preserves the involution, then it is called a \(*\)-homomorphism. We will usually refer to a \(*\)-homomorphism between a C*-algebra and the bounded operators on some Hilbert space as a representation of that C*-algebra.

Consequently, the following remarks can be made about \(*\)-homomorphisms.

**Remark.**

1. If \( A \) and \( B \) are unital C*-algebras and \( \pi : A \rightarrow B \) is a \(*\)-homomorphism then
   \[ ||\pi(a)||_B \leq ||a||_A. \]
   In particular, \( \pi \) is automatically continuous.

2. If \( \pi \) is a \(*\)-isomorphism, then it is isometric, i.e. the above inequality is an equality. In particular, there is at most one norm that makes a Banach algebra with isometric involution into a C*-algebra.

The proofs of these useful facts are left to standard texts on C*-algebras such as [16].

We conclude our brief aside on C*-algebras by mentioning the Gelfand-Naimark-Segal (GNS) construction that shows that all C*-algebras can essentially be viewed as the bounded linear operators on some appropriately chosen Hilbert space. In order to do this, we require the concept of a state on a C*-algebra.

**Definition 2.2.8.** A state \( \phi \) on a C*-algebra \( A \) is a positive linear functional with norm 1. In other words, \( \phi \) is given by

\[ \phi : A \rightarrow \mathbb{C} \text{ such that } \phi(a^*a) \geq 0 \text{ and } ||\phi|| = \sup\{|\phi(a)| : ||a|| = 1\} = 1. \]

We are now ready to explain the GNS construction.

**Gelfand-Naimark-Segal Construction.** Given a C*-algebra \( A \) and a state \( \phi \) on \( A \), the formula

\[ \langle a, b \rangle = \phi(a^*b) \]

defines a semi-definite inner product. The set \( I := \{ a \in A : \phi(a^*a) = 0 \} \) is a left ideal of \( A \) and therefore the inner product is positive definite on the quotient \( A/I \). Completing \( A/I \) with respect to the inner product norm produces a Hilbert space \( \mathcal{H}_\phi \) associated to the state \( \phi \). Consider now
the representation \( \pi_{\phi} : A \to B(\mathcal{H}_\phi) \) given by

\[
\pi_{\phi}(a)[x] = [ax]
\]

where \([a]\) is the element \([a] = a + I\) in the quotient \(A/I\). This indicates that all \(C^*\)-algebras can be considered as bounded operators on a Hilbert space.

**Proof.** We show that \(\langle \cdot, \cdot \rangle\) is an inner product, by showing that

\[
\langle a, b \rangle = \phi(a^*b) = \phi((b^*a)^*) = \overline{\phi(b^*a)} = \overline{\langle b, a \rangle}
\]

and also

\[
\langle a, a \rangle = \phi(a^*a) \geq 0.
\]

Using the Cauchy Schwarz, gives that \(I\) is a left ideal of \(A\), because

\[
|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b) = 0 \quad \Rightarrow \phi(a^*b) = 0 \text{ for } b \in I.
\]

Thus, it is also well-defined on equivalence classes \([a]\) of \(A/I\) as follows:

\[
\langle [a], [b] \rangle = \langle a + x, b + y \rangle \quad \text{where } x, y \in I
\]

\[
= \phi((a + x)^*(b + y))
\]

\[
= \phi(a^*b + a^*y + x^*(b + y))
\]

\[
= \phi(a^*b) + \phi(a^*y) + \phi((b + y)^*x)
\]

\[
= \phi(a^*b)
\]

\[
= \langle a, b \rangle
\]

Moreover, \(\langle [a], [b] \rangle = 0\) if and only if \([a] = [0]\), so \(\langle \cdot, \cdot \rangle\) defines an inner product on \(A/I\) and the completion of an inner product space is by definition a Hilbert space.

Note, that the map \(\pi_{\phi}\) is a well-defined map from \(A/I\) to itself and is clearly linear. Moreover,

\[
|\pi_{\phi}(a)[x]|^2 = |[ax]|^2
\]

\[
= \langle [ax], [ax] \rangle
\]

\[
= \phi(x^*a^*ax)
\]

\[
\leq |a^*a|\phi(x^*x)
\]

\[
\leq |a|^2\langle [x], [x] \rangle
\]

\[
= |a|^2|[x]|^2
\]

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shows that $\pi_\phi$ is continuous on $\mathcal{A}/I$, which can be extended to $\mathcal{H}_\phi$. One can show that this map is also a *-homomorphism with involution $\pi_\phi(a)^* = \pi_\phi(a^*)$. Thus, $\pi_\phi$ is a representation of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H}_\phi)$ as required.

\[ \square \]

### 2.3 Hecke algebras

As the motivation behind the construction of the Bost-Connes system has already been described, I will get straight into explaining the construction itself. The system is based on the C*-algebra of a Hecke algebra and thus, this is where the discussion begins. The following is a brief summary of the work contained in [5] and is meant merely as an introduction to the ideas behind the original work.

Given a discrete group $\Gamma$ and an almost normal subgroup $\Gamma_0$, we can define what a Hecke algebra is. First, we need a quick explanation of what an almost normal subgroup is.

**Definition 2.3.1.** The subgroup $\Gamma_0$ of $\Gamma$ is *almost normal* if the orbits of $\Gamma_0$ acting on the left of $\Gamma/\Gamma_0$ are finite. Equivalently one can think of $\Gamma_0$ as a normal subgroup of a subgroup of finite index in $\Gamma$.

Another definition that we need from group theory is the concept of a double coset.

**Definition 2.3.2.** Let $H$ and $K$ be subgroups of a group $G$. A subset $L$ of $G$ is a **double coset** for $H$ and $K$ if there exists an element $g \in G$ such that

$$L = HgK.$$

When $H = K$ we call these just double cosets of $H$. The set of double cosets is written $H \backslash G / K$.

Armed with this, we can finally define the Hecke algebra.

**Definition 2.3.3.** The **Hecke algebra** $\mathcal{H}(\Gamma, \Gamma_0)$ is defined as the convolution algebra of functions with finite support in $\Gamma_0 \backslash \Gamma / \Gamma_0$. The **convolution** of two functions $f_1, f_2 \in \mathcal{H}(\Gamma, \Gamma_0)$ is given by

$$(f_1 * f_2)(\gamma) = \sum_{\gamma_1 \in \Gamma_0 \backslash \Gamma} f_1(\gamma \gamma_1^{-1}) f_2(\gamma_1) \quad \forall \gamma \in \Gamma.$$

We now need to complete this algebra in an appropriate norm to get a C*-algebra. Here we will follow the work of Bost and Connes rather closely. We first find a regular representation of $\mathcal{H}(\Gamma, \Gamma_0)$ into $\ell^2(\Gamma_0 \backslash \Gamma)$. 

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Lemma 2.3.4. The following defines an involutive representation $\lambda$ of $\mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0\backslash\Gamma)$.

$$(\lambda(f)\xi)(\gamma) = \sum_{\gamma, \gamma_1 \in \Gamma_0\backslash\Gamma} f(\gamma\gamma_1^{-1})\xi(\gamma_1) \quad \forall \gamma \in \Gamma_0\backslash\Gamma, f, \xi \in \mathcal{H}(\Gamma, \Gamma_0), \|\xi\|_{\mathcal{H}(\Gamma, \Gamma_0)} = 1.$$ 

One checks that $\lambda(f)$ is bounded for any $f \in \mathcal{H}(\Gamma, \Gamma_0)$. The involution on $\mathcal{H}(\Gamma, \Gamma_0)$ such that $\lambda(f^*) = \lambda(f)^*$ $\forall f \in \mathcal{H}(\Gamma, \Gamma_0)$ is given by the following equality:

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad \forall \gamma \in \Gamma_0\backslash\Gamma/\Gamma_0.$$ 

Anyone familiar with the process of defining the C*-algebra of a group will find these expressions quite familiar. So, we are now able to take the C*-algebra norm closure of $\mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0\backslash\Gamma)$. We shall call this $\mathcal{A} = \overline{\lambda(\mathcal{H}(\Gamma, \Gamma_0))} = C^*_r(\Gamma, \Gamma_0)$.

The last expression indicates that this is considered the reduced C*-algebra of the Hecke algebra.

We have constructed the C*-algebra $\mathcal{A}$ and we are left to describe the time evolution on $\mathcal{A}$. We shall do this through the following lemma.

Lemma 2.3.5. There exists a unique one parameter group of automorphisms $\sigma_t \subset \text{Aut}(C^*_r(\Gamma, \Gamma_0))$ such that

$$(\sigma_t(f))(\gamma) = \left(\begin{array}{c} L(\gamma) \\ R(\gamma) \end{array}\right)^{-it} f(\gamma) \quad \gamma \in \Gamma_0\backslash\Gamma/\Gamma_0.$$ 

We define $L(\gamma)$ and $R(\gamma)$ as the cardinality of the image of $\Gamma_0\gamma\Gamma_0$ in $\Gamma/\Gamma_0$ and $\Gamma_0\backslash\Gamma$ respectively.

Note, that since each orbit of $\Gamma_0$ on $\Gamma/\Gamma_0$ is finite, $L(\gamma), R(\gamma) \in \mathbb{N}^+$, $R(\gamma) = L(\gamma^{-1})$, and $L$ and $R$ are both $\Gamma_0$-bi-invariant functions.

We are now ready to work with the original Bost-Connes system. We consider the Hecke algebra for the groups:

$$\Gamma = P^+_Q, \quad \Gamma_0 = P^+_Z$$

where $P^+_K$ is the group of $2 \times 2$ matrices

$$P^+_K = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \mid a, b \in K, aa^{-1} = a^{-1}a = 1, a > 0 \right\}.$$ 

We immediately see
Lemma 2.3.6. $P^+_Z$ is almost normal in $P^+_Q$.

This leads us to the main results of Bost and Connes’ paper.

Theorem 2.3.7. Let $(A, \sigma_t)$ be the C*-dynamical system associated to the almost normal subgroup $P^+_Z$ of $P^+_Q$. Then

1. For $0 < \beta \leq 1$, there exists a unique KMS$\beta$ (see Chapter 7) state $\phi_\beta$ on $(A, \sigma_t)$.
2. For $\beta > 1$, the KMS$\beta$ states on $(A, \sigma_t)$ form a simplex whose extreme points $\phi_{\beta, \chi}$ are parametrized by the complex imbeddings $\chi : \mathbb{Q}^{\text{cycl}} \to \mathbb{C}$ of the subfield $\mathbb{Q}^{\text{cycl}}$ of $\mathbb{C}$ generated by the roots of unity.
3. The partition function is the Riemann zeta function.

These facts will be proven later in this thesis (throughout Chapters 5, 6 and 7) after we have rewritten the Bost-Connes system into other forms. We shall also see that $\text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ acts naturally as a group of automorphisms of $A$ commuting with the time evolution for $\beta > 1$.

One fact that will be imperative to understanding this system and showing it is identical to the other forms presented later is Theorem 2.3.9 about the presentation of the system. Recall that $\mathcal{H}(P^+_Q, P^+_Z)$ is a *-algebra. Consider a basis $e_X$ for the algebra, which is indexed by the double cosets $X \in P^+_Z \backslash P^+_Q / P^+_Z$. We follow the notation of [5].

Definition 2.3.8. Define

1. $\mu_n = n^{-\frac{1}{2}} e_{X_n}$ for $n \in \mathbb{N}^*$, where $X_n$ is the double coset class of $\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ of $P^+_Q$,
2. $e(\gamma) = e_{X^\gamma}$ for $\gamma \in \mathbb{Q}/\mathbb{Z}$, where $X^\gamma$ is the double coset class of $\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ of $P^+_Q / P^+_Z$.

This is enough to state the theorem. This theorem has been made more efficient since it was first stated by Bost and Connes. I present the form given in [3].

Theorem 2.3.9. The elements $\mu_n, e(\gamma)$ for $n \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z}$ generate $C^*(P^+_Q, P^+_Z)$ and the following relations give a presentation for it.

1. $\mu_n^* \mu_n = 1 \quad \forall n \in \mathbb{N}^*$,
2. $\mu_m \mu_n = \mu_{mn} \quad \forall m, n \in \mathbb{N}^*$,
3. $e(0) = 1, e(\gamma_1)^* = e(-\gamma_1), e(\gamma_1 + \gamma_2) = e(\gamma_1)e(\gamma_2) \quad \forall \gamma_1, \gamma_2 \in \mathbb{Q}/\mathbb{Z}$.
\[ \mu_n e(\gamma) \mu^*_n = \frac{1}{n} \sum_{n \delta = \gamma} e(\delta) \quad \forall n \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z}. \]

The proof of this will not be reproduced as it is analogous to the work shown in Theorem 5.5.2. Jean-Benoit Bost and Alain Connes in [5] show that these four conditions hold and that these elements generate \( C^*(P_{\mathbb{Q}^+}^*, P_{\mathbb{Z}^+}^*) \). In Chapter 6.3, it is explained that the first two conditions imply that \( \mu \) is an isometric representation of \( \mathbb{N}^* \), the third condition provides \( e \) as a unitary representation of \( \mathbb{Q}/\mathbb{Z} \) and the last condition is equivalent to the covariance of \((\pi_e, \mu)\) where \( \pi_e \) is a representation of \( C^*(\mathbb{Q}/\mathbb{Z}) \) induced from \( e \).

### 2.4 Kronecker-Weber Theorem and Hilbert’s 12th Problem

In Theorem 2.3.7 of the previous section, Benoît Bost and Alain Connes find that for low temperatures, there are multiple equilibrium states at each temperature. It is a theorem about equilibrium states (see Remark 7.1) that the equilibrium states form a convex simplex and so one can look at the extremal equilibrium states. All of the equilibrium states can then be written in terms of these extremal ones. In the case of the Bost-Connes system, it has been found that these extremal equilibrium states form a set equivalent to \( \text{Gal}(\mathbb{Q}_{\text{cycl}}^+ / \mathbb{Q}) \). Moreover, evaluated on an arithmetic subalgebra, they generate \( \mathbb{Q}_{\text{cycl}}^+ \). This is an incredible result.

To get a feel for what this means, I introduce briefly the number theoretic flavour to the thesis and also why this result is so tantalising. I will provide further details in Chapter 3.

Firstly, this result of Bost-Connes systems has direct implications to the Kronecker-Weber Theorem. Here are three equivalent statements of the Kronecker-Weber Theorem:

**Kronecker-Weber Theorem.** The following are equivalent statements of this theorem:

1. Every abelian extension of the rationals is contained within a cyclotomic field.
2. The maximal cyclotomic extension of the rationals \( \mathbb{Q}_{\text{cycl}}^+ \) is equal to the maximal abelian extension of the rationals \( \mathbb{Q}_{\text{ab}}^+ \).

This theorem uses terminology that is defined in Section 3.1. Essentially, the Kronecker-Weber Theorem allows us to replace all mention of \( \mathbb{Q}_{\text{cycl}}^+ \) by \( \mathbb{Q}_{\text{ab}}^+ \). The special thing about the Kronecker-Weber Theorem is that it tells us that to generate all abelian extensions of the rationals it is enough to adjoin roots of unity to the rationals. This is a very specific solved case of Hilbert’s 12th Problem. Hilbert’s 12th Problem asks for an analogue of the roots of unity for other number
fields. More specifically,

**Hilbert’s 12th Problem.** What are all the algebraic numbers necessary to construct all abelian extensions of a number field $K$? Alternatively, provide a transcendental function, whose special values would generate the maximal abelian extension $K^\text{ab}$ of $K$.

As one of the few of the 23 famous Hilbert problems still unsolved, it is exciting to see a completely new idea with potential to make an impact on this problem. Currently, we are only able to claim that we have solved this problem for a few fields, including the rationals, imaginary quadratic fields and complex multiplication fields. But this barely scratches the surface of all the different types of number fields out there. It is worth considering whether or not Bost-Connes systems may be the key to solving this problem. This is the main inspiration for future work, which mostly all works to a common goal that has been coined the Bost-Connes Problem.

### 2.5 Bost-Connes Problem

Following the work of Bost and Connes and the amazing properties their system had, it made sense to consider whether or not this system can be generalised to find similar properties for other number fields. Namely, we wish to retain some important useful properties of the system as exhibited in Theorem 2.3.7; the partition function should be a zeta function, there are unique equilibrium states at high temperature and at low temperature the extremal equilibrium states generate the maximal abelian Galois group of the number field when evaluated on a ‘nice’ subalgebra of the system. The first researchers to express this in terms of a problem to be solved were Alain Connes, Matilde Marcolli and Niranjan Ramachandran in [11].

The Bost-Connes problem is as follows: for a number field $K$ construct a $C^*$-dynamical system

$$A_K = (A_t, (\sigma_t)_{t \in \mathbb{R}})$$

with the following seven properties. I note that some of the following terms will come with no explanation or motivation. This has been intentional in order to maintain the introductory brevity aimed for in chapter 2. As I present more and more technical aspects of the Bost-Connes system in chapters 5, 6 and 7, all of these terms will be properly explained and detailed.

**Bost-Connes Problem.** The original Bost-Connes system of [5] exhibited many interesting properties of the class field theory of $\mathbb{Q}$. A solution of the Bost-Connes problem would achieve
similar results for any arbitrary number field $K$. In particular, such a solution $(A, \sigma_t)$ must exhibit the following properties.

1. The partition function is given by the Dedekind zeta function $\zeta_K$.
2. The maximal abelian Galois group $\text{Gal}(K^{ab}/K)$ of $K$ acts as symmetries on $A_K$.
3. For each inverse temperature $0 < \beta \leq 1$ there is a unique $\text{KMS}_\beta$-state.
4. For each $\beta > 1$ the action of the symmetry group $\text{Gal}(K^{ab}/K)$ on the set of extremal $\text{KMS}_\beta$-states is free and transitive.

When a system satisfies these four properties it is called an analytic Bost-Connes system for $K$.

5. There exists a $K$-rational subalgebra $A^{\text{arith}}$ such that for every extremal $\text{KMS}_\infty$-state $\rho$ and every $f \in A^{\text{arith}}$, we have
   $$\rho(f) \in K^{ab}$$
   and further $K^{ab}$ is generated over $K$ by these values.

6. If we denote by $\nu_\rho$ the action of a symmetry $\nu \in \text{Gal}(K^{ab}/K)$ on an extremal $\text{KMS}_\infty$-state $\rho$ (given by pull-back), we have for every element $f \in A^{\text{arith}}$ the following compatibility relation
   $$\nu_\rho(f) = \nu^{-1}(\rho(f)).$$

7. The $C^*$-algebra $A^{\text{arith}} \otimes_K \mathbb{C}$ is dense in $A$.

A system $A_K$ which satisfies these last properties is called a full Bost-Connes system for $K$ and $A^{\text{arith}}$ is called an arithmetic subalgebra.

There has been much progress in this direction. The following timeline paints an approximate picture of the major breakthroughs in this area.

- (1995) Full Bost-Connes system for $K = \mathbb{Q}$ (Bost, Connes)
- (2005) Full Bost-Connes system for $K$ an imaginary quadratic number field (Connes, Marcolli, Ramachandran)
- (2005) Analytic Bost-Connes systems $A_K$ for $K$ arbitrary - properties 1 and 2 (Ha, Paugam)
- (2009) Full Bost-Connes system for arbitrary $K$, called Connes-Marcolli systems (Laca, Larsen, Neshveyev)

There have been numerous other contributions to this problem and sincere apologies to
those who are not mentioned. Since the problem was first identified, many mathematicians have worked in related tangential areas also worthy of mention. There has been research on Bost-Connes systems for function fields [22, 36] and numerous contributions in rewriting these systems in different forms such as in terms of Hecke Algebras [5, 3], $\mathbb{Q}$-lattices [12], Shimura varieties [11], groupoid $C^*$-algebras [29], semigroup crossed products [28, 21], endomotives [9, 41] and many others.

Depending on what the intention of the author is, there are many reasons to find new descriptions for this system. At first, there was difficulty in finding a nice generalisation of the system. Alain Connes, Matilde Marcolli and Niranjan Ramachandran [8] found that the Hecke algebra description suffered in this way and found a very geometrical description utilising a lattice type structure, which they called $\mathbb{Q}$-lattices. This allowed them to find a useful description that could generalise to negative imaginary fields. I will give a brief account of this work in Chapter 5.

Some of the ideas used by Connes, Marcolli and Ramachandran allowed others to rewrite the system first as the enveloping $C^*$-algebra of a groupoid, which mirrors the inconvenient algebra of the $\mathbb{Q}$-lattices and allows for easier computation, and second, as the semigroup crossed product of $C^*$-algebras. This latter form has been most convenient in form for many generalisations. I provide an account of these two forms in Chapters 5 and 6. In Chapter 7, I will be explaining the concept of Kubo-Martin-Schwinger (KMS) equilibrium states, but I will have enough tools to give an explanation of what is considered to be the best solution to the Bost-Connes problem to date for any number field.

Since 1995 and the very first published work on the Bost-Connes system, there has been much progress. Accounts of this work are provided in Chapters 5, 6 and 7, whereas I will attempt to add to this research, my own work on the Bost-Connes system in Chapters 8, 9 and 10.
Part II

Number Theory and the
Kronecker-Weber Theorem
Chapter 3

Field Theory

This chapter contains the number theory required to understand some of the theory of the Artin map and class field theory required when discussing the equilibrium states of the Bost-Connes systems as they correspond to important number theoretic objects.

In Section 3.1, the relevant field theory is introduced. For readers familiar with undergraduate field theory this can be skipped. Section 3.2 introduces ramification theory essential to understanding class field theory. Sections 3.3 and 3.4 introduce valuations and hence adèle rings, as well as class field theory relevant to $\mathbb{Q}$. Armed with these concepts a brief look at the Artin map is considered. In Chapter 4, two elegant number theory proofs for the Kronecker-Weber Theorem are exhibited. The first by Eknath Ghate uses a lot of powerful results from Class Field Theory, while Greenberg’s proof uses much more elementary means and only depends on ramification theory. The main steps of his proof are elucidated, but the fiddly details are left for the reader to find in Greenberg’s excellent article.

3.1 Field Theory Preliminaries

We will be using the notation and terminology of [31] unless explicitly mentioned.
3.1.1 Fields

**Definition 3.1.1.** A ring is a set $R$ with addition $(x, y) \mapsto x + y$ and multiplication $(x, y) \mapsto xy$ from $R \times R$ into $R$ satisfying

1. Under addition, $R$ is an abelian group. Denote the additive identity by $0_R$.
2. For all $x, y, z \in R$ we have
   \[
   x(y + z) = xy + xz \quad \text{and} \quad (y + z)x = yx + zx.
   \]
3. For all $x, y, z \in R$ we have $(xy)z = x(yz)$.
4. There exists an element $1_R \in R$ (called the multiplicative identity) such that $1_Rx = x1_R = x$ for all $x \in R$.

A ring $R$ is **commutative** if $xy = yx$ for all $x, y \in R$. A ring with a multiplicative identity element is a **ring with unity**; the multiplicative identity is called unity.

**Definition 3.1.2.** If $a$ and $b$ are two nonzero elements of a ring $R$ such that $ab = 0$ then $a$ and $b$ are **divisors of 0**. An **integral domain** is a commutative ring with unity $1 \neq 0$ and containing no divisors of 0.

**Definition 3.1.3.** An additive subgroup $N$ of a ring $R$ is an **ideal** if it satisfies

\[
aN \subseteq N \quad \text{and} \quad Nb \subseteq N \quad \text{for all} \ a, b \in R
\]

An ideal $N \neq R$ in a commutative ring $R$ is a **prime ideal** if $ab \in N$ implies $a \in N$ or $b \in N$ for $a, b \in R$. A **maximal ideal** of a ring $R$ is an ideal $M$ different from $R$ such that there is no proper ideal $N$ of $R$ properly containing $M$. If $R$ is a commutative ring with unity and $a \in R$, the ideal \{ra : r \in R\} of all multiples of $a$ is the **principal ideal** generated by $a$ and is denoted \{a\}. An ideal is a principal ideal if it is of the form \{a\} for some $a \in R$.

**Definition 3.1.4.** A commutative ring $R$ such that the subset of non-zero elements of $R$ forms a multiplicative group and $0_R \neq 1_R$ is called a **field**.

**Definition 3.1.5.** A subset $K$ of a field $F$ is a **subfield** of $F$ if it is a field under the induced operations from $F$. Let $U$ be a subset of $F$. We define $K(U)$ as the smallest subfield of $F$ containing $K$ and $U$. We can analogously define $K[U]$ as the smallest subring of $F$ containing $K$ and $U$. If $U = \{u_1, \ldots, u_n\}$, then we can write $K(U)$ (and $K[U]$) as $K(u_1, \ldots, u_n)$ (and $K[u_1, \ldots, u_n]$).
Remark. Let $R$ be a commutative ring with unity. Then

1. An ideal $M$ of $R$ is maximal if and only if $R/M$ is a field.
2. An ideal $N$ of $R$ is prime if and only if $R/N$ is an integral domain.
3. Every maximal ideal of $R$ is a prime ideal.

The proof of these results can be found in [18], §27.

We can now define the concepts of ring and field homomorphisms analogously to the group homomorphisms.

**Definition 3.1.6.** Let $F,F'$ be fields. A field homomorphism $f : F \to F'$ is a non-zero map having the following properties. For all $x,y \in F$

1. $f(xy) = f(x)f(y)$, and
2. $f(x + y) = f(x) + f(y)$.

If a field homomorphism is bijective, we call it a field isomorphism and we say that the two groups are isomorphic and write $F \cong F'$. This definition applies to ring homomorphisms if we replace $F$ and $F'$ with two rings.

### 3.1.2 Polynomials

We can also develop some facts about polynomials that will be useful in the development of more advanced field theory later in this chapter. This work is again based on [18] and [31].

In this section, we will assume $R$ is a ring, $F$ and $K$ are fields and $G$ is a group.

**Definition 3.1.7.** Let $R$ be a commutative ring. A polynomial $f(x)$ with coefficients in $R$ is an infinite formal sum

$$
\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n + \cdots
$$

where $a_i = 0$ for all but finitely many values of $i$ and $a_i \in R$. With this definition, the set of all polynomials in $x$ with coefficients in $R$ is a ring $R[x]$ with the usual addition and multiplication. We can define polynomials in $n$ indeterminates $x_i$ inductively by considering polynomials in $x_n$ with coefficients in $R[x_1, \ldots, x_{n-1}]$. In other words, $R[x_1, \ldots, x_n] = (R[x_1, \ldots, x_{n-1}])[x_n]$. 

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Let $f$ be a polynomial in $n$ indeterminates and write
\[ f(x_1, \ldots, x_n) = \sum c(i)x_1^{i_1} \cdots x_n^{i_n} \]
We call each term $c(i)x_1^{i_1} \cdots x_n^{i_n}$ a monomial and if $c(i) \neq 0$, we define the degree of the monomial to be
\[ \deg(x_1^{i_1} \cdots x_n^{i_n}) = i_1 + \cdots + i_n \]
f is said to be homogeneous of degree $d$ if all the monomials with $c(i) \neq 0$ have the property that
\[ i_1 + i_2 + \cdots + i_n = d \]

**Definition 3.1.8.** The degree of a polynomial in one indeterminate $x$ is the largest integer $n$ such that the coefficient of $x^n$ is non-zero. The coefficient of this term is also called the leading coefficient. The zero polynomial has no leading coefficient, so we define the degree of the zero polynomial to be $-\infty$.

**Definition 3.1.9.** Let $f \in R[x]$. If $f(\alpha) = 0$ for $\alpha \in R$, then $\alpha$ is a root of $f$. Let $K$ be a subfield of $F$. Then $\alpha$ is algebraic over $K$ if it is the root of some non-zero polynomial $f \in K[x]$. Otherwise, $\alpha$ is transcendental over $K$.

An important tool in algebra is the concept of the minimal polynomial for algebraic elements over a field. In order to define the minimal polynomial we first need the concepts of monic polynomials and irreducible polynomials.

**Definition 3.1.10.** If the leading coefficient of a polynomial $f \in R[x]$ is the multiplicative identity element of $R$, then we call $f$ a monic polynomial. If $f \in K[x]$ then multiplying every coefficient by the inverse of the leading coefficient will produce a monic polynomial $g \in K[x]$. As $f$ is a constant multiple of $g$, they have the same roots.

From this point on in this chapter, we assume that all polynomials with coefficients in a field are monic.

**Definition 3.1.11.** A polynomial $f \in K[x]$ is said to be irreducible over $K$ if it is of degree at least one and if, given a factorisation $f = gh$ with $g, h \in K[x]$ then $\deg g$ or $\deg h$ is zero. Thus, the only monic divisors of $f$ are 1 and $f$ itself.

**Remark.** Let $F$ be a field. Every ideal in $F[x]$ is principal. To see this, consider the following train of thought. The trivial ideal $\{0\} = (0)$ is principal. Suppose that an ideal $N \neq \{0\}$. Let $g(x)$ be a nonzero element of $N$ of minimal degree. If this degree is zero, then $N = F[x] = (1)$ is principal. If the degree of $g(x)$ is greater than 1, let $f(x)$ be any element of $N$. Then applying
If \( f \in \text{fields} \) and this section is the Fundamental Theorem of Galois Theory. This theory follows an approximate

This section deals with the theory of field extensions and field automorphisms. The aim of

Motivated by Lemma 3.1.12, we can define the minimal polynomial \( f \in K[x] \) of \( \alpha \) algebraic over \( K \). The minimal polynomial of \( \alpha \) is the unique monic irreducible polynomial \( f \in K[x] \) such that \( \alpha \) is a root of \( f \).

If \( \alpha \) and \( \beta \) are roots of the same minimal polynomial over \( K \), then we say that \( \alpha \) and \( \beta \) are conjugate over \( K \) (also called \( K \)-conjugate).

3.1.3 Field extensions and automorphisms

This section deals with the theory of field extensions and field automorphisms. The aim of

Remark. A non-trivial ideal \( \langle p(x) \rangle \) of \( F[x] \) is maximal if and only if \( p(x) \) is irreducible over \( F \).

Conversely, if \( p(x) \) is irreducible over \( F \), suppose there is an ideal \( N \) of \( F \) with \( \langle p(x) \rangle \subset N \subset F[x] \). As \( N \) is an ideal of \( F[x] \) it is a principal ideal. Let \( N = \langle g(x) \rangle \) for some \( g(x) \in N \). But \( p(x) \in N \) implies \( p(x) = f(x)g(x) \) for some \( f(x) \in F[x] \). As \( p(x) \) is irreducible either \( f(x) \) or \( g(x) \) is of degree 0. If \( g(x) \) is of degree 0, then \( g(x) \) is a unit of \( F \) and hence \( N = \langle g(x) \rangle = F[x] \).

If \( f(x) \) has degree 0, then \( g(x) \) is a scalar multiple of \( p(x) \) and hence is an element of \( \langle p(x) \rangle \) and thus \( N = \langle p(x) \rangle \). So, \( \langle p(x) \rangle \) is maximal in \( F[x] \).

Lemma 3.1.12. Let \( \alpha \) be algebraic over \( K \). Then there exists a unique monic irreducible polynomial \( p \in K[x] \) with root \( \alpha \). Moreover, this polynomial is minimal in the sense that it divides all other monic polynomials in \( K[x] \) which have \( \alpha \) as a root.

Definition 3.1.13. Motivated by Lemma 3.1.12, we can define the minimal polynomial \( f \in K[x] \) of \( \alpha \) algebraic over \( K \). The minimal polynomial of \( \alpha \) is the unique monic irreducible polynomial \( f \in K[x] \) such that \( \alpha \) is a root of \( f \).

Definition 3.1.14. If \( \alpha \) and \( \beta \) are roots of the same minimal polynomial over \( K \), then we say that \( \alpha \) and \( \beta \) are conjugate over \( K \) (also called \( K \)-conjugate).

3.1.3 Field extensions and automorphisms
for later work.

**Definition 3.1.15.** Let $K$ be a subfield of a field $F$. Then $F$ is a **field extension** of $K$. We write $F/K$. If $U$ is a subset of $F$, then $K(U)$ denotes the smallest subfield of $F$ containing $K$ and $U$. Moreover, if every element $x \in F$ is algebraic over $K$, then we call $F$ an **algebraic extension** of $K$.

**Definition 3.1.16.** A field $F$ is said to be **algebraically closed** if every polynomial $f \in F[x]$ of degree at least 1, has a root in $F$.

**Definition 3.1.17.** Let $K$ be a subfield of $F$. Then $F$ can be viewed as a vector space over $K$. $F$ is a **finite extension** of $K$ if $F$ is a finite dimensional vector space over $K$. If $F$ is a finite extension of $K$, we denote by $[F : K]$ the dimension of $F$ viewed as a vector space over $K$, and call it the **degree of $F$ over $K$**.

**Theorem 3.1.18.** Let $K \subset E \subset F$ be fields. If $[F : E]$ and $[E : K]$ is finite, then $[F : K]$ is finite and 

$$[F : K] = [F : E][E : K].$$

Conversely, if $[F : K]$ is finite, then so are $[F : E]$ and $[E : K]$.

We can now define the concept of an algebraic number and a number field.

**Definition 3.1.19.** An **algebraic number** is any complex number algebraic over $\mathbb{Q}$. Thus, an algebraic number is the root of a non-zero polynomial $f \in \mathbb{Q}[x]$. The field of all algebraic numbers is denoted $\overline{\mathbb{Q}}$.

**Definition 3.1.20.** A **number field** is a finite field extension of the field of rational numbers.

We now consider field automorphism theory whose goal is the fundamental theorem of Galois theory.

**Definition 3.1.21.** A **field automorphism** of $K$ is a field isomorphism $\sigma : K \to K$. The set of all automorphisms of $K$ form a group with the operation of composition, called the **automorphism group** of $K$. We denote the automorphism group of $K$ by $\text{Aut}(K)$. We denote the image by $\sigma \in \text{Aut}(K)$ of $k \in K$ by $k^\sigma$. If $f \in K[x]$ and $\sigma \in \text{Aut}(K)$ then $f^\sigma$ is the polynomial constructed by applying $\sigma$ to the coefficients of $f$. Let $K$ be a subfield of $F$. Then an automorphism $\sigma$ of $F$ is a $K$-**automorphism** if $x^\sigma = x$ for all $x \in K$. We denote the set of all $K$-automorphisms of $F$ by $\text{Aut}_K(F)$. 

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Definition 3.1.22. Let $G$ be a group of automorphisms of a field $K$. The fixed field of $G$ is the set $K^G$ defined by

$$K^G := \{ x \in K \mid x^\sigma = x \text{ for all } \sigma \in G \}.$$ 

Remark. If $x, y \in K^G$, then $(x + y)^\sigma = x + y, (xy)^\sigma = xy, (x^{-1})^\sigma = x^{-1}$. Since $0, 1 \in K^G$, we find that $K^G$ is indeed a field.

We now provide a few results about field extensions and field automorphisms.

Theorem 3.1.23. Let $u \in F$ be transcendental (not algebraic) over $K$. Then there exists a $K$-isomorphism of fields

$$\phi : K(u) \to K(x)$$

where $x$ is a variable, so $K(x)$ is the field of rational functions over $K$.

Theorem 3.1.24. If $F$ is an extension of $K$ and $u \in F$ is algebraic over $K$ then

2. $K(u) \cong K[x]/(f)$ where $f \in K[x]$ is the minimal polynomial of $u$.
3. $[K(u) : K] = n$ where $f$ has degree $n$.
4. $\{1, u, u^2, \ldots, u^{n-1}\}$ is a basis for $K(u)$ over $K$.
5. Any $v \in K(u)$ can be written uniquely in the form $\sum_{i=0}^{n-1} a_i u^i$, where $a_i \in K$.

Corollary 3.1.25. If $F$ is a finite dimensional extension of $K$, then $F$ is algebraic over $K$.

Theorem 3.1.26. Let $\sigma : K \to L$ be a field isomorphism. Let $u$ be a root of an irreducible polynomial $f \in K[x]$ and let $v$ be a root of $f^\sigma \in L[x]$. Then there exists an isomorphism $\tau : K(u) \to L(v)$ with $\tau(u) = v$ and $\tau|_K = \sigma$.

Corollary 3.1.27. Let $E$ and $F$ be extensions of $K$ and $u \in E, v \in F$ be algebraic over $K$. Then $u, v$ are conjugate over $K$ if and only if there exists a $K$-isomorphism $\tau : K(u) \to K(v)$ with $u^\tau = v$.

Corollary 3.1.28. Let $F$ be an extension of $K$ and suppose $u \in F$ is algebraic over $K$. Then any $K$-automorphism of $F$ will send $u$ to $v$ where $v$ is a $K$-conjugate of $u$.

We will now introduce a generalisation of this isomorphism extension in Theorem 3.1.26 to the Isomorphism Extension Theorem, but first we need to introduce the concept of a splitting field.
Definition 3.1.29. Let $f$ be a polynomial of degree $n > 0$ in $K[x]$. An extension field $F$ of $K$ is a splitting field for $f$ over $K$ if it is the smallest field such that $f$ is a product of linear factors in $F[x]$. We say that $f$ splits in $F$, if $F$ is a splitting field for $f$.

Remark. If $f \in K[x]$ has roots $x_1, \ldots, x_n$, then the splitting field $F$ for $f$ over $K$ is $F = K(x_1, \ldots, x_n)$.

Isomorphism Extension Theorem. Let $\sigma : K \to L$ be an isomorphism, $S = \{f_i : i = 1, \ldots, s\}$ be a set of polynomials in $K[x]$ of positive degree and $S' = \{f_i' : i = 1, \ldots, s\} \subset L[x]$. If $F$ is a splitting field for $S$ over $K$ and $M$ a splitting field for $S'$ over $L$, then $\sigma$ can be extended to an isomorphism from $F$ to $M$.

Now, we introduce special types of extensions such as normal, separable and Galois extensions to a field $F$.

Definition 3.1.30. An algebraic extension $F$ of $K$ is normal over $K$ if every irreducible polynomial in $K[x]$ either has no roots in $F$, or splits in $F$.

Theorem 3.1.31. Let $F$ be a finite extension of $K$. Then $F$ is normal over $K$ if and only if $F$ is the splitting field for some polynomial $f \in K[x]$.

Definition 3.1.32. A polynomial $f \in F[x]$ is separable if the number of distinct roots of $f$ equals the degree of $f$. An element $\alpha$ of $E \supset F$ is separable over $F$ if its minimal polynomial over $F$ is separable. A finite extension $E$ of $F$ is separable if all $\alpha \in E$ are separable over $F$.

Definition 3.1.33. An algebraic extension $F$ of a field $K$ is a Galois extension if it is normal and separable. To indicate $F$ is Galois over $K$, one can write $F/K$.

Theorem 3.1.34. Let $F$ be a finite extension of $K$. Then the following are equivalent:

1. The fixed field of $\text{Aut}_K(F)$ is $K$.
2. $F$ is a Galois extension of $K$.
3. $F$ is a splitting field for a separable polynomial $f \in K[x]$.

For a Galois extension $F/K$, one writes $\text{Gal}(F/K)$ for $\text{Aut}_K(F)$.

Theorem 3.1.35. Let $F$ be a Galois extension of $K$ and $K \subset E \subset F$. Then
1. \( F \) is a Galois extension of \( E \).

2. Each \( \sigma \in \text{Aut}_K(E) \) can be extended to \( \sigma' \in \text{Gal}(F/K) \).

**Remark.** Let \( G \) be a finite subgroup of \( \text{Aut}(F) \) and \( K \subset F \) is the fixed field of \( G \). Then \( |G| = [F : K] \). Also, if \( \sigma_i \in \text{Aut}(F) \) are distinct for \( i = 1, 2, \ldots, n \) and if \( L \) is the fixed field of \( \{\sigma_i\} \), then \( [F : L] \geq n \). The proof of these two remarks is omitted, as it requires tedious and extensive manipulations of equations involving linearly independent automorphisms of \( F \). The proof can be found in [18], §53.

**Corollary 3.1.36.** Let \( G \) be a finite subgroup of \( \text{Aut}(F) \). Then

1. If \( K \) is the fixed field of \( G \) in \( F \) then for each \( \sigma \in \text{Aut}(F) \), which leaves \( K \) fixed, we have \( \sigma \in G \).
2. There are no distinct finite subgroups of \( \text{Aut}(F) \) with the same fixed subfield of \( F \).

**Theorem 3.1.37.** Let \( E \) be an extension field of \( F \). Then

\[
\overline{F}_E = \{ \alpha \in E : \alpha \text{ is algebraic over } F \}
\]

is a subfield of \( E \).

**Definition 3.1.38.** Motivated by Theorem 3.1.37, if \( E \) is an extension field of \( F \), we can define the *algebraic closure* \( \overline{F}_E \) of \( F \) in \( E \) as the subfield of \( E \) given by

\[
\overline{F}_E = \{ \alpha \in E : \alpha \text{ is algebraic over } F \}.
\]

We say that a field \( F \) is *algebraically closed* if every non-constant polynomial in \( F[x] \) has a zero in \( F \).

**Remark.** It can be shown using Zorn’s Lemma that every field \( F \) has an algebraic closure, that is, an algebraic extension \( \overline{F} \) that is algebraically closed. For a proof, see [18] §31.

**Theorem 3.1.39.** Let \( u, v \) be \( K \)-conjugate with \( u \) algebraic over \( K \) and consider the normal closure \( L \) of \( K(u) \) over \( K \). Then there exists an automorphism \( \sigma \) of \( L \) such that \( \sigma(u) = v \).

**Fundamental Theorem of Galois Theory.** Let \( F \) be a finite Galois extension of \( K \) and let \( G := \text{Gal}(F/K) \). Then

1. There is a one-to-one correspondence between the intermediate fields \( E \) (\( K \subset E \subset F \)) and the subgroups \( H \) of \( G \).
2. We have \([E : K] = [G : H] = |G|/|H|\) and \([F : E] = |H|\).

3. \(E\) is a normal extension of \(K\) if and only if \(H \triangleleft G\). In this case, \(\text{Aut}_K(E) \cong G/H\).

### 3.2 Ramification Theory

Ramification theory in algebraic number theory is the study of how primes factorise into prime ideals. Recall the following definitions.

**Definition 3.2.1.** A prime \(p\) in a ring \(R\) satisfies the following conditions:

- \(p \in R\) is non-zero nor a unit.
- if \(p|xy\) then \(p|x\) or \(p|y\) for \(x, y \in R\).

**Definition 3.2.2.** A prime ideal \(P\) of a ring \(R\) generalises the concept of a prime. In particular, \(P \neq R\) and satisfies the condition that if \(x, y \in R\) yield \(xy \in P\), then either \(x \in P\) or \(y \in P\).

**Definition 3.2.3.** The ring of integers is the set of integers making an algebraic structure \(\mathbb{Z}\) with the operations of integer addition, negation, and multiplication. It is a commutative ring. More generally the ring of integers of an algebraic number field \(K\), often denoted by \(\mathcal{O}_K\) is the ring of algebraic integers (roots of monic integer polynomials) contained in \(K\). With this notation, \(\mathbb{Z} = \mathcal{O}_\mathbb{Q}\) since \(\mathbb{Z}\) as above is the ring of integers of the field \(\mathbb{Q}\).

Consider a finite Galois extension \(L/K\) of degree \(n\) and Galois group \(G\) over a number field \(K\). Let \(\mathcal{O}_L\) be the ring of all algebraic integers in \(L\). If \(b\) is a prime ideal of \(\mathcal{O}_L\), its intersection with \(K\) is a prime ideal \(p\) in the ring \(\mathcal{O}_K\) of integers of \(K\). Let \(\overline{L} = \mathcal{O}_L/b, \overline{K} = \mathcal{O}_K/p\) be the residue fields. \(\overline{L}\) and \(\overline{K}\) are finite fields of orders \(q^f\) and \(q\) respectively.

The rest of this section follows the famous work of Zariski and Samuel. Proofs of the following propositions can be found on pages 290-305 of their book [42]. The following proposition allows for the definition of the ramification index, which is the basis of ramification theory.

**Proposition 3.2.4.** The ideal \(p\mathcal{O}_L\) generated by \(p\) is equal to a product

\[p = (b_1b_2\ldots b_g)^e,\]

where the \(b_i\) are images of \(b\) under automorphisms of \(G\) and \(b_1 = b\). Moreover, \(n = efg\).

This proposition allows for the definition of the ramification index of \(b\) over \(p\).
Definition 3.2.5. Given the setting of Proposition 3.2.4, recall the definition of the ramification index of \( b \) over \( p \) as the number \( e \). If \( e = 1 \) then \( p \) is unramified in \( L \), while when \( e = n \), then \( p \) is totally ramified in \( L \). Moreover, the decomposition group \( Z \) of \( b \) is the group of all automorphisms \( \sigma \in G \), such that \( \sigma(b) = b \). Each \( \sigma \) induces an automorphism \( \sigma_j \) of the ring \( \mathcal{O}_L/b^{j+1} \). The map \( \sigma \to \sigma_j \) is a homomorphism of \( Z \) and define \( V_j \) as the kernel of this map. Hence, there is a descending chain of normal subgroups

\[ V_0 \triangleright V_1 \triangleright \ldots \triangleright V_j \triangleright \ldots. \]

\( T = V_0 \) is called the inertia group of \( b \) and the other \( V_j \) are the higher ramification groups of \( b \).

These definitions give rise to the following propositions that will be used later to prove the Kronecker-Weber Theorem. It is based on [42] (pp. 290-305).

Proposition 3.2.6. For \( j = 0 \), the homomorphism \( \sigma \to \sigma_0 \) induces an isomorphism of \( Z/T \) onto the Galois group \( G \) of the residue extension \( \overline{L}/\overline{K} \). In particular, \( Z/T \) is cyclic, generated by the coset of an automorphism called the Frobenius element \( \sigma \) such that \( \sigma(x) \equiv x^q \pmod{b} \) for all \( x \in A \).

Proposition 3.2.7. Let \( e \) be the ramification index of \( b \) over \( p \). Then \( e \) is the order of the inertia group \( T \). If \( L_T \) is the fixed field of \( T \), and \( b_T = b \cap L_T \), then \( b_T \) has ramification index 1 (unramified over \( p \)), while \( b \) is totally ramified over \( b_T \).

Proposition 3.2.8. \( T/V_1 \) is isomorphic to a subgroup of the multiplicative group \( \overline{L}^* \) of \( \overline{L} \), hence is cyclic and its order divides \( q^j - 1 \). For each \( j \geq 1 \), \( V_j/V_{j+1} \) is isomorphic to a subgroup of the additive group of the residue field \( \overline{L} \); hence if \( K \) has characteristic \( p \), then \( V_j/V_{j+1} \) is either trivial or a direct product of cyclic groups of order \( p \). For \( j \) sufficiently large, \( V_j \) itself is trivial.

Proposition 3.2.9. (Minkowski's Theorem). For every number field \( K \neq \mathbb{Q} \), there exist primes which ramify in \( K \) and there are only finitely many ramified primes.

Proposition 3.2.10. If \( p \) is an odd prime, then for all \( r \), \( \mathbb{Q}(\zeta_{p^r}) \) is cyclic of order \( p^{r-1}(p-1) \), whereas for \( r \geq 3 \), \( \mathbb{Q}(\zeta_{2^r}) \) is the direct product of two cyclic groups, one of order \( 2^{r-2} \) and the other of order 2 generated by the automorphism \( \zeta \to \zeta^{-1} \). For any prime \( p \) and any \( r \), \( p \) is the only ramified prime in \( \mathbb{Q}(\zeta_{p^r}) \) and it is totally ramified. For any \( m > 2 \), the ramified primes in \( \mathbb{Q}(\zeta_m) \) are the primes dividing \( m \).

Proposition 3.2.11. If \( L, M \) are Galois extensions of a field \( K \) with Galois groups \( G, H \), and \( LM \) is the compositum of \( L \) and \( M \), then \( LM \) is a Galois extension of \( K \), whose Galois group is
canonically isomorphic to the subgroup of $G \times H$ consisting of those pairs $(\sigma, \tau)$ such that $\sigma$ and $\tau$ have the same restriction to $L \cap M$. The isomorphism assigns to an automorphism $\rho$ of $LM$ over $K$ its pair of restrictions $(\rho|_L, \rho|_M)$.

**Theorem 3.2.12.** Let $L_1$ and $L_2$ be two finite Galois extensions of $K$, and let $S_1$ and $S_2$ denote the sets of primes of $K$ which split completely in $L_1$ and $L_2$ respectively. Then $S_1 \subset S_2$ (except for a finite subset) iff $L_2 \subset L_1$.

As this thesis focuses on the noncommutative geometric aspects of Bost-Connes systems, the proofs of these statements will be omitted.

### 3.3 Valuations

**Definition 3.3.1.** Let $K$ be a field. A **valuation** on $K$ is a map $|\cdot|_v : K \to \mathbb{R}_{\geq 0}$ such that for all $a, b \in K$

- $|a|_v = 0 \iff a = 0$,
- $|ab|_v = |b|_v|a|_v$, and
- $|a + b|_v \leq |a|_v + |b|_v$.

The **trivial valuation** is the valuation $|a|_v = 1$ for all $a \in K^*$.

**Definition 3.3.2.** If $|\cdot|_v$ is a non-trivial valuation on $K$ satisfying $|a + b|_v \leq \max(|a|_v, |b|_v)$ then $|\cdot|_v$ is called **non-archimedean**. Otherwise, it is **archimedean**.

**Definition 3.3.3.** Consider the image $K^* \xrightarrow{|\cdot|_v} \mathbb{R}^+ \xrightarrow{\log} \mathbb{R}$. If the image is discrete, we say $|\cdot|_v$ is a **discrete valuation**.

**Definition 3.3.4.** Two valuations $|\cdot|_1$ and $|\cdot|_2$ are **equivalent** if $|\cdot|_1 = |\cdot|_2^c$ for some $c > 0$.

**Lemma 3.3.5.** Any valuation gives rise to a metric. Define $d(a, b) := |a - b|_v$. Equivalent valuations induce the same topology.

**Definition 3.3.6.** Let $K$ be fixed. A **place** of $K$ is an equivalence class of non-trivial valuations on $K$. 
3.3.1 Rational $p$-adics

Let $K = \mathbb{Q}$. Fix prime $p$. We can write any $a \in \mathbb{Q}^\ast$ as $a = \text{sign}(a) \prod p_i^{e_i}$ where $p_i$ are prime numbers, $e_i$ are integers.

Define $v_p(a) = e$, where $e = \begin{cases} e_i & \text{if } p = p_i, \\ 0 & \text{otherwise.} \end{cases}$

Define $|a|_p = p^{-v_p(a)}$. This is a discrete, non-archimedean valuation.

Define $|a|_\infty := |a|, a \in \mathbb{Q}^\ast$. This is an archimedean valuation.

Lemma 3.3.7. $\left( \prod_p |a|_p \right) |a|_\infty = 1$.

Ostrowski’s Theorem. The only places of $\mathbb{Q}$ are $| \cdot |_p$ and $| \cdot |_\infty$.

Definition 3.3.8. More generally, let $F$ be a number field, and $\alpha \in F^\ast$. Thus, we can write $\langle \alpha \rangle = \prod P_i^{e_i}$ where $P_i$ are prime ideals in $\mathcal{O}_F$.

Define $v_P(\alpha) = e$, where $e = \begin{cases} e_i & \text{if } P = P_i, \\ 0 & \text{otherwise.} \end{cases}$

Define $|\alpha|_P = N(P)^{-v_P(\alpha)}$, where $N(P)$ is the absolute norm of $P$ (see Definition 3.3.9). This is a discrete, non-archimedean valuation on $F$.

Let $\omega$ be an embedding of $F$ into $\mathbb{C}$. If image is contained in $\mathbb{R}$, then we say $\omega$ is real, otherwise $\omega$ is complex. If there are $r$ real, $s$ conjugate pairs of complex embeddings, then $r + 2s = n = [F : \mathbb{Q}]$.

Define $| \cdot |_\omega = \begin{cases} | \cdot | & \text{if } \omega \text{ is real,} \\ | \cdot |^2 & \text{if } \omega \text{ is complex.} \end{cases}$

Definition 3.3.9. Let $K$ be a number field with ring of integers $\mathcal{O}_K$, and $\alpha$ a nonzero ideal of $\mathcal{O}_K$. Then the absolute norm of $\alpha$ is defined to be

$$N(\alpha) = [\mathcal{O}_K : \alpha] = |\mathcal{O}_K/\alpha|.$$ 

By convention, the absolute norm of the zero ideal is taken to be zero.

Ostrowski’s Theorem. The places of $F$ are $| \cdot |_P$ where $P$ is a prime ideal of $\mathcal{O}_F$ and $| \cdot |_\omega$ where $\omega$ is an embedding of $F$ into $\mathbb{C}$.

Proposition 3.3.10. Let $\nu$ be a place of a number field $F$. As a metric space $F$ is not complete.
We can embed $F$ in a field $F_v$ that is complete with respect to $|\cdot|_v$. For example,

<table>
<thead>
<tr>
<th>$F = \mathbb{Q}$</th>
<th>$F_v = Q_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = p$</td>
<td></td>
</tr>
<tr>
<td>$v = \infty$</td>
<td>$F_v = \mathbb{R}$</td>
</tr>
</tbody>
</table>

$F \subseteq \overline{\mathbb{Q}}$ |
| $v = \mathfrak{p}$ | $F_v$ is finite extension of $\mathbb{Q}_p$ |
| $v = \omega$     | $F_v = \mathbb{R}$ or $\mathbb{C}$. |

### 3.4 Class Field Theory

To begin the discussion of class field theory, one requires the concepts of the adele ring and the idele group. Throughout this section $K$ is an algebraic number field, and $\mathfrak{p}$ is a prime of $K$. Denote by $K_\mathfrak{p}$ the completion of $K$ at the prime $\mathfrak{p}$. From Section 3.3, recall that if $\mathfrak{p}$ is a finite place, then $K_\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$ whereas if $\mathfrak{p}$ is infinite, then $K_\mathfrak{p}$ is either $\mathbb{R}$ or $\mathbb{C}$. The set of integral elements is denoted $\mathcal{O}_p$.

**Definition 3.4.1.** The **adèle ring** of $K$ is

$$\mathbb{A}_K = \prod_{\mathfrak{p}} K_\mathfrak{p}$$

where this product is restricted with respect to $\mathcal{O}_p$. The topology on $\mathbb{A}_K$ is seen by setting $A \subseteq \mathbb{A}_K$ open iff

$$A \cap \left( \prod_{p} K^*_p \times \prod_{p<\infty} \mathcal{O}_p^* \right)$$

is open in the product topology for all $a \in \mathbb{A}_K$.

**Remark.** One can embed $K$ in $\mathbb{A}_K$ by $x \mapsto (x)_\mathfrak{p}$ as $|x|_\mathfrak{p} \leq 1$ for all but finitely many primes. Moreover, the quotient is compact. In particular, $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is called the solenoid and $\mathbb{A}_K/K = \mathbb{A}_\mathbb{Q}/\mathbb{Q} \otimes \mathbb{Q} K$.

**Definition 3.4.2.** The **idele group** of $K$ is

$$\mathbb{A}^*_K = \left\{ (x)_\mathfrak{p} \in \prod_{\mathfrak{p}} K^*_\mathfrak{p} : |x|_\mathfrak{p} = 1 \text{ for all but finitely many } \mathfrak{p} \right\}.$$ 

$A$ in $\mathbb{A}^*_K$ is open iff

$$aA \cap \left( \prod_{p} K^*_p \times \prod_{p<\infty} \mathcal{O}^*_p \right)$$

is open in the product topology for all $a \in \mathbb{A}^*_K$.

**Definition 3.4.3.** The **idele class group** of $K$ is

$$C_K = \mathbb{A}^*_K/K^*.$$
3.4.1 The Artin Map

Let \( L/K \) be an abelian extension of number fields. Let \( J_K \) denote the group of fractional ideals of \( K \). Let \( S \) denote a finite set of prime ideals of \( K \), including all the primes that ramify in \( L \), and let \( J_{K,S} \) denote the subgroup of \( J_K \) generated by all the prime ideals outside \( S \). For each fractional ideal \( A \) in \( J_{K,S} \), write

\[
A = \prod_p p^{a(p)}.
\]

An important part of class field theory is the Artin map which maps fractional ideals in \( J_{K,S} \) to elements in the Galois group of \( L/K \).

**Definition 3.4.4.** The Artin map for the extension \( L/K \) is the homomorphism \( r_{L/K} : J_{K,S} \to \text{Gal}(L/K) \) given by

\[
r_{L/K}(A) = \prod_p \left[ \frac{L/K}{p} \right]^{a(p)}
\]

where \( A \) is as described above and \( \left[ \frac{L/K}{p} \right] \in \text{Gal}(L/K) \) is the Frobenius element at \( p \) (defined in Section 3.2).

One of the most important results about the Artin map is that it is surjective.

**Theorem 3.4.5.** The Artin map is surjective.

For a proof consult [38].

It is important to consider the kernel of the Artin map. Not only does this give more information about the Artin map, but it would also allow the Artin map to be turned into an isomorphism (by taking the quotient by the kernel).

Consider a prime ideal \( p \in J_{K,S} \). Then \( p \) is in the kernel of the Artin map iff the Frobenius element at \( p \) is trivial

\[
\left[ \frac{L/K}{p} \right] = 1.
\]

By definition, the Frobenius element has order \( f(b/p) \) and as it is trivial \( f(b/p) = 1 \). Due to the choice of \( S \), \( e(b/p) = 1 \). Thus, \( g(b/p) = [L:K] \). Hence \( p \) splits completely in \( L \).

To further investigate the kernel, recall the following definitions.
Definition 3.4.6. Given a number field \( K \), a modulus for \( K \) is the product
\[
m = \prod_p p^{n(p)}
\]
where the product is taken over all primes of \( K \) and \( n(p) = 1 \) for a finite number of real primes and \( n(p) = 0 \) for all other primes.

The modulus \( m \) in general can be written as \( m_f m_\infty \) with the first factor only divisible by finite places and the second factor only divisible by infinite places.

Definition 3.4.7. Here we recall the definition of the ray class group. First, set
\[
K_m = \{a/b : a, b \in \mathcal{O}_K, (a), (b) \text{ relatively prime to } m_f\}
\]
\[
K_{m,1} = \{x \in K_m : x \equiv 1 \pmod{m}\}
\]
Note, that \( x \equiv 1 \pmod{m} \) if for each \( p | m_f \), \( v_p(x - 1) \geq n(p) \) and for each \( p | m_\infty \), \( \omega(x) > 0 \) (from Definition 3.3.8). Call \( J_{K,m} \) the group \( J_{K,S} \) where \( S \) is the set of primes dividing \( m_f \). As before, assume \( S \) contains all the primes that ramify in \( L \). Recall that the ray class group modulo \( m \) is the quotient
\[
\frac{J_{K,m}}{K_{m,1}}.
\]
For \( m = 1 \), the ray class group is the same as the usual class group.

From Proposition 3.2.4, each prime in \( K \) may also be viewed as a product of primes in \( L \). In this way \( m \) is also a modulus for \( L \). Thus, \( J_{L,m} \) is well defined and there is a natural norm map \( N_{L/K} : J_{L,m} \to J_{K,m} \) given by
\[
N_{L/K}(b) = p^f.
\]
Here \( f \) is \( f(b/p) \).

This allows more to be concluded about the kernel of the Artin map.

Proposition 3.4.8. Let \( L/K \) be a finite abelian extension, and let \( m \) be any modulus of \( K \) such that \( m_f \) is divisible by all the primes of \( K \) which ramify in \( L \) as in all the work above. Then
\[
N_{L/K}(J_{L,m}) \subseteq \ker r_{L/K}.
\]

Proof. This follows from the fact that the Artin map sends \( p^f \) to \( \left[ \frac{L/K}{p} \right]^f = 1 \). □

This leads onto the big theorem about the kernel called the Artin Reciprocity Theorem.

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Artin Reciprocity Theorem. Given a finite abelian extension \( L/K \) there exists a modulus \( m \) divisible by at least the primes of \( K \) which ramify in \( L \) such that the kernel of the Artin map is
\[
\ker r_{L/K} = N_{L/K}(J_{L,m}) \cdot K_{m,1}.
\]

Remark. If a modulus \( m \) satisfies the Artin Reciprocity Theorem, then any modulus \( m' \) divisible by \( m \) also satisfies the conditions of the theorem. Thus, one can speak of the greatest common divisor of all such moduli. It is called the conductor of \( L/K \).

Remark. It can also be shown that the kernel of the Artin map is equal to the maximal connected component of the identity. A proof of this can be found in standard texts on class field theory such as [38].

3.4.2 Class Field Theory for \( \mathbb{Q} \)

In this section, the class field theory related to \( \mathbb{Q} \) is discussed to the extent that is needed for the purposes of this thesis. These are standard results and can be found in many texts including [33].

Lemma 3.4.9. The primes of \( \mathbb{Q} \) are all the normal integer primes and \( \infty \). Thus, \( \mathbb{Q} \) has only real places.

Remark. The completion of \( \mathbb{Q} \) with respect to \( \infty \) is the completion in the usual absolute value which means
\[
\mathbb{Q}_\infty = \mathbb{R}.
\]

\( \mathbb{Q}_p \) is the \( p \)-adic completion of the rationals with respect to the prime \( p \). It can be considered as the field of fractions for \( \mathbb{Z}_p \) (the \( p \)-adic completion of the integers). In particular, an element \( a \in \mathbb{Q}_p \setminus \mathbb{Z}_p \) is of the form \( p^{-n}z \) where \( n \in \mathbb{N}, z \in \mathbb{Z}_p^* \).

Corollary 3.4.10. Writing \( \mathbb{p}^\mathbb{N} = \{p^n | n \in \mathbb{N}\} \), yields the isomorphism of the \( p \)-adics to the field of fractions
\[
\mathbb{Q}_p \cong (\mathbb{p}^\mathbb{N})^{-1} \mathbb{Z}_p.
\]

Thus,
\[
\mathbb{Q}_p^* \cong (p) \times \mathbb{Z}_p^* \cong \mathbb{Z} \times \mathbb{Z}_p^*
\]
as one can identify \( (p) \) with \( \mathbb{Z} \).

Remark. It is worth considering the adèle ring of \( \mathbb{Q} \) from Definition 3.4.1. Recall that \( \mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_p \mathbb{Q}_p \) where this product is restricted over \( \mathbb{Z}_p \). Explicitly one may write this as
\[
\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_p \mathbb{Q}_p = \{a = (a_\infty, a_2, a_3, \ldots) \in \mathbb{A}_\mathbb{Q} : a_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}.
\]
The next thing to investigate is the idèle group $\mathbb{A}_Q^*$. From the definition of $\mathbb{A}_Q$, the idèle group is

$$\mathbb{A}_Q^* \cong \mathbb{R}^\times \times \prod_p \mathbb{Q}_p^\times \cong \mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times \times \oplus_p \mathbb{Z}^\times.$$

Stripping the sign from $\mathbb{R}^\times$ and combining with the $\oplus_p \mathbb{Z}$ term, reveals a term isomorphic to $\mathbb{Q}^*$ by the map $\text{sgn}(r) \prod_p p^{n(p)} \mapsto (\text{sgn}(r), n(p)_p)$. Thus, $\mathbb{Q}^* \cong \{\pm 1\} \times \oplus_p \mathbb{Z}$. Thus,

$$\mathbb{A}_Q^* \cong \mathbb{Q}^* \times \mathbb{R} \times \prod_p \mathbb{Z}_p^\times,$$

where $\mathbb{R}^\times \cong \mathbb{R}$ by logarithm.

Thus, we reach the identity

**Theorem 3.4.11.** $C_Q \cong \mathbb{A}_Q^*/\mathbb{Q}^* \cong \mathbb{R} \times \prod_p \mathbb{Z}_p^\times$.

It is fairly clear that the maximal connected component of the identity is $\mathbb{R}$ and thus from the discussion in Section 3.4.1, the kernel of the Artin map is $\mathbb{R}$. Thus, there is an isomorphism from $\prod_p \mathbb{Z}_p^\times$ and $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$.

This result will be compared to the maximal cyclotomic Galois group, which will be shown in Chapter 4 to be isomorphic to the maximal abelian extension.

Recall that a cyclotomic extension is an extension of $\mathbb{Q}$ by a root of unity, such as $\mathbb{Q}(\zeta_n)$. Recall that

$$\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/n\mathbb{Z}^\times$$

Thus,

$$\text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \cong \varprojlim \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$$

$$\cong \varprojlim \mathbb{Z}/n\mathbb{Z}^\times$$

$$\cong \varprojlim \prod_p (\mathbb{Z}/p^n\mathbb{Z})^\times$$

$$\cong \prod_p \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times$$

$$\cong \prod_p \mathbb{Z}_p^\times = \mathbb{Z}_\mathbb{A}^\times.$$

This verifies that $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$.

This section concludes with a discussion of the kernel of the Artin map and the conductor of a cyclotomic extension of the rationals.
Lemma 3.4.12. \( \ker r_{Q(\zeta_n)/Q} = Q_{m,1} \). Additionally, \( n\infty \) is the conductor of \( Q(\zeta_n)/Q \).

Proof. For any \( p \nmid n \), \((p)\) is unramified in \( Q(\zeta_n) \). \( \sigma_p \) which sends \( \zeta_n \) to \( \zeta_p \) satisfies the Frobenius categorisation, so

\[
r_{Q(\zeta_n)/Q}(p) = \sigma_p
\]

and it is not hard to see how this extends to

\[
r_{Q(\zeta_n)/Q}(\frac{a}{b}) = \sigma_{ab'},
\]

where \( b' \) is a positive integer such that \( bb' \equiv 1 \pmod{n} \). Thus, the kernel of the Artin map occurs when \( a \equiv b \pmod{n} \), which is the set \( Q_{m,1} \). This gives the first result.

Given that

\[
N_{Q(\zeta_n)/Q}(J_{Q(\zeta_n),n\infty}) = 1
\]

this implies

\[
\ker r_{Q(\zeta_n)/Q} = N_{Q(\zeta_n)/Q}(J_{Q(\zeta_n),n\infty})Q_{m,1}.
\]

It can also be shown that \( n\infty \) is in fact the greatest common divisor of all such moduli. Thus \( n\infty \) is the conductor of \( Q(\zeta_n)/Q \). \( \square \)
Chapter 4

Kronecker-Weber Theorem

In Section 2.4, The Kronecker-Weber Theorem is introduced. For clarity, here is the statement of the theorem again.

Kronecker-Weber Theorem. Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field.

Remark. 1. It is worth remarking that given that the maximal abelian extension of $\mathbb{Q}$ can be written as the inductive limit of finite abelian extensions of $\mathbb{Q}$, thus, $\mathbb{Q}^{ab}$ is contained within an inductive limit of cyclotomic fields. As cyclotomic fields are abelian, this fact ensures that in fact $\mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$.

2. Therefore, this solves Hilbert’s 12th Problem (see Section 2.4) for the rationals. In particular, the maximal abelian extension of $\mathbb{Q}$ can be generated by concrete roots of unity.

To get a flavour for the type of techniques necessary to solve the Kronecker-Weber Theorem, two proofs will be discussed in detail, both of which draw on the theory discussed earlier in this chapter. Both of these proofs motivate ideas on how to recover the Kronecker-Weber Theorem from partial Connes-Marcolli Systems constructed in Chapter 8. Chapter 10 details how this is done.

4.1 Class field theory proof

The first proof follows the methodology recommended by Eknath Ghate in [19]. It relies heavily on the machinery of class field theory, taking this very far from what is normally considered
elementary number theory.

Proof. Consider an abelian extension $L$ of $\mathbb{Q}$. From the Artin Reciprocity Theorem, there exists a modulus $m$ that is divisible by at least the primes of $\mathbb{Q}$ that ramify in $L$, such that the kernel of the Artin map is given by

$$\ker r_{L/Q} = N_{L/Q}(J_{L,m})K_{m,1}.$$  

From work done in Section 3.4.2, we may assume the modulus is given by some $m = n \cdot \infty$, for some natural number $n$. From Section 3.4.2,

$$\ker r_{Q(\zeta_n)/Q} = Q_{m,1},$$

and hence,

$$N_{Q(\zeta_n)/Q}(J_{Q(\zeta_n),m}) \subset N_{Q(\zeta_n)/Q}(J_{Q(\zeta_n),m}) \cdot Q_{m,1} = Q_{m,1} \subset N_{L,m}(J_{L,m}) \cdot Q_{m,1} = \ker r_{L/Q}.$$  

If $p$ is a prime of $K$ that does not divide $n$ and splits in $Q(\zeta_n)$ then it is given by the norm of a prime of $Q(\zeta_n)$. Then $p \in N_{Q(\zeta_n)/Q}(J_{Q(\zeta_n),m})$ and by the Artin Reciprocity Theorem, $p \in \ker r_{L/K}$. So $p$ splits completely in $L$. So, every prime that splits in $Q(\zeta_n)$ also splits in $L$. Thus, by Theorem 3.2.12, $L \subset Q(\zeta_n)$. 

4.2 Ramification theory proof

The flavour of the second proof relies more on ramification theory and reducing the situation to field extensions that are unramified over $\mathbb{Q}$. This follows the proof of [20]. The notation is the same as that used in Section 3.2. This proof relies heavily on the fact that it is sufficient to prove that every finite abelian extension is contained within a cyclotomic extension.

Proof. The proof can be summarised by a collection of sequential lemmas that lead to the final proof. Given a finite abelian extension $L/Q$, prime $p$ of $Q$, prime $b$ of $L$ over $p$ (with $p = (b_1b_2 \ldots b_g)^e$ by Proposition 3.2.4), denote the ramification index of $b/p$ as $e(b/p)$, decomposition group of $b$ as $Z$, higher ramification groups $V_j$ and inertia group $T$.

Lemma 4.2.1. If $Z/V_1$ is abelian, then $T/V_1$ is cyclic of order dividing $q - 1$.  

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Due to the fact that finite abelian groups can be decomposed uniquely into cyclic subgroups and Lemma 4.2.1, the following lemma can be proved.

**Lemma 4.2.2.** If the Kronecker-Weber Theorem holds for cyclic extensions that have prime power order, then the theorem is true for all abelian extensions. In other words, if cyclic extensions of prime power order are contained within a cyclotomic extension, then this is also true for all finite abelian extensions.

To ease the calculation, it is worth reducing the abelian extensions to ones where every prime is unramified.

**Lemma 4.2.3.** Suppose $K$ is an abelian extension of $\mathbb{Q}$ of prime power degree $\lambda^m$. If $K$ is contained within a cyclotomic extension under the additional assumption that every prime $p \neq \lambda$ is unramified in $K$, then it is also true for $K$ when this assumption is relaxed.

This implies the following important corollary.

**Corollary 4.2.4.** If $K$ is an abelian extension of $\mathbb{Q}$ of prime power degree $\lambda^m$, and $p \neq \lambda$ is the only prime ramified in $K$, then it can be shown that $p$ is totally ramified in $K$,

$$p \equiv 1 \pmod{\lambda^m}$$

and $K$ is the unique subfield of $\mathbb{Q}(\zeta_{p\lambda})$ of degree $\lambda^m$. Therefore, $K/\mathbb{Q}$ is cyclic.

Before we proceed, we consider the prime 2. Because it behaves a little differently to odd primes, we need the following corollary, which follows from Corollary 4.2.4.

**Corollary 4.2.5.** If $K$ is an abelian extension of $\mathbb{Q}$ of odd degree, then 2 is unramified in $K$.

Combining Lemmas 4.2.2, 4.2.3 and Corollaries 4.2.4, 4.2.5 yields the following lemma.

**Lemma 4.2.6.** Let $K$ be an abelian extension of $\mathbb{Q}$ of degree $\lambda^m$, $\lambda$ an odd prime, in which $p$ is the only ramified prime. Then $K/\mathbb{Q}$ is cyclic.

Having reduced the problem this far, it can now be shown that abelian extensions of odd prime power degree satisfy the Kronecker-Weber Theorem.

**Lemma 4.2.7.** The Kronecker-Weber Theorem holds for abelian extensions of $\mathbb{Q}$ of degree $\lambda^m$, where $\lambda$ is an odd prime. More specifically, if $\lambda$ is the only ramified prime in $K$, then $K$ is the unique subfield of $\mathbb{Q}(\zeta_{\lambda^{m+1}})$ of degree $\lambda^m$. 

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Lastly, we need to deal with the prime 2.

**Lemma 4.2.8.** Every quadratic extension of $\mathbb{Q}$ is cyclotomic.

This is used to prove the stronger result:

**Lemma 4.2.9.** Every cyclic extension $K$ of $\mathbb{Q}$ of degree $2^m$ is cyclotomic.

The key results that we can now combine are Lemmas 4.2.2, 4.2.7 and 4.2.9. Together, they combine to complete the proof of the Kronecker-Weber Theorem. $\square$
Part III

Bost-Connes System
Chapter 5

Q-lattices, Groupoids and their C*-algebras

In the previous chapter, the concept of the Bost-Connes system was introduced including an overview of the motivation and construction of the Bost-Connes system via Hecke algebras. As explained in Section 2.5, mathematicians realised that there is significant merit in constructing such a Bost-Connes type system for other number fields. In other words, they searched for a system that would solve the Bost-Connes problem (see Section 2.5) for number fields other than the rationals.

There was significant work done in generalising the Hecke algebra system to other number fields, but as shown by Marcelo Laca and Machiel van Frankenhuijsen in [30], the action of the symmetry group on these systems corresponds to the class field theory of the number field if and only if \( K = \mathbb{Q} \). One can think of this as saying that while the symmetry groups acting on these systems are often isomorphic to the maximal abelian Galois group \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) as required, the values of the extremal equilibrium states evaluated on the corresponding arithmetic Hecke subalgebra are only enough to generate \( K^{cycl} \) and not enough to generate \( K^{ab} \). These two fields are equal only in the case of \( K = \mathbb{Q} \).

In [11], Alain Connes, Matilde Marcolli and Niranjan Ramachandran construct a C*-dynamical system, called a Bost-Connes-Marcolli system, based on the geometry of Q-lattices. In this chapter I present this construction, show that it describes the same dynamical system and derive some of the more important properties of the system. Section 5.1 derives some of the more important results about Q-lattices, Section 5.2 introduces the notion of an étale groupoid,
Section 5.3 gives the construction for the C*-algebra of an étale groupoid and in Section 5.5 we show that the quantum statistical mechanical system for a special type of groupoid C*-algebra (described in Section 5.4) is the same system as the Bost-Connes system, evaluate some of its properties and give a brief discussion how one can generalise this system to other number fields.

5.1 Q-lattices

After Bost and Connes’ original work, the next major contribution came from a collaboration between Alain Connes, Matilde Marcolli and Niranjan Ramachandran in [11]. They described a new system that they then showed was isomorphic to the original, but arose from a much more geometric approach. Here we first consider Q-lattices and with them build a non-commutative geometry that will be equivalent to the Bost-Connes System for the set of rationals. While in later work, this was often generalised to larger number fields, we shall focus primarily on the Bost-Connes System for \( \mathbb{Q} \).

**Definition 5.1.1.** A Q-lattice in \( \mathbb{R}^n \) consists of a pair \((\Lambda, \phi)\) of a lattice \( \Lambda \subset \mathbb{R}^n \) (a cocompact free abelian subgroup of \( \mathbb{R}^n \) of rank \( n \)) together with a system of labels of its torsion points given by a homomorphism of abelian groups

\[
\phi : \mathbb{Q}^n / \mathbb{Z}^n \to \mathbb{Q} \Lambda / \Lambda.
\]

In order to set up an algebraic structure with the Q-lattices, it is worth defining concepts of equivalence between the lattices.

**Definition 5.1.2.** A Q-lattice is invertible if \( \phi \) is an isomorphism. Two Q-lattices \((\Lambda_1, \phi_1)\) and \((\Lambda_2, \phi_2)\) are commensurable iff \( \mathbb{Q} \Lambda_1 = \mathbb{Q} \Lambda_2 \) and \( \phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2 \).

We would like to have well-defined equivalence classes of the Q-lattices. It turns out that while not immediately obvious, commensurability is an equivalence relation. In order to prove this, we need the following useful lemma.

**Lemma 5.1.3.** Given two lattices \( \Lambda_1, \Lambda_2 \subset \mathbb{R}^n \) such that \( \mathbb{Q} \Lambda_1 = \mathbb{Q} \Lambda_2 \), we have that \( \Lambda_1 + \Lambda_2 \) is also a lattice that contains the lattices \( \Lambda_1 \) and \( \Lambda_2 \). Moreover, \( \Lambda_1 \) and \( \Lambda_2 \) are sublattices of finite index.

**Proof.** Consider the following forms for the lattices \( \Lambda_1 \) and \( \Lambda_2 \). Note, the sets \( \{v_i\}_{i=1}^n \) and \( \{w_i\}_{i=1}^n \)
are bases for \( \mathbb{R}^n \).

\[
\Lambda_1 = \left\{ \sum a_i v_i : a_i \in \mathbb{Z} \right\} \quad \text{and} \quad \Lambda_2 = \left\{ \sum b_i w_i : b_i \in \mathbb{Z} \right\},
\]

then

\[
Q\Lambda_1 = \left\{ \sum a_i v_i : a_i \in \mathbb{Q} \right\} \quad \text{and} \quad Q\Lambda_2 = \left\{ \sum b_i w_i : b_i \in \mathbb{Q} \right\}.
\]

Thus, in particular, given \( Q\Lambda_1 = Q\Lambda_2 \), for every \( 1 \leq j \leq n \) there exist \( a_{ij} \in \mathbb{Q} \) such that

\[
w_j = \sum \limits_i a_{ij} v_i.
\]

This yields

\[
\Lambda_2 = \left\{ \sum_i b_i \left( \sum_j a_{ji} v_j \right) : b_i \in \mathbb{Z} \right\}
\]

\[
= \left\{ \sum_i \sum_j b_i a_{ji} v_j : b_i \in \mathbb{Z} \right\}
\]

\[
= \left\{ \sum_j \left( \sum_i b_i a_{ji} \right) v_j : b_i \in \mathbb{Z} \right\}
\]

One can easily show that \( \sum b_i a_{ji} \) can be written as \( b_j r_j \) where \( r_i \) is found by first multiplying through by the lowest common multiple of the denominators of \( a_{i,j} \), finding the greatest common divisor of the resultant numbers and then dividing back by the lowest common multiple of the denominators.

So,

\[
\Lambda_1 + \Lambda_2 = \left\{ \sum \left( a_i + b_i r_i \right) v_i : a_i, b_i \in \mathbb{Z} \right\}.
\]

Writing \( r_i = a/b \) in reduced form gives

\[
\mathbb{Z} + \frac{a}{b} \mathbb{Z} = \frac{1}{b} \left( b \mathbb{Z} + a \mathbb{Z} \right) = \frac{1}{b} \gcd(a, b) \mathbb{Z} = \frac{1}{b} \mathbb{Z}.
\]

Therefore,

\[
\Lambda_1 + \Lambda_2 = \left\{ \sum \left( \frac{a_i}{\text{den}(r_i)} \right) v_i : a_i \in \mathbb{Z} \right\}
\]

where \( \text{den} \) returns the denominator. This expression for \( \Lambda_1 + \Lambda_2 \) is clearly a lattice and \( \Lambda_1 \) has index \( \prod \text{den}(r_i) \) in \( \Lambda_1 + \Lambda_2 \).

Now, we have the ingredients to show it is an equivalence relation.

**Lemma 5.1.4.** The commensurability relation is indeed a well-defined equivalence relation on \( \mathbb{Q} \)-lattices. Thus, we can write \( (\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2) \) to mean commensurability.
Proof. Clearly, reflexivity and symmetry are immediate from the symmetry in the definition of the relation. Consider, now the transitivity relation. In particular, suppose

\[(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2) \text{ and } (\Lambda_2, \phi_2) \sim (\Lambda_3, \phi_3)\].

First, we see that \(QA_1 = QA_2 = QA_3\). Hence, in particular, the sum of any two or all three of these lattices is another lattice. From the first lemma,

\[\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3\]

is a lattice containing any \(\Lambda_i\) or \(\Lambda_i + \Lambda_j\) as sublattices of finite index.

Consider now the homomorphism of torsion points. We have

\[\phi_1 - \phi_2 = 0 \mod \Lambda_1 + \Lambda_2 \quad \text{and} \quad \phi_2 - \phi_3 = 0 \mod \Lambda_2 + \Lambda_3\].

As, \(\Lambda_1 + \Lambda_2 + \Lambda_3\) is a larger lattice containing \(\Lambda_1 + \Lambda_2\) and \(\Lambda_2 + \Lambda_3\), we can say

\[\phi_1 - \phi_2 = \phi_2 - \phi_3 = 0 \mod \Lambda\].

So, \(\phi_1 - \phi_3 = 0 \mod \Lambda\). Now, we know from homological algebra, that if we have the following commutating diagram of homomorphisms

\[
\begin{array}{c}
\ker \phi \\
\downarrow f \\
M \\
\downarrow \phi \\
N
\end{array}
\]

then if \(\phi \circ f = 0\), then there is a homomorphism \(h\) that commutes with the diagram.

In our scenario, we have the following commutating diagram:

\[
\begin{array}{c}
Q^n/Z^n \\
\downarrow g \\
\ker g \\
\downarrow g \\
QA_1/(\Lambda_1 + \Lambda_3) \\
\downarrow g \\
QA_1/(\Lambda)
\end{array}
\]

where we think of \(g\) as modulo the lattice \(\Lambda\). In this case, \(g \circ (\phi_1 - \phi_3) = 0\) and \(\ker g = \Lambda/(\Lambda_1 + \Lambda_3)\).

This satisfies the conditions of the above result and thus there exists a homomorphism

\[h : Q^n/Z^n \to \Lambda/(\Lambda_1 + \Lambda_3)\]
such that the diagram commutes. $Q^n/Z^n$ is infinitely divisible while $\Lambda/(\Lambda_1 + \Lambda_3)$ is finite. So, if $h(c) = k \neq 0$ for some $c$, then because $Q^n/Z^n$ is divisible, $c = nd$ and therefore,

$$nh(d) = k \neq 0$$

but this is a contradiction, because the order of the group is $n$ and hence $na = 0$ for all $a$. So, $h = 0$ and hence by commutivity

$$\phi_1 - \phi_3 = 0 \mod \Lambda_1 + \Lambda_3.$$  

This equivalence relation induces a quotient that is best described through noncommutative geometry. For this we consider non-invertible $Q$-lattices.

In fact, we can show

**Lemma 5.1.5.** Two invertible $Q$-lattices are commensurable if and only if they are equal.

**Proof.** If two invertible $Q$-lattices $(\Lambda_1, \phi_1)$ and $(\Lambda_2, \phi_2)$ are commensurable, work from Lemma 5.1.3 allows us to write

$$\Lambda_1 = \left\{ \sum a_i v_i : a_i \in \mathbb{Z} \right\}$$

and

$$\Lambda_2 = \left\{ \sum b_i r_i v_i : b_i \in \mathbb{Z} \right\}$$

where $\{v_i\}_{i=1}^n$ forms a basis for $\mathbb{R}^n$ and $r_i \in Q$ is fixed and in simplest form. Additionally, $\Lambda_1$ and $\Lambda_2$ are sublattices of finite index in $\Lambda_1 + \Lambda_2$. For this reason, $Q\Lambda_1 = Q\Lambda_2 = Q(\Lambda_1 + \Lambda_2)$ and in particular,

$$\Lambda_1 + \Lambda_2 = \left\{ \sum \frac{c_i v_i}{\text{den}(r_i)} : c_i \in \mathbb{Z} \right\}.$$  

Showing $\Lambda_1 = \Lambda_2$ is equivalent to showing $r_i = 1$ for all $i \in \{1, \ldots, n\}$. To do this, choose a particular $i \in \{1, \ldots, n\}$ and without loss of generality assume $r_i < 1$. Set $x = \frac{v_i}{2\text{den}(r_i)}$. As $x < \frac{v_i}{\text{den}(r_i)}$ and $x \in Q\Lambda_1 = Q\Lambda_2$, it follows that $x \in Q(\Lambda_1 + \Lambda_2)$ and similarly $x \in Q(\Lambda_1 + \Lambda_2)$ mod $(\Lambda_1 + \Lambda_2)$ and similarly $x \in Q(\Lambda_1 + \Lambda_2)$ mod $(\Lambda_1 + \Lambda_2)$. The commensurability condition gives $\phi_1 = \phi_2$ mod $(\Lambda_1 + \Lambda_2)$. This combined with the fact that invertibility means $\phi_1$ and $\phi_2$ are isomorphisms implies there exists $p \in Q^n/Z^n$ such that $\phi_1(p) = \phi_2(p) = x$.

Consider now $p' = 2r_i\text{den}(r_i)p$. As $r_i\text{den}(r_i)$ is just the numerator of $r_i$, $2r_i\text{den}(r_i)$ is integral, thus

$$\phi_1(p') = 2r_i\text{den}(r_i)\phi_1(p) = r_i v_i \neq 0$$

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as \( r_i < 1 \). On the other hand,

\[
\phi_2(p') = 2r_i \text{den}(r_i)\phi_2(p) = r_i v_i = 0
\]

By taking the inverse of each isomorphism, \( p' \) is both zero and non-zero which is a contradiction. Thus, \( r_i = 1 \) for all \( i \in \{1, \ldots, n\} \) and \( \Lambda_1 = \Lambda_2 \).

The reverse follows from Lemma 5.1.4 as commensurability is an equivalence relation and thus satisfies the axiom of reflexivity. \( \square \)

From now on, I will concentrate only on the 1-dimensional \( \mathbb{Q} \)-lattices. Any mention of \( \mathbb{Q} \)-lattices unless otherwise mentioned assumes the dimension is one.

**Lemma 5.1.6.** A \( \mathbb{Q} \)-lattice is of the form \((\lambda \mathbb{Z}, \phi)\) and is thus completely determined by the element \( \rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \) where \( \phi = \lambda \rho \).

**Proof.** This is fairly simple. Clearly, there exists an element \( v \in \Lambda \), such that

\[
\Lambda = \{av : a \in \mathbb{Z}\} = v \mathbb{Z}.
\]

Thus, setting \( \lambda = v \) yields \( \Lambda = \lambda \mathbb{Z} \). Moreover,

\[
\phi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}\Lambda/\Lambda = \mathbb{Q}(\lambda \mathbb{Z})/(\lambda \mathbb{Z}) = \lambda \mathbb{Q}\mathbb{Z}/\mathbb{Z} = \lambda \mathbb{Q}/\mathbb{Z}.
\]

Thus, if we let \( \phi = \lambda \rho \), then \( \rho : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \). Also, \( \rho \) is also clearly a homomorphism, hence \( \rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \). Thus, up to scaling, a \( \mathbb{Q} \)-lattice is determined by \( \rho \). \( \square \)

In order to get a better understanding of what structure these commensurability classes of \( \mathbb{Q} \)-lattices has, we consider \( \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \). From Section 3.4.2, it is worth noting that

**Lemma 5.1.7.** \( \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \hat{\mathbb{Z}} = \lim \frac{\mathbb{Z}}{n} \).

It is worth finding an explicit presentation of commensurable \( \mathbb{Q} \)-lattices.

**Proposition 5.1.8.** If \( \Lambda_1 \) is a \( \mathbb{Q} \)-lattice of the form \((\lambda_1 \mathbb{Z}, \lambda_1 \rho_1)\), and \( \Lambda_2 \) is a \( \mathbb{Q} \)-lattice commensurable to \( \Lambda_1 \), then \( \Lambda_2 = (r\lambda_1 \mathbb{Z}, \lambda_1 \rho_1) \) for some \( r \in \mathbb{Q}^* \).

**Proof.** Consider the following two \( \mathbb{Q} \)-lattices:

\[
\Lambda_1 = (\lambda_1 \mathbb{Z}, \lambda_1 \rho_1), \quad \Lambda_2 = (\lambda_2 \mathbb{Z}, \lambda_2 \rho_2)
\]
such that $\lambda_1, \lambda_2 \in \mathbb{R}, \rho_1, \rho_2 \in \hat{\mathbb{Z}}$.

Given that $\Lambda_1 \sim \Lambda_2$, immediately we get $Q\lambda_1 Z = Q\lambda_2 Z$. This is equivalent to $\lambda_1 = r\lambda_2$, where $r \in \mathbb{Q}^+$. Additionally,

$$\lambda_1 \rho_1 = \lambda_2 \rho_2 \mod \lambda_1 Z + \lambda_2 Z$$

Writing $r = \frac{a}{b}$ in simplest terms and knowing (from previous work) that $Z + rZ = Z + \frac{a}{b}Z = \frac{1}{b}Z$, this becomes

$$\lambda_1 \rho_1 = \frac{a}{b} \lambda_1 \rho_2 \mod \lambda_1 \frac{1}{b}Z$$

$$\Rightarrow \rho_1 = \frac{a}{b} \rho_2 \mod \frac{1}{b}Z.$$  

Clearing denominators, this yields $b\rho_1 = a\rho_2 \mod \mathbb{Z}$, however, given that $\rho_1, \rho_2 : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, this implies $b\rho_1 = a\rho_2$. So, $\rho_2 = \frac{1}{r}\rho_1$.

Hence, commensurale lattices are of the form

$$\Lambda_1 = (\lambda_1 Z, \lambda_1 \rho_1); \quad \Lambda_2 = (r\lambda_1 Z, \lambda_1 \rho_1) \quad \text{for } r \in \mathbb{Q}^+, \lambda_1 \in \mathbb{R}^+, \rho_1 \in \hat{\mathbb{Z}}.$$

This means that commensurable $\mathbb{Q}$-lattices have the same system of labels and their lattices are the same modulo a scaling by a rational. I can now follow the work of [10] and define the graph of commensurability $\mathbb{Q}$ of $\mathbb{Q}$-lattices. Note, we shall use the notation that $\lambda(\Lambda, \phi) = (\lambda \Lambda, \lambda \phi)$.

**Definition 5.1.9.** The graph of commensurability of $\mathbb{Q}$-lattices $\mathbb{Q}$ consists of all pairs $(\Lambda_1, \Lambda_2)$ of commensurable $\mathbb{Q}$-lattices $\Lambda_1$ and $\Lambda_2$. Moreover, define $\mathcal{U} = \mathbb{Q}/\mathbb{R}^+$ to be the graph of commensurability of $\mathbb{Q}$-lattices modulo scaling by the positive reals. Note, two commensurable pairs of $\mathbb{Q}$-lattices $(\Lambda_1, \Lambda_2), (\Lambda_3, \Lambda_4)$ are equivalent modulo scaling by the positive reals if $\Lambda_1 = \lambda \Lambda_3$ and $\Lambda_2 = \lambda \Lambda_4$ for some $\lambda \in \mathbb{R}^+$.

It turns out that we can turn $\mathbb{Q}$ and $\mathcal{U}$ into groupoids. This will be explained in Section 5.4. Note, elements of $\mathcal{U}$ are of the form

$$((Z, \rho), (rZ, \rho)) \quad \text{for } r \in \mathbb{Q}^+, \rho \in \hat{\mathbb{Z}}.$$  

**5.2 Étale Groupoids**

In Section 5.4, I will show that the equivalence classes of $\mathbb{Q}$-lattices up to a scaling by the reals form an étale groupoid. This allows for the building of a $\mathbb{C}^*$-algebra out of this groupoid, and
hence, a $C^*$-dynamical system. First, I provide a brief overview of étale groupoids based on the conventions and work of Marius Crainic and Ieke Moerdijk in [14].

**Definition 5.2.1.** A groupoid $G$ is a small category of objects $G_0$ and morphisms $G_1$ where all morphisms are isomorphisms.

I shall unpack this definition a little more for clarity. The fact that a groupoid is a small category, means that $G_0$ and $G_1$ are simply sets. Additionally, the fact that all the morphisms are isomorphisms can equivalently be expressed by saying that the morphisms have inverses. The set of morphisms is the union of the (possibly empty) sets $G_1(x,y)$ of morphisms from $x$ to $y$, which are in standard practice written as $f : x \to y$, where $x, y \in G_0$. It is required that there be one element $I_x$ in $G_1(x,x)$ for each $x \in G_0$, which is called the unit at $x$. The structure of a groupoid can be represented by this diagram:

$$G_1 \times G_0 \xrightarrow{m} G_1 \xrightarrow{s} G_0 \xrightarrow{u} G_1 \xrightarrow{i} G_1$$

There are five structure maps in this diagram, specifically called the source ($s$), target ($t$), inverse ($i$), unit ($u$) and multiplication ($m$) maps. Recall their definition: continuing to write $f : x \to y$, then $s(f) = x$, $t(f) = y$ and $i(f) = f^{-1} : y \to x$. Also, $u(x) = I_x$. Finally, define the multiplication map of morphisms as $m(f,g) = f \circ g : s(g) \to t(f)$ on the set of composable pairs $G_1 \times G_0 \to G_1 = \{(f,g) : s(f) = t(g)\}$. With these structure maps, and $f$ defined as above, the following conditions must also hold:

- **associativity:** $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in G_1$ assuming this multiplication is defined;
- **identity:** $f \circ I_x = f = I_y \circ f$;
- **inverse:** $f \circ f^{-1} = I_y$ and $f^{-1} \circ f = I_x$.

At this point it is worth noting that a groupoid is sometimes referred to simply as $G_1$ if the set of objects is clear. Here, we follow the convention of referring to the groupoid as $G$. Simple checking will verify that $G_1$ with the operation of morphism multiplication satisfies the group axioms of associativity, inversion and identity (although not unique), but it is not a group, because multiplication is not always defined. So, one can think of a groupoid as generalising the notion of a group, and in fact, if $G_0$ were to only contain a single element, then $G_1$ becomes a group (unique identity and multiplication is always defined). In much the same way, for any set of objects $G_0$, the subset $G_1(x,x) \subset G_1$ is a group, called the isotropy group at $x$. To illustrate groupoids in practice, I present three important examples.
Examples.  

1. (General Linear Groupoid) The general linear groupoid $GL^*_\mathbb{F}(F)$ for a field $F$ given by all invertible matrices with entries in $F$ is a groupoid. The set of objects is $G_0 = \mathbb{N}$ and the morphisms from $m$ to $n$ are given by

$$G_1(m,n) = \{ \text{set of } m \times n \text{ invertible matrices with entries in } F \}$$

whenever $m = n$. $I_n$ is the $n \times n$ identity matrix.

2. (Graph of an Equivalence Relation) Consider a set $X$ with an equivalence relation $\sim$. A groupoid on this equivalence relation can be constructed by setting $G_0 = X$, and a single morphism $f_{x,y} : x \to y \in G_1$ if $x \sim y$ in $X$. The axioms of equivalence: reflexivity, symmetry and transitivity imply the identity exists for each $x \in G_0$, each morphism has an inverse and multiplication is well defined, respectively. The fact that we have a single morphism in $G_1(x,y)$ whenever $x \sim y$ implies associativity and forces all the compatibility (inverse and identity) conditions to hold. The Bost-Connes-Marcolli system uses this exact construction.

3. (Group Action Groupoid) Given a group $G$ acting on a set $X$, we construct the action groupoid by setting $G_0 = X$ and a single morphism $f_{(g,x)} : x \to y \in G_1(x,y)$ for every $g \in G$ such that $gx = y$. Note, that each $f_{(g,x)}$ is completely defined by $g$ and $x$ and so $f_{(g,x)}$ is written $(g,x)$. For this reason, we associate the action groupoid with the set $G \times X$ with

$$s(g,x) = x,$$

$$t(g,x) = gx,$$

$$i(g,x) = (g^{-1}, y),$$

$$u(x) = (e, x),$$

$$m((g,x),(h,y)) = (gh, y).$$

In order to construct a dynamical system from a groupoid, it needs certain properties.

Definition 5.2.2. A topological groupoid $\mathcal{G}$ is a groupoid that has $G_0$ and $G_1$ both being topological spaces and all five structure maps being continuous. An étale groupoid is a topological groupoid with the source map $s : \mathcal{G}_1 \to \mathcal{G}_0$ a local homeomorphism. This means that there exists a neighbourhood of any point in the domain of $s$ where $s$ is an open and continuous bijection.

We can immediately conclude that all the other structure maps of an étale groupoid are local homeomorphisms.

Lemma 5.2.3. If $\mathcal{G}$ is an étale groupoid, then all structure maps are local homeomorphisms.

Proof. I shall provide a brief overview of the proof of this statement.

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First, notice that $i$ is its own inverse:

$$(f^{-1})^{-1} = (f^{-1})^{-1} \circ I_x = (f^{-1})^{-1} \circ (f^{-1} \circ f) = (f^{-1})^{-1} \circ f = I_y \circ f = f$$

Hence, $i$ is a continuous bijection and is its own inverse, thus it is always a homeomorphism.

By definition, $t = s \circ i$, and the composition of local homeomorphisms is a local homeomorphism.

To show $u$ is a local homeomorphism, we will use the fact that $s$ is a local homeomorphism. For $u$, let $x \in G_0$. By definition, there exists an open neighborhood $V$ of $u(x)$ such that $s|_V : V \to s(V)$ is a homeomorphism. $u(s(V)) \subset s^{-1}(s(V)) = V$, as $s \circ u \circ s(V) = s(V)$ (in fact, $s \circ u$ is the identity on $G_0$). But since all sources of $V$ must be distinct for $s|_V$ to be a bijective, $u(s(V)) = V$. Hence $u$ is a local homeomorphism. Moreover, $u(G_0)$ is open in $G_1$.

Finally for $m$, for $(f,g) \in G_1 \times_{G_0} G_1$, there exists an open neighborhood $U$ of $f$ and $V$ of $g$ such that the restriction of $s$ to $U$ or $V$ respectively is a homeomorphism. Let $U \times_{G_0} V = \{(f,g) : f \in U, g \in V \text{ such that } t(g) = s(f)\}$. Such sets, for open $U$ and $V$, form the basis of the induced topology on $G_1 \times_{G_0} G_1$. $m$ is injective on this set: if $f_1 \circ g_1 = f_2 \circ g_2$, since all elements in $V$ have distinct sources, $g_1 = g_2$. Since $t|_U$ is also a homeomorphism, we know that all elements in $U$ have distinct targets, so $f_1 = f_2$. Hence $m|_U \times_{G_0} V$ is a continuous bijection with continuous inverse. Finally note that $m(U \times_{G_0} V)$ is open, as $m(U \times_{G_0} V) = s^{-1}(s(V)) \cap t^{-1}(t(U))$ (all arrows $f : x \to y$ come from a multiplication, as $f \circ u(x) = f$).

In order to define a convolution algebra of an étale groupoid without needing a Haar measure, it is enough to show that the fibres of all structure maps are discrete.

**Lemma 5.2.4.** In an étale groupoid, the fibres of all five structure maps are discrete.

**Proof.** By the last proposition, all of the groupoid structure maps are local homeomorphisms. So consider a local homeomorphism $\phi : X \to Y$ between topological spaces $X$ and $Y$. Let $y \in Y$ and consider the fibre $\phi^{-1}(y) = \{x \in X : \phi(x) = y\}$. For each $x \in \phi^{-1}(y)$, there exists an open neighborhood $U_x$ of $x$ such that $\phi|_{U_x}$ is a homeomorphism. Consider $U_x \cap \phi^{-1}(y)$. Clearly $x \in U_x \cap \phi^{-1}(y)$, but say $x_0 \in U_x \cap \phi^{-1}(y)$ too. $\phi(x_0) = \phi(x)$, but $\phi$ is injective on $U_x$, so $x_0 = x$, and $U_x \cap \phi^{-1}(y) = \{x\}$. Since all single point sets are open in $\phi^{-1}(y)$, the subspace is discrete.

We are now ready to define a groupoid C*-algebra.
5.3 Groupoid C*-algebras

In the same way that C*-algebras of groups have been well known for decades, C*-algebras of groupoids have also been well understood. In particular, in the case of étale groupoids, the construction of this C*-algebra is very similar to the construction of a discrete group C*-algebra. I begin by recalling the definition of the convolution algebra of an étale groupoid. I am following the work of [10, 4].

**Definition 5.3.1.** Given an étale groupoid $G$, the convolution algebra $\mathcal{C}(G)$ is the *-algebra of all continuous complex-valued functions on $G_1$ with compact support under pointwise addition and scalar multiplication with a convolution product given by

$$f \cdot g(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2)$$

and involution defined as

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$  

It is important to make the point here that this is a well-defined *-algebra. First, I make the point that because of Lemma 5.2.4, the fibre of $m$ is discrete and hence, $m^{-1}(\gamma) = \{(\gamma_1, \gamma_2) : \gamma_1 \circ \gamma_2 = \gamma\}$ is a discrete set. Moreover, as we are working with functions of compact support, this implies the convolution product is always a finite sum. This is the reason Lemma 5.2.4 avoids the use of an appropriate Haar measure in the definition of the convolution algebra.

Going one step further, $f \cdot g$ is compactly supported, as it is a finite sum of compactly supported products. The other conditions of a *-algebra are met as the convolution product is associative:

$$(f \cdot g) \cdot h(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} (f \cdot g)(\gamma_1)h(\gamma_2)$$

$$= \sum_{\gamma_1 \circ \gamma_2 = \gamma} \left( \sum_{\gamma_3 \circ \gamma_4 = \gamma_1} f(\gamma_3)g(\gamma_4) \right)h(\gamma_2)$$

$$= \sum_{\gamma_3 \circ \gamma_4 \circ \gamma_2 = \gamma} f(\gamma_3)g(\gamma_4)h(\gamma_2)$$

$$= \sum_{\gamma_3 \circ \gamma_5 = \gamma} f(\gamma_3) \left( \sum_{\gamma_4 \circ \gamma_2 = \gamma_5} g(\gamma_4)h(\gamma_2) \right)$$

$$= \sum_{\gamma_3 \circ \gamma_5 = \gamma} f(\gamma_3)(g \cdot h)(\gamma_5)$$

$$= f \cdot (g \cdot h)(\gamma)$$
It is also distributive over addition with scalar multiplication and the involution is also distributive and satisfies:

\[ f^{**}(\gamma) = f^*(\gamma^{-1}) = f((\gamma^{-1})^{-1}) = f(\gamma) \]

and

\[
(f \cdot g)^*(\gamma) = (f \cdot g)(\gamma^{-1}) = \sum_{\gamma_2 \circ \gamma_1 = \gamma} f(\gamma_2^{-1})g(\gamma_1^{-1}) \\
= \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_2^{-1})g(\gamma_1^{-1}) \\
= \sum_{\gamma_1 \circ \gamma_2 = \gamma} f^*(\gamma_2)g^*(\gamma_1) = (g^* \cdot f^*)(\gamma).
\]

Thus, Definitions 2.2.1 and 2.2.2 are satisfied and the convolution algebra is well-defined.

Following [10], we will now define a Hilbert space for each \( x \in G_0 \), equip it with an inner product and define a representation of \( C(G) \) into the bounded operators on this Hilbert space. By taking the supremum value of the norm of these representations over all \( x \in G_0 \) gives us a \( C^* \)-algebra norm and when one takes the completion of the convolution algebra in this norm, the resulting completion is a \( C^* \)-algebra.

In particular, for any \( x \in G_0 \), consider the standard Hilbert space \( \ell^2 \) associated to the isotropy group \( G_1(x,x) \):

\[
\ell^2(G_1(x,x)) = \left\{ \alpha : G_1(x,x) \to \mathbb{C} : \sum_{\gamma \in G_1(x,x)} |\alpha(\gamma)|^2 < \infty \right\}.
\]

There is a natural inner product on this Hilbert space:

\[
<\alpha_1, \alpha_2> = \sum_{\gamma \in G_1(x,x)} \overline{\alpha_1(\gamma)}\alpha_2(\gamma).
\]

This is well-known, so I shall avoid repeating proofs of standard results. For any \( f \in C(G) \), consider the operator \( B_f : \ell^2(G_1(x,x)) \to \ell^2(G_1(x,x)) \) given by \( B_f(\alpha) = \hat{f}_\alpha \), where \( \hat{f}_\alpha : G_1(x,x) \to \mathbb{C} \) is given by

\[
\hat{f}_\alpha(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)\alpha(\gamma_2).
\]

Remembering from Lemma 5.2.4, that the fibres of \( m \) are discrete and adding to the fact that \( f \) is compactly supported implies once more that the sum is finite. This implies \( \hat{f}_\alpha \) is also
compactly supported, so it is an element of $\ell^2(G_1(x,x))$. Therefore, $B_f$ is properly defined as an operator on $\ell^2(G_1(x,x))$ and it is no surprise it is denoted with a B as it is also bounded in the operator norm:

$$\|B_f\|_{op} = \sup_{\|\alpha\|_{2^1} = 1} \left\{ \|\hat{f}_\alpha\|_{2^1} \right\}$$

$$= \sup_{\|\alpha\|_{2^1} = 1} \left\{ \sum_{\gamma \in \tilde{G}_1(x,x)} |\hat{f}_\alpha(\gamma)|^2 \right\}$$

$$= \sup_{\|\alpha\|_{2^1} = 1} \left\{ \sum_{\gamma \in \tilde{G}_1(x,x)} \left| \sum_{\gamma_1 \gamma_2 \in \tilde{G}_1(x,x)} f(\gamma_1)\alpha(\gamma_2) \right|^2 \right\}$$

$$\leq \sup_{\|\alpha\|_{2^1} = 1} \left\{ \sum_{\gamma \in \tilde{G}_1(x,x)} \left| f(\gamma_1)\alpha(\gamma_2) \right|^2 \right\}$$

$$= \sup_{\|\alpha\|_{2^1} = 1} \left\{ \sum_{\gamma \in \tilde{G}_1(x,x)} \left| f(\gamma_1) \right|^2 \left| \alpha(\gamma_2) \right|^2 \right\}$$

$$= \sup_{\|\alpha\|_{2^1} = 1} \left\{ \sum_{\gamma \in \tilde{G}_1(x,x)} \left| f(\gamma_1) \right|^2 \right\}$$

$$= \sum_{\gamma \in \tilde{G}_1(x,x)} \left| f(\gamma_1) \right|^2 < \infty$$

All sums including the last one are all finite, due to the discrete fibres of the étale groupoid multiplication and the fact that $f$ is compactly supported. Now, consider the map $\pi_x : C(G) \to \ell^2(G_1(x,x))$ given by $\pi_x(f) = B_f$.

See [4] for the details of showing this is a $^*$-homomorphism (the work is very similar to what is shown above). Hence, $\pi_x$ is a representation of $C(G)$ into $\ell^2(G_1(x,x))$. Define the norm

$$\|f\|_{\text{new}} = \sup_{x \in G_0} \{ \|\pi_x(f)\|_{op} \}$$

on $C(G)$. This is as intended a C*-algebra norm.

**Lemma 5.3.2.** The norm $\| \cdot \|_{\text{new}}$ on $C(G)$ satisfies the C*-algebra condition.

The details of the proof of this statement can be found in [17]. While unnecessary for our purposes, Ruy Exel has found that this norm actually attains its supremum [17].
These representations $\pi_x$ are in fact the regular representations of $C(G)$, so $\|\cdot\|_{\text{new}}$ is called the reduced C*-algebra norm and completing the convolution algebra with respect to this norm produces the reduced C*-algebra $C^*(G)$ of the groupoid $G$. Due to the fact that $G$ is amenable (see [10] for details), this C*-algebra is the same as the C*-algebra obtained by completing in the full norm (the norm found by taking the supremum over all nondegenerate representations of $C(G)$).

5.4 The Bost-Connes-Marcolli System

After all this work on étale groupoids and their enveloping C*-algebra, it should come as no surprise that the graph of commensurable $\mathbb{Q}$-lattices modulo scaling by the reals is an étale groupoid.

**Proposition 5.4.1.** $\mathcal{U}$ is a locally compact étale groupoid with the following set up: let $\mathcal{U}_1 = \mathcal{U}$, $\mathcal{U}_0 = \hat{\mathbb{Z}}$. Define the five structure maps:

\[
\begin{align*}
    s((\mathbb{Z}, \rho), (r\mathbb{Z}, \rho)) &= \rho, \\
    t((\mathbb{Z}, \rho), (r\mathbb{Z}, \rho)) &= r\rho, \\
    u(\rho) &= ((\mathbb{Z}, \rho), (\mathbb{Z}, \rho)), \\
    i((\mathbb{Z}, \rho), (r\mathbb{Z}, \rho)) &= \left((\mathbb{Z}, \rho), \left(\frac{1}{r}\mathbb{Z}, \rho\right)\right), \\
    m(((\mathbb{Z}, \rho_1), (r_1\mathbb{Z}, \rho_1)), ((\mathbb{Z}, \rho_2), (r_2\mathbb{Z}, \rho_2))) &= ((\mathbb{Z}, \rho_2), (r_1 r_2\mathbb{Z}, \rho_2)) \text{ when } \rho_1 = r_2\rho_2
\end{align*}
\]

I delay the proof here for a moment, as the notation above is both tedious and hard on the eyes. However, immediately one can see that each element of $\mathcal{U}_1$ is totally described by the pair $(r, \rho)$. In fact the function $\phi : \mathbb{Q}^+ \times \hat{\mathbb{Z}} \to \mathcal{U}_1$ when restricted to the pairs $(r, \rho)$ where $r\rho \in \hat{\mathbb{Z}}$ given by

\[
\phi(r, \rho) = ((\mathbb{Z}, \rho), (r\mathbb{Z}, \rho))
\]

is a bijection of sets.

When we translate the above groupoid into the language of these pairs, we will get a groupoid and $\phi$ becomes a groupoid isomorphism. In particular,

**Proposition 5.4.2.** $\mathcal{U}^*$ is a locally compact étale groupoid with the following set up: let $\mathcal{U}_1^* = \left((\mathbb{Z}, \rho), (r\mathbb{Z}, \rho)\right)$ when $\rho_1 = r_2\rho_2$.
\{(r, \rho) : r \in \mathbb{Q}^+, \rho, r\rho \in \hat{\mathbb{Z}}\}, \mathcal{U}^*_0 = \hat{\mathbb{Z}}. Define the five structure maps:

\begin{align*}
s(r, \rho) &= \rho,
t(r, \rho) &= r\rho,
u(\rho) &= (1, \rho),
i(r, \rho) &= \left(\frac{1}{r}, r\rho\right),
m((r_1, \rho_1), (r_2, \rho_2)) &= (r_1r_2, \rho_2) \text{ when } \rho_1 = r_2\rho_2
\end{align*}

Moreover, \( \phi \) defined above is a groupoid isomorphism between \( \mathcal{U} \) and \( \mathcal{U}^* \).

**Proof.** First, \( \phi \) is clearly a surjection due to Proposition 5.1.8. It is also clearly injective and one can show without any difficulty that \( \phi \) preserves the groupoid structural maps. Showing that these structural maps define a groupoid can be verified by checking the conditions from Definition 5.2.1. In particular,

\[
[(r_1, \rho_1) \circ (r_2, \rho_2)] \circ (r_3, \rho_3) = (r_1r_2, \rho_2) \circ (r_3, \rho_3) \quad \text{if } \rho_1 = r_2\rho_2
\]

\[
= (r_1r_2r_3, \rho_3) \quad \text{if } \rho_1 = r_2\rho_2, \rho_2 = r_3\rho_3
\]

\[
= (r_1, \rho_1) \circ (r_2r_3, \rho_3) \quad \text{if } \rho_2 = r_3\rho_3
\]

\[
= (r_1, \rho_1) \circ [(r_2, \rho_2) \circ (r_3, \rho_3)]
\]

and

\[
(r, \rho) \circ (1, \rho) = (r, \rho) = (1, r\rho) \circ (r, \rho)
\]

and

\[
(r, \rho) \circ \left(\frac{1}{r}, r\rho\right) = (1, r\rho) \quad \text{and} \quad \left(\frac{1}{r}, r\rho\right) \circ (r, \rho) = (1, \rho)
\]

\( \hat{\mathbb{Z}} \) is compact from the profinite topology (see Section 3.4.2) and \( \mathbb{Q} \times \hat{\mathbb{Z}} \) is given the product topology assuming \( \mathbb{Q} \) to be discrete. With the topologies identified, note that all of the structural maps are continuous (standard algebraic functions) and the source map is simply a projection onto the second coordinate, which is a local homeomorphism. So, \( \mathcal{U} \cong \mathcal{U}^* \) are étale. \( \square \)

We will now refer to \( \mathcal{U}^* \) simply as \( \mathcal{U} \) and use its presentation rather than the one involving \( \mathbb{Q} \)-lattices for ease of notation.

Using the work from Section 5.3, we can construct \( C^*(\mathcal{U}) \). In order to turn this into a dynamical system, I need to define the time evolution on \( C^*(\mathcal{U}) \). Fortunately, there is a natural time evolution that arises from the ratio of covolumes of commensurable lattices.
Definition 5.4.3. The **covolume** of a lattice $\Lambda$ is the determinant of the matrix that has columns which are the vectors $v_i$ where $\Lambda$ can be thought of as the $\mathbb{Z}$-span of linearly independent vectors $v_i$. This is denoted $d(\Lambda)$ and is the volume of the fundamental region of the lattice. Moreover, it can be shown to be independent of the basis $\{v_i\}$.

Hence, we can define the time evolution. A pair $(r, \rho)$ corresponds to the commensurable lattices $(\mathbb{Z}, \rho)$ and $(r\mathbb{Z}, \rho)$. The ratio of their covolumes is clearly $r$.

Hence, we get the natural time evolution

$$\sigma_t(f)(r, \rho) = r^{it} f(r, \rho)$$

Lemma 5.4.4. The time evolution $\sigma_t$ is well-defined.

**Proof.** We need to show that $\sigma_t$ is an automorphism of $C(U)$. Immediately, one can see that addition and scalar multiplication are satisfied. For involution,

$$\sigma_t(f)^*(r, \rho) = \overline{\sigma_t(f)(r^{-1}, r\rho)} = \overline{r^{-it} f(r^{-1}, r\rho)} = r^{it} f(r^{-1}, r\rho) = r^{it} f^*(r, \rho) = \sigma_t(f^*)(r, \rho)$$

For convolution,

$$\sigma_t(f \cdot g)(r, \rho) = r^{it} (f \cdot g)(r, \rho)$$

$$= r^{it} \sum_{r_1 r_2 = r} f(r_1, r_2\rho) g(r_2, \rho)$$

$$= \sum_{r_1 r_2 = r} r_1^{it} f(r_1, r_2\rho) r_2^{it} g(r_2, \rho)$$

$$= \sum_{r_1 r_2 = r} \sigma_t(f)(r_1, r_2\rho) \sigma_t(g)(r_2, \rho)$$

$$= (\sigma_t(f) \cdot \sigma_t(g))(r, \rho)$$

Definition 5.4.5. The **Bost-Connes-Marcolli system** is $(C^*(U), \sigma_t)$.

5.5 **Comparison to Bost-Connes System**

In this section, we show that these two $C^*$-dynamical systems are indeed $*$-isomorphic. This will be done by showing an equivalence between the presentations of the two systems.

I bring your attention to the presentation of the Bost-Connes system of Theorem 2.3.9. In a similar vein, first note the following definitions:
**Definition 5.5.1.** Define

1. For \( n \in \mathbb{N}^* \), let
   \[
   \mu_n(r, \rho) = \begin{cases} 
   1 & r = n \\
   0 & \text{otherwise}
   \end{cases}
   \]

2. For \( \gamma \in \mathbb{Q}/\mathbb{Z} \), let
   \[
   e_\gamma(r, \rho) = \begin{cases} 
   \exp(2\pi i \rho(\gamma)) & r = 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]

In this exponentiation, we view the result of \( \rho(\gamma) \) as lying in \( \mathbb{Q} \) and abuse notation by continuing to call this \( \rho \).

This is enough to state the theorem.

**Theorem 5.5.2.** The elements \( \mu_n, e(\gamma) \) for \( n \in \mathbb{N}^* \), \( \gamma \in \mathbb{Q}/\mathbb{Z} \) generate \( C^*(\mathcal{U}) \) and the following relations give a presentation for it.

1. \( \mu_n^* \circ \mu_n = 1 \quad \forall n \in \mathbb{N}^* \),
2. \( \mu_n \circ \mu_m = \mu_{mn} \quad \forall m, n \in \mathbb{N}^* \),
3. \( e_0 = 1, e_\gamma^* = e_{-\gamma}, e_{\gamma_1+\gamma_2} = e_{\gamma_1} \circ e_{\gamma_2} \quad \forall \gamma_1, \gamma_2 \in \mathbb{Q}/\mathbb{Z} \),
4. \( \mu_n \circ e_\gamma \circ \mu_n^* = \frac{1}{n} \sum_{n|d=\gamma} e_d \quad \forall n \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z} \).

**Proof.** Marcelo Laca in [29] shows that these elements generate a dense subset of \( C^*(\mathcal{U}) \), and moreover, the completion of the algebras generated by the basis elements \( \mu_n \) and \( e_\gamma \) are \( C^*(\mathbb{N}^*) \) and \( C^*(\mathbb{Q}/\mathbb{Z}) \). Now, I shall show these equalities hold.

1. For all \( n \in \mathbb{N}^* \),
   \[
   \mu_n^* \circ \mu_n(r, \rho) = \sum_{r_1 r_2 = r} \mu_n^*(r_1, r_2 \rho) \mu_n(r_2, \rho) = \sum_{r_1 r_2 = r} \mu_n(r_1^{-1}, r \rho) \mu_n(r_2, \rho) = \begin{cases} 
   1 & r = 1 \\
   0 & \text{otherwise}
   \end{cases} = \mu_1(r, \rho)
   \]
   As both terms in the summation are non-zero iff \( r_1 = n^{-1} \) and \( r_2 = n \), hence, \( r = 1 \).
2. For all \( m, n \in \mathbb{N}^* \),
   \[
   \mu_n \circ \mu_m(r, \rho) = \sum_{r_1 r_2 = r} \mu_n(r_1, r_2 \rho) \mu_m(r_2, \rho) = \begin{cases} 
   1 & r = mn \\
   0 & \text{otherwise}
   \end{cases} = \mu_{mn}(r, \rho)
   \]
3. These follow immediately from exponential laws.
4. For all $n \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z}$,

$$\mu_n \circ e_\gamma \circ \mu_n^*(r, \rho) = \sum_{r_1 r_2 = r} (\mu_n \circ e_\gamma \circ (r_1, r_2 \rho)) \mu_n^*(r_2, \rho)$$

$$= \sum_{r_1 r_2 = r} (\mu_n \circ e_\gamma \circ (r_1, r_2 \rho)) \mu_n(r_2^{-1}, r_2 \rho)$$

$$= (\mu_n \circ e_\gamma \circ (nr, \frac{1}{n}) \rho) \quad \text{as this vanishes unless } r_2 = n^{-1}$$

$$= \sum_{r_3 r_4 = nr} \mu_n(r_3, r_4 \frac{1}{n}) \rho) e_\gamma(r_4, \frac{1}{n} \rho)$$

$$= e_\gamma(r, \frac{1}{n} \rho) \quad \text{as this vanishes unless } r_3 = n$$

$$= \begin{cases} 
\exp(2\pi i \frac{1}{n} \rho(\gamma)) & r = 1, \rho \in n\hat{\mathbb{Z}} \\
0 & \text{otherwise}
\end{cases}$$

Note, that in $\mathbb{Q}/\mathbb{Z}$ it doesn’t make sense to divide by $n \in \mathbb{Z}$. For $\gamma \in \mathbb{Q}/\mathbb{Z}$, there are $n$ values $\delta \in \mathbb{Q}/\mathbb{Z}$ such that $n\delta = \gamma$. Fix a representative $\gamma_0 \in \mathbb{Q}$ for $\gamma$. Then

$$\delta_k := \frac{\gamma_0 + k}{n} \quad \text{for } k \in \{0, 1, \ldots, n - 1\}$$

satisfies $n\delta_k = \gamma \in \mathbb{Q}/\mathbb{Z}$, where $\gamma_0$ was lifted to $\mathbb{Q}$, $\delta_k$ was evaluated in $\mathbb{Q}$ and then reduced modulo $\mathbb{Z}$.

Then, for $k \in \{0, 1, \ldots, n - 1\}$,

$$\exp(2\pi i \rho(\delta_k)) = \exp(2\pi i \rho(\frac{\gamma_0 + k}{n})) = \exp(2\pi i \frac{1}{n} \rho(\gamma_0)) \exp(2\pi i \frac{1}{n} \rho(k)).$$

The key is that

$$\mu_n \circ e_\gamma \circ \mu_n^*(r, \rho) = e_\gamma(r, \frac{1}{n} \rho)$$

implying that this is zero, unless $\rho \in n\hat{\mathbb{Z}}$. In this case, $\exp(2\pi i \frac{1}{n} \rho(k)) = 1$, whereas if $n \nmid \rho$, then $\sum_{k=0}^{n-1} \exp(2\pi i \frac{1}{n} \rho(k)) = 0$ as it is the sum of all powers of a root of unity. Hence,

$$\mu_n \circ e_\gamma \circ \mu_n^*(r, \rho) = \begin{cases} 
\exp(2\pi i \frac{1}{n} \rho(\gamma)) & r = 1, \rho \in n\hat{\mathbb{Z}} \\
0 & \text{otherwise}
\end{cases}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \exp(2\pi i \rho(\frac{2k}{n})) \quad r = 1$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \exp(2\pi i \rho(\delta)) \quad r = 1$$

$$= \frac{1}{n} \sum_{n\delta \in \gamma} e_\delta(r, \rho)$$

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In this previous equation, whenever $\gamma$ was used, it was assumed that a choice of representative had been made.

Moreover, the functions $\mu_m \circ e_{\gamma} \circ \mu_n^*$ span $C^*(\mathcal{U})$ over $\mathbb{C}$ for all $m, n$ coprime and $\gamma \in \mathbb{Q}/\mathbb{Z}$. This is because if we follow the same working as we did for part 4 above, we find that

$$\mu_m \circ e_{\gamma} \circ \mu_n^* = \begin{cases} \exp \left(2\pi i \frac{1}{n} \rho(\gamma)\right) & r = \frac{m}{n} \\ 0 & \text{otherwise} \end{cases}$$

Hence, we can say

**Theorem 5.5.3.** The Bost-Connes-Marcolli system is $*$-isomorphic to the Bost-Connes system.

**Proof.** The two systems have the same generators satisfying the same presentation conditions. (See Theorem 2.3.9 and Theorem 5.5.2)

The Bost-Connes-Marcolli system has two interesting generalisations. First, the Connes-Marcolli systems arise from this same (identical analysis) using one dimensional $K$-lattices for some number field $K$. One can also generalise to two dimensional $\mathbb{Q}$-lattices. The former solves the Bost-Connes Problem for $K$ an imaginary quadratic number field.
Chapter 6

Semigroups and their C*-algebras

The theory of Bost-Connes systems has progressed every year since Benoît Bost and Alain Connes’ initial 1995 article [5] in which they describe their system as a Hecke C*-algebra dynamical system. The research done since to generalise the system to any number field and retain the key properties that are closely tied to class field theory have spurred on many key insights into related theories. In particular, there has been a wealth of work done in the area of Hecke C*-algebras and also groupoid C*-algebras. Additionally, and this is the focus of this chapter, the original Bost-Connes system was written as a semigroup crossed product C*-algebra system. In 2005, Eugene Ha and Frédéric Paugam in [21] found that by studying the Shimura data for a number field, one can construct a natural semigroup crossed product C*-dynamical system that solves what is now known as the Bost-Connes Problem.

The concepts involved in the Shimura data approach will not be explained, but in this chapter I will endeavour to bring together the relevant theory for the semigroup crossed product presentation of the Bost-Connes system. In Section 6.1, there is a brief discussion of some semigroup definitions and basic facts. In Section 6.2, I discuss the definition of the semigroup crossed product and some relevant theories.

Using the theory of Sections 6.1 and 6.2 and our knowledge of the Bost-Connes system, in Section 6.3, I show the semigroup crossed product form for the Bost-Connes system and introduce the Connes-Marcolli system for arbitrary number field $K$ (in the semigroup crossed product form) that solves the Bost-Connes Problem. In the same chapter, it is shown that the Connes-Marcolli system for $K = \mathbb{Q}$ is isomorphic to the original Bost-Connes system.
6.1 Semigroups

In this section I will be recalling standard definitions of semigroups. This may be found in work by Gerard Murphy [35] or Mamoon Ahmed [2].

**Definition 6.1.1.** A *semigroup* is a set of $S$ with an associative binary operation $\ast$. We normally write $ab$ for $a \ast b$ and omit mentioning $\ast$ in cases where the context makes it clear.

**Remark.** A semigroup need not have an identity. A semigroup with an identity is called a *monoid*. A semigroup with identity has a unique identity, for if it had two identities $e$ and $f$, then $e = ef = f$. A semigroup $S$ with no identity can be embedded in a monoid by adjoining an element $e \notin S$ and defining $e \ast s = s \ast e = s$ for all $s \in S \cup \{e\}$. It is standard to call the monoid thus obtained $S^1$.

I add here a few important definitions pertaining to the properties of semigroups.

**Definition 6.1.2.** Analogously to subgroups, we call a subset $A$ of a semigroup $S$ a *subsemigroup* if $ab \in A$ for all $a, b \in A$.

**Definition 6.1.3.** A *semigroup homomorphism* is a function $f : S_1 \rightarrow S_2$ between semigroups that preserves semigroup structure. In other words, $f$ satisfies

$$f(ab) = f(a)f(b) \text{ for all } a, b \in S_1.$$

If $f$ is also a bijection, then it is a *semigroup isomorphism*.

For our purposes, we are most interested in the case where a semigroup ‘comes from’ a group. The following definitions illustrate the motivation for such an approach. Let us first define a partial order on a discrete group. Unless otherwise mentioned, all groups and semigroups will be assumed discrete and nonempty.

**Definition 6.1.4.** Let $G$ be a discrete group. Define a *partial order* on $G$ as a binary operation $\leq$ on $G$ which satisfies (for $g, h, k \in G$)

1. $g \leq g$ (reflexivity)
2. $g \leq h$ and $h \leq g$ implies $g = h$ (antisymmetry)
3. $g \leq h$ and $h \leq k$ implies $g \leq k$ (transitivity)
4. $g \leq h$ implies $kg \leq kh$ (left invariant)

A nonempty group $G$ with a partial order is called a *partially ordered group*.
Definition 6.1.5. Given a partially ordered group $G$, we define the positive cone $G_+$ of $G$ as the semigroup of all positive elements (ie. all elements $g \in G$ such that $g \geq e$, $e$ the identity of $G$).

Remark. The positive cone $G_+$ is indeed a semigroup as it inherits an associative binary operation from $G$ and for $g, h \in G_+$, we have

$$g \geq e \text{ implies } gh \geq h \geq e.$$ 

Thus, $gh \in G_+$.

An important example of a partial order on a group comes from a specific semigroup.

Proposition 6.1.6. Given a subsemigroup $G_+$ of a group $G$, such that $G_+ \cap G^{-1}_+ = \{e\}$ (where $A^{-1} = \{a^{-1} : a \in A\}$), a partial order $\leq$ can be defined for $G$ where $g \leq h$ if $g^{-1}h \in G_+$. We denote by $(G,G_+)$ the partially order group $G$ with partial order determined by $G_+$.

Proof. We show the four conditions of a partial order from Definition 6.1.4 are satisfied.

1. $g^{-1}g = e \in G_+$ by definition.
2. If $g \leq h$ and $h \leq g$, then $g^{-1}h, h^{-1}g \in G_+$. But as $h^{-1}g = (g^{-1}h)^{-1}$ it belongs to $G^{-1}_+$, so $g^{-1}h = e$. Hence, $g = h$
3. If $g \leq h$ and $h \leq k$, then $g^{-1}h, h^{-1}k \in G_+$. Then $(g^{-1}h)(h^{-1}k) = g^{-1}k \in G_+$. So, $g \leq k$.
4. If $g \leq h$, then $(kg)^{-1}(kh) = g^{-1}k^{-1}kh = g^{-1}h \in G_+$. So, $kg \leq kh$.

There are a few critical examples of semigroups and partially ordered groups worth discussing.

Examples. 1. A partially ordered group $(G,G_+)$ is quasi-lattice ordered iff

- every element of $G$ with upper bound in $G_+$ has a least upper bound in $G_+$,
- every two elements of $G$ with a common upper bound in $G_+$, have a least upper bound in $G_+$.

We will denote the least upper bound of $g, h \in G$ as $g \lor h$. A greatest lower bound is analogously denoted $g \land h$.

2. A partially ordered group $(G,G_+)$ is lattice ordered iff every two elements of $G$ have an upper bound in $G$. Note, Ahmed shows in [2] that every lattice ordered group is a quasi-lattice ordered group.

3. An Ore semigroup $G_+$ is a cancellative semigroup which is right reversible. Left cancellative means that for $g, h, k \in G_+$ if $gh = gk$, then $h = k$. Note, right cancellative is defined
analogously. *Right reversible* means that for all \( g, h \in G_+ \) then \( G_+ g \cap G_+ h \neq \emptyset \). This is an important example of a semigroup that can be embedded in a group. In particular, there is a unique (up to canonical isomorphism) enveloping group \( G \) such that \( G = G_+^{-1} G_+ \).

4. \((\mathbb{N},+)\) is an Ore semigroup \((0 \in \mathbb{N})\). It can be naturally embedded in \((\mathbb{Z},+)\). We can also see that \((\mathbb{Z},\mathbb{N})\) is a lattice-ordered group and hence a quasi-lattice ordered group. Similarly, \((\mathbb{N}^*,\times)\) is an Ore semigroup, naturally embedded in \((\mathbb{Q}^*,\times)\).

## 6.2 Semigroup Crossed Products

In this section, we recall the definitions of the semigroup crossed product as it is found in the literature. We will closely follow the work of Gerard Murphy \[35\], Alexandru Nica \[39\], Marcelo Laca and Iain Raeburn \[27\] and also Mamoon Ahmed \[2\].

**Definition 6.2.1.** Let \( B \) be a unital C*-algebra. An element \( u \in B \) is said to be an *isometry* if \( u^*u = 1_B \). Denote the set of isometries of \( B \) as \( \text{Isom}(B) \).

It is trivial to see that the isometries of a unital C*-algebra form a semigroup. In particular, identity and associativity follow from the unital C*-algebra. Also, closure follows from the following calculation. If \( u, v \in \text{Isom}(B) \) then

\[
(vu)^* (uv) = v^* u^* uv = v^* 1_B v = v^* v = 1_B.
\]

This naturally allows us to define an isometric representation of a semigroup into a unital C*-algebra.

**Definition 6.2.2.** Let \( G_+ \) be a subsemigroup of a group \( G \), \( B \) a unital C*-algebra and \( V \) a map from \( G_+ \) to \( B \). Then \( V \) is said to be an *isometric representation* of \( G_+ \) if it satisfies the following three conditions:

(i) \( V_e = 1_B \);
(ii) \( V_x^* V_x = 1_B \) for any \( x \in G_+ \);
(iii) \( V_x V_y = V_{xy} \) for any \( x, y \in G_+ \).

In order to define a semigroup crossed product, we also need the concept of a semigroup dynamical system.

**Definition 6.2.3.** A *semigroup dynamical system* is a triple \((A, G_+, \alpha)\) where \( A \) is a C*-algebra
and $\alpha$ is an action of the semigroup $G_+$ on $A$ by endomorphisms (i.e. $\alpha : G_+ \to \text{End}(A)$ is a homomorphism). Two dynamical systems $(A, G_+, \alpha)$ and $(B, G_+, \alpha)$ are isomorphic if there is an isomorphism $\phi : A \to B$ such that $\phi \circ \alpha_x = \beta_x \circ \phi$ for all $x \in G_+$.

In order to turn a semigroup dynamical system into a semigroup crossed product $C^*$-algebra, we need to define the concept of a covariant representation for the semigroup dynamical system.

**Definition 6.2.4.** A **covariant representation** of a dynamical system $(A, G_+, \alpha)$ is a pair $(\pi, V)$, where $\pi$ is a non-degenerate representation of $A$ on a Hilbert space $\mathcal{H}$, and $V$ is an isometric representation of $G_+$ on $\mathcal{H}$ satisfying

$$\pi(\alpha_x(a)) = V_x \pi(a) V_x^* \quad \text{for all } x \in G_+, a \in A.$$ 

We are now ready to recall the definition of the semigroup crossed product.

**Definition 6.2.5.** A **crossed product** for a dynamical system $(A, G_+, \alpha)$ is a $C^*$-algebra $B$ together with a non-degenerate homomorphism $i_A : A \to B$ and a homomorphism $i_{G_+}$ of $G_+$ into the semigroup of isometries in $\mathcal{M}(B)$ (the multiplier algebra of $B$) such that:

1. $i_A(\alpha_x(a)) = i_{G_+}(x) i_A(a) i_{G_+}(x)^*$ for $x \in G_+$ and $a \in A$;
2. for every covariant representation $(\pi, V)$ of $(A, G_+, \alpha)$ there is a non-degenerate representation $\pi \times V$ of $B$ such that $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_{G_+} = V$;
3. $B$ is generated by $\{i_A(a) i_{G_+}(x) : a \in A, x \in G_+\}$.

We write $A \rtimes_{\alpha} G_+$ to denote the crossed product for the dynamical system $(A, G_+, \alpha)$. The homomorphisms $(i_A, i_{G_+})$ are called the **universal representation**.

**Remark.** The **multiplier algebra** of a $C^*$-algebra $B$ is defined as the $C^*$-algebra $\mathcal{M}(B)$, such that for all $C^*$-algebras $D$ for which $B$ is an ideal, there exists a unique *-homomorphism $\phi : D \to \mathcal{M}(B)$ such that the identity homomorphism is extended and $\phi$ sends elements in the orthogonal complement of $B$ to $\{0\}$. Note, that when $B$ is unital, $\mathcal{M}(B) = B$. The multiplier algebra always exists and one can construct it in the following way. A **double centralizer** of a $C^*$-algebra $A$ is a pair $(L, R)$ of bounded linear maps on $A$ to itself such that

$$aL(b) = R(a)b, \quad L(ab) = L(a)b, \quad R(ab) = aR(b) \quad \text{for all } a, b \in A.$$ 

This implies that $|L| = |R|$ and we define $|(L, R)| := |L|$. The set of double centralizers of $A$ can be given a $C^*$-algebra structure by setting $(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1)$, $a(L_1, R_1) =$
b(L_2, R_2) := (aL_1 + bL_2, aR_1 + bR_2) as well as \((L, R)^* := (L^*, R^*)\). This C*-algebra contains \(\mathcal{A}\) as an ideal and is the multiplier algebra \(\mathcal{M}(\mathcal{A})\).

**Remark.** Sriwulan Adji, Marcelo Laca, May Nilsen and Iain Raeburn in [1] show that if \(\mathcal{A}\) is unital, the representation \(\pi\) of Definition 6.2.4 and the homomorphism \(i_{\mathcal{A}}\) of Definition 6.2.5 must both be unital, and condition (2) of Definition 6.2.5 reduces to the existence of a unital representation \(\pi \times V\) of \(\mathcal{B}\) such that

\[(\pi \times V) \circ i_{\mathcal{A}} = \pi \quad \text{and} \quad (\pi \times V) \circ i_{G_+} = V.\]

It is worth considering whether or not there always exists such an universal representation, and hence, a crossed product for a given semigroup dynamical system. To this end, we state the following important proposition [2].

**Proposition 6.2.6.** If \((\mathcal{A}, G_+, \alpha)\) is a semigroup dynamical system, then there exists a crossed product for the system that is unique up to isomorphism, in the following two separate situations:

1. \(\mathcal{A}\) is unital and the system has a non-trivial covariant representation.
2. \(G_+\) is an Ore semigroup and the system has a non-zero covariant representation and is a system with extendible homomorphisms.

**Proof.**

1. We follow the proof outlined in [28]. It is well-known (see [35] Theorem 5.1.3) that a nondegenerate representation of a C*-algebra decomposes as a direct sum of cyclic representations. For any covariant pair \((\pi, V)\), the C*-algebra \(C^*(\pi(\mathcal{A}) \cup V(G_+))\) decomposes into cyclic representations, therefore, so does the covariant pair into a direct sum of cyclic pairs. Now, consider the collection of all covariant pairs \((\pi_\ell, V_\ell)\) on a Hilbert space \(\mathcal{H}_\ell\) such that \(C^*(\pi_\ell(\mathcal{A}) \cup V_\ell(G_+))\) acts cyclically on \(\mathcal{H}_\ell\). The pair \(i_{\mathcal{A}} = \bigoplus_\ell \pi_\ell\) and \(i_{G_+} = \bigoplus_\ell V_\ell\) is covariant, and nontrivial because of the assumed existence of a covariant pair. Let \(\mathcal{B}\) be the C*-algebra generated by \(i_{\mathcal{A}}(\mathcal{A}) \cup i_{G_+}(G_+)\). Therefore, the pair \((i_{\mathcal{A}}, i_{G_+})\) has the universal property.

Suppose \((\mathcal{B}, i_{\mathcal{A}}, i_{G_+})\) and \((\mathcal{B}', i'_{\mathcal{A}}, i'_{G_+})\) are triples satisfying the conditions of a semigroup crossed product for \((\mathcal{A}, G_+, \alpha)\). Since \((\mathcal{B}, i_{\mathcal{A}}, i_{G_+})\) satisfies condition (2) of Definition 6.2.5 there is a homomorphism \(i'_{\mathcal{A}} \times i'_{G_+} : \mathcal{B} \rightarrow \mathcal{B}'\) which is onto by condition (3) of Definition 6.2.5, and it is easy to see that it has an inverse given by \(i_{\mathcal{A}} \times i_{G_+} : \mathcal{B}' \rightarrow \mathcal{B}\), proving that \((\mathcal{B}, i_{\mathcal{A}}, i_{G_+})\) is unique up to isomorphism.

2. This is the focus of the paper [32] by Nadia Larsen.
6.3 Bost-Connes System

This section is the focus of the chapter and aims to show that the original Bost-Connes system can be alternatively viewed as a semigroup crossed product. In Section 6.4, I will present a Bost-Connes type system, called the Connes-Marcolli system that solves the Bost-Connes Problem for an arbitrary number field. The easiest way to describe the system is in terms of a semigroup crossed product C*-dynamical system. So, the work presented in this section aims to be easily replicable with the more general system in mind.

Following the notation of Marcelo Laca and Iain Raeburn in [29], consider the discrete group \( \mathbb{Q}/\mathbb{Z} \). Denote by \( i: \mathbb{Q}/\mathbb{Z} \to C^{*}(\mathbb{Q}/\mathbb{Z}) \) the canonical embedding of \( \mathbb{Q}/\mathbb{Z} \) as unitaries in its group C*-algebra. Note, we also mean \( i(r) \) to mean \( i(r + \mathbb{Z}) \). Consider the following action of \( \mathbb{N}^{*} \) by endomorphisms of \( C^{*}(\mathbb{Q}/\mathbb{Z}) \):

**Lemma 6.3.1.** For \( n \in \mathbb{N}^{*} \), define the action \( \alpha: \mathbb{N}^{*} \to \text{End}(C^{*}(\mathbb{Q}/\mathbb{Z})) \) by

\[
\alpha_{n}(i(r)) = \frac{1}{n} \sum_{j=1}^{n} i \left( \frac{r + j}{n} \right) \quad \text{for } r \in \mathbb{Q}.
\]

This is an action of \( \mathbb{N}^{*} \) by endomorphisms of \( C^{*}(\mathbb{Q}/\mathbb{Z}) \) and satisfies:

- for the projections \( \alpha_{n}(1) := \alpha_{n}(i(0)) \),

\[
\alpha_{m}(1)\alpha_{n}(1) = \alpha_{[m,n]}(1),
\]

- the endomorphism \( \beta_{n}: i(r) \mapsto i(nr) \) is a left inverse for \( \alpha_{n} \) such that \( \alpha_{n} \circ \beta_{n} \) is multiplication by \( \alpha_{n}(1) \).

**Proof.** To show this is an action by endomorphisms, we must show that \( \alpha_{n} \) is an endomorphism of \( C^{*}(\mathbb{Q}/\mathbb{Z}) \) and \( \alpha: \mathbb{N}^{*} \to \text{End}(C^{*}(\mathbb{Q}/\mathbb{Z})) \) is a homomorphism.

First, note that the operators \( \alpha_{n}(i(r)) \) are multiplicative in \( r + \mathbb{Z} \):
\[ \alpha_n(i(r))\alpha_n(i(s)) = \frac{1}{n^2} \sum_{j,k=1}^{n} i \left( \frac{r}{n} + \frac{s}{n} + \frac{j}{n} + \frac{k}{n} \right) \]
\[ = \frac{1}{n^2} \sum_{j,k=1}^{n} \sum_{i=1}^{n} i \left( \frac{r + s + j + k}{n} \right) \]
\[ = \frac{1}{n} \sum_{j=1}^{n} \alpha_n(i(r + s)) \]
\[ = \frac{1}{n} \sum_{j=1}^{n} \alpha_n(i(r + s)) \]
\[ = \alpha_n(i(r + s)) \]

Moreover,
\[ \alpha_n(i(r))^* = \frac{1}{n} \sum_{j=1}^{n} i \left( -\frac{r}{n} - \frac{j}{n} \right) = \frac{1}{n} \sum_{k=1}^{n} i \left( -\frac{r}{n} + \frac{k}{n} \right) = \alpha_n(i(-r)) = \alpha_n(i(r)^*) \]

Thus, \( \alpha_n(1) \) is a projection. The two calculations imply that \( \alpha_n(i(r)) = \alpha_n(i(0))\alpha_n(i(r)) \) is a unitary element of the corner \( \alpha_n(1)C^*(\mathbb{Q}/\mathbb{Z}) = \alpha_n(1)C^*(\mathbb{Q}/\mathbb{Z})\alpha_n(1) \), and hence the universal property of \( (C^*(\mathbb{Q}/\mathbb{Z}), i) \) gives a homomorphism \( \alpha_n : C^*(\mathbb{Q}/\mathbb{Z}) \to C^*(\mathbb{Q}/\mathbb{Z}) \).

We now need to show \( \alpha \) is a homomorphism of \( \mathbb{N}^+ \) into \( \text{End}(C^*(\mathbb{Q}/\mathbb{Z})) \). This is true because

\[ \alpha_m(\alpha_n(i(r))) = \frac{1}{mn} \sum_{j,k=1}^{m} \sum_{i=1}^{n} i \left( \frac{r}{mn} + \frac{j}{mn} + \frac{k}{n} \right) \]
\[ = \frac{1}{mn} \sum_{i=1}^{n} i \left( \frac{r}{mn} + \frac{l}{mn} \right) \]
\[ = \alpha_{mn}(i(r)) \]

So, \( \alpha \) is an action by endomorphisms. To show the two remaining results of the lemma, first consider the case where \( (m, n) = 1 \). Then

\[ \alpha_m(1)\alpha_n(1) = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} i \left( \frac{j}{m} \right) i \left( \frac{k}{n} \right) \]
\[ = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} i \left( \frac{jm + km}{mn} \right) \]
\[ = \frac{1}{mn} \sum_{l=1}^{mn} i \left( \frac{l}{mn} \right) \]
\[ = \alpha_{mn}(1) \]
Now, let \( m = (m, n)a \) and \( n = (m, n)b \) (with \( (a, b) = 1 \)). Then
\[
\alpha_m(1)\alpha_n(1) = \alpha_{(m, n)}(\alpha_a(1)\alpha_b(1)) \\
= \alpha_{(m, n)}(\alpha_{ab}(1)) \\
= \alpha_{(m, n)ab}(1) \\
= \alpha_{[m, n]}(1).
\]

The last result can easily be seen from the definition of \( \alpha \) and \( \beta \). In particular,
\[
\beta_n(\alpha_n(i(r))) = \beta_n\left(\frac{1}{n} \sum_{k=1}^{n} i\left(\frac{r + k}{n}\right)\right) \\
= \frac{1}{n} \sum_{k=1}^{n} i(r + k) \\
= \frac{i}{n} \sum_{k=1}^{n} i(r) \\
= i(r)
\]

Also,
\[
\alpha_n(\beta_n(i(r))) = \alpha_n(i(nr)) \\
= \frac{1}{n} \sum_{k=1}^{n} i\left(\frac{nr + k}{n}\right) \\
= \frac{1}{n} \sum_{k=1}^{n} i\left(\frac{r + k}{n}\right) \\
= \frac{1}{n} \sum_{k=1}^{n} i(r)i\left(\frac{k}{n}\right) \\
= \alpha_n(1)i(r)
\]

Combining Lemma 6.3.1 and Proposition 6.2.6 implies that the system \((C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)\) from Lemma 6.3.1 will have a semigroup crossed product if the semigroup dynamical system has nontrivial covariant pairs. Unsurprisingly, this system has nontrivial covariant pairs. In particular, the left regular representations into \(\ell^2(\mathbb{N}^*)\) or \(\ell^2(\mathbb{Q}/\mathbb{Z})\) are nontrivial (see [29]). So by Proposition 6.2.6, we have a semigroup crossed product.

**Definition 6.3.2.** The Connes-Marcolli system for \( \mathbb{Q} \) is the C*-dynamical system \((\mathcal{A}_\mathbb{Q}, \sigma_t) = (C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_{\alpha} \mathbb{N}^*, \sigma_t)\), where the C*-algebra is as discussed above and the time evolution is defined
by
\[ \sigma_t(\mu_n) = n^\gamma \mu_n, \quad \sigma_t(e_\gamma) = e_{-\gamma} \]
for all \( n \in \mathbb{N}, \gamma \in \mathbb{Q}/\mathbb{Z} \),
where \( \mu_n, e_\gamma \) are basis elements for \( C^*(\mathbb{N}) \) and \( C^*(\mathbb{Q}/\mathbb{Z}) \).

From Chapter 2, we know that the Hecke algebra is a universal involutive algebra generated by \( \{ \mu_n : n \in \mathbb{N}^* \} \) and \( \{ e(r) : r \in \mathbb{Q}/\mathbb{Z} \} \) that satisfies the following conditions (from Theorem 2.3.9)

1. \( \mu_n^* \mu_n = 1 \quad \forall n \in \mathbb{N}^* \),
2. \( \mu_n \mu_m = \mu_{mn} \quad \forall m, n \in \mathbb{N}^* \),
3. \( e(0) = 1, e(\gamma_1)^* = e(-\gamma_1), e(\gamma_1 + \gamma_2) = e(\gamma_1) e(\gamma_2) \quad \forall \gamma_1, \gamma_2 \in \mathbb{Q}/\mathbb{Z} \),
4. \( \mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n \delta = \gamma} e(\delta) \quad \forall n \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z} \).

Remark. Following the work of Marcelo Laca and Iain Raeburn in [29], if we suppose \( \{ \mu_n : n \in \mathbb{N}^* \} \) and \( \{ e(r) : r \in \mathbb{Q}/\mathbb{Z} \} \) are operators on a Hilbert space \( \mathcal{H} \) satisfying the above relations, then there is a representation \( \pi_e \) of \( C^*(\mathbb{Q}/\mathbb{Z}) \) on \( \mathcal{H} \) such that \( \pi_e(i(r)) = e(r) \) and \( (\pi_e, \mu) \) is a covariant representation of the semigroup dynamical system \( (C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha) \).

This follows from the fact that the first two conditions imply that \( \mu \) is an isometric representation of \( \mathbb{N}^* \), while similarly the third condition implies that \( e \) is a unitary representation of \( \mathbb{Q}/\mathbb{Z} \) by definition. \( e \) can therefore be integrated up to a representation \( \pi_e \) of \( C^*(\mathbb{Q}/\mathbb{Z}) \) such that \( e(r) = \pi_e(i(r)) \). The last condition is equivalent to the definition of covariance of the pair \( (\pi_e, \mu) \).

We conclude the section by drawing all of this work into the following corollary that is the basis of this chapter.

**Corollary 6.3.3.** The original Bost-Connes system is isomorphic to the \( C^* \)-dynamical system \( (A_Q, \sigma_t) \) (defined in this section.)

**Proof.** As explained in Chapter 2, the Bost-Connes \( C^* \)-algebra is the universal \( C^* \)-algebra generated by elements \( \{ \mu_n : n \in \mathbb{N}^* \} \) and \( \{ e(r) : r \in \mathbb{Q}/\mathbb{Z} \} \) subject to the above relations.

Given Lemma 6.3.1 and the fact that for every covariant pair \( (\pi, V) \) of \( (C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha) \), the pair \( \{ V_n, \pi(i(r)) \} \) satisfies the conditions from Proposition 2.3.9, this implies that there is an isomorphism between the Bost-Connes system and the Connes-Marcolli system for \( \mathbb{Q} \).
6.4 The Connes-Marcolli system

One can follow the ideas used in the construction of the Connes-Marcolli system for $\mathbb{Q}$ and generalise certain elements to be the number field $K$ analogue of the original.

In Definition 6.4.1, the definition of the Connes-Marcolli system is presented. Some of the ideas used in the Connes-Marcolli system will be used to motivate the partial Connes-Marcolli systems defined in Chapter 8.

**Definition 6.4.1.** The *Connes-Marcolli system* $(A_K, \sigma_t)$ for a number field $K$ is defined as follows. Using notation from Chapter 3, denote by $\mathcal{O}_K$ the ring of integers of $K$, $\mathbb{A}_K$ the adeles of $K$, $\hat{\mathcal{O}}_K$ the integral adeles of $K$ and $J_K^+$ the integral ideals of $K$.

Define

$$Y_K = \text{Gal}(K^{ab}/K) \times \hat{\mathcal{O}}_K,$$

balanced by the diagonal action of $\hat{\mathcal{O}}_K^*$ on $\text{Gal}(K^{ab}/K) \times \hat{\mathcal{O}}_K$ given by

$$g(\gamma, x) = (\gamma r_K(g)^{-1}, gx),$$

where $r_K$ is the Artin map $r_K: \mathbb{A}_K^* \to \text{Gal}(K^{ab}/K)$ restricted to $\hat{\mathcal{O}}_K^*$. This action naturally induces an action of $J_K^+$ on $Y_K$. Thus,

$$A_K := C(Y_K) \rtimes J_K^+.$$

The dynamics is given by the formula

$$\sigma_t(f\mu_g) = N_K(g)^t f\mu_g \text{ for } f \in C(Y_K), g \in J_K^+.$$
Chapter 7

KMS States of the Bost-Connes System

Thus far, the Bost-Connes system has been analysed via different constructions, which have allowed the Bost-Connes system to be written in different forms. In particular, it has given impetus to the question of generalising the Bost-Connes system to arbitrary number fields. From this, an important step has been the construction of the Connes-Marcolli system, which aims to generalise the Bost-Connes system and satisfy the aims of the Bost-Connes Problem. In other words, it has to satisfy conditions about equilibrium states and dynamics that are analogous to those of the original Bost-Connes system.

Any mention of the Bost-Connes Problem has been set aside thus far as the presentation and form of the Bost-Connes system has been the focus. At this point, enough exposition has been demonstrated that one can now show that the Bost-Connes system (and Connes-Marcolli systems) does in fact satisfy the Bost-Connes Problem. To this end, in this chapter there will be a discussion about KMS$_\beta$ states (Section 7.1) and properties of these states. With an understanding of KMS$_\beta$ states for C*-dynamical systems, there is discussion of the properties of the Bost-Connes system relevant to the Bost-Connes Problem. This includes in particular, a discussion of the Hamiltonian and partition function (Section 7.2). In Sections 7.3 and 7.4, the KMS$_\beta$ states will be calculated. This will culminate in showing that the Bost-Connes system satisfies the properties Benoît Bost and Alain Connes were initially so excited about.

Note, in Chapter 8, a Bost-Connes type system that generalises the Connes-Marcolli system is constructed. These are called partial Connes-Marcolli systems and are a further extension of the Connes-Marcolli systems. This allows me in Chapter 9 to show that it satisfies a generalised modified Bost-Connes Problem. Thus it is shown that the Connes-Marcolli system satisfies the
Bost-Connes Problem in Chapter 9, while this chapter is devoted to doing the same for the original Bost-Connes system.

7.1 KMS Condition

Quantum statistical mechanics is an area of mathematics largely inspired by and continues to be a large area of research in physics, particularly mathematical physics. Many of the concepts discussed about quantum statistical mechanical systems come from processes and objects observed in nature. One such concept is that of an equilibrium state. This section will not concentrate, however, on the physics and instead there will be a discussion of the theory of equilibrium states, which in quantum statistical mechanical systems are characterised by the KMS condition.

Recall that a quantum statistical mechanical system is made up of an algebra of observables \( \mathcal{A} \), which is a C*-algebra. The observables of a system are given expectation values through linear functionals called states \( \phi : \mathcal{A} \to \mathbb{C} \) must satisfy normalisation and positivity, namely,

\[
\phi(1) = 1, \quad \phi(a^*a) \geq 0.
\]

There is a time evolution on \( \mathcal{A} \) given by a one-parameter family of automorphisms \( \sigma_t \in \text{Aut}(\mathcal{A}) \). The time evolution is implemented infinitesimally through a self-adjoint operator \( H \) called the Hamiltonian. This is usually dependent on the representation of \( \mathcal{A} \) as a concrete algebra of operators in a Hilbert space \( \mathcal{H} \). Thus,

\[
\sigma_t(a) = e^{itH}ae^{-itH}
\]

or alternatively, one can think of \( H \) as the operator

\[
H = \frac{d}{dt}\sigma_t \bigg|_{t=0}.
\]

The quantum statistical mechanical system, then may have equilibrium states. These will be dependent on the inverse temperature \( \beta = 1/kT \), a thermodynamic parameter (although \( k \) here is usually set to 1). The classical concept of Gibbs equilibrium is extended to the quantum mechanical setting, by considering equilibrium states given by

\[
\phi(a) = \frac{1}{Z(\beta)} \text{Tr}(ae^{-\beta H})
\]

with the normalising factor here called the partition function and given by

\[
Z(\beta) = \text{Tr}(e^{-\beta H}).
\]
Immediately, one notices that this makes little sense unless $e^{-\beta H}$ is of trace class, which is not guaranteed. Ryogo Kubo, Paul Martin and Julian Schwinger [6] came up with the notion of the KMS condition which generalises this notion, while also satisfying states in the form given by Definition 7.1.1.

**Definition 7.1.1.** Given the notation described above, the triple $(\mathcal{A}, \sigma_t, \phi)$ satisfies the Kubo-Martin-Schwinger (KMS) condition at inverse temperature $0 \leq \beta < \infty$ if

- for all $x, y \in \mathcal{A}$, there exists a holomorphic function $F_{x,y}(z)$ on the strip $0 < \text{im}(z) < \beta$,
- $F_{x,y}(z)$ extends as a continuous function on the boundary of the strip $0 < \text{im}(z) < \beta$, and
- $F_{x,y}(t) = \phi(x \sigma_t(y))$ and $F_{x,y}(t + i\beta) = \phi(\sigma_t(x))$, $\forall t \in \mathbb{R}$.

Such a state $\phi$ is then called a KMS$_\beta$ state.

*Remark.* It turns out that set of all KMS$_\beta$ states forms a Choquet simplex [6], which allows every such state to be expressed uniquely in terms of a specific subset of extremal KMS$_\beta$ states. The set of extremal KMS$_\beta$ states will be denoted by $\mathcal{E}_\beta$.

*Remark.* At temperature 0, ($\beta = \infty$), only the first equation relating $F$ and $\phi$ from Definition 7.1.1 makes any sense and this by itself is a fairly weak condition, satisfied trivially in a significant amount of cases. States that satisfy just this condition are normally called ground states. As this is fairly weak, a stronger notion of KMS$_\infty$ states will be used.

**Definition 7.1.2.** A state $\phi$ is a KMS$_\infty$ state in a quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ if it satisfies

$$\phi_\infty(a) = \lim_{\beta \to \infty} \phi_\beta(a) \quad \forall a \in \mathcal{A}.$$  

With this definition, the set of all KMS$_\infty$ states can be shown to be a weakly compact convex set, thus there is again a well-defined notion of extremal points analogously denoted $\mathcal{E}_\infty$.

There is an important notion in quantum statistical mechanical systems concerning symmetries of the system.

**Definition 7.1.3.** A symmetry group of a quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ is a compact group of automorphisms $G \subset \text{Aut}(\mathcal{A})$ ($\alpha : G \to \text{Aut}(\mathcal{A})$ continuous and faithful) that commutes with the time evolution

$$\sigma_t \alpha_g = \alpha_g \sigma_t \quad \forall g \in G, t \in \mathbb{R}.$$  

More about symmetries will be mentioned in Section 7.2, but when considering KMS$_\beta$ states
there is a concept called symmetry breaking that occurs when the symmetry does not act trivially on the extremal equilibrium states. In the case where the system has a unique phase transition (like the Bost-Connes systems do), there is usually a single point where the temperature is above some critical temperature ($\beta < \beta_c$). At this point, there is spontaneous symmetry breaking.

An important property of a KMS$\beta$ state is that it is $\sigma_t$ invariant.

**Proposition 7.1.4.** A KMS$\beta$ state $\phi$ in quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ is $\sigma_t$ invariant, ie.

$$\phi(\sigma_t(a)) = \phi(a).$$

**Proof.** From the definition of a KMS$\beta$ state, there exists a holomorphic function $F_{x,y}(z)$ on the strip $0 < \text{im}(z) < \beta$ that extends continuously to the boundary, such that

$$F_{x,y}(t) = \phi(x \sigma_t(y)) \quad \text{and} \quad F_{x,y}(t + i\beta) = \phi(\sigma_t(y)x) \quad \forall t \in \mathbb{R}.$$  

Thus,

$$F_{1,y}(t) = \phi(\sigma_t(y)) = F_{1,y}(t + i\beta).$$

$F_{1,y}(z)$ is holomorphic because $F_{x,y}(z)$ is and it is also bounded as

$$|F_{1,y}(z)| \leq |\sigma_z(y)| = |\sigma_{\text{im}(z)}(y)|$$

which takes on a maximum value as $\text{im}(z)$ is bounded. Therefore $F_{1,y}(z)$ is a bounded, periodic function. So, by Liouville’s Theorem it is constant and hence

$$\phi(\sigma_t(a)) = \phi(\sigma_0(a)) = \phi(a).$$

Moreover, the definition for the KMS condition is actually rather unwieldy. Fortunately, Bratteli and Robinson [6] have found an equivalent formulation for the KMS$\beta$ condition that is more convenient.

**Theorem 7.1.5.** Let $(\mathcal{A}, \sigma_t)$ be a $C^*$-dynamical system. The following are equivalent:

- $\phi$ is a KMS$\beta$ state, and
- $\phi$ is a state that satisfies

$$\phi(a \sigma_{i\beta}(b)) = \phi(ba)$$

for all $a, b$ in a norm dense $\sigma$-invariant $*$-subalgebra of $\mathcal{A}$.

**Proof.** See [6] Proposition 5.3.7.
7.2 Dynamics and Partition Function

To begin with, recall that the time evolution on the Bost-Connes system is given by
\[ \sigma_t(\mu_n) = n^t \mu_n \quad \text{and} \quad \sigma_t(e_\gamma) = e_{\gamma} \quad \forall n \in \mathbb{N}, \gamma \in \mathbb{Q}/\mathbb{Z}. \]

Consider the representation of the Bost-Connes system \( \mathcal{A} \) into the Hilbert space \( \mathcal{B}(\ell^2(\mathbb{N})) \).

Note, that the operator \( \mu_n \) maps into \( \mathcal{B}(\ell^2(\mathbb{N})) \) by \( \mu_n \varepsilon_k = \varepsilon_{nk} \).

Thus, the Hamiltonian can be calculated. We know that
\[ n^t \varepsilon_{kn} = n^t \mu_n \varepsilon_k = \sigma_t(\mu_n) \varepsilon_k = e^{itH} \mu_n e^{-itH} \varepsilon_k. \]

To get an idea of what the Hamiltonian could be, let \( H \varepsilon_k = f(k) \varepsilon_k \) for some nice function \( f \).

Therefore, notice that
\[ e^{itH} \mu_n e^{-itH} \varepsilon_k = e^{itH} \mu_n (e^{f(k)})^{-it} \varepsilon_k = e^{itH} (e^{f(k)})^{-it} \varepsilon_{nk} = (e^{f(nk)})^{it} (e^{f(k)})^{-it} \varepsilon_{nk} = (e^{f(nk) - f(k)})^t \varepsilon_{nk}. \]

We thus require \( f(nk) - f(k) \) to be equal to \( \log(n) \), so clearly \( f \) is logarithmic and in fact, we set \( H \varepsilon_k = \log(k) \varepsilon_k \).

Since, the time evolution is invariant on the \( e_\gamma \), this can be extended to the whole system. This allows us to calculate the partition function of the Bost-Connes system \( (\mathcal{A}, \sigma_t) \).

**Theorem 7.2.1.** The partition function of the Bost-Connes system is the Riemann zeta function.

**Proof.** As defined in Section 7.1, the partition function is defined as \( Z(\beta) = \text{Tr}(e^{-\beta H}) \). Thus,
\[ Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_{n=1}^{\infty} e^{-\beta H} \varepsilon_n, \varepsilon_n > = \sum_{n=1}^{\infty} n^{-\beta} \varepsilon_n, \varepsilon_n > = \sum_{n=1}^{\infty} n^{-\beta} < \varepsilon_n, \varepsilon_n > = \zeta(\beta) \]
is the Riemann zeta function. \( \square \)
At this point it is worth considering the symmetries of the Bost-Connes system.

Recall the definition of a symmetry $G$ of the quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ given in Section 7.1. One can consider the fixed point sub-algebra corresponding to a symmetry $G$ given by

$$\mathcal{A}^G = \{ a \in \mathcal{A} \mid \alpha_g(a) = a \}.$$  

This allows for the definition of a projection of $\mathcal{A}$ onto $\mathcal{A}^G$ given by $E$:

$$E(a) = \int \alpha_g(a) d\mu(g)$$

where $\mu$ is a Haar measure (see Section 9.2) on $G$ that can be defined given that $G$ is compact.

We also say that a symmetry $G$ acts on the KMS$_\beta$ state $\phi$ by automorphism via the pullback

$$g \cdot \phi(a) = \phi(\alpha_g(a)).$$

Importantly, because the action of the symmetry commutes with the time evolution, the action of the symmetry on a KMS$_\beta$ state is again a KMS$_\beta$ state. This is because

$$g \cdot \phi(a \sigma_{i \beta}(b)) = \phi(\alpha_g(a \sigma_{i \beta}(b)))$$

$$= \phi(\alpha_g(a) \alpha_g(\sigma_{i \beta}(b)))$$

$$= \phi(\alpha_g(a)) \alpha_g(\sigma_{i \beta}(b))$$

$$= \phi(\alpha_g(b)) \alpha_g(a)$$

$$= \phi(\alpha_g(ba))$$

$$= g \cdot \phi(ba)$$

Moreover, it sends extremal KMS$_\beta$ states to extremal KMS$_\beta$ states. The concept of symmetry breaking that was mentioned in Section 7.1 occurs when the action of a symmetry on an extremal equilibrium state is not trivial, i.e. $g \cdot \phi \neq \phi$. In this case, the symmetry is broken to a smaller subgroup called the isotropy subgroup $G_\phi = \{ g \in G \mid g \cdot \phi = \phi \}$.

With these definitions, it can be shown that $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ is in fact a symmetry group of the Bost-Connes system. This will be seen in Section 7.4.
7.3 KMS states for $\beta < 1$

Alain Connes and Benoît Bost classified the equilibrium $\text{KMS}_\beta$ states of the original Bost-Connes system. As we will look at this in more detail for the partial Connes-Marcolli system (which will cover the original Bost-Connes system too) in Chapter 9, in this section, I will merely mention the main result and discuss the method by which the original authors came to their conclusions. In particular,

**Theorem 7.3.1.** For $0 < \beta < 1$, there is a unique $\text{KMS}_\beta$ state for the original Bost-Connes system $(\mathcal{A}, \sigma_t)$.

**Proof.** The method is somewhat technical and a little different than what will be used in Chapter 9, so a brief explanation will be provided. First, $\text{KMS}_\beta$ states for $0 < \beta < 1$ are projected down onto the fixed-point subalgebra of the symmetry group $\hat{\mathbb{Z}}^*$, which is $C^*(\mathbb{N}^*)$. The projection maps $\text{KMS}_\beta$ states of the Bost-Connes system faithfully to $\text{KMS}_\beta$ states on $C^*(\mathbb{N}^*)$.

One can now define a spectral subspace of the Bost-Connes system as follows. For $\chi : \hat{\mathbb{Z}}^* \to S^1$, a character of the symmetry group, define the subspace

$$\mathcal{A}_\chi = \{ a \in \mathcal{A} : ga = \chi(g)a \text{ for all } g \in \hat{\mathbb{Z}}^* \}.$$ 

The fixed-point subalgebra of the Bost-Connes system corresponds to the trivial character 1. Amazingly, since $\hat{\mathbb{Z}}^*$ is compact, one can show that the direct sum of all spectral subspaces is dense in $\mathcal{A}$.

Moreover, the projection of a $\text{KMS}_\beta$ state of the Bost-Connes system to spectral subspaces corresponding to non-trivial characters vanishes completely. Thus, the $\text{KMS}_\beta$ states for the Bost-Connes system are in correspondence with $\text{KMS}_\beta$ states on $C^*(\mathbb{N}^*)$ for which there is a unique such state for each $0 < \beta < 1$. \qed

7.4 KMS states for $\beta \geq 1$

The method used here closely follows the method used by Jean-Benoit Bost and Alain Connes in their original work [5].

Consider the Bost-Connes system $(\mathcal{A}, \sigma_t)$. As in Section 7.2, we shall consider a represen-
Consider the representation \( \pi_1 : \mathcal{A} \to \mathcal{B}(\ell^2(\mathbb{N})) \) given by

\[
\pi_1(\mu_n)\varepsilon_k = \varepsilon_{nk} \quad \forall n \in \mathbb{N} \\
\pi_1(e_\gamma)\varepsilon_k = e^{2\pi ik\gamma}\varepsilon_k \quad \forall \gamma \in \mathbb{Q}/\mathbb{Z}
\]

Important,

\[
\pi_1(\mu_n)^*\varepsilon_k = \begin{cases} 
\varepsilon_{k/n} & \text{if } n|k \\
0 & \text{otherwise.}
\end{cases}
\]

This has been shown in Section 6.3 to be a representation. This representation can be extended to a family of representations indexed by \( \alpha \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) via

\[
\pi_\alpha(\mu_n)\varepsilon_k = \varepsilon_{nk} \quad \forall n \in \mathbb{N} \\
\pi_\alpha(e_\gamma)\varepsilon_k = \alpha(e^{2\pi ik\gamma})\varepsilon_k
\]

First and foremost, it is important to note here that we have used the Kronecker-Weber Theorem. The theorem tells us that \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) = \text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \), so \( \alpha \) acts on \( e^{2\pi ik\gamma} \) which are roots of unity. Second of all \( \pi_\alpha(\mu_n) = \pi_1(\mu_n) \) and \( \pi_\alpha(e_\gamma)^* = \pi_\alpha(e_{-\gamma}) \), so it follows from the fact that \( \pi_1 \) was a representation that \( \pi_\alpha \) is also a representation.

**Proposition 7.4.1.** The states \( \psi_{\beta,\alpha} \) defined by

\[
\psi_{\beta,\alpha}(a) = \frac{1}{\zeta(\beta)} \text{Tr}(e^{-\beta H}\pi_\alpha(a))
\]

on \( \mathcal{A} \) for all \( \alpha \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) and \( \beta > 1 \) are KMS\(_\beta\) states.

**Proof.** First, note that for all \( y \in C^*(\mathbb{Q}/\mathbb{Z}) \), \( \pi_\alpha(y) \) commutes with the Hamiltonian \( H \) from the definition of \( \pi_\alpha \) and \( \sigma_1(y) = y \). On the other hand, for \( x \in C^*(\mathbb{N}^*) \), \( \pi_\alpha(x) = e^{-itH}\pi_\alpha(\sigma_1(x))e^{itH} \) from the definition of \( \sigma_1(x) \) for \( x \in \mathbb{N}^* \) found in Chapter 7.1.

Thus, following through the definition of a KMS\(_\beta\) state from Theorem 7.1.1, it will be seen that this defines a KMS\(_\beta\) state. First, for \( b \in C^*(\mathbb{Q}/\mathbb{Z}) \),

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For $b \in C^*(\mathbb{N}^*)$,

$$
\psi_{\beta,\alpha}(a\sigma_i\beta(b)) = \frac{1}{\zeta(\beta)} \text{Tr}(e^{-\beta H} \pi_\alpha(a\sigma_i\beta(b))) \\
= \frac{1}{\zeta(\beta)} \text{Tr}(e^{-\beta H} \pi_\alpha(a)\pi_\alpha(b)) \\
= \frac{1}{\zeta(\beta)} \text{Tr}(\pi_\alpha(b)e^{-\beta H} \pi_\alpha(a)) \\
= \frac{1}{\zeta(\beta)} \text{Tr}(e^{-\beta H} \pi_\alpha(b)\pi_\alpha(a)) \\
= \frac{1}{\zeta(\beta)} \text{Tr}(e^{-\beta H} \pi_\alpha(ba)) \\
= \psi_{\beta,\alpha}(ba).
$$

as required.

**Remark.** Alternatively, note that it is in the form of a Gibbs equilibrium state as the Hamiltonian is of trace class and hence, this is automatically an equilibrium state.

**Remark.** The KMS$_\beta$ states $\psi_{\beta,\alpha}$ are in fact extremal equilibrium states. This is because the representation $\pi_\alpha$ is irreducible.

What is required here is to show that these are all the KMS$_\beta$ states. This is accomplished through the following theorem.

**Theorem 7.4.2.** The map $\alpha \rightarrow \psi_{\beta,\alpha}$ is a homeomorphism of the Galois group $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ with the space of extremal KMS$_\beta$ states $\mathcal{E}_\beta$.

**Proof.** For a complete proof of this, consult page 442 in [5]. The proof begins by showing the map is injective. From a KMS$_\beta$ state $\psi_{\beta,\alpha}$ one can canonically determine a KMS$_\infty$ state $\psi_{\infty,\alpha}(x) = \ldots$
\( (\pi_\alpha(x)\epsilon_1, \epsilon_1) \). One can then restrict these extremal equilibrium states to the arithmetic subalgebra \( \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \) which injectively imbed into the field \( \mathbb{Q}^{gcd} \) of roots of unity in \( \mathbb{C} \). To prove surjection, consider the projection of the Bost-Connes system to \( C^*(\mathbb{N}^*) \). All KMS\( \beta \) states project into this algebra as KMS\( \beta \) states injectively. One can also project the KMS\( \beta \) states of the form given above into \( C^*(\mathbb{N}^*) \) injectively. Considering the inverse of the second projection provides the necessary surjective result. Note, that one can also show \( \alpha \to \psi_{\beta,\alpha} \) is continuous and hence, a homeomorphism.

**Corollary 7.4.3.** The Galois group \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) acts as a symmetry on the Bost-Connes system \((\mathcal{A}, \sigma_t)\).

**Proof.** The Galois group \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) is a compact group. Define the following action of \( g \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) on the generators of \((\mathcal{A}, \sigma_t)\):

\[
\begin{align*}
g\mu_n &= \mu_{ng} \quad n \in \mathbb{N}^* \\
g e_\gamma &= e_{g(\gamma)} \quad \gamma \in \mathbb{Q}/\mathbb{Z}
\end{align*}
\]

This is the action embedded in the representation from Chapter 7.4. The composition \( g(\gamma) \) is calculated by identifying \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) with \( \text{End}(\mathbb{Q}/\mathbb{Z}) \). Here, composition is continuous and faithful. One can also easily see that this commutes with the time evolution. Thus, \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) acts as a symmetry on \((\mathcal{A}, \sigma_t)\). \(\square\)
Part IV

Partial Connes-Marcolli Systems
Chapter 8

Construction of the Partial Connes-Marcolli systems

In Chapter 6, the Bost-Connes system is introduced in the form of a discrete semigroup crossed product. This form showed great promise and since the work of Marcelo Laca, Nadia Larsen and Sergey Neshveyev [26], there is now a system (the Connes-Marcolli system) that satisfies the Bost-Connes Problem of Section 2.5 for any arbitrary number field and is also written as a discrete semigroup crossed product. Immediately, it must be asked, does this system provide us with a description of the maximal abelian extension of any number field in a way that satisfies the requirements of Hilbert’s 12th Problem?

Technically, no. Remember that a system that satisfies the Bost-Connes Problem must contain an arithmetic subalgebra, such that when the extremal zero temperature equilibrium states are evaluated on this subalgebra, the result is isomorphic to the maximal abelian extension of the number field. Marcelo Laca, Nadia Larsen and Sergey Neshveyev’s construction is beautiful and nicely encapsulates the class field theory described in Chapter 3 for each number field, however, there are aspects to it that are not as convenient as desired. Most importantly, even with such a description for the system, explicitly calculating the values of the equilibrium states evaluated on the arithmetic subalgebra is very difficult. This difficulty is of course tied very closely to the difficulty that mathematicians have experienced for over a century when trying to solve Hilbert’s 12th Problem. This difficulty translates directly into this Connes-Marcolli system. Quite simply, it can be shown that the whole set of extremal states generates the maximal abelian extension without identifying what each state evaluates to.
Hilbert’s 12th Problem has been solved for only a handful of number fields, the simplest being the field of rationals \( \mathbb{Q} \), and negative imaginary quadratic fields \( \mathbb{Q}(\sqrt{-d}) \). The generators for the maximal abelian extension of the rationals are the roots of unity, while for negative imaginary fields, we need to additionally add values of the Weierstrass \( j \) function. It can be expected, therefore, that for arbitrary number fields, there is not just a single analytical function whose values would generate the maximal abelian extension.

This motivates a new approach to tackling Hilbert’s 12th Problem. Instead of considering the full Bost-Connes and examining the equilibrium states, one could consider a partial version of the system, where the equilibrium states generate part of the desired maximal abelian extension. Finding a way to recover the original full Connes-Marcolli system from the partial would then produce the required descriptions of the generators of the maximal abelian extension. This lines up a lot more with traditional class field theory proof techniques as seen in Chapter 4.

What will be exhibited in this chapter, as well as Chapter 9 and Chapter 10 is original work to define and construct a partial Connes-Marcolli system, by considering the Bost-Connes type system of an arbitrary abelian extension rather than the maximal abelian extension. There are many advantages to this approach. Firstly, it should simplify the calculation of all the relevant properties, namely, the extremal KMS\( _\beta \) states and their action on the arithmetic sub-algebra. Secondly, if the finite extensions are chosen in such a way as to allow the removability of ramified primes, one can follow the standard technique of reducing the problem to the case where the field extension is unramified and thus, has nicer arithmetic properties. Section 8.3 shows an example of this technique in action and this motivates us to try to include a similar flexibility to our system.

In Section 8.1, I formally define the relevant partial Connes-Marcolli Problem, which the partial Connes-Marcolli systems must satisfy. In Section 8.2, there is a description of the naïve approach to solving the Partial Connes-Marcolli Problem. Section 8.3 follows the thinking of Marcelo Laca, Nadia Larsen and Sergey Neshveyev [26] to motivate the system via a group action groupoid C*-dynamical system. In Section 8.4, this same system is described as a direct semigroup crossed product, which shows a clear similarity to the Connes Marcolli system. In Section 8.5, there is a detailed description of this partial Connes-Marcolli system for the original scenario of the field of rationals as the underlying field. These particular systems we shall call the partial Connes-Marcolli systems and will be concentrating heavily on in Chapters 9 and 10.
8.1 Partial Connes-Marcolli Problem

The idea for the Partial Connes-Marcolli system is simple. Rather than consider a system that is based on the concept of generating the maximal abelian extension, the system is to satisfy these same conditions for a (possibly finite) abelian extension.

Namely, for a number field $K$ and abelian extension $L/K$ and a (possibly empty) finite set $S$ of primes of $K$, one must construct a $C^*$-dynamical system

$$(A_{L/K,S}, (\sigma_t)_{t \in \mathbb{R}})$$

that satisfies the following problem.

**Modified Bost-Connes Problem.** $(A_{L/K,S}, \sigma_t)$ must exhibit the following properties.

1. The partition function is given by the Dedekind zeta function $\zeta_{K,S} = \sum_{s \in J_{K,S}^*} N(s)^{-\beta}$ where $J_{K,S}^*$ is the subset of the integral ideals of $K$ generated by finite primes not in $S$.
2. The finite abelian Galois group $\text{Gal}(L/K)$ of $K$ acts as symmetries on $A_{L/K,S}$.
3. For each inverse temperature $0 < \beta \leq 1$ there is a unique KMS$_\beta$-state.
4. For each $\beta > 1$ the action of the symmetry group $\text{Gal}(L/K)$ on the set of extremal KMS$_\beta$-states is free and transitive.

When a system satisfies these four properties it is called an analytic Partial Connes-Marcolli system for $K$.

5. There exists a $K$-rational subalgebra $A_{\text{arith}}$ such that for every extremal KMS$_\infty$-state $\rho$ and every $f \in A_{\text{arith}}$, we have

$$\rho(f) \in L$$

and further $L$ is generated over $K$ by these values.
6. If we denote by $\nu_\rho$ the action of a symmetry $\nu \in \text{Gal}(L/K)$ on an extremal KMS$_\infty$-state $\rho$ (given by pull-back), we have for every element $f \in A_{\text{arith}}$ the following compatibility relation

$$\nu_\rho(f) = \nu^{-1}(\rho(f)).$$

7. The $C^*$-algebra $A_{\text{arith}} \otimes_K \mathbb{C}$ is dense in $A_{L/K,S}$.

A system $A_{L/K,S}$ which satisfies these last properties is called a Partial Connes-Marcolli system for $K$ and $A_{\text{arith}}$ is called an arithmetic subalgebra.
8.2 Naïve Approach to the construction of a finite Bost-Connes system

Chapters 2, 5, 6 and 7, provide a brief exposition of the work that has been carried out on the Bost-Connes system. In particular, it is shown in great detail how the original Bost-Connes system satisfies the Bost-Connes Problem and how knowledge of the Kronecker-Weber result allows for the conclusion of many interesting properties about the system’s equilibrium states. It is conceptually possible that one might try to think of this argument in reverse. If all the properties of the Bost-Connes system can be deduced without making use of the Kronecker-Weber Theorem, then it should be able to prove the Kronecker-Weber Theorem.

This would be an exciting prospect for two reasons:

• this is exactly the sort of thing the Bost-Connes system was developed for. The main goal is to solve Hilbert’s 12th Problem using a generalisation of the Bost-Connes system. Proving the Kronecker-Weber Theorem, would be achieving this goal for the field of rationals.
• this would be a new proof for the Kronecker-Weber Theorem with minimal use of classical number theory. It would also give more credence to non-commutative geometry being a powerful tool for solving number theoretic problems.

The inspiration for how to attempt the above came from a question at a conference talk I gave. The curious listener was interested if there is a finite analogue of \( \mathbb{Q} \)-lattices. The concept of a finite analogue was too interesting to ignore and eventually led to considering solving the partial Connes-Marcolli Problem. The known results of the Kronecker-Weber Theorem, give a viable way of trying to build up the Bost-Connes system from finite systems corresponding to roots of unity that make up the maximal abelian Galois extension of \( \mathbb{Q} \).

It is worth having a brief aside into the attempts made in constructing this partial Connes-Marcolli system.

First, and foremost, it is not unreasonable to try and model this partial system on the original full system. In particular, it will be written as a semigroup crossed product (using the definitions in Section 6.1). The original Bost-Connes system looked as follows:

\[
C^\ast(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^* 
\]

with an appropriate semigroup action \( \alpha \) by endomorphisms from \( \mathbb{N}^* \) (see Section 6.2). Also, \( \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \) is isomorphic to the Galois group of \( \mathbb{Q}^{ab} \) and \( \mathbb{N}^* \) is the integral subsemigroup
of ideals in \(Q\).

Given that the Galois group of the cyclotomic extension of \(Q\) corresponding to a concrete \(n^{th}\) root of unity is isomorphic to \((\mathbb{Z}/n\mathbb{Z})^*\), it seems right to consider the C*-dynamical system

\[
C^*((\mathbb{Z}/n\mathbb{Z})^*) \rtimes_\alpha \mathbb{N}^*
\]

with an appropriate action corresponding to the one above. Similarly, in order to consider the same analysis for any finite abelian extension \(L\) of \(Q\), construct the system

\[
C^*(\text{Gal}(L/Q)) \rtimes_\alpha \mathbb{N}^*
\]

This did not reveal anything useful as the obstruction faced by this construction amounted to the fact that the action \(\alpha\) is essentially trivial due to the finiteness of the Galois group \(\text{Gal}(L/Q)\).

More motivation can be found in the work done by Marcelo Laca, Nadia Larsen and Sergey Neshveyev [26] in which they describe the full Connes-Marcolli system that satisfies the Bost-Connes Problem. In particular, they describe the system as a semigroup crossed product in which that C*-algebra being acted upon by endomorphisms of a semigroup is a quotient of a group action closely related to the Artin homomorphism from class field theory. Giving more credence to their work is the later work by Sergey Neshveyev and Simen Rustad [37], which constructs and analyses a ‘finite-type’ Bost-Connes system for abelian extensions of finite fields. The system discussed in this chapter is closely related to the one provided by Neshveyev and Rustad, but based on number fields, rather than function fields. My contribution to this research is to adapt their work to the case of number fields and provide the details that are ommitted in [37] of the Neshveyev-Rustad construction.

To describe the system motivated in this way, consider the following standard notation, first described in Chapter 3. For any number field \(K\), and \(p\) a prime of \(K\), denote by \(K_p\) the corresponding \(p\)-adic completion of \(K\), and if \(p\) is finite denote by \(O_p \subset K_p\) its maximal compact subring, also known as the \(p\)-adic integers corresponding to the finite prime \(p\). Denote by \(J_K\) the group of fractional ideals of \(K\), by \(\mathbb{A}_{K,f}\) the finite adele ring of \(K\), and by \(\mathring{O}_K\) the maximal compact subring of \(\mathbb{A}_{K,f}\). We have that \(J_K\) is the free abelian group generated by the finite primes of \(K\) and \(\mathbb{A}_{K,f} = \prod_{p < \infty} K_p\) is the restricted product of the \(p\)-adic completions with respect to \(O_p\). From Chapter 3, one has \(\mathring{O}_K = \prod_{p < \infty} O_p\).

At this point, consider a finite set \(S\) of finite primes in \(K\). Now, let \(J_{K,S}\) be the free abelian group generated by finite primes in the complement of \(S\), \(\mathbb{A}_{K,S,f} = \prod_{p \in S^c, p < \infty} K_p\) and \(\mathring{O}_{K,S} = \prod_{p \in S^c, p < \infty} O_p\)
Consider also the finite abelian extension $L/K$. Set $Y_{L/K}$ as the balanced product

$$Y_{L/K,S} = \text{Gal}(L/K) \times \hat{\mathcal{O}}_{K,S}$$

where the action of $\hat{\mathcal{O}}_{K,S}$ on $\text{Gal}(L/K) \times \hat{\mathcal{O}}_{K,S}$ is given by

$$g(x,y) = (x r_{L/K}(g)^{-1}, gy)$$

for $g \in \hat{\mathcal{O}}_{K,S}^\times$, $x \in \text{Gal}(L/K)$, $y \in \hat{\mathcal{O}}_{K,S}$, and $r_{L/K} : \mathbb{A}_{K,f}^\times \to \text{Gal}(L/K)$ is the Artin map.

The action defined above is a restriction of the action of $\mathbb{A}_{K,S,f}^\times$ on $Y_{L/K,S}$ given by the same formula. This induces an action $\alpha$ by endomorphisms of $J_{K,S}^+$ of integral elements of $J_{K,S}$ on $Y_{L/K,S}$. Hence, we can consider the semigroup crossed product $C^*$-algebra

$$\mathcal{A}_{L/K,S} = C(Y_{L/K,S}) \rtimes_{\alpha} J_{K,S}^+$$

The dynamics of this system are given by

$$\sigma_t (f) = f, \quad \sigma_t (\mu_j) = N_K(j)^t \mu_j$$

for $f \in C(Y_{L/K,S}), j \in J_{K,S}$, where $N(j) = |O_K/j|$ is the norm of $j$.

### 8.3 Group Action Groupoid and Semigroup Crossed Product $C^*$-algebras

In Chapter 5, the concept of a groupoid was introduced and how one can naturally construct a groupoid $C^*$-algebra. This was motivated by a physical example of $Q$-lattices that yielded a groupoid whose $C^*$-algebra was the $C^*$-algebra of observables of a dynamical system that went some way to generalise the original Bost-Connes system. In [26], the Bost-Connes Problem is solved and the system used uses the $C^*$-algebra of a certain groupoid motivated by the commensurability groupoids from Chapter 5. While this seems desirable as motivation, most authors including Marcelo Laca, Nadia Larsen and Sergey Neshveyev in [26], treat this system in its isomorphic form; as a semigroup crossed product.

This section aims to show this isomorphism which will shed light on the construction of the partial Connes-Marcolli system as a semigroup crossed product in Section 8.4. It follows the work of Marcelo Laca in [25].
First, it will be shown that a semigroup action by endomorphisms can be extended to a group action by automorphisms. This will rely heavily on a fact of Richard Kadison and John Ringrose (Proposition 11.4.1 in [24]) about inductive systems of C*-algebras. In particular it says:

**Proposition 8.3.1.** Consider a direct system of C*-algebras. Namely, for a directed set \((I, \leq)\), consider the family of C*-algebras \(A_i\) indexed by \(I\) with homomorphisms \(f_{ij}: A_i \to A_j\) for all \(i \leq j\) such that

- \(f_{ii}\) is the identity homomorphism on \(A_i\) for all \(i \in I\), and
- \(f_{ik} = f_{jk} \circ f_{ij}\) for all \(i, j, k \in I\) and \(i \leq j \leq k\).

Then there exists a unique C*-algebra \(A_\infty\) which is the inductive limit of the directed system \((A_i, f_{ij})\) together with embeddings \(\alpha_x: A_x \to A_\infty\) such that

1. \(\alpha^x = \alpha^y \circ f_{xy}\) whenever \(x \leq y\), and
2. \(\cup_{x \in I} \alpha^x(A_x)\) is dense in \(A_\infty\).

Now, one can consider how a semigroup dynamical system by endomorphisms can be extended to a group dynamical system by automorphisms.

**Theorem 8.3.2.** Let \(S\) be an Ore semigroup with enveloping group \(G = S^{-1}S\) and let \(\alpha\) be an action of \(S\) by injective endomorphisms of a unital C*-algebra \(A\). Then there exists a C*-dynamical system \((B, G, \beta)\), unique up to isomorphism, consisting of an action \(\beta\) of \(G\) by automorphisms of a C*-algebra \(B\) and an embedding \(i: A \to B\) such that

- \(\beta\) dilates \(\alpha\), that is, \(\beta_s \circ i = i \circ \alpha_s\) for \(s \in S\), and
- \((B, G, \beta)\) is minimal, that is, \(\cup_{s \in S} \beta_s^{-1}(i(A))\) is dense in \(B\).

**Proof.** Using Proposition 8.3.1, consider the directed system of C*-algebras determined by the maps \(f_{xy} = \alpha_{yx^{-1}}\) from \(A_x := A\) into \(A_y := A\), for \(x \in S\) and \(y \in Sx\), (i.e. \(x \leq y\) in \(S\) in the natural partial order described in Chapter 6.1).

It is left to show that \(\alpha_x\) can be extended to an automorphism of \(A_\infty\). Again, this will borrow from Proposition 8.3.1 and here will be provided a brief explanation of how to apply this. As \(S\) is a quasi-lattice ordered semigroup, for any \(x \in S\), there exists \(y \in Ss\) for a fixed \(s \in S\) such that \(x \leq y\). So, by repeating the work above, it is seen that \(A_\infty\) is also the inductive limit of the directed subsystem \((A_x, f_{xy})_{x \in Ss}\). For this subsystem, there are also embeddings \(\psi^x: A_x \to A_\infty\) defined by \(\psi^x(a) = \alpha_{xs^{-1}}(a)\) for \(x \in Ss\) and \(a \in A_x\). Proposition 8.3.1 states that there is an
automorphism $\beta_s$ of $A_\infty$ such that $\beta_s \circ \alpha_x = \psi_x$ for every $x \in Ss$.

It can now be shown that the conditions of this theorem are met with the newly defined $\beta$ and $i : \alpha^1 : A_1 \rightarrow A_\infty$. The first condition is met because

$$\beta_s \circ i = \beta_s \circ \alpha^1 = \beta_s \circ \alpha^s \circ \alpha_s^1 = \psi^s \circ \alpha_s^{-1} \circ \alpha_s = \alpha^1 \circ \alpha_s$$

The second condition is met, because

$$\beta_s^{-1}(i(A)) = \beta_s^{-1} \circ \alpha^1(A_1) = \beta_s^{-1} \circ \psi^s(A_s) = \alpha^s(A_s)$$

along with the density construction from earlier.

Note, Proposition 8.3.1 also guarantees $A_\infty$ with this embedding is unique up to isomorphism.

**Definition 8.3.3.** A system $(B, G, \beta)$ satisfying the conditions of Theorem 8.3.2 is called the *minimal automorphic dilation* of $(A, S, \alpha)$.

Now one needs to be able to extend covariant representations of a semigroup crossed product to covariant representations of the dilated system (the one defined in Theorem 8.3.2). This is achieved through the following proposition.

**Proposition 8.3.4.** Let $(\pi, V)$ be a covariant representation for the system $(A, S, \alpha)$ on the Hilbert space $H$. Then there exists a covariant representation $(\bar{\pi}, \bar{V})$ for the minimal automorphic dilation $(B, G, \beta)$ on a Hilbert space $H'$ containing a copy of $H$, such that $\bar{\pi} \circ i = \pi$ on $H$, where $i : A \rightarrow B$ is the embedding from Theorem 8.3.2.

**Proof.** For the sake of completeness, a sketch of the proof of this proposition is provided. In Theorem 1.2.1 [25], Marcelo Laca shows that for an isometric representation $V$ of $S$ into some Hilbert space $H$, there exists what is called the minimal unitary dilation $\bar{V}$ which is a unitary representation of $S$ into a Hilbert space $H'$ containing a copy of $H$ such that

- $\bar{V}_s$ leaves $H$ invariant and $\bar{V}_s|_H = V_s$ for all $s \in S$, and
- $H_0 = \cup_{s \in S} \bar{V}_s^*H$ is dense in $H'$.

We construct $\bar{\pi}$ as follows. Following Lemma 2.1.3 in [25], consider the dense subspace $H_0 = \cup_{s \in S} \bar{V}_s^*H$ (that is defined and shown to exist in the previous paragraph) of $H'$ and the dense subalgebra $B_0 = \cup_{s \in S} \beta_s^{-1}(i(A))$ of $B$ from Theorem 8.3.2. So, for $\xi \in H_0$ and $b = \beta_t^{-1}(i(a)) \in B_0$
One can check the details of the rest of the proof in [25]. By showing that this definition for \( \tilde{\pi} \) is independent of choice of \( a \) and \( s \), one sees that \( \tilde{\pi} \) is in fact an operator. It is also linear and bounded by injectivity of the endomorphisms, which means it can be uniquely extended to a bounded linear operator on \( H' \). Moreover, a simple calculation shows that \( (\tilde{\pi}, \tilde{V}) \) is a covariant pair.

All of this is leading to Theorem 8.3.6, where one shows that a specific type of semigroup crossed product is isomorphic to a full corner of a group crossed product. First, we need the definition of a full corner.

**Definition 8.3.5.** Given a C*-algebra \( A \) and a projection \( p \) in \( A \), then the algebra \( pA \) is a corner of \( A \). A full corner is a corner \( pA \) where the linear span of \( A \) is dense in \( A \).

The theorem introduced earlier can now be stated and proved.

**Theorem 8.3.6.** Given the semigroup dynamical system \((A, S, \alpha)\) where \( S \) is an Ore semigroup acting by injective endomorphisms, let \((B, G, \beta)\) be the minimal automorphic dilation, with embedding \( i: A \to B \). Then \( A \rtimes_{\alpha} S \) is canonically isomorphic to the full corner \( i(1)(B \rtimes_{\beta} G)i(1) \).

**Proof.** This proof follows [25]. Let \( U \) be the unitary representation of \( G \) in the multiplier algebra of \( B \rtimes_{\beta} G \), and notice that \( i(1)U_s i(1) = U_s i(1) \) for \( s \in S \) as \( i(A) \) is invariant under \( \beta_s \). Define \( v_s = U_s i(1) \). Then

\[
v_s^* v_s = i(1)U_s^* U_s i(1) = i(1)
\]

and

\[
v_s v_t = U_s i(1)U_t i(1) = U_s U_t i(1) = U_{st} i(1) = v_{st},
\]

so \( v \) is an isometric representation of \( S \). Since \( i(1)(B \rtimes_{\beta} G)i(1) \) is generated by the elements \( i(1)U_s^*i(a)U_t i(1) = v_s^*i(a)v_t \), the isomorphism will be established by uniqueness of the crossed product once we show that the pair \((i, v)\) is universal.

Suppose \((\pi, V)\) is a covariant representation for the system \((A, S, \alpha)\), and let \((\tilde{\pi}, \tilde{V})\) be the corresponding dilated covariant representation of \((B, G, \beta)\) given by Proposition 8.3.4. By the universal property of \( B \rtimes_{\beta} G \) there is a homomorphism \((\tilde{\pi} \times \tilde{V}): B \rtimes_{\beta} G \to C^*(\tilde{\pi}, \tilde{V}) \) such that

\[
\tilde{\pi}(b)\tilde{V}_s = (\tilde{\pi} \times \tilde{V})(i_B(b)U_s).
\]
Let $\rho$ be the restriction of $(\tilde{\pi} \times \tilde{V})$ to $i(1)(B \rtimes \beta G)i(1)$ to the invariant subspace $\mathcal{H}$. By Proposition 8.3.4

$$
\rho(i(a)) = \tilde{\pi} \circ i(a) = \pi(a) \quad \rho(v_s) = \tilde{V}_s \pi(1) = V_s,
$$

for $s \in S$ and $a \in A$. Thus $(i, \nu)$ is universal for $(A, S, \alpha)$.

So, this shows that $A \rtimes S$ is isomorphic to a full corner of $B \rtimes \beta G$ where $G = S^{-1}S$ and $(B, G, \beta)$ is the minimal automorphic dilation of $(A, S, \alpha)$.

Let us consider the C*-algebra of a group action groupoid.

Suppose that $G$ is a countable discrete group acting on a second countable, locally compact, Hausdorff topological space $X$ and that $Y$ is a clopen subset of $X$ satisfying $GY = X$. It is a well-known fact that the C*-algebra $C_0(X) \rtimes_r G$ is the reduced C*-algebra of the group action groupoid $G \times X$.

Consider the subgroupoid $G \boxtimes Y = \{(g, x) \mid x \in Y, gx \in Y\}.$ and denote by $C^*_r(G \boxtimes Y)$ (as described in Chapter 5) its reduced C*-algebra. Given that $G \boxtimes Y$ is the projection of $G \times X$ to $Y$ yields, $C^*_r(G \boxtimes Y) = 1_Y(C_0(X) \rtimes_r G)1_Y$, where $1_Y = i(1)_Y$ is the projection defined in Theorem 8.3.6.

Therefore, these facts and Theorem 8.3.6 induce the following corollary.

**Corollary 8.3.7.** Given $G, X$ and $Y$ as above,

$$
C^*_r(G \boxtimes Y) \cong 1_Y(C_0(X) \rtimes_r G)1_Y \cong C^*(Y) \rtimes_\alpha S.
$$

Thus, if one can fit the framework of the partial Connes-Marcolli system into the same framework as is used in Corollary 8.3.7, one can immediately deduce that the system exists and is well defined. Moreover this corollary provides a groupoid-centric avenue to motivate such a system. This will not be attempted but this would be analogous to how $\mathbb{Q}$-lattices (and indeed $K$-lattices) motivated the construction of the Connes-Marcolli systems.

Let us define the partial Connes-Marcolli system as in Section 8.2.

**Definition 8.3.8.** Using the same notation as in Section 8.2, define the *partial Connes-Marcolli system* for an abelian extension $L/K$ and (possibly empty) finite set $S$ of finite primes of $K$ to
be \((A_{L/K,S}, \sigma)\) where

\[ A_{L/K,S} = C(Y_{L/K,S}) \rtimes J_{K,S}^+ \]

and

\[ \sigma_l(f) = f, \quad \sigma_l(\mu_j) = N_K(j) \mu_j \quad \text{for} \quad f \in C(Y_{L/K,S}), \mu_j \in J_{K,S}, \]

where \(N_K(j) = |\mathcal{O}_K/j|\) is the norm of \(j\).

Here we define

\[ Y_{L/K,S} = \text{Gal}(L/K) \times \hat{O}_{K,S} \]

where the action of \(\hat{O}_{K,S}\) on \(\text{Gal}(L/K) \times \hat{O}_{K,S}\) is given by

\[ g(x, y) = (x r_{L/K}(g)^{-1}, g y) \]

for \(g \in \hat{O}_{K,S}, x \in \text{Gal}(L/K), y \in \hat{O}_{K,S}\) and \(r_{L/K} : A_{K,f}^+ \to \text{Gal}(L/K)\) is the Artin map. By restriction, this action induces an action by endomorphisms of \(J_{K,S}^+\) into \(Y_{L/K,S}\) denoted by \(\alpha\).

**Remark.** This is a well-defined C*-dynamical system, because one can apply Corollary 8.3.7 to Definition 8.3.8. In particular, define

\[ X_{L/K,S} = \text{Gal}(L/K) \times \hat{O}_{K,S}^\ast A_{K,S,f} \]

where the product is balanced over the same action of \(\hat{O}_{K,S}^\ast\) on \(\text{Gal}(L/K) \times A_{K,S,f}\) as described above. Then setting \(X = X_{L/K,S}, Y = Y_{L/K,S}, G = J_{K,S}\) satisfies the same conditions as are required, because \(X\) is a second countable, locally compact Hausdorff topological space, \(G\) is a countable discrete group. \(Y\) is the restriction of \(X\) in its second coordinate to a maximal compact subgroup, thus \(Y\) is clopen in \(X\). Moreover, as \(J_{K,S} \cong A_{K,S,f;j}^\ast/\hat{O}_{K,S}^\ast\), it can be shown that \(GY = X\). In particular, if \(g \in G = J_{K,S}\) and \((x, y) \in Y = \text{Gal}(L/K) \times \hat{O}_{K,S}^\ast A_{K,S,f}\), then

\[ g(x, y) = (x r_{L/K} h^{-1}, g y) \]

for \(g \in \hat{O}_{K,S}^\ast, x \in \text{Gal}(L/K), y \in \hat{O}_{K,S}^\ast\). Additionally, if \((x, y) \sim (z, w)\), then \((z, w) = (x r_{L/K} h^{-1}, g y)\) for some \(h \in \hat{O}_{K,S}^\ast\). Then,

\[ g(x, y) = (x r_{L/K} h^{-1}, g y) \sim (x r_{L/K} h^{-1} r_{L/K} h^{-1}, g h y) = (x r_{L/K} h^{-1} r_{L/K} h^{-1}, g h y) = g(z, w). \]

Thus, \(GY \subset X\). Similarly, for any \((x', y') \in X = \text{Gal}(L/K) \times \hat{O}_{K,S}^\ast A_{K,S,f}\), there exists an element \(g \in A_{K,S,f;j}^\ast/\hat{O}_{K,S}^\ast\) which can be identified with an element \(g' \in \hat{O}_{K,S}^\ast\), such that \(g^{-1} y' \in \hat{O}_{K,S}^\ast\). Then, there exists \((x'', y'') \in \text{Gal}(L/K) \times \hat{O}_{K,S}^\ast \hat{O}_{K,S}\), such that \(g'(x'', y'') = (x', y')\). As in the other direction, the balanced product is preserved. So, \(X \subset GY\). Thus, \(GY = X\).

So, the partial Connes-Marcolli system satisfies the conditions of Corollary 8.3.7. Hence, one can start with the C*-algebra of an appropriate transformation groupoid, then project down to the partial Connes-Marcolli system proving that it exists and is well-defined.

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8.4 The partial Connes-Marcolli semigroup crossed product

In this section, it will be seen that the Partial Connes-Marcolli systems can be alternatively motivated as a semigroup crossed product in its own right, rather than as a full corner of a group C*-algebra.

To show that this system \( (A_{L/K,S}, \sigma_t) \) (see Definition 8.3.8) is indeed well-defined, the following propositions must first be proven.

**Proposition 8.4.1.** The triple \( (C(Y_{L/K,S}), J^+_{K,S}, \alpha) \) is a well-defined semigroup dynamical system.

**Proof.** This amounts to showing that \( \alpha \) acts by endomorphisms on \( C(Y_{L/K,S}) \). The action of \( J^+_{K,S} \) on \( Y_{L/K,S} \) is given by
\[
g(x, y) = (x r_{L/K}(g)^{-1}, gy) \quad \text{for all} \quad g \in J^+_{K,S}, x \in \text{Gal}(L/K), y \in \hat{O}_{K,S}.
\]

We can turn this into a homomorphism \( \alpha : J^+_{K,S} \to C(Y_{L/K,S}) \) in a natural way. First denote by \( i_Y : Y_{L/K,S} \to C(Y_{L/K,S}) \) the canonical embedding of \( Y_{L/K,S} \) as unitaries in its group C*-algebra. Define
\[
\alpha_g(i_Y(x, y)) = \frac{1}{N_K(g)} \sum_{g(x', y') = (x, y)} i(x', y'),
\]
where \( N_K \) is the ideal norm of \( K \).

One can show that \( \alpha \) is a homomorphism and \( \alpha_g \) is an endomorphism by following the proof of Lemma 6.3.1 analogously. All of the steps work in the same way.

**Proposition 8.4.2.** \( C(Y_{L/K,S}) \rtimes_{\alpha} J^+_{K,S} \) defined as above is a well-defined semigroup crossed product.

**Proof.** The above proposition shows that \( (C(Y_{L/K,S}), J^+_{K,S}, \alpha) \) is a well-defined semigroup dynamical system where \( J^+_{K,S} \) is an Ore semigroup (from Section 6.1). Moreover, there is a natural left regular representation of \( Y_{L/K,S} \) into \( \ell^2(J^+_{K,S}) \) that is non-degenerate. This *-representation \( \pi \) of \( A_{L/K,S} \) into \( B(\ell^2(J^+_{K,S})) \) is given by
\[
\pi(\mu_g) \epsilon_k = \epsilon_{gk} \quad \text{for} \quad g, k \in J^+_{K,S}
\]
\[
\pi(i_Y(x, y)) \epsilon_k = \chi(x r_{L/K}(k)^{-1}) \epsilon_k \quad \text{for} \quad (x, y) \in Y_{L/K,S}, \chi : \text{Gal}(L/K) \to S^1
\]
The representation maps the unitary elements of $Y_{L/K,S}$ to the evaluation of a character of $\text{Gal}(L/K)$ via the canonical embedding of $\text{Gal}(L/K)$ into $S^1$ via the double Pontryagin dual, where the Pontryagin dual of a group $X$ is the character group of $X$ (group of homomorphisms of $X$ into $S^1$ with pointwise multiplication of functions). The existence of such a character is proven by Jane Arledge, Marcelo Laca and Iain Raeburn in [3].

These relations can be shown to be a representation as they satisfy the relations in Theorem 5.5.2 and this is analogous to the left regular representation of the original Bost-Connes system into $\ell^2(\mathbb{N}^*)$.

So, by Proposition 6.2.6 this semigroup crossed product exists.

Of course, one can see that this semigroup crossed product is well-defined by following the motivating group action groupoid approach of Section 8.3. The corner of the C*-algebra of that groupoid is isomorphic to this semigroup crossed product. One only has to show the corner is nontrivial, but we shall leave that as it is no longer necessary.

8.5 The Bost-Connes System as a Partial Connes-Marcolli system

In this section, I will first adapt the Partial Connes-Marcolli system to the situation where the underlying field is the field of rationals. In this case, $K = \mathbb{Q}$. Then setting $L = \mathbb{Q}^{ab}$ and $S = \emptyset$ should recover the original Bost-Connes system.

Consider the set of rational numbers $\mathbb{Q}$. The primes of $\mathbb{Q}$ correspond to the integer primes. For every prime $p$ denote by $\mathbb{Q}_p$ the corresponding $p$-adic completion of $\mathbb{Q}$, and by $\mathbb{Z}_p = \mathcal{O}_p \subset \mathbb{Q}_p$ its maximal compact subring for finite primes $p$, also known as the $p$-adic integers corresponding to the prime $p$. Denote by $J_{\mathbb{Q}}$ the group of fractional ideals of $\mathbb{Q}$, by $\mathbb{A}_{\mathbb{Q},f}$ the finite adele ring of $\mathbb{Q}$, and by $\mathbb{A}_{\mathbb{Z}}$ its maximal compact subring.

So $J_{\mathbb{Q}} \cong \mathbb{A}_{\mathbb{Q},f}^*/\mathbb{A}_{\mathbb{Z}}^*$, where $\mathbb{A}_{\mathbb{Q},f}$ is the restricted product $\prod'_{p} \mathbb{Q}_p = \mathbb{Q} \otimes_{\mathbb{Z}} (\prod_{p} \mathbb{Z}_p)$ of the fields $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p \subset \mathbb{Q}_p$, and

$$\mathbb{A}_{\mathbb{Z}} = \prod_{p} \mathbb{Z}_p \subset \mathbb{A}_{\mathbb{Q},f}.$$ 

Hence, for $\mathbb{Q}$, $J_{\mathbb{Q}} = \mathbb{Q}_0^*$ and $J_{\mathbb{Q}}^* = \mathbb{N}^*$.

For a finite set $S$ of finite primes in $\mathbb{Q}$ define $\mathbb{A}_{\mathbb{Q},S,f} = \prod_{p \in S^c, p < \infty} \mathbb{Q}_p$ and $\mathbb{A}_{\mathbb{Z},S} = \prod_{p \in S^c, p < \infty} \mathbb{Z}_p$. 

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Let $L/\mathbb{Q}$ be an abelian extension, finite or infinite, and $S$ be a finite, possibly empty, set of primes in $\mathbb{Q}$. Consider the quotient space

$$X_{L/Q,S} = \text{Gal}(L/\mathbb{Q}) \times \mathbb{A}_{\mathbb{Z},S}^* \overline{\mathbb{A}_{\mathbb{Q},S,f}}.$$ 

Here the action of $\mathbb{A}_{\mathbb{Z},S}^*$ on $\text{Gal}(L/\mathbb{Q})$ is defined using the Artin map $r_{L/\mathbb{Q}} : \mathbb{A}_{\mathbb{Q},f}^* \rightarrow \text{Gal}(L/\mathbb{Q})$. Identify $J_{\mathbb{Q},S}$ with $\mathbb{A}_{\mathbb{Q},S,f}^*/\mathbb{A}_{\mathbb{Z},S}^*$, then the diagonal action of $\mathbb{A}_{\mathbb{Q},S,f}^*$ on $\text{Gal}(L/\mathbb{Q}) \times \mathbb{A}_{\mathbb{Q},S,f}^*$ defines an action of $J_{\mathbb{Q},S}$ on $X_{L/Q,S}$.

By restricting $X_{L/Q,S}$ in the second coordinate to its maximal compact subring of $\mathbb{A}_{\mathbb{Z},S}$, the result is $Y_{L/Q,S}$, which can be written as

$$Y_{L/Q,S} = \text{Gal}(L/\mathbb{Q}) \times \mathbb{A}_{\mathbb{Z},S}^* \overline{\mathbb{A}_{\mathbb{Z},S}^*} \subset X_{L/Q,S}$$

Now, the partial Connes-Marcolli system is the $C^*$-algebra completion of the group action groupoid arising from the action of $J_{\mathbb{Q},S}^*$ on $Y_{L/Q,S}$, in the way described in Chapter 5. As a semigroup crossed product it can be written as

$$\mathcal{A}_{L/Q,S} = C(Y_{L/Q,S}) \rtimes J_{\mathbb{Q},S}^*.$$ 

The action of $\text{Gal}(L/\mathbb{Q})$ by translations on itself defines an action of $\text{Gal}(L/\mathbb{Q})$ on $X_{L/Q,S}$, which in turn defines an action on $\mathcal{A}_{L/Q,S}$. Define a dynamics $\sigma$ on $\mathcal{A}_{L/Q,S}$. It is easier to explain its extension to the multiplier algebra of the whole crossed product $C_0(X_{L/Q,S}) \rtimes J_{\mathbb{Q},S}^*$, which will continue to be denoted by $\sigma$. We put $\sigma_t(f) = f$ for $f \in C_0(X_{L/Q,S})$ and $\sigma_t(u_n) = |n|^tu_n$ for $n \in J_{\mathbb{Q},S}^*$.

This partial Connes-Marcolli system will be extensively used in Chapter 10, where the Kronecker-Weber Theorem will be recovered from an inductive system of partial Connes-Marcolli systems.

The link between the partial Connes-Marcolli systems and the original Bost-Connes system is nicely encapsulated in the following theorem.

**Theorem 8.5.1.** Setting $S = \emptyset$ and $L = \mathbb{Q}^{ab}$, then the partial Connes-Marcolli system recovers the original Bost-Connes system.
Proof. First and foremost, \( J_{Q,S}^+ = J_Q^+ = \mathbb{N}^+ \). Secondly,

\[
\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \lim_{\longrightarrow n} \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \\
\cong \lim_{\longrightarrow n} (\mathbb{Z}/n\mathbb{Z})^* \\
\cong \lim_{\longrightarrow n} \prod_p (\mathbb{Z}/p^n\mathbb{Z})^* \\
\cong \prod_p \lim_{\longrightarrow n} (\mathbb{Z}/p^n\mathbb{Z})^* \\
\cong \prod_p \mathbb{Z}_{p}^* = A_Z^*.
\]

by the Kronecker-Weber Theorem.

Hence,

\[
\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \times_{A_Z^*} A_Z \cong A_Z^* \times_{A_Z^*} A_Z \cong A_Z.
\]

Moreover, from Chapter 3, we saw that \( C(A_Z) \cong C^*(\mathbb{Q}/\mathbb{Z}) \). All the other properties of the systems follow from these equalities and hence, in this particular way, one can recover the original Bost-Connes system from the new partial Connes-Marcolli system.

\[\square\]

In Chapter 9, I will show certain properties of this system including the partition function, symmetries and KMS\(_\beta\) states.
Chapter 9

Properties of Partial Connes-Marcolli Systems

Chapter 8 saw the detailed construction of the Partial Connes-Marcolli System. It is constructed as a full corner of a group crossed product and hence, can also be written as a semigroup crossed product where the semigroup is the Ore semigroup whose enveloping group is the original action group. With a flurry of papers appearing soon after one another it is difficult to ascertain who came up with what ideas first, however, chronologically by date of publishing the Partial Connes-Marcolli systems borrow the concept of taking arbitrary abelian Galois extension (rather than the full maximal abelian extension) from the work of Sergey Neshveyev and Simen Rustad in [37] and the concept of subsets of the primes from the work of Cornelissen and Marcolli in [13]. Sincere apologies if any of these ideas are misattributed.

I observed that combining both ideas gives a Bost-Connes type system that still satisfies a Bost-Connes type problem. By this, it is meant that if one replaces some of the basic ingredients in the Bost-Connes system and makes the same changes in the statement of the Bost-Connes Problem, then the new system should still solve the problem. More specifically, the partial Connes-Marcolli system as described in Definition 8.3.8 satisfies the modified Bost-Connes Problem described in Section 8.1. This chapter is devoted to proving that this is the case. Recall the modified Bost-Connes Problem for a partial Connes-Marcolli system \((A_{L/K,S}, \sigma_t)\):

**Modified Bost-Connes Problem.** \((A_{L/K,S}, \sigma_t)\) must exhibit the following properties.

1. The partition function is given by the Dedekind zeta function \(\zeta_{K,S} = \sum_{s \in J_{K,S}} N(s)^{-\beta}.\)
2. The finite abelian Galois group $\text{Gal}(L/K)$ of $K$ acts as symmetries on $A_{L/K,S}$.

3. For each inverse temperature $0 < \beta \leq 1$ there is a unique KMS$_\beta$-state.

4. For each $\beta > 1$ the action of the symmetry group $\text{Gal}(L/K)$ on the set of extremal KMS$_\beta$-states is free and transitive.

When a system satisfies these four properties it is called an analytic Partial Connes-Marcolli system for $K$.

5. There exists a $K$-rational subalgebra $A^{\text{arith}}$ such that for every extremal KMS$_\infty$-state $\rho$ and every $f \in A^{\text{arith}}$, we have

$$\rho(f) \in L$$

and further $L$ is generated over $K$ by these values.

6. If we denote by $\nu_\rho$ the action of a symmetry $\nu \in \text{Gal}(L/K)$ on an extremal KMS$_\infty$-state $\rho$ (given by pull-back), we have for every element $f \in A^{\text{arith}}$ the following compatibility relation

$$\nu_\rho(f) = \nu^{-1}(\rho(f)).$$

7. The $C^*$-algebra $A^{\text{arith}} \otimes_K C$ is dense in $A_{L/K,S}$.

A system $A_{L/K,S}$ which satisfies these last properties is called a Partial Connes-Marcolli system for $K$ and $A^{\text{arith}}$ is called an arithmetic subalgebra.

A partial Connes-Marcolli system as described in Chapter 8 seems to have some of the ingredients to make up larger Bost-Connes type systems. More importantly, it is the author’s intent to try and build the original Bost-Connes system using partial Connes-Marcolli systems (this being the reason for the naming of these systems). In order to do this, however, many of the properties of the system must be understood to see how the structure of the systems is preserved when manipulating the systems. Therefore, the key focus of this chapter is to elucidate the key properties of partial Connes-Marcolli systems. This will be done by calculating the partition function, describing the symmetry groups (Section 9.1) and the KMS$_\beta$ states for all inverse temperatures $\beta$ (see Sections 9.3, 9.4, 9.5 and 9.6). In order to calculate the KMS$_\beta$ states, we follow the work of Marcelo Laca, Nadia Larsen and Sergey Neshveyev, which gives some more general results from which the calculations will follow. There will be an attempt in Section 9.2 to explain their work in more detail and show how it will be applied to the partial Connes-Marcolli systems. These results contain some useful theorems about KMS$_\beta$ states in corners of C*-algebra crossed products. In particular, there is a particular emphasis in showing that there is a correspondence between KMS$_\beta$ states and Radon measures on an underlying space of the crossed product.
9.1 Symmetries and Partition Function

Recall the partial Connes-Marcolli system \((A_{L/K,S}, \sigma_t)\).

**Definition 9.1.1.** For an abelian extension \(L/K\) of a number field \(K\), call the subring of integers of \(K\), \(\mathcal{O}_K\). Denote by \(A_{K,f}\) the adeles of \(K\), \(\hat{\mathcal{O}}_K\) the integral adeles, \(J_K\) are the fractional ideals of \(K\) and \(J^+_K\) the subsemigroup of integral ideals. Consider a set \(S\), a possibly empty finite set of primes of \(K\). Recall the following analogous definitions: \(A_{K,S,f} = \prod_{p \in S^c, p < \infty} K_p\) and \(\hat{\mathcal{O}}_{K,S} = \prod_{p \in S^c, p < \infty} \mathcal{O}_p\) are defined similarly to \(A_K\) and \(\hat{\mathcal{O}}_K\) but restricted to primes in the complement of \(S\). Let \(J_{K,S}\) be the free abelian group generated by primes in the complement of \(S\). There is a natural diagonal action of \(A^*_{K,S,f}\) on \(\text{Gal}(L/K) \times \hat{\mathcal{O}}_{K,S}\) given by the diagonal action

\[g(x, y) = (x r_{L/K}(g)^{-1}, gy)\]

for \(g \in A^*_{K,S,f}, x \in \text{Gal}(L/K), y \in \hat{\mathcal{O}}_{K,S}\) and \(r_{L/K} : A^*_{K,f} \to \text{Gal}(L/K)\) is the Artin map. This induces an action \(J_{K,S} \cong A^*_{K,S,f}/\hat{\mathcal{O}}^*_{K,S,}\) on \(\text{Gal}(L/K) \times \hat{\mathcal{O}}_{K,S}\). Then, the **partial Connes-Marcolli system** is given by

\[A_{L/K,S} = C(\text{Gal}(L/K) \times \hat{\mathcal{O}}^*_{K,S}) \times J^+_{K,S}\]

and

\[\sigma_t(f) = f \quad \text{and} \quad \sigma_t(\mu_n) = N_K(n)^it \mu_n \quad \forall f \in C_0(\text{Gal}(L/K) \times \hat{\mathcal{O}}^*_{K,S}), n \in J_{K,S}.\]

For ease of notation, we shall set

\[X_{L/K,S} = \text{Gal}(L/K) \times \hat{\mathcal{O}}^*_{K,S}, A_{K,S,f}\]

\[Y_{L/K,S} = \text{Gal}(L/K) \times \hat{\mathcal{O}}^*_{K,S}.\]

Consider the representation of \(A_{L/K,S}\) into the Hilbert space \(B(\ell^2(J^+_{K,S}))\) as described in Section 8.4:

\[\pi(\mu_g)\epsilon_k = \epsilon_{gk} \quad \forall g, k \in J^+_{K,S}\]

\[\pi(iy(x,y))\epsilon_k = \chi(x r_{L/K}(k)^{-1})\epsilon_k \quad \forall (x, y) \in Y_{L/K,S}, \chi : \text{Gal}(L/K) \to S^1\]

In a similar way to Chapter 7, one can investigate the Hamiltonian corresponding to this representation. Given that the time evolution is invariant on \(X_{L/K,S}\), we can focus on the effect on \(J^+_{K,S}\). All of the following work assumes ideals are represented by concrete choice of representative.
We know that
\[ N_K(n)^i \varepsilon_{nk} = N_K(n)^i \mu_n \varepsilon_k = \sigma(t(\mu_n)) \varepsilon_k = e^{itH} \mu_n e^{-itH} \varepsilon_k. \]

To get an idea of what the Hamiltonian could be, let \( H \varepsilon_k = f(k) \varepsilon_k \) for some nice function \( f \).

Therefore, notice that
\[ e^{itH} \mu_n e^{-itH} \varepsilon_k = e^{itH} \mu_n (e^{f(k)})^{-it} \varepsilon_k = e^{itH} (e^{f(k)})^{-it} \varepsilon_{nk} = (e^{f(nk)})^{it} (e^{f(k)})^{-it} \varepsilon_{nk} = (e^{f(nk)-f(k)})^{it} \varepsilon_{nk}. \]

We thus require \( f(nk) - f(k) \) to be equal to \( \log(N_K(n)) \), so clearly \( f \) is logarithmic and in fact, we set \( H \varepsilon_k = \log(N_K(k)) \varepsilon_k \).

Since, the time evolution is invariant on the \( e_\gamma \), this can be extended to the whole system. This allows us to calculate the partition function of the partial Connes-Marcolli system \( (A_{L/K}, \sigma_t) \).

**Theorem 9.1.2.** The partition function of the partial Connes-Marcolli system is the Dedekind zeta function for \( K \) restricted by \( S \).

**Proof.** As defined in Section 7.2, the partition function is defined as \( Z(\beta) = \text{Tr}(e^{-\beta H}) \). Thus,
\[
Z(\beta) = \text{Tr}(e^{-\beta H}) \\
= \sum_{n \in J_{L/K,S}} < e^{-\beta H} \varepsilon_n, \varepsilon_n > \\
= \sum_{n \in J_{L/K,S}} < N_K(n)^{-\beta} \varepsilon_n, \varepsilon_n > \\
= \sum_{n \in J_{L/K,S}} N_K(n)^{-\beta} < \varepsilon_n, \varepsilon_n > \\
= \sum_{n \in J_{L/K,S}} N_K(n)^{-\beta} \\
= \zeta_K(\beta, S)
\]

is the Dedekind zeta function restricted by \( S \) defined by
\[
\zeta_K(\beta, S) := \sum_{s \in J_{K,S}} N_K(s)^{-\beta}.
\]

In this case, the norm \( N_K \) is the absolute norm in \( K \). \( \Box \)

At this point it is worth considering the symmetries of the partial Connes-Marcolli system. It will be shown in Section 9.5 that \( \text{Gal}(L/K) \) is a symmetry group for the partial Connes-Marcolli system.
9.2 Properties of KMS$_\beta$ states in Bost-Connes-like systems

This section follows the work of Laca, Larsen and Neshveyev in Chapter 1 of [26]. Specifically, Propositions 1.1, 1.2 and 1.3 will be discussed as they will also apply to the partial Connes-Marcolli systems. Thus, we develop an understanding of what KMS$_\beta$ states are like on C*-algebra crossed products that resemble Bost-Connes type systems. The idea is to lay the groundwork for calculating all the KMS$_\beta$ states simply as specific instances of what is described in this section. The work elucidated in this section follows closely that of [26]. The author strongly recommends reading the proofs of these propositions in [26], as all three proofs are very clean and well explained. There is really no opportunity to add anything to the proofs, so I will merely summarise the proofs to get a feeling of what sort of machinery is being used. Nevertheless, I will provide some detail in adapting this method to the partial Connes-Marcolli systems.

Following the notation in [26], suppose that $G$ is a countable discrete group acting on a second countable, locally compact, Hausdorff topological space $X$ and that $Y$ is a clopen subset of $X$ satisfying $GY = X$. The C*-algebra $C_0(X) \rtimes_r G$ is the reduced C*-algebra of the group action groupoid $G \times X$ as has been seen in Section 8.3.

Consider the subgroupoid

$$G \bowtie Y = \{(g, x) \mid x \in Y, gx \in Y\}$$

of $G \times X$ and denote by $C^*_r(G \bowtie Y)$ its reduced C*-algebra. This is constructed as described in Section 5.3. By making sure that $gx \in Y$, this becomes the largest group action groupoid of $G$ acting on $Y$. The definition of $Y$ means that

$$C^*_r(G \bowtie Y) = \mathbb{1}_Y(C_0(X) \rtimes_r G)\mathbb{1}_Y,$$

where $\mathbb{1}_Y$ is the characteristic function of $Y$ as shown in Corollary 8.3.7.

This form should be reminiscent of the construction of the partial Connes-Marcolli system. Further in this vein, give $C^*_r(G \bowtie Y)$ the dynamics $\sigma_{t \in \mathbb{R}}$ associated to a given homomorphism $N : G \to (0, +\infty)$. Hence,

$$\sigma_t(f)(g, x) = N(g)^it f(g, x)$$

for $t \in \mathbb{R}$ and $f \in C_c(G \bowtie Y) \subset C^*_r(G \bowtie Y)$.

**Definition 9.2.1.** $E$ is the conditional expectation from $C_0(X) \rtimes_r G$ onto $C_0(X)$. In particular,

$$E(f\mu_g) = \begin{cases} f & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

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Of course, the corner $C^r_\gamma(G \boxtimes Y)$ under $E$ is $C_0(Y)$.

Recall the definition of a Radon measure.

**Definition 9.2.2.** Given a Hausdorff topological space $X$ with a $\sigma$-algebra of Borel sets (sets that can be formed from open sets through the process of countable union, countable intersection and complement). Then a measure $m$ on the $\sigma$-algebra of Borel sets is a **Radon measure** if it is inner regular and locally finite. In other words,

- $m(B) = \sup_{K \subset B, \text{compact}} m(K)$ (inner regular),
- every point of $X$ has a neighbourhood on which the measure is finite (locally finite).

**Remark.** When the space $X$ is locally compact, Radon measures correspond to positive linear functionals on the space of continuous functions with compact support.

Using the above remark and the fact that $C_0(Y)$ is the space of all compactly supported functions on locally compact space $Y$, by restriction to $C_0(Y)$, a state $\phi$ on $C^r_\gamma(G \boxtimes Y)$ yields a Radon probability measure $\mu$. Similarly, a Radon probability measure on $Y$, which defines a state $\mu_*$ on $C_0(Y)$ can be extended via $E$ to a state on $C^r_\gamma(G \boxtimes Y)$. Here, we abuse notation somewhat and consider $E$ going up and down according to Definition 9.2.1. One can therefore, see that

$$ (\mu_* \circ E)|_{C_0(Y)} = \mu_*, $$

however, if the restriction is applied before the projection, there is a potential loss of information and one cannot guarantee that $(\phi|_{C_0(Y)}) \circ E$ is the same as $\phi$.

The goal of the following three propositions is to describe a situation where the above problem is no longer and KMS states can be extended via $E$ from their restriction. In such a situation, these states will be in one-to-one correspondence with a class of measures on $Y$ characterized by a scaling condition.

**Proposition 9.2.3** ([26] Proposition 1.1). **Under the general assumptions on $G, X, Y$ and $N$ listed above, suppose there exists a sequence $\{Y_n\}_{n=1}^\infty$ of Borel subsets of $Y$ and a sequence $\{g_n\}_{n=1}^\infty$ of elements of $G$ such that**

(i) $\bigcup_{n=1}^\infty Y_n$ contains the set of points in $Y$ with nontrivial isotropy;

(ii) $N(g_n) \neq 1$ for all $n \geq 1$;

(iii) $g_n Y_n = Y_n$ for all $n \geq 1$.

Then for each $\beta \neq 0$ the map $\mu \rightarrow \phi = (\mu_* \circ E)|_{C^r_\gamma(G \boxtimes Y)}$ is an affine isomorphism between Radon
measures $\mu$ on $X$ satisfying $\mu(Y) = 1$ and the scaling condition

$$\mu(gZ) = N(g)^{-\beta}\mu(Z)$$

for Borel $Z \subset X$, and KMS$_\beta$ states $\phi$ on $C^*_r(G \boxtimes Y)$.

Proof. Note, that the combination of the conditional probability and restriction is an affine map, so it is sufficient to show the isomorphism. First, Marcelo Laca et al. notices that a positive linear functional on $C_0(Y)$ corresponding to a measure satisfying the scaling condition can be extended via $E$ to a KMS$_\beta$ state.

To go other way, let $\phi$ be a KMS$_\beta$ state. One can easily show that the restriction to $C_0(Y)$ satisfies the scaling condition. It is just a matter of showing that $\phi = \mu_* \circ E$, where $\mu_*$ is the measure corresponding to the restriction of $\phi$ to $C_0(Y)$. This boils down to showing that $\phi(f \mu_g) = 0$ for $g \neq e$. One can split this up into the cases where $\text{supp } f \cap Y_g = \emptyset$ or $\text{supp } f \subset Y_g$, where $Y_g$ is the set of points of $Y$ invariant to a left action by $g$. Both of these cases yield $\phi(f \mu_g) = 0$ as required.

At this point it is worth mentioning that one can give an explicit form for the extension of the measure $\mu$ (which will continue to be called $\mu$). We can write

$$\mu(Z) = \sum_i N(h_i)^\beta \mu(h_i Z \cap Z_i),$$

where $h_i \in G$ and $Z_i \subset Y$ are such that $X$ is the disjoint union of the sets $h_i^{-1}Z_i$. $\square$

Our next goal is to classify measures satisfying the scaling condition.

**Proposition 9.2.4 ([26] Proposition 1.2).** Assume the hypotheses of Proposition 9.2.3. Let $\beta \neq 0$, $S$ be a subset of $G$, and $Y_0 \subset Y$ a nonempty Borel set such that

(i) $gY_0 \cap Y_0 = \emptyset$ for $g \in G \backslash \{e\}$;

(ii) $SY_0 \subset Y$;

(iii) if $gY_0 \cap Y \neq \emptyset$ then $g \in S$;

(iv) $Y \backslash SU \subset \bigcup_n Y_n$ for every open set $U$ containing $Y_0$;

(v) $\zeta_K(\beta, S) < \infty$.

Then

(1) the map $\phi = \mu_* \circ E \mapsto \zeta_K(\beta, S)\mu|_{Y_0}$ is an affine isomorphism between the KMS$_\beta$ states on $C^*_r(G \boxtimes Y)$ and the Borel probability measures on $Y_0$; the inverse map is given by $\nu \mapsto \mu_* \circ E$, 

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where the measure $\mu$ on $Y$ is defined by

$$\mu(Z) = \zeta_K(\beta, S)^{-1} \sum_{s \in S} N(s)^{-\beta} \mu(s^{-1}Z \cap Y_0);$$

(2) if $\mu$ is the measure on $Y$ defined by a probability measure $\nu$ on $Y_0$, and $H_S$ is the subspace of $L^2(Y, d\mu)$ consisting of functions $f$ such that $f(sy) = f(y)$ for $y \in Y_0$ and $s \in S$, then for $f \in H_S$ we have

$$\|f\|_2^2 = \zeta_K(\beta, S) \int_{Y_0} |f(y)|^2 d\mu(y);$$

furthermore, the orthogonal projection $P : L^2(Y, d\mu) \to H_S$ is given by

$$Pf \mid_{s_y} = \zeta_K(\beta, S)^{-1} \sum_{s \in S} N(s)^{-\beta} f(sy)$$

for $y \in Y_0$.

Proof. To begin with note that by Proposition 9.2.3 any $\text{KMS}_\beta$-state is determined by a Radon measure $\mu$ such that $\mu(Y) = 1$ and $\mu$ satisfies the scaling condition in Proposition 9.2.3. Using the assumptions of the problem yields $\zeta_S(\beta) \mu(Y_0) = 1$, from which it follows that $\mu$ is completely determined by its restriction to $Y_0$. To construct the inverse map $\nu$, one uses the result from Proposition 9.2.3 and notices that $\zeta_K(\beta, S) \mu$ extends $\nu$. Checking through, this satisfies the desired equation above.

Part (2) follows from the following equation

$$\int_{sY_0} |f|^2 d\mu(y) = N(s)^{-\beta} \int_{Y_0} |f(s\cdot)|^2 d\mu(y),$$

valid for $f \in L^2(Y, d\mu)$, on summing over $s \in S$ given that $SY_0$ is a subset of full measure in $Y$. Additionally one can show that the right hand side of the second identity in (2) is a contraction. Given that it also sends $f$ to $f$ for $f \in H_S$, this shows that the last identity is given by the projection as required.

In our applications the set $S$ will be a subsemigroup of $\{ g \in G \mid N(g) \geq 1 \}$ and $Y_0$ the complement of the union of the sets $gY$, for $g \in S \setminus \{e\}$.

We next give a similar classification of ground states. Ground states are equilibrium states which only satisfy the condition that if the holomorphic function $z \mapsto \phi(a\sigma_z(b))$ is bounded on the upper half-plane for $a$ and $b$ in a set of $\sigma$-analytic elements spanning a dense subspace. One recalls this is one of two conditions that a $\text{KMS}_\beta$ state must satisfy. $\text{KMS}_\infty$ states are those ground states that are equal to the weak *-limit of $\text{KMS}_\beta$ states.

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Proposition 9.2.5 ([26] Proposition 1.3). Under the general assumptions on $G, X, Y$ and $N$ listed before in Proposition 9.2.3, define $Y_0 = Y \cup \cup_{g: N(g) > 1} gY$. Assume $Y_0$ has the property that if $gY_0 \cap Y_0 \neq \emptyset$ for some $g \in G$ then $g = e$. Then the map $\mu \mapsto \mu_* \circ E$ is an affine isomorphism between the Borel probability measures on $Y$ supported on $Y_0$ and the ground states on $C^*_r(G \boxtimes Y)$.

Proof. See [26]. The method used here mirrors the proof of Proposition 9.2.3. □

This is sufficient to now classify the KMS$_\beta$ states on the partial Connes-Marcolli systems. Using Proposition 9.2.3 yields the following corollary.

Corollary 9.2.6. With the definition of the partial Connes-Marcolli system given in Definition 8.3.8, for $\beta \neq 0$, KMS$_\beta$ states correspond to measures $\mu$ on $X_{L/K,S}$ such that $\mu(Y_{L/K,S}) = 1$ and $\mu$ satisfies the scaling condition

$$\mu(gZ) = N(g)^{-\beta} \mu(Z)$$

for Borel $Z \subset X_{L/K,S}$, and KMS$_\beta$ states $\phi$ on $(A_{L/K,S}, \sigma_t)$.

Proof. We apply Proposition 9.2.3 to the following setting. Set $X = X_{L/K,S}, Y = Y_{L/K,S}, G = J_{K,S}$. One can easily see that $X$ is a discrete second countable locally compact Hausdorff space and given that $Y$ is the restriction of $X$ in the second coordinate to its maximum compact subring, $Y$ is a clopen set of $X$. One needs to consider the set of points of $Y$ of nontrivial isotropy. Suppose $(\alpha, a) \in \text{Gal}(L/K) \times \hat{A}_{K,S,f}$ in $X$ has nontrivial isotropy then one can find an ideal $n$, such that $an = 0$ (this is found by considering two such ideals whose action fixes $a$ and subtracting them). Set $\{(g_v, Y_v)\}_n = (p_v, Y_v)$ indexed by the finite places $v$, where $p_v$ is the prime ideal of $\mathcal{O}_{K,S}$ corresponding to $v$ and $Y_v \subset Y$ consists of the images in $X$ of all pairs $(\alpha, a) \in \text{Gal}(L/K) \times \hat{O}_{K,S}$ with $av = 0$. The result then follows. □

9.3 KMS-states for $\beta < 0$

To systematically classify the KMS$_\beta$ states, we begin with $\beta < 0$.

Lemma 9.3.1. For the system $(A_{L/K,S}, \sigma_t)$ and $\beta < 0$ there are no KMS$_\beta$-states.

Proof. Using Corollary 9.2.6, we see that given ideal $a \in G$ such that $aY \subset Y$, then the scaling condition on a measure $\mu$ would imply $N_K(a)^{-\beta} \leq 1$, which is a contradiction as $N_K(a) \geq 1$. Such an $a$ exists from Corollary 9.2.6. □
The proofs provided in [26] Theorem 2.1 will work for the partial Connes-Marcolli systems by making the simple transformation of replacing $K^{ab}/K$ with $L/K$ and restricting the action of the integral ideals to those generated by finite primes outside of a finite set of primes $S$. Nevertheless, details will still be shown.

9.4 KMS-states for $0 < \beta \leq 1$

**Lemma 9.4.1.** For the system $(A_{L/K,S}, \sigma)$ and for every $0 < \beta \leq 1$ there is a unique KMS$_\beta$-state.

**Proof.** The proof for this lemma is done analogously to that of Theorem 2.1 in [26]. From Corollary 9.2.6, KMS$_\beta$ states correspond to measures $\mu$ on $X_{L/K,S}$ that satisfy the conditions of Corollary 9.2.6. This proof relies on explicitly describing such a unique measure for each $\beta \in (0, 1]$.

First, consider the set of places $V$ that correspond to prime ideals in $S^c$. In particular, for each place $v$, there is a corresponding prime ideal $p_v$. Consider the following definition for a KMS$_\beta$ state. Define $\mu_\beta$ as the image of the measure $\mu_G \times \prod_{v \in V} \mu_{\beta, v}$ on $\text{Gal}(L/K) \times A_{K,S,f}$ under the quotient map $\text{Gal}(L/K) \times A_{K,S,f} \to X_{L/K,S}$, where $\mu_G$ is the normalized Haar measure on $\text{Gal}(L/K)$ and $\mu_{\beta, v}$ is the measure on $K_v$ defined as follows. The measure $\mu_{1, v}$ is the Haar measure on $K_v$ such that $\mu_{1, v}(O_v) = 1$. The measure $\mu_{\beta, v}$ is absolutely continuous with respect to $\mu_{1, v}$ and satisfies

$$\frac{d\mu_{\beta, v}}{d\mu_{1, v}}(a) = \frac{1 - N(p_v)^{-\beta}}{1 - N(p_v)^{-1}} |a|_v^{\beta - 1}$$

where $| \cdot |_v$ is the normalised valuation of $v$.

Following the method of Marcelo Laca, Nadia Larsen and Sergey Neshveyev in [26], to prove uniqueness, it is sufficient to show that all measures on $X_{L/K,S}$ that satisfy the conditions of Corollary 9.2.6 are ergodic for the action of $J_{K,S}$ on $(X_{L/K,S}, \mu)$. This boils down to showing that the subspace of $L^2(Y_{L/K,S}, d\mu)$ of $J_{K,S}^+$-invariant functions consists of scalars. From here, the proof is identical. First, consider a finite subset $A \subset V$ of finite places of $K$. Consider $J_{K,A}^+$ to be the unital subsemigroup of $J_{K,S}^+$ generated by $p_v$ for $v \in A$. In [26], it is shown that showing that the subspace of $L^2(Y_{L/K,S}, d\mu)$ of $J_{K,S}^+$-invariant functions consisting of scalars is equivalent to showing that the subspace of $L^2(Y_{L/K,S}, d\mu)$ of $J_{K,A}^+$-invariant functions consists of scalars. This is done by applying the second part of Proposition 9.2.4 to $G = J_{K,A}, S = J_{K,A}^+, Y_0 =$
Gal(L/K) × \mathcal{O}_A^* \times \tilde{\mathcal{O}}_A). This shows that the projection onto this subspace of \( J_{K,A}^+ \)-invariant functions is either constant or zero. Hence, the subspace required consists of scalars only. Thus, the action of \( J_{K,S} \) on \((X_{L/K,S}, \mu)\) is ergodic for all measures \( \mu \) that satisfy the conditions of Corollary 9.2.6. Thus, there can only be one such measure. So, the measure explicitly defined is the only one. Hence, there is only one KMS\( _\beta \) state for each \( \beta \).

Note, that Sergey Neshveyev and Simen Rustad in [37] used this method similarly for the case of a finite extensions, however for a function field \( K \).

### 9.5 KMS-states for \( 1 < \beta < \infty \)

For low temperatures \((1 < \beta < \infty)\), we get the following lemma.

**Lemma 9.5.1.** For the system \((A_{L/K,S}, \sigma_t)\) and for every \(1 < \beta < \infty\) the extremal KMS\( _\beta \)-states are indexed by the points of the subset

\[
Y_{L/K,S}^0 = \text{Gal}(L/K) \times \mathcal{O}_{K,S}^* \tilde{\mathcal{O}}_{K,S}^* \approx \text{Gal}(L/K)
\]

of \( Y_{L/K,S} \), with the state corresponding to \( \omega \in Y_{L/K,S}^0 \) given by

\[
\phi_{\beta, \omega}(f) = \frac{1}{\zeta_K(\beta, S)} \sum_{a \in J_{K,S}^+} N_K(a)^{-\beta} f(a\omega).
\]

**Proof.** Here we apply Proposition 9.2.4 with \( S = J_{K,S}^+ \) (the \( S \) on the left is from the proposition, the one in the subscript is the finite set of primes), and \( Y_0 \) from Proposition 9.2.5 is equal to \( Y_{L/K,S}^0 \) as described in this lemma. Clearly the conditions (i), (ii), (iii) and (v) are satisfied. One just needs to show that condition (iv) is satisfied. We do this analogously to Marcelo Laca et al. in [26] Theorem 2.1. Take a set \( S' \), such that \( S \subset S' \). Then consider the subset

\[
\text{Gal}(L/K) \times \mathcal{O}_{K,S}^* (\tilde{\mathcal{O}}_{K,S'}^* \times \mathcal{O}_{K,S \cap S'}^*).
\]

It is open in \( Y \) and the intersection of all such subsets over finite sets \( S' \) is \( \text{Gal}(L/K) \times \mathcal{O}_{K,S}^* \tilde{\mathcal{O}}_{K,S}^* = Y_0 \).

Moreover, the above subset is also closed, while \( Y \) is compact, so by definition every neighbourhood of \( Y_0 \) contains a subset like the one described above for some set \( S' \). Choose an \( S' \) and call the above subset \( W_{S'} \). Then the complement of \( J_{K,S}^+ W_{S'} \) in \( Y \) is the set of images of points \((\alpha, a) \in \text{Gal}(L/K) \times \tilde{\mathcal{O}}_{K,S} \) such that \( av = 0 \) for some \( v \in S' \). Thus, \( Y \setminus J_{K,S}^+ W_{S'} \) is covered by \( Y_v \) for
Thus, applying Proposition 9.2.4 yields a correspondence between the KMS_β states and the measures on Y_0. Moreover, the extremal KMS_β states are in direct correspondence with the points of Y_0. The formula for the KMS_β comes from adapting the formula in Proposition 9.2.4.

\[ v \in S' \text{ as described.} \]

\section{KMS-states for \( \beta = \infty \)}

\begin{lemma}
For the system \((A_{L/K,S}, \sigma_t)\) the extremal ground states are indexed by \(Y_{L/K,S}^0\), with the state corresponding to \(\omega \in Y_{L/K,S}^0\) given by
\[
\phi_{\infty, \omega}(f) = f(\omega),
\]
and all ground states are KMS_∞-states.
\end{lemma}

\begin{proof}
For this, one needs to just apply Proposition 9.2.5. Much like the work so far, this mirrors the work done in [26] Theorem 2.1.
\end{proof}

\section{Summary}

One can conclude this chapter by summarising the results in the following theorem:

\begin{theorem}
For the system \((A_{L/K,S}, \sigma_t)\) we have:

(i) for \(\beta < 0\) there are no KMS_β-states;

(ii) for every \(0 < \beta \leq 1\) there is a unique KMS_β-state;

(iii) for every \(1 < \beta < \infty\) extremal KMS_β-states are indexed by the points of the subset
\[
Y_{L/K,S}^0 = \text{Gal}(L/K) \times \hat{\mathfrak{O}}_{K,S}^* \cong \text{Gal}(L/K)
\]
of \(Y_{L/K,S}\), with the state corresponding to \(\omega \in Y_{L/K,S}^0\) given by
\[
\phi_{\beta, \omega}(f) = \frac{1}{\zeta_{K,S}(\beta)} \sum_{a \in J_{K,S}} N_K(a)^{-\beta} f(a\omega).
\]

(iv) the extremal ground states are indexed by \(Y_{L/K,S}^0\), with the state corresponding to \(\omega \in Y_{L/K,S}^0\) given by
\[
\phi_{\infty, \omega}(f) = f(\omega),
\]
and all ground states are KMS_∞-states.
\end{theorem}
Proof. This follows from combining the results of Sections 9.3, 9.4, 9.5 and 9.6.
Chapter 10

Partial Connes-Marcolli Systems and the Kronecker-Weber Theorem

As explained in Chapter 8, the partial Connes-Marcolli systems were constructed in order to create a dictionary between finite extensions of number fields on the one hand and Bost-Connes type systems on the other. In particular, when the base number field is \( \mathbb{Q} \), the Kronecker-Weber Theorem ascertains that the maximal abelian extension of the rationals is equal to the maximal cyclotomic extension of the rationals.

Recall that the maximal cyclotomic extension of the rationals is defined as the field of rationals adjoined with all roots of unity:

\[
\mathbb{Q}^{cyc} = \mathbb{Q}(\{\zeta_n; n \in \mathbb{N}\}).
\]

There are a few remarks that must be made about \( \mathbb{Q}(\zeta_n) \).

**Remark.**

1. The cyclotomic extension \( \mathbb{Q}(\zeta_n) \) corresponding to adjoining the primitive \( n^{th} \) root of unity \( \zeta_n \) is the splitting field of \( f(x) = x^n - 1 \) over \( \mathbb{Q} \).

2. \( \mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_n^1, \zeta_n^2, \ldots, \zeta_n^n) \) where \( \zeta_n^i \) are all \( n^{th} \) roots of unity corresponding to powers of a primitive \( n^{th} \) root of unity.

3. Any automorphism of \( \mathbb{Q}(\zeta_n) \) sends \( \zeta_n \) to some other primitive \( n^{th} \) root of unity. Hence, \( \mathbb{Q}(\zeta_n) \) is independent of choice of \( \zeta_n \). We shall denote \( \zeta_n \) as a concrete \( n^{th} \) root of unity chosen to generate \( \mathbb{Q}(\zeta_n) \).

4. The composite of two cyclotomic fields is a cyclotomic field. Moreover, \( \mathbb{Q}(\zeta_n)\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{[m,n]}) \) where \([m,n] = \text{lcm}(m,n)\). One can see this by realising that both \( \mathbb{Q}(\zeta_n) \) and \( \mathbb{Q}(\zeta_m) \) lie in \( \mathbb{Q}(\zeta_{[m,n]}) \), hence, \( \mathbb{Q}(\zeta_m)\mathbb{Q}(\zeta_n) \subset \mathbb{Q}(\zeta_{[m,n]}) \). Moreover, multiplying totally
coprime primitive $n^{th}$ and $m^{th}$ roots of unity yields a $[m, n]^{th}$ root of unity, hence $\mathbb{Q} (\zeta_{[m,n]}) \subset \mathbb{Q} (\zeta_{m}) \mathbb{Q} (\zeta_{n})$.

Combining these remarks, allows us to rewrite the maximal cyclotomic extension of the rationals as

$$\mathbb{Q}^{cycl} = \bigcup_{n} \mathbb{Q} (\zeta_{n}) = \lim_{\rightarrow} \mathbb{Q} (\zeta_{n}).$$

Note, that cyclotomic extensions are abelian because of the following remark.

**Remark.** Recall from Section 3.4 that $\text{Gal}(\mathbb{Q}(\zeta_{n})/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^{\ast}$, which is abelian simply because multiplication of integers is commutative.

Thus, because of the Kronecker-Weber Theorem, the maximal abelian extension of the rationals is the inductive limit of cyclotomic extensions of $\mathbb{Q}$.

Thus, in order to consider the partial Connes-Marcolli system for $\mathbb{Q}^{ab}/\mathbb{Q}$, which is the Connes-Marcolli system for $\mathbb{Q}$ and also the original Bost-Connes system, one can also attempt to consider the inductive limit of partial Connes-Marcolli systems corresponding to cyclotomic extensions of the rationals. In Section 10.1, the author will recall the definition of the partial Connes-Marcolli system and attempt to describe how an embedding of number fields can yield an embedding of partial Connes-Marcolli systems. Special consideration will be given to how this affects the partition function, the symmetries and the equilibrium states.

Section 10.2 will outline the effect of taking the inductive limit of appropriately chosen partial Connes-Marcolli systems. The analysis is then completed by discussing the relationship between the inductive limit of cyclotomic partial Connes-Marcolli systems and the Bost-Connes system in Section 10.3.

### 10.1 Partial Connes-Marcolli system for $\mathbb{Q}$

We begin by recalling the partial Connes-Marcolli system for an abelian extension $L/K$ and a finite set of primes $S$ of $K$.

Let $J_{K,S}$ be the free abelian group generated by primes in the complement of $S$, $\mathfrak{A}_{K,S,f} = \prod_{p \in S^{c}, p < \infty} K_p$ and $\hat{O}_{K,S} = \prod_{p \in S^{c}, p < \infty} \mathcal{O}_p$ are defined similarly to $\mathfrak{A}_{K}$ and $\hat{O}_{K}$ but restricted to primes in the complement of $S$. Note, $J_{K,S} \cong \mathfrak{A}_{K,S,f}/\hat{O}_{K,S}^{\ast}$. 

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Set $Y_{L/K,S}$ as the balanced product

$$Y_{L/K,S} = \text{Gal}(L/K) \times \hat{\mathcal{O}}_{K,S}^* \hat{\mathcal{O}}_{K,S}$$

where the action of $\hat{\mathcal{O}}_{K,S}^*$ on $Y_{L/K,S}$ is given by

$$g(x, y) = (x r_{L/K}(g)^{-1}, y)$$

for $g \in \hat{\mathcal{O}}_{K,S}^*, x \in \text{Gal}(L/K), y \in \hat{\mathcal{O}}_{K,S}$ and $r_{L/K} : \mathcal{A}_{K,f} \rightarrow \text{Gal}(L/K)$ is the Artin map.

The action defined above is a restriction of the action of $\mathcal{A}_{K,S,f}$ on $Y_{L/K,S}$ given by the same formula. This induces an action $\alpha$ by endomorphisms of $J_{K,S}^+$ of integral elements of $J_{K,S}$ on $Y_{L/K,S}$. Hence, we can consider the semigroup crossed product $C^*$-algebra

$$A_{L/K,S} = C^*(Y_{L/K,S}) \rtimes_{\alpha} J_{K,S}^+$$

The dynamics of this system are given by

$$\sigma_t(f) = f \text{ for } f \in C(Y_{L/K,S}) \text{ and } \sigma_t(\mu_j) = N(j)^t \mu_j \text{ for } \mu_j \in J_{K,S},$$

where $N(j) = |\mathcal{O}_K/j|$ is the norm of $j$.

Now, we consider this definition for $K = \mathbb{Q}$.

Consider the set of rational numbers $\mathbb{Q}$. The primes of $\mathbb{Q}$ correspond to the integer primes. For every prime $p$ denote by $\mathbb{Q}_p$ the corresponding $p$-adic completion of $\mathbb{Q}$, and by $\mathbb{Z}_p = \mathcal{O}_p \subset \mathbb{Q}_p$ its maximal compact subring for finite primes $p$, also known as the $p$-adic integers corresponding to the prime $p$. Denote by $J_{\mathbb{Q}}$ the group of fractional ideals of $\mathbb{Q}$, by $\mathcal{A}_{\mathbb{Q},f}$ the finite adele ring of $\mathbb{Q}$, and by $\mathcal{A}_{\mathbb{Z}}$ its maximal compact subring.

So $J_{\mathbb{Q}} \cong \mathcal{A}_{\mathbb{Q},f}^*/\mathcal{A}_{\mathbb{Z}}^*$, where $\mathcal{A}_{\mathbb{Q},f}$ is the restricted product $\prod_{p < \infty} \mathbb{Q}_p = \mathbb{Q} \otimes_{\mathbb{Z}} (\prod_p \mathbb{Z}_p)$ of the fields $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p \subset \mathbb{Q}_p$, and

$$\mathcal{A}_{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \mathcal{A}_{\mathbb{Q},f}.$$

Hence, for $\mathbb{Q}$, $J_{\mathbb{Q}} = \mathbb{Q}^*_0$ and $J_{\mathbb{Q}}^* = \mathbb{N}^*$.

For a finite set $S$ of primes in $\mathbb{Q}$ define $\mathcal{A}_{\mathbb{Q},S,f} = \prod_{p \notin S} \mathbb{Q}_p$ and $\mathcal{A}_{\mathbb{Z},S} = \prod_{p \notin S} \mathbb{Z}_p$. Let $L/\mathbb{Q}$ be an abelian extension, finite or infinite, and $S$ be a finite, possibly empty, set of primes in $\mathbb{Q}$. Consider the quotient space

$$X_{L/\mathbb{Q},S} = \text{Gal}(L/\mathbb{Q}) \rtimes \mathcal{A}_{\mathbb{Z},S} \mathcal{A}_{\mathbb{Q},S,f}.$$
Here the action of $\mathbb{A}_{Z,S}^*$ on $\text{Gal}(L/Q)$ is defined using the Artin map $r_{L/Q} : \mathbb{A}_{Q,f}^* \to \text{Gal}(L/Q)$. Identify $J_{Q,S}$ with $\mathbb{A}_{Q,S,f}^*/\mathbb{A}_{Z,S}^*$. Then the diagonal action of $\mathbb{A}_{Q,S,f}^*$ on $\text{Gal}(L/Q) \times \mathbb{A}_{Q,S,f}^*$, $g(x,y) = (x r_L/K(y)^{-1}, g y)$, defines an action of $J_{Q,S}$ on $X_{L/Q,S}$.

By restricting $X_{L/Q,S}$ in the second coordinate to its maximal compact subring of $\mathbb{A}_{Z,S}^*$, the result is $Y_{L/Q,S}$, which can be written as

$$Y_{L/Q,S} = \text{Gal}(L/Q) \times_{\mathbb{A}_{Z,S}^*} \mathbb{A}_{Z,S} \subset X_{L/Q,S}$$

Now, the partial Connes-Marcolli system is the C*-algebra completion of the group action groupoid arising from the action of $J_{Q,S}$ on $Y_{L/Q,S}$, in the way described in Chapter 5. As a semigroup crossed product it can be written as

$$\mathcal{A}_{L/Q,S} = C(Y_{L/Q,S}) \rtimes J_{Q,S}^*.$$ 

The action of $\text{Gal}(L/Q)$ by translations on itself defines an action of $\text{Gal}(L/Q)$ on $X_{L/Q,S}$, which in turn defines an action on $\mathcal{A}_{L/Q,S}$. Define a dynamics $\sigma$ on $\mathcal{A}_{L/Q,S}$. It is easier to explain its extension to the multiplier algebra of the whole crossed product $C_0(X_{L/Q,S}) \rtimes J_{Q,S}$, which will continue to be denoted by $\sigma$. We put $\sigma_t(f) = f$ for $f \in C_0(X_{L/Q,S})$ and $\sigma_t(u_n) = |n|^{it} u_n$ for $n \in J_{Q,S}$.

This system satisfies the modified Bost-Connes Problem:

**Modified Bost-Connes Problem.** $(\mathcal{A}_{L/Q,S}, \sigma_t)$ must exhibit the following properties.

1. The partition function is given by the Riemann zeta function $\zeta_S = \sum_{s \in Q, s \notin S} N(s)^{-\beta}$.
2. The finite abelian Galois group $\text{Gal}(L/Q)$ of $Q$ acts as symmetries on $\mathcal{A}_{L/Q,S}$.
3. For each inverse temperature $0 < \beta \leq 1$ there is a unique KMS$_\beta$-state.
4. For each $\beta > 1$ the action of the symmetry group $\text{Gal}(L/Q)$ on the set of extremal KMS$_\beta$-states is free and transitive.

When a system satisfies these four properties it is called an analytic Partial Connes-Marcolli system for $K$.

5. There exists a $Q$-rational subalgebra $\mathcal{A}^{arith}$ such that for every extremal KMS$_\infty$-state $\rho$ and every $f \in \mathcal{A}^{arith}$, we have $\rho(f) \in L$. 

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and further $L$ is generated over $Q$ by these values.

6. If we denote by $\nu_\rho$ the action of a symmetry $\nu \in \text{Gal}(L/Q)$ on an extremal $\text{KMS}_{\infty}$-state $\rho$ (given by pull-back), we have for every element $f \in A^{\text{arith}}$ the following compatibility relation

$$\nu_\rho(f) = \nu^{-1}(\rho(f)).$$

7. The $C^*$-algebra $A^{\text{arith}} \otimes_Q \mathbb{C}$ is dense in $A_{L/Q,S}$.

A system $A_{L/Q,S}$ which satisfies these last properties is called a Partial Connes-Marcolli system for $Q$ and $A^{\text{arith}}$ is called an arithmetic subalgebra.

Moreover, the $\text{KMS}_\beta$ states satisfy the classification theorem from Section 9.7.

**Theorem 10.1.1.** For the system $(A_{L/Q,S}, \sigma_t)$ we have:

(i) for $\beta < 0$ there are no $\text{KMS}_\beta$-states;

(ii) for every $0 < \beta \leq 1$ there is a unique $\text{KMS}_\beta$-state;

(iii) for every $1 < \beta < \infty$ extremal $\text{KMS}_\beta$-states are indexed by the points of the subset

$$Y_{L/Q,S}^0 = \text{Gal}(L/Q) \times_{\hat{O}_{Q,S}^*} \hat{O}_{Q,S}^\times \cong \text{Gal}(L/Q)$$

of $Y_{L/Q,S}$, with the state corresponding to $\omega \in Y_{L/Q,S}^0$ given by

$$\phi_{\beta,\omega}(f) = \frac{1}{\zeta_S(\beta)} \sum_{a \in \mathcal{J}_{Q,S}} |a|^{-\beta} f(a\omega).$$

(iv) the extremal ground states are indexed by $Y_{L/Q,S}^0$, with the state corresponding to $\omega \in Y_{L/Q,S}^0$ given by

$$\phi_{\infty,\omega}(f) = f(\omega),$$

and all ground states are $\text{KMS}_\infty$-states.

With so much known about the partial Connes-Marcolli systems for $K = Q$, it is worth considering whether or not there exists a homomorphism between partial Connes-Marcolli systems. In particular, one would need a homomorphism of $C^*$-algebras that intertwines with the dynamics of the systems. Moreover, it would be ideal for the homomorphism to induce a homomorphism between equilibrium states. This is a requirement for the inductive system that is being considered in Section 10.2. Having a homomorphism that preserves as much of the system as possible is a key ingredient in being able to induct a system of partial Connes-Marcolli systems to another partial Connes-Marcolli system. The author believes that in the formulation being considered here with a fixed base number field, there will exist a sufficiently all-encompassing homomorphism. The formulas for the equilibrium states are most promising in this regard.
10.2 Inductive limit of partial Connes-Marcolli systems

In Section 10.1, there was discussion about finding a C*-dynamical system homomorphism between two partial Connes-Marcolli systems. The reason such a homomorphism is worth investigating is because such a homomorphism can provide the ingredients to create an inductive system of partial Connes-Marcolli systems.

Recall Proposition 8.3.1 from the book by Richard Kadison and John Ringrose (Proposition 11.4.1 in [24]).

**Proposition 10.2.1.** Consider a direct system of C*-algebras. Namely, for a directed set \((I, \leq)\), consider the family of C*-algebras \(A_i\) indexed by \(I\) with homomorphisms \(f_{ij}: A_i \to A_j\) for all \(i \leq j\) such that

1. \(f_{ii}\) is the identity homomorphism on \(A_i\) for all \(i \in I\), and
2. \(f_{ik} = f_{jk} \circ f_{ij}\) for all \(i, j, k \in I\) and \(i \leq j \leq k\).

Then there exists a unique C*-algebra \(A_\infty\) which is the inductive limit of the directed system \((A_i, f_{ij})\) together with embeddings \(\alpha^x : A_x \to A_\infty\) such that

1. \(\alpha^x = \alpha^y \circ f_{xy}\) whenever \(x \leq y\), and
2. \(\bigcup_{x \in I} \alpha^x(A_x)\) is dense in \(A_\infty\).

The important point to take away from this proposition is that it guarantees the existence of such a C*-algebra. Remember also that if we are building up to the maximal abelian extension of the rationals, our base field is constantly \(\mathbb{Q}\), so the dynamics on the systems are defined identically. This means it is easier to consider C*-algebra homomorphisms between partial Connes-Marcolli systems that will also preserve other properties of the system such as the equilibrium states, dynamics or partition function.

One can think of a few ways of constructing an inductive system of partial Connes-Marcolli systems:

- inductive system of partial Connes-Marcolli systems of arbitrary finite abelian extensions of \(\mathbb{Q}\),
- inductive system of partial Connes-Marcolli systems of arbitrary cyclotomic extensions of \(\mathbb{Q}\),
- inductive system of partial Connes-Marcolli systems of cyclotomic extensions of \(\mathbb{Q}\) that are
in a chain where each one is contained in the next one,

- inductive system of partial Connes-Marcolli systems of abelian extensions of \( \mathbb{Q} \) where each next system is unramified in the previous one.

These four inductive systems are in descending order of complexity and in ascending order of likelihood that such an inductive system exists. Due to the deadline of this thesis, this induction has not been completed.

### 10.3 Isomorphism of Connes-Marcolli systems

Supposing one can set up an inductive system of partial Connes-Marcolli systems whose limit is also a partial Connes-Marcolli system, it would be advantageous to be able to compare this inductive limit with the original Bost-Connes system.

There is a theorem by Gunther Cornelissen and Matilde Marcolli in [13] about the equivalence of the isomorphism of number fields and isomorphism of quantum statistical mechanical systems. In particular,

**Theorem 10.3.1.** Let \( K \) and \( L \) denote arbitrary number fields. Then the following conditions are equivalent:

(i) **Field isomorphism** \( K \) and \( L \) are isomorphic as fields;
(ii) **QSM isomorphism** there is an isomorphism \( \phi \) of QSM systems \( (A_K, \sigma_K) \) and \( (A_L, \sigma_L) \)

that respects the dagger subalgebras:

\[
\phi(A_K^\dagger) = A_L^\dagger.
\]

Here the dagger subalgebra is defined as \( A_K^\dagger \) is the C*-algebra generated by functions in \( C(X_K) \) and partial isometries in \( J_K^+ \).

A little bit of further work is required. For example this theorem’s setting involved the Connes-Marcolli systems, not partial Connes-Marcolli systems. The statement has to be checked to see if it holds for the partial Connes-Marcolli system case.
10.4 Further work

The theory of Bost-Connes systems is rich and fascinating and every year, something new has been created and analysed. So, I believe there will always be potential avenues of research to look at. In particular, most importantly, further work includes taking this idea of taking an inductive system of partial Connes-Marcolli systems and inducting to get an inductive limit. There are numerous ways of attempting this as explained in Section 10.2. For example, one could try an inductive system of partial Connes-Marcolli systems corresponding to finite abelian extensions. Instead of finite abelian extensions, this could be replaced with cyclotomic extensions to try to show that the inductive limit of the cyclotomic partial Connes-Marcolli systems generates the Bost-Connes system. However, there are other possibilities. One could choose cyclotomic extensions that are consecutively extensions of one another. Moreover, through the use of the set $S$, one could consider unramified extensions of $\mathbb{Q}$. The author thinks these ideas are promising and regrets that there was not enough time available to get this done before the submission deadline.

Of course, the next major further work that is planned is to consider this same idea for an arbitrary number field rather than for just $\mathbb{Q}$. Based on the difficulty of Hilbert’s 12th Problem, the author believes that a straight induction is unlikely to work for other number fields. It is well known that the maximal abelian extension of negative imaginary quadratic extensions requires adjoining roots of unity and values of the Weierstrass $j$-function. Perhaps use of the set $S$ and being able to choose which primes are ignored can get past the hurdle that other formulas suffer from. They have the system written in terms of $\text{Gal}(K^{ab}/K)$ without the possibility of it being broken up corresponding to what elements generate which part of the maximal abelian extension.

The third bit of further work being considered is further research into another formulation being used nowadays which is in terms of endomotives. This was ignored in favour of groupoid $C^*$-algebras and semigroup crossed product formulations.
Bibliography


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