Noncommutative Geometry Methods in Number Theory

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Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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Abstract

Noncommutative geometry deals with many natural spaces for which the classical set-theoretic tools of analysis, such as measure theory, topology, calculus, and metric ideas lose their pertinence, but which correspond very naturally to a noncommutative algebra. While looking for new ways to get information about the Riemann zeta function (as an essential ingredient of the Riemann Hypothesis), Alain Connes and Benoît Bost published a paper [5] in 1995 discussing the use of certain quantum statistical mechanical systems (noncommutative geometric objects) called Bost-Connes systems to recover the class field theory for number fields. The Bost-Connes system surprisingly relates to Galois theory and the maximal abelian extensions of number fields. This has led to speculation that it may have use in trying to get a deeper understanding of abelian extensions of number fields and thus has been seen as one approach to solving Hilbert's 12^{th} problem, which aims to generalise an incredible result called the Kronecker-Weber Theorem, which asserts that the maximal abelian extension of the rationals, $\mathbb{Q}^{ab}$, is generated by the roots of unity. Hilbet's 12^{th} problem asks, for any given number field K, to describe the generators of the maximal abelian extension K^{ab} of K in terms of algebraic values of a transcendental function of K. While there has been progress in this area using modern number theoretic methods, it remains unsolved.

This thesis is predominantly an exposition of the research that has been done in this very young and exciting area. In particular, the Bost-Connes system will be reformulated as an enveloping C*-algebra of commensurability classes of $\mathbb{Q}$-lattices, as a groupoid C*-dynamical system and as a semigroup crossed product system.

In the thesis a new Bost-Connes type system is constructed, called the partial Connes-Marcolli system and it generalises the systems that are currently studied. It also satisfies a generalised Bost-Connes Problem statement. At the end of the thesis there is also a discussion on how these systems could be used to try to deduce the Kronecker-Weber Theorem from purely noncommutative geometry objects.