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RESEARCH ARTICLE

# Functions of Multivector Variables

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## Abstract

As is well known, the common elementary functions defined over the real numbers can be generalized to act not only over the complex number field but also over the skew (non-commuting) field of the quaternions. In this paper, we detail a number of elementary functions extended to act over the skew field of Clifford multivectors, in both two and three dimensions. Complex numbers, quaternions and Cartesian vectors can be described by the various components within a Clifford multivector and from our results we are able to demonstrate new inter-relationships between these algebraic systems. One key relationship that we discover is that a complex number raised to a vector power produces a quaternion thus combining these systems within a single equation. We also find a single formula that produces the square root, amplitude and inverse of a multivector over one, two and three dimensions. Finally, comparing the functions over different dimension we observe that  $Cl(\mathbb{R}^3)$  provides a particularly versatile algebraic framework.



## OPEN ACCESS

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## Introduction

Clifford algebras are associative non-commutative algebras developed by William K. Clifford around 1878 building on the exterior algebras developed earlier by Hermann Grassmann. Specifically, denoting  $\wedge \mathbb{R}^n$  as the exterior algebra of  $\mathbb{R}^n$  then we produce the space of multivectors  $\mathbb{R} \oplus \mathbb{R}^n \oplus \dots \oplus \wedge^n \mathbb{R}^n$  denoted by  $Cl(\mathbb{R}^n)$  with unity 1. These algebras can be either simple, hence isomorphic to matrix algebras over the reals, complex numbers, or quaternions, or semi-simple, and hence isomorphic to the direct sum of two matrix algebras over the reals or quaternions [1–5].

In this paper, we firstly describe some general results applicable in  $Cl(\mathbb{R}^n)$ , before exploring the elementary functions based on multivectors in two and three dimensions, which then finally allows us to identify several unifying relationships. Clifford multivectors form a generalization of the elementary functions over complex and quaternionic numbers [6] that can be recovered as special cases.

## Analysis

### General results for multivectors in $Cl(\mathfrak{R}^n)$

Within  $Cl(\mathfrak{R}^n)$  we form a multivector  $\mathfrak{R} \oplus \mathfrak{R}^2 \oplus \dots \oplus \wedge^n \mathfrak{R}^n$  that we can write as

$$M = A_0 + A_1 + A_2 + A_3 + \dots + A_n, \tag{1}$$

where  $A_0 \in \mathfrak{R}, A_1 \in \mathfrak{R}^2, A_2 \in \wedge^2 \mathfrak{R}^n, \dots, A_n \in \wedge^n \mathfrak{R}^n$ . The following definitions for the general case of multivectors over  $Cl(\mathfrak{R}^n)$  are essentially as found in [5].

**Definition.** (Grade selection) We define the grade selection operation  $\langle M \rangle_k = A_k \in \wedge^k \mathfrak{R}^n$ . The number of elements in each grade  $A_k$  follows the Pascal triangle relation  $\frac{n!}{k!(n-k)!}$  with the  $n+1$  grades forming a  $2^n$ -dimensional real vector space.

**Definition.** (Orthonormal basis) For a set of basis elements  $\{e_k: 1 \leq k \leq n\}$  for  $\mathfrak{R}^n$ , we define the properties

$$e_k e_k = 1, e_j e_k = -e_k e_j, j, k, 1 \leq j, k \leq n. \tag{2}$$

These  $n$  elements generate a basis of  $2^n$  elements for  $Cl(\mathfrak{R}^n)$  with the highest grade element being the pseudoscalar  $e_1 e_2 \dots e_n$ .

For example, in  $Cl(\mathfrak{R}^3)$  we have the basis elements  $e_1, e_2, e_3$  forming a multivector  $M = A_0 + A_1 + A_2 + A_3$  with  $A_0 = a_0, A_1 = a_1 e_1 + a_2 e_2 + a_3 e_3, A_2 = a_4 e_1 e_2 + a_5 e_3 e_1 + a_6 e_2 e_3$  and  $A_3 = a_7 e_1 e_2 e_3$ , where  $a_0, \dots, a_7 \in \mathfrak{R}$ . In order to abbreviate notation we often write  $e_{12} \equiv e_1 e_2$  and  $e_{123} \equiv e_1 e_2 e_3$  etc.

**Definition.** (Multivector involutions) We define three involutions on a multivector  $M$ : firstly **space inversion** written as  $M^*$  defined by  $e_k \rightarrow -e_k$ , secondly **reversion** written as  $M^\dagger$  that reverses the order of all products,  $e_1 e_2 \dots e_n \rightarrow e_n e_{n-1} \dots e_1$  and thirdly a composition of the first two that forms **Clifford conjugation** written as  $\bar{M} = M^*$ . This produces a variation in signs over the different grades as follows

$$\begin{aligned} M^* &= A_0 - A_1 + A_2 - A_3 + A_4 - A_5 + A_6 - A_7 + A_8 \dots + (-)^n A_n \\ M^\dagger &= A_0 + A_1 - A_2 - A_3 + A_4 + A_5 - A_6 - A_7 + A_8 + \dots + (-)^{\lfloor n/2 \rfloor} A_n \\ \bar{M} &= A_0 - A_1 - A_2 + A_3 + A_4 - A_5 - A_6 + A_7 + A_8 + \dots + (-)^{n+\lfloor n/2 \rfloor} A_n. \end{aligned} \tag{3}$$

Addition and subtraction of multivectors involves adding and subtracting the corresponding terms of the algebra and multiplication is through the formal application of the law of the distribution of multiplication over addition, that is explicated in the sections on two and three dimensional multivectors to follow. We find that **reversion** and **Clifford conjugation** are anti-automorphisms producing  $(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger$  and  $M_1 \bar{M}_2 = \bar{M}_2 \bar{M}_1$  whereas **space inversion**  $(M_1 M_2)^* = M_1^* M_2^*$  is an automorphism.

Note, that using the reversion involution, calculating the corresponding grades in  $MM^\dagger$  we find that all products are of the form  $e_1 e_2 \dots e_n e_n e_{n-1} \dots e_1 = +1$ . Hence we can use the reversion involution to form a positive definite scalar  $\langle MM^\dagger \rangle_0$ . This leads us to define an inner product for multivectors.

**Definition.** (Inner product) We define for two multivectors  $M_1$  and  $M_2$  the product

$$\langle M_1 M_2^\dagger \rangle_0 = \langle M_2 M_1^\dagger \rangle_0, \tag{4}$$

which can be shown to have the required properties for an inner product. This induces a norm on

a multivector

$$||M||^2 = \langle MM^\dagger \rangle_0, \tag{5}$$

which is positive definite as required. Conventional results now follow, such as a triangle inequality for multivectors.

**Definition.** (Square root) A square root of a multivector  $Y$  is a multivector  $M$  such that  $Y = M^2$  and we write  $M = Y^{1/2}$ .

We reserve the square root symbol  $\sqrt{\phantom{x}}$  to act over the reals and complex-like numbers, with its conventional definition, producing a value within the complex-like numbers. We define a complex-like number as numbers of the form  $a + Ib$ , where  $a, b \in \mathfrak{R}$  and  $I$  is any algebraic quantity that squares to minus one. For example, we will find that the bivectors and trivectors will square to minus one. Naturally  $I$  will commute with  $a$  and  $b$  and so we therefore have an isomorphism with conventional complex numbers and so we have available the results from complex number theory in these cases.

**Definition.** (Multivector amplitude) We define the amplitude of a multivector  $M$  as

$$|M| = (M\bar{M})^{\frac{1}{2}}. \tag{6}$$

Note that in the general case for multivectors in  $Cl(\mathfrak{R}^n)$ ,  $M\bar{M}$  may produce a multivector of various grades and so the square root may not exist in all cases. However, for multivectors of grade less than or equal to three, which is the case primarily dealt with in this paper, we will find that  $M\bar{M}$  always produces a complex-like number, and so we are then entitled to use the square root symbol that we reserved for this case. That is, we can write

$$|M| = \sqrt{M\bar{M}}, \text{ for } n \leq 3. \tag{7}$$

Note that the amplitude in these cases is in general also a complex-like number.

**Definition.** (Multivector amplitude) We define the amplitude of a multivector  $M$  as

$$|M| = \sqrt{M\bar{M}}. \tag{8}$$

Note that  $M\bar{M}$  is not positive definite and does not have a value in the reals in general and hence the amplitude may not exist in all cases.

**Definition.** (Multivector exponential) The exponential of a multivector is defined by constructing the Taylor series

$$e^M = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots, \tag{9}$$

which is absolutely convergent for all multivectors [5].

Convergence is easily demonstrated because  $\|M^n\| < \|M\|^n$ . The infinite sequence  $\{M_n\}$  of multivectors  $M_1, M_2, M_3, \dots, M_n, \dots$  approaches the multivector  $L$  as a limit, that is  $M_n \rightarrow L$ , if  $\|L - M_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Definition.** (Logarithm) The logarithm of a multivector is defined as the inverse of the exponential function. For a given multivector  $Y$  we find  $M$ , such that  $Y = e^M$  and we write  $M = \log Y$ , which is multivalued in general. Hence we have  $e^{\log Y} = Y$ . The principal value of the logarithm can be defined as the multivector  $M = \log Y$  with the smallest norm.

In even dimensional spaces  $Cl(\mathfrak{R}^{2n})$  the pseudoscalar is non-commuting with some components of the algebra, whereas in odd dimension the pseudoscalar is commuting with all elements. Additionally spaces of dimension 2,3,6,7,10,11, . . . have a pseudoscalar that squares to minus one whereas 4,5,8,9,12,13, . . . the pseudoscalar squares to plus one. Hence spaces that

have a commuting pseudoscalar that squares to minus one lie in spaces of dimension  $3, 7, 11, 15, \dots, 4n-1, \dots$ , where  $n \in \mathbb{N}$ . As we can see, in general, these pseudoscalar properties have period four.

**Definition.** (Hyperbolic trigonometric functions) *Splitting the exponential series, as shown in Eq. (9), into odd and even terms we define the hyperbolic trigonometric functions*

$$\begin{aligned} \cosh M &= \sum_{n=0}^{\infty} \frac{M^{2n}}{(2n)!} = \frac{1}{2}(e^M + e^{-M}), \\ \sinh M &= \sum_{n=0}^{\infty} \frac{M^{2n+1}}{(2n+1)!} = \frac{1}{2}(e^M - e^{-M}). \end{aligned} \tag{10}$$

The exponential form immediately implies  $e^M = \cosh M + \sinh M$  and we can then easily confirm the usual results that  $\sinh 2M = 2\sinh M \cosh M$  and  $\cosh^2 M - \sinh^2 M = 1$ .

**Definition.** (Trigonometric functions) *We define the trigonometric functions with the alternating series*

$$\cos M = \sum_{n=0}^{\infty} \frac{(-)^n M^{2n}}{(2n)!}, \sin M = \sum_{n=0}^{\infty} \frac{(-)^n M^{2n+1}}{(2n+1)!}. \tag{11}$$

This definition then implies  $\cos^2 M + \sin^2 M = 1$ .

We can write the trigonometric functions in an exponential form, such as  $\cos M = \frac{1}{2}(e^{jM} + e^{-jM})$  for example, provided we have a commuting pseudoscalar with  $J^2 = -1$ . This is only true though in spaces of dimension  $3, 7, 11, \dots$ , as previously discussed.

For the multivector finite series  $S_n = 1 + M + M^2 + \dots + M^n$  we find  $MS_n = M + M^2 + \dots + M^{n+1}$  and so  $S_n - MS_n = (1-M)S_n = 1 - M^{n+1}$ . Multiplying on the left with the inverse of  $(1-M)$  we find for the sum

$$S_n = (1 - M)^{-1}(1 - M^{n+1}), \tag{12}$$

provided the inverse exists.

### Clifford's geometric algebra of two dimensions

Within Clifford's geometric algebra  $Cl(\mathfrak{R}^2)$ , we form a multivector  $M \in \mathfrak{R} \oplus \mathfrak{R}^2 \oplus \wedge^2 \mathfrak{R}^2$  that can be expressed in terms of an orthonormal basis as

$$M = a + xe_1 + ye_2 + be_{12}, \tag{13}$$

where  $a, x, y, b$  are real scalars and the bivector defined as  $e_{12} = e_1e_2$ . We then find for the bivector that  $e_{12}^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1$ .

We note that the space of multivectors in  $Cl(\mathfrak{R}^2)$  is isomorphic to the matrix algebra  $Cl(\mathfrak{R}^2) \cong \text{Mat}(2, \mathfrak{R})$ . We also note that the subalgebra of  $Cl_2$  spanned by 1 and  $e_{12}$ , consisting of scalar and bivector components forming the even subalgebra, with  $e_{12}$  taking the role of the unit imaginary, is isomorphic to  $\mathbb{C}$ . Hence the even subalgebra in two dimensions, given by  $a + be_{12}$ , is isomorphic to the complex field, and so we can assume the results from complex number theory when the multivector lies within this restricted domain. For example, the log of a multivector in the even subalgebra  $\log(a + e_{12}b) = \log\sqrt{a^2 + b^2} + \theta e_{12}$ , with the multivalued  $\theta = \arctan(b/a)$ , as found in complex number theory. In addition to the even subalgebra representing the complex numbers, we also have the subalgebra  $a + xe_1$  forming the one-dimensional Clifford algebra  $Cl(\mathfrak{R}^1)$ .

The sum or difference of two multivector numbers  $M_1 = x_1e_1 + y_1e_2 + b_1e_{12}$  and  $M_2 = a_2 + x_2e_1 + y_2e_2 + b_2e_{12}$  is defined by

$$M_3 = M_1 \pm M_2 = a_1 \pm a_2 + (x_1 \pm x_2)e_1 + (y_1 \pm y_2)e_2 + (b_1 \pm b_2)e_{12}. \tag{14}$$

The product  $M_3$  of multivectors  $M_1$  and  $M_2$  is found through the formal application of the distributive law of multiplication over addition

$$M_3 = M_1M_2 = a_1a_2 + x_1x_2 + y_1y_2 - b_1b_2 + (a_1x_2 + a_2x_1 + b_1y_2 - y_1b_2)e_1 + (a_1y_2 + y_1a_2 + x_1b_2 - b_1x_2)e_2 + (a_1b_2 + b_1a_2 + x_1y_2 - y_1x_2)e_{12}. \tag{15}$$

In two dimensions the *conjugation* involution produces

$$\bar{M} = a - xe_1 - ye_2 - be_{12}. \tag{16}$$

In terms of multiplication and additions we can write

$\bar{M} = -\frac{1}{2}(M - e_1Me_1 - e_2Me_2 + e_{12}Me_{12})$ . We then have the scalar part of a multivector  $\langle M \rangle_0 = \frac{1}{2}(M + \bar{M})$  and the sum of vector and bivector components  $\langle M \rangle_1 + \langle M \rangle_2 = \frac{1}{2}(M - \bar{M})$ . If required, we can also isolate the vector components of  $M$  as  $\langle M \rangle_1 = \frac{1}{2}(M + e_{12}Me_{12}) = v_1e_1 + v_2e_2$ . Using Clifford conjugation we then find

$$\begin{aligned} M\bar{M} &= \bar{M}M \\ &= (a + xe_1 + ye_2 + be_{12})(a - xe_1 - ye_2 - be_{12}) \\ &= a^2 - x^2 - y^2 + b^2, \end{aligned} \tag{17}$$

producing a real number, though not necessarily non-negative.

**Definition.** (Negative square root) *We define the principal square root of negative numbers in two dimensions as follows: given a real number  $a \in \Re \geq$  we define*

$$\sqrt{-a} = e_{12}\sqrt{a}, \tag{18}$$

using the property that the bivector squares to minus one.

The *amplitude* of a multivector in two dimensions becomes

$$|M| = \sqrt{M\bar{M}} = \sqrt{a^2 - x^2 - y^2 + b^2}. \tag{19}$$

Note that that the special case of  $x = y = 0$  produces the magnitude of a complex-like number.

The *reversion* involution on the multivector  $M$  in two dimensions produces

$$M^\dagger = a + xe_1 + ye_2 - be_{12}, \tag{20}$$

which we can also write algebraically as  $M^\dagger = \frac{1}{2}(M + e_1Me_1 + e_2Me_2 + e_{12}Me_{12})$ . From [Eq. \(5\)](#), we then find the norm of a multivector in two dimensions

$$\|M\| = \langle MM^\dagger \rangle_0^{1/2} = \langle a^2 + x^2 + y^2 + b^2 + 2a(xe_1 + ye_2) \rangle_0^{1/2} = \sqrt{a^2 + x^2 + y^2 + b^2}. \tag{21}$$

Also, this definition of the product and the definition of the amplitude in [Eq. \(17\)](#) produces the homomorphism

$$|M_1M_2|^2 = M_1M_2\bar{M}_1\bar{M}_2 = M_1M_2\bar{M}_2\bar{M}_1 = |M_1|^2|M_2|^2. \tag{22}$$

Expanding this expression in full we have

$$\begin{aligned}
 & (a_1^2 - a_2^2 - a_3^2 + a_4^2)(b_1^2 - b_2^2 - b_3^2 + b_4^2) \\
 &= (a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4)^2 - (a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3)^2 \\
 & \quad - (a_1b_3 + a_2b_4 + a_3b_1 - a_4b_2)^2 + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)^2
 \end{aligned} \tag{23}$$

and so is a variation of Euler’s four-square identity. It should be noted that  $|M_1|^2|M_2|^2 = \pm(|M_1||M_2|)^2$  and taking roots we find that  $|M_1M_2| = \pm |M_1| |M_2|$ .

Also, from Eq. (17) we can see that because  $|M|^2 = M\bar{M}$  is a real scalar, we can define the inverse multivector as

$$M^{-1} = \bar{M}/|M|^2. \tag{24}$$

This gives  $MM^{-1} = M\bar{M}/|M|^2 = |M|^2/|M|^2 = 1$  as required. This now allows us to define the division operation  $M_1/M_2 = M_1M_2^{-1}$ . Clearly, a multivector fails to have an inverse if  $M\bar{M} = a^2 - x^2 - y^2 + b^2 = 0$  and so fails to form a division algebra in these cases. This expression for the inverse is analogous to the formula for the inverse of a complex number  $z^{-1} = \bar{z}/|z|^2$ , that can be recovered as a special case from Eq. (24) for  $M$  in the even subalgebra.

Now, for more complex manipulations to follow it is preferable to write the general multivector as

$$M = a + \mathbf{v} + ib, \tag{25}$$

where  $\mathbf{v} = xe_1 + ye_2$  defines a vector, with the bivector  $i = e_1e_2$ . We also define  $F = \mathbf{v} + ib$  so that we can write  $M = a + F$ . We have used the symbol  $i$  for the pseudoscalar that is also commonly used for the unit imaginary  $\sqrt{-1}$ . This notation is adopted because complex numbers also lie in a two-dimensional space analogous to the even subalgebra of the two-dimensional multivector. Note that the pseudoscalar  $i$  is non-commuting with the vector component  $\mathbf{v}$  of the multivector. In general, the pseudoscalar refers to the highest dimensional element of the algebra, which is of dimension  $n$  for a Clifford algebra  $Cl(\mathbb{R}^n)$ . We have the important result that  $\mathbf{v}^2 = (xe^1 + ye^2)(xe^1 + ye^2) = x^2 + y^2$  and so a real scalar giving the Pythagorean length. Hence, using this notation, the condition for a multivector inverse to exist is given by  $a^2 + b^2 \neq \mathbf{v}^2$ .

**The square root.** The square roots of a multivector in  $Cl(\mathbb{R}^2)$  are given by

$$M^{\frac{1}{2}} = \frac{1}{\sqrt{2(a \pm |M|)}} (M \pm |M|) = \frac{1}{\sqrt{M + \bar{M} \pm 2|M|}} (M \pm |M|), \tag{26}$$

with the conditions that  $|M|$  is real for the square root using the plus sign, with an extra condition that  $a > |M|$  when selecting the negative sign in order ensure a positive argument for the square root function in the denominator. We require these conditions because of the non-commuting pseudoscalar that will be generated from the root of a negative number. Proof: Given a multivector  $S = c + \mathbf{w} + id$  we find  $S^2 = c^2 + \mathbf{w}^2 - d^2 + 2c(\mathbf{w} + id)$ . Hence, provided  $c \neq 0$  implying vector or bivector components are present in  $MM$ , the root of a multivector  $M = a + \mathbf{v} + ib$  must be of the form  $S = c + \frac{1}{2c}(\mathbf{v} + ib) = \frac{1}{2c}(2c^2 + \mathbf{v} + ib) = \frac{1}{2c}(2c^2 - a + M)$ . It just remains now to find  $c$ . The scalar component of the equation  $S^2 = M$  gives us  $c^2 = \frac{1}{2}(a \pm |M|)$ . Substituting this expression we find  $S = \frac{1}{\sqrt{2(a \pm |M|)}} (M \pm |M|)$  as required. For the case where  $c = 0$ , returning to the first line of the proof we see that this implies that  $M$  is just a real number and provided we choose the positive sign Eq. (26) produces  $\sqrt{a}$  as required. However, because with  $c = 0$ ,  $S^2 = \mathbf{w}^2 - d^2$  a scalar, and so we can see that we now have available a new set of roots. If

we are seeking the square root of a negative real- $a$ , where  $a \geq 0$ , then we have the equation  $\mathbf{w}^2 - d^2 = -a$ , and solving for  $d$ , we find the root

$$(-a)^{\frac{1}{2}} = \mathbf{w} \pm i\sqrt{a + \mathbf{w}^2}, \tag{27}$$

which is satisfied for all vectors  $\mathbf{w} = w_1e_1 + w_2e_2$ . The special case with  $\mathbf{w} = 0$  produces the principal root defined earlier in Eq. (18). Additionally for the case  $a = 1$ , we now produce  $(-1)^{1/2}$  and we therefore find an infinite number of possible roots. The possible roots of minus one in Clifford multivectors has been further investigated elsewhere as in [8]. For the roots of positive reals it is preferable to solve instead for the vector length giving  $a^{\frac{1}{2}} = \pm\sqrt{a + d^2} \hat{\mathbf{v}} + id$ . This last expression shows the need to distinguish the square root of reals given by  $\sqrt{a}$  and the more general square roots over the domain of multivectors shown as  $a^{\frac{1}{2}}$  in order to avoid circular definitions. The principal values though will correspond with each other.

As general comments, inspecting Eq. (26) we can see that it can produce two distinct roots, each of which though can also be negative, so therefore in general produces four possible square roots. The last version on the right has the advantage of being expressed in  $M$  alone and not in components.

From Eq. (26), for the special case of a multivector in the form of a complex-like number  $z = a + ib$  we have

$$z^{\frac{1}{2}} = \frac{z \pm |z|}{\sqrt{z + \bar{z} \pm 2|z|}}, \tag{28}$$

which agrees with results from complex number theory.

**Trigonometric form of a multivector. Definition.** (Multivector argument) *We define the argument of a multivector  $M = a + \mathbf{v} + ib = a + F$  as*

$$\arg M = \arctan\left(\frac{|F|}{a}\right), \tag{29}$$

for  $a \neq 0$ , where  $|F| = \sqrt{b^2 - \mathbf{v}^2}$ . For  $a = 0$  we have  $\arg M = \pi/2$ . The function is multivalued modulo  $2\pi$  and also depends on which quadrant the point  $(a, |F|)$  is in. We define the principal value of the argument  $-\pi < \phi \leq \pi$ .

**Theorem** (Trigonometric form) *A two dimensional multivector can be written in the form*

$$M = a + \mathbf{v} + ib = \left(\cos \phi + \frac{\mathbf{v} + ib}{\sqrt{b^2 - \mathbf{v}^2}} \sin \phi\right) |M| = (\cos \phi + \hat{F} \sin \phi) |M|, \tag{30}$$

provided  $|M|, |F| \neq 0$ , where  $\phi = \arg M$ , and we have defined  $\hat{F} = F/|F|$ .

*Proof.* Assuming  $b^2 > \mathbf{v}^2$  and  $|M| \neq 0$ , we have  $\cos \phi = a/|M|$  and  $\sin \phi = \sqrt{b^2 - \mathbf{v}^2}/|M|$ . Substituting we find  $M = a + \mathbf{v} + ib$  as required. Alternatively if  $b^2 < \mathbf{v}^2$  then  $\sqrt{b^2 - \mathbf{v}^2}$  becomes a bivector but because it will cancel with the same term in  $\sin \phi$  the multivector will be returned. Likewise if  $|M|$  is a bivector, then this will also cancel with  $|M|$  in the  $\sin \phi$  and  $\cos \phi$  terms. Hence Eq. (30) applies provided  $|M| \neq 0$  and  $b^2 \neq \mathbf{v}^2$ , as required.

The order of the factors in Eq. (30) is important because  $\phi$  and  $|M|$  can lie in the even sub-algebra and so will not necessarily commute with  $\mathbf{v} + ib$ , in general. Notationally, it is also important to note that we define  $\hat{F} = \frac{\mathbf{v} + ib}{\sqrt{b^2 - \mathbf{v}^2}} \equiv (\mathbf{v} + ib)/\sqrt{b^2 - \mathbf{v}^2}$ , where the denominator always follows the numerator, due to commutativity issues. Note, it turns out that we can rearrange the factors to produce an equivalent form  $M = |M| \left(\cos \phi + \sin \phi (b^2 - \mathbf{v}^2)^{-\frac{1}{2}} (\mathbf{v} + ib)\right)$ .

Now, assuming the trigonometric form in Eq. (30) exists, we find for integer powers  $p$  that

$$M^p = (\cos \phi + \hat{F} \sin \phi)^p |M|^p = (\cos p\phi + \hat{F} \sin p\phi) |M|^p, \tag{31}$$

a generalization of de Moivre’s theorem for multivectors, valid for  $|F|, |M| \neq 0$ .

Now, because multivector multiplication is associative we can find the rational powers  $M^{\frac{p}{q}}$ , where  $p, q$  are integers. We will now see how this relation can be written in polar form using the exponential map, which will allow us to calculate more general multivector powers using logarithms.

**Exponential map of a multivector.** Given a two-dimensional multivector  $a + \mathbf{v} + ib = a + F$ , we find  $F^2 = (\mathbf{v} + ib)^2 = \mathbf{v}^2 - b^2$  and so  $|F|^2 = F\bar{F} = -F^2$ . Hence, given the exponential map in Eq. (9), we find

$$\begin{aligned} e^{a+\mathbf{v}+ib} &= e^a e^{\mathbf{v}+ib} = e^a e^F \\ &= e^a \left( 1 + F - \frac{|F|^2}{2!} - \frac{F|F|^2}{3!} + \frac{|F|^4}{4!} + \dots \right). \end{aligned} \tag{32}$$

If  $|F| = 0$ , then referring to the last line of the derivation above, we see that all terms following  $F$  will be zero, and so, in this case  $e^{a+\mathbf{v}+ib} = e^a(1+\mathbf{v}+ib)$ . Now, assuming the power series definitions given in Eq. (11) for the trigonometric functions, we can then find the closed form

$$e^{a+\mathbf{v}+ib} = e^a (\cos|F| + \hat{F} \sin|F|), \tag{33}$$

a result that remains valid even if  $|F| = \sqrt{b^2 - \mathbf{v}^2}$  is a bivector, because as we know from complex number theory the trigonometric functions will simply become hyperbolic trigonometric functions.

We can thus rearrange this result, to write a multivector in polar form as

$$M = a + \mathbf{v} + ib = |M| e^{\hat{F}\phi} = |M| e^{(\mathbf{v}+ib)/|F|\phi}, \tag{34}$$

where  $\phi = \arg M$ . We find that an exponential form is only possible if  $|M|$  is real, even though the trigonometric form, shown previously in Eq. (30), is valid generally. This is because  $|e^{a+\mathbf{v}+ib}| = (e^{a+\mathbf{v}+ib} e^{a-\mathbf{v}-ib})^{1/2} = e^a$ , a result that is always real, whereas in general  $|a + \mathbf{v} + ib|$  can become a bivector. This also explains why the square root fails to exist in these cases. Eq. (34) is a generalization of the exponential form for complex numbers. That is, for  $\mathbf{v} = 0$ , we have  $M = a + ib = |M| e^{\hat{F}\phi} = |M| e^{i\phi} = \sqrt{a^2 + b^2} (\cos\phi + i\sin\phi)$ , where  $i$  is the bivector, reducing to the polar form of a complex number.

Hence the logarithm of a multivector  $M$  becomes

$$\log M = \log|M| + \hat{F}\phi, \tag{35}$$

$\phi = \arg M$ . The logarithm multivaluedness coming from the argument function.

We can now also define the multivector power  $M^P = e^{\log(M)P}$ , where  $P$  is a also general multivector and, due to non-commutativity, alternatively as  $e^{P\log(M)}$ .

**Trigonometric functions of a multivector.** In two dimensions, the expressions for the hyperbolic trigonometric functions given in Eq. (10) can be simplified to give

$$\begin{aligned} \cosh M &= \frac{1}{2} (e^{a+F} + e^{-a-F}) = \cos|F| \cosh a + \hat{F} \sin|F| \sinh a \\ \sinh M &= \frac{1}{2} (e^{a+F} - e^{-a-F}) = \cos|F| \sinh a + \hat{F} \sin|F| \cosh a. \end{aligned} \tag{36}$$

**Table 1. Multivector functions in two dimensions**  $M \in Cl(\mathfrak{R}^2)$ .

Main results	Notes ( $i = e_{12}$ non-commuting)
$M = a + \mathbf{v} + ib = a + F$	Define $\mathbf{v} \in \mathfrak{R}^2, a, b \in \mathfrak{R}, F = \mathbf{v} + ib$
$\bar{M} = a - \mathbf{v} - ib = a - F$	Conjugation
$ M  = \sqrt{M\bar{M}} = \sqrt{a^2 - \mathbf{v}^2 + b^2}$	Amplitude
$M^{-1} = \bar{M}/(M\bar{M}):  M  \neq 0$	Inverse
$\phi = \arg M = \arctan\left(\frac{ F }{a}\right)$	Argument, $ F  = \sqrt{b^2 - \mathbf{v}^2}$
$M = (\cos \phi + \hat{F} \sin \phi) M :  M ,  F  \neq 0$	Trigonometric form, $\hat{F} = F/ F $
$M^p = (\cos p\phi + \hat{F} \sin p\phi) M ^p$	Integer powers, $p \in \mathbb{N}$
$e^M = e^a(\cos F  + \hat{F}\sin F )$	Exponential, $e^M \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!}$
	If $ F  = 0$ then $e^M = e^a(1 + F)$
$M^{\frac{1}{2}} = \frac{M \pm  M }{\sqrt{M + M \pm 2 M }}:  M  \in \mathfrak{R}$	Square root
$M =  M e^{\hat{F}\phi}:  M  \in \mathfrak{R} \neq 0$	Polar form, $\hat{F}^2 = -1$
$M^x =  M ^x(\cos x\phi + \hat{F}\sin x\phi):  M  \in \mathfrak{R}$	Real powers $x \in \mathfrak{R}$
$\log_e M = \log_e  M  + \hat{F}\phi$	Logarithm
$M^p = e^{\log(M)P}$ or $e^{P\log(M)}$	General powers
Hyperbolic trigonometric functions	
$\cosh M = \cos F \cosh a + \hat{F}\sin F \sinh a$	$M = a + F = a + \hat{F} F $ $e^M = \cosh M + \sinh M$
$\sinh M = \cos F \sinh a + \hat{F}\sin F \cosh a$	$\cosh^2 M - \sinh^2 M = 1$
$\operatorname{arcsinh} M = \log\left(M + (1 + M^2)^{\frac{1}{2}}\right)$	Inverse hyperbolic sin
$\operatorname{arccosh} M = \log(M + (M^2 - 1)^{1/2})$	Inverse hyperbolic cos

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We can view these relations as a generalization of the results for complex numbers. For example, for complex numbers we have  $\cosh(a + ib) = \cos b \cosh a + i \sin b \sinh a$ , whereas for the case of multivectors we can write  $\cosh(a + \mathbf{v} + ib) = \cosh(a + F) = \cosh(a + \hat{F}|F|)$ , and so produce the results of Eq. (36), where  $\hat{F}$  now takes the role of the unit imaginary, because  $\hat{F}^2 = -1$ . These results also remaining valid if  $|F|$  is a bivector.

Now, because the pseudoscalar  $i$  in two dimensions is not commuting there is no way to generate the alternating series shown in Eq. (11) for the trigonometric functions from the exponential series using the pseudoscalar and so these will be developed in the next section in three dimensions.

Our complete list of results for multivectors in  $Cl(\mathfrak{R}^2)$  are tabulated in Table 1. The inverse hyperbolic trigonometric functions are also shown in Table 1, using the algebraic procedure shown next in three dimensions. In conclusion, we have identified several limitations in two dimensions, such as the lack of a commuting pseudoscalar, the nonexistence of the square root and exponential representation in a significant class of multivectors, however, we now produce the corresponding expressions with multivectors in the more general three-dimensional space where these limitations are absent.

### The Multivector in Three Dimensions

In three dimensions we have the three basis elements  $e_1e_2$  and  $e_3$ , the three bivectors  $e_1e_2, e_3e_1$  and  $e_2e_3$ , as well as the trivector  $j = e_1e_2e_3 = e_{123}$  and we form the three dimensional geometric algebra  $Cl(\mathfrak{R}^3)$ . In order to assist the readers intuition we note an isomorphism with matrix

algebra that  $Cl(\mathfrak{R}_3) \cong Mat(2, C)$ . This isomorphism also implies that Clifford algebra shares the non-commuting and associativity properties of matrix algebra. However it should be noted that the Clifford algebra we have defined over  $\mathfrak{R}^3$  has more structure than is the case with the matrix definition, for example, we have a graded structure in  $Cl(\mathfrak{R}_3)$  of scalars, vectors, bivectors and trivectors. In three dimensions the trivector squares to minus one and commutes with all quantities and so in close analogy to the unit imaginary. Indeed, using the trivector we can also form what are called the dual relations,  $e_1e_2 = je_3$ ,  $e_3e_1 = je_2$  and  $e_2e_3 = je_1$ . Hence, we can write a general multivector in three dimensions as

$$M = a + \mathbf{v} + j\mathbf{w} + jt, \tag{37}$$

where  $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3$  and  $\mathbf{w} = w_1e_1 + w_2e_2 + w_3e_3$ , which thus produces a multivector of eight dimensions. The Clifford algebra  $Cl(\mathfrak{R}^3)$  contains the element  $j = e_{123}$  as a pseudoscalar such that the two dimensional subalgebra generated by  $j$  is the center  $Z(A)$  of the algebra  $A = Cl(\mathfrak{R}^3)$ . That is, every element of  $A$  commutes with every element of the center  $Z(A)$  that can be represented as  $a + jt$ . Thus  $A$  is isomorphic to an algebra over the complex field. This is in contrast to  $Cl(\mathfrak{R}^2)$  where the imaginary element  $i = e_{12}$  is not commuting with other elements of the algebra and so does not belong to the center  $Cen(Cl(\mathfrak{R}^2))$ .

Before proceeding to a general multivector product it is instructive to firstly calculate the special case of the product of two vectors  $\mathbf{v}$  and  $\mathbf{w}$ . Assuming the distributive property of multiplication over addition we find

$$\begin{aligned} \mathbf{v}\mathbf{w} &= (v_1e_1 + v_2e_2 + v_3e_3)(w_1e_1 + w_2e_2 + w_3e_3) \\ &= v_1w_1 + v_2w_2 + v_3w_3 + (v_2w_3 - w_2v_3)e_2e_3 + (v_1w_3 - v_3w_1)e_1e_3 + (v_1w_2 - w_1v_2)e_1e_2 \\ &= \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \wedge \mathbf{w}, \end{aligned} \tag{38}$$

consisting of the sum of the dot and wedge products, being a scalar and a bivector respectively. In three dimensions we in fact have the relation  $\mathbf{v} \wedge \mathbf{w} = j\mathbf{v} \times \mathbf{w}$ , where  $j$  is the trivector and  $\times$  is the vector cross product. For a vector squared, that is  $\mathbf{v}^2 = \mathbf{v}\mathbf{v}$ , we have  $\mathbf{v} \wedge \mathbf{v} = 0$  and so  $\mathbf{v}\mathbf{v} = \mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2$  producing a scalar equal to the Pythagorean length squared.

Now, defining  $Z = a + jt$  and  $F = \mathbf{v} + j\mathbf{w}$ , we can write  $M = Z + F$ , which splits the multivector into a component  $Z$  isomorphic to the complex number field and a multivector  $F$ .

For the multivector  $M$ , we then have *Clifford conjugation*

$$\bar{M} = a - \mathbf{v} - j\mathbf{w} + jt = Z - F, \tag{39}$$

that produces the *amplitude* of a multivector in three dimensions

$$|M| = \sqrt{M\bar{M}} = \sqrt{a^2 - \mathbf{v}^2 + \mathbf{w}^2 - t^2 + 2j(at - \mathbf{v} \cdot \mathbf{w})}, \tag{40}$$

that in general is a complex-like number. We note that is well behaved with  $|M_1M_2|^2 = |M_1|^2|M_2|^2$  and  $|M_1M_2| = \pm |M_1||M_2|$ .

**Definition** (Negative square root) *We define the principal square root when acting over negative reals in  $Cl(\mathfrak{R}^3)$  as follows: given a positive real number  $a \in \mathfrak{R}$  we define*

$$\sqrt{-a} = e_{123}\sqrt{a} = j\sqrt{a}. \tag{41}$$

In three dimensions the pseudoscalar  $j$  is commuting and so closely analogous to the scalar unit imaginary  $\sqrt{-1}$ . The subalgebra, consisting of quantities of the form  $a + jb$  form an isomorphism with the commuting complex numbers and we can therefore assume the results

from complex number theory when restricted to this domain. We can now investigate whether we can identify the root for a complex number within the expanded domain of multivectors. That is, we have the equation

$$M^2 = a^2 + \mathbf{v}^2 - \mathbf{w}^2 - t^2 + 2(\mathbf{a}\mathbf{v} - t\mathbf{w}) + 2j(\mathbf{t}\mathbf{v} + \mathbf{a}\mathbf{w}) + 2j(\mathbf{a}t + \mathbf{v} \cdot \mathbf{w}) = c + jd, \quad (42)$$

where  $c, d \in \mathfrak{R}$ . Solving this equation for  $M$  we find two distinct cases, either  $\mathbf{v} = \mathbf{w} = 0$  that corresponds to the conventional square root over the complex numbers and  $a = t = 0$  that provides a different set of roots over the domain of vectors and bivectors. That is, we find  $(\mathbf{v} + j\mathbf{w})^2 = c + jd$ , where  $c = \mathbf{v}^2 - \mathbf{w}^2$  and  $d = 2\mathbf{v} \cdot \mathbf{w}$ . Hence we have an alternative set of roots for complex numbers as

$$(c + jd)^{\frac{1}{2}} = \mathbf{v} + j\mathbf{w}. \quad (43)$$

As a special case we can find for  $c = -1$  and  $d = 0$

$$(-1)^{\frac{1}{2}} = \sinh \theta \hat{w}^\perp + j \cosh \theta \hat{w}, \quad (44)$$

where  $\hat{w}$  is a unit vector and  $\hat{w}^\perp$  is a unit vector perpendicular to  $\hat{w}$  and  $\theta \in \mathfrak{R}$ . This equation also provides an alternative root of minus one to the trivector in Eq. (41). The investigation of roots within Clifford multivectors has been previously studied [8], and roots are simpler to analyze using the polar form of a multivector, investigated shortly.

In three dimensions we have the *reversion* involution

$$M^\dagger = a + \mathbf{v} - j\mathbf{w} - jt, \quad (45)$$

giving

$$MM^\dagger = a^2 + \mathbf{v}^2 + \mathbf{w}^2 + t^2 + 2(\mathbf{a}\mathbf{v} - j\mathbf{v} \wedge \mathbf{w} + t\mathbf{w}), \quad (46)$$

with the norm  $\|M\| = \langle MM^\dagger \rangle_0^{1/2} = \sqrt{a^2 + \mathbf{v}^2 + \mathbf{w}^2 + t^2}$ . Also when representing complex numbers in three dimensions using  $z = a + jb$  then the norm produces  $\sqrt{zz^\dagger} = \sqrt{a^2 + b^2}$ .

Now, because  $M\bar{M}$  is a commuting complex-like number, we can find the inverse multivector to  $M$  as

$$M^{-1} = \bar{M} / (M\bar{M}), \quad (47)$$

which is the same definition as in the two-dimensional case. The multivector inverse now fails to exist when  $M\bar{M} = 0$  or when  $a^2 + \mathbf{w}^2 = \mathbf{v}^2 + t^2$  and  $at = \mathbf{v} \cdot \mathbf{w}$ , which we can write as the single condition  $(\mathbf{v} + j\mathbf{w})^2 = (a + jt)^2$  or  $F^2 = Z^2$ . The inverse of a vector is a special case of this general multivector inverse,  $\mathbf{v}^{-1} = \mathbf{v}/\mathbf{v}^2$ . The inverse obeys the relations  $(M^{-1})^{-1} = M$  and  $(MN)^{-1} = N^{-1}M^{-1}$ .

Hamilton's quaternions  $i, j, k$ , satisfying  $i^2 = j^2 = k^2 = ijk = -1$ , can be shown to be isomorphic to the even subalgebra of  $Cl(\mathfrak{R}^3)$ , so that a quaternion

$q = a + w_1i - w_2j + w_3k \cong a + j\mathbf{w} = a + w_1je_1 + w_2je_2 + w_3je_3$ . Hamilton in fact originally conceived the quaternions as the quotient of two vectors, and indeed using Clifford algebra vectors we can explicate this idea, finding the quotient of two vectors

$$\mathbf{v}/\mathbf{w} = \mathbf{v}\mathbf{w}/\mathbf{w}^2 = \frac{1}{\mathbf{w}^2}(\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \wedge \mathbf{w}) \quad (48)$$

that lies on the even subalgebra and so isomorphic to the quaternions as asserted by Hamilton.

### The square root

We find that the same expression for square root of a multivector in two dimensions

$$M^{\frac{1}{2}} = \frac{M \pm |M|}{\sqrt{M + \bar{M} \pm 2|M|}}, \tag{49}$$

produces the square root in three dimensions, however, in three dimensions all the restrictions on  $M$  in order for the root to exist can be lifted except that the denominator

$$\frac{1}{2}(M + \bar{M} \pm 2|M|) = Z \pm |M| \neq 0.$$

The full algebraic analysis of roots in three dimensions is quite extensive, however, as in complex number theory roots are more easily handled using the polar form of a number and we will find that the positive sign above will correspond to the principal value in the polar form  $e^{0.5 \log M}$ , calculated using logarithms that are defined shortly.

### Trigonometric form of a multivector

**Definition** (Multivector argument) *We define the argument of a multivector*

$\phi = \arg M = \arctan\left(\frac{|F|}{Z}\right)$ , where  $\arg(M), Z, |F|$  are complex like numbers. The real part of  $\phi = \arg M = a + jb$  is multivalued modulo  $2\pi$ , where  $a, b \in \mathbb{R}$ , and so we can once again define a principal value  $-\pi < a \leq \pi$ .

It is helpful to remember that complex number theory proves that functional identities that are true for all real values of the variable are also true for complex values of the variable [7].

Now, a multivector in  $Cl(\mathbb{R}^3)$  can be written in the form

$$M = Z + F = |M|(\cos \phi + \hat{F} \sin \phi), \tag{50}$$

where  $\phi = \arg M$ , provided  $|M|, |F| \neq 0$ . We have defined  $\hat{F} = F/|F|$  that has the key property that  $\hat{F}^2 = -1$ . This result can be confirmed by substituting  $\phi$ , using the fact that  $\cos \phi = \frac{Z}{|M|}$  and  $\sin \phi = \frac{|F|}{|M|}$ . Specifically, with  $F = \mathbf{v} + j\mathbf{w}$  we find  $|F| = \sqrt{F\bar{F}} = \sqrt{-\mathbf{v}^2 + \mathbf{w}^2 - 2j\mathbf{v} \cdot \mathbf{w}}$ . The order of the factors is not as significant in three dimensions compared to two dimensions because the pseudoscalar  $j$  is commuting. We can use here the conventional trigonometric results from complex number theory because  $Z, |F|$  and  $|M|$  are all complex-like numbers.

We will then find for integer powers  $p$  that

$$M^p = |M|^p (\cos \phi + \hat{F} \sin \phi)^p = |M|^p (\cos p\phi + \hat{F} \sin p\phi), \tag{51}$$

an extension of de Moivre’s theorem for multivectors in three dimensions.

### Exponential map of a 3D multivector

Now, given a three-dimensional multivector  $a + \mathbf{v} + j\mathbf{w} + jt = Z + F$ , we find

$F^2 = (\mathbf{v} + j\mathbf{w})^2 = \mathbf{v}^2 - \mathbf{w}^2 + 2j\mathbf{v} \cdot \mathbf{w} = -F\bar{F} = -|F|^2$ . Now using Eq. (9) and the fact that  $F = \hat{F}|F|$  we find

$$\begin{aligned} e^M &= e^{Z+F} = e^Z e^F \\ &= e^Z \left( 1 + F + \frac{F^2}{2!} + \frac{F^3}{3!} + \frac{F^4}{4!} + \dots \right) \\ &= e^Z \left( 1 + \hat{F}|F| - \frac{|F|^2}{2!} - \frac{\hat{F}|F|^3}{3!} + \frac{|F|^4}{4!} + \dots \right) \\ &= e^Z (\cos|F| + \hat{F} \sin|F|), \end{aligned} \tag{52}$$

and thus in a closed form. If  $F^2 = 0$ , then referring to the second line of the derivation above, we see that all terms following  $F$  are zero, and so, in this case  $e^M = e^{Z+F} = e^Z(1 + F)$ . The exponential function will also have the expected properties that  $(e^M)^- = e^{\bar{M}}$  and likewise for reversion and space inversion operations, as defined in Eq. (3). A corollary of this result is that  $|e^M| = |e^Z e^F| = |e^Z| |e^F| = e^Z$ .

We can thus write quite generally a multivector in polar form

$$M = a + \mathbf{v} + j\mathbf{w} + jt = |M|e^{\phi\hat{F}}, \tag{53}$$

where  $\phi = \arg M$ , provided  $|M|, |F| \neq 0$ , where clearly the exponent is multivalued. The polar form can also be expanded as  $|M|e^{\phi\hat{F}} = |M|(\cos \phi + \hat{F} \sin \phi)$  and so equivalent to the trigonometric form shown in Eq. (50). We can therefore write a multivector  $M = e^{\log|M|} e^{\phi\hat{F}} = e^{\log|M| + \phi\hat{F}}$  and defining the logarithm as the inverse of the exponential function, obtain the logarithm of a multivector

$$\log M = \log|M| + \phi\hat{F}, \tag{54}$$

where  $\phi = \arg M$  and  $\hat{F} = \frac{\mathbf{v} + j\mathbf{w}}{|F|}$ , provided  $|M|, |F| \neq 0$ . Naturally, this will also coincide with the power series expansion of

$\log M = \log(1 + (M - 1)) = (M - 1) - \frac{1}{2}(M - 1)^2 + \frac{1}{3}(M - 1)^3 \dots$  This leads to analogous results, as for complex numbers, that  $\log jM = \log M + j\pi/2$  and  $\log(-M) = \log M - \pi\hat{F}$ . Some properties of the logarithm include  $\log(-1) = j\pi$  as well as the log of the trivector  $\log j = \frac{\pi}{2}j$ ,  $\log(bj) = \log b + \frac{\pi}{2}j$ , and the log of a unit vector  $\log e_1 = j(1 - e_1)\frac{\pi}{2}$  generalizing to  $\log \hat{\mathbf{v}} = j(1 - \hat{\mathbf{v}})\frac{\pi}{2}$  and finally for a general vector  $\log \mathbf{v} = \log|\mathbf{v}| + j(1 - \hat{\mathbf{v}})\frac{\pi}{2}$ .

The multivector logarithm is naturally a generalization of the well known result for quaternions, that can be recovered by setting  $\mathbf{v} = t = 0$  giving a multivector  $M = a + j\mathbf{w}$ , with

$$\log(a + j\mathbf{w}) = \log|q| + \phi j\hat{\mathbf{w}} = \log \sqrt{a^2 + \mathbf{w}^2} + \arctan\left(\frac{\sqrt{\mathbf{w}^2}}{a}\right) \frac{j\mathbf{w}}{\sqrt{\mathbf{w}^2}}, \tag{55}$$

where  $\phi = \arctan\left(\frac{\sqrt{\mathbf{w}^2}}{a}\right)$ , producing the quaternion logarithm as required. If we now set  $e_3 = 0$  we find

$$\log(a + w_3 e_{12}) = \log \sqrt{a^2 + w_3^2} + \arctan\left(\frac{w_3}{a}\right) \frac{w_3 e_{12}}{\sqrt{w_3^2}} = \log \sqrt{a^2 + w_3^2} + \arctan\left(\frac{w_3}{a}\right) e_{12}, \tag{56}$$

the definition of the log of a complex number  $z = a + iw_3$ .

The nesting of real, complex numbers and quaternions within a multivector can be used to illustrate the Cayley-Dickson construction. In the Cayley-Dickson construction, complex numbers are generated from pairs of real numbers, and subsequently quaternions are then generated from pairs of complex numbers, etc.

Now, the quaternions are the even subalgebra of  $Cl(\mathbb{R}^3)$  and so we can write a quaternion

$$q = a + j\mathbf{w} = (a + w_3 e_{12}) + (w_2 + w_1 e_{12})e_{31} = z_1 + z_2 e_{31} \tag{57}$$

consisting now of a pair of complex-like numbers  $z_1 = a + w_3 e_{12}$  and  $z_2 = w_2 + w_1 e_{12}$ . We can then find the norm  $|q|^2 = \sqrt{|z_1|^2 + |z_2|^2}$ , and so derived from the norm of the constituent complex numbers. Also, given two quaternions  $p = x_1 + x_2 e_{31}$  and  $q = y_1 + y_2 e_{31}$ , where  $x_1, x_2,$

$y_1, y_2$  are complex like numbers in the form  $a + e_{12}b$ , we find their product

$$pq = (x_1y_1 - x_2\bar{y}_2) + (x_1y_2 + x_2\bar{y}_1)e_{31}. \tag{58}$$

This allows us to implement non-commutative quaternion multiplication using only commuting complex number arithmetic, which has advantages in numerical applications that utilize the already efficient implementation of complex number arithmetic.

Also, re-arranging the multivector in Eq. (37) to  $(a + j\mathbf{w}) + j(t - j\mathbf{v}) = q_1 + q_2$ , where  $q_1 = a + j\mathbf{w}$  and  $q_2 = t - j\mathbf{v}$  are quaternions, we have now written the multivector as a pair of quaternions. Though this is analogous to the Cayley-Dickson construction that will then produce the octonions, in our case we have formed rather the complexified quaternions, though both being eight dimensional spaces. Hence in  $Cl(\mathfrak{R}^3)$  we can identify the full multivector with the field of complexified quaternions, the even subalgebra  $a + j\mathbf{w}$  with the real quaternions,  $a + jt$  with the commuting complex numbers and the subalgebra  $a + v_1e_1 + v_2e_2 + w_3e_1e_2$  with  $Cl(\mathfrak{R}^2)$ .

The multivector logarithm highlights both the issue of multivaluedness and the non-commuting nature of multivectors. Firstly, the non-commutativity implies that  $e^A e^B \neq e^{A+B}$  and hence  $\log AB \neq \log A + \log B$ . Also  $A^n B^n \neq (AB)^n$ , unless  $A$  and  $B$  commute, where  $n$  an integer.

Secondly, the issue of multivaluedness is typically addressed through defining the principle value of the logarithm and the use of Riemann surfaces, however with the multivector logarithm the multivaluedness can expand into two domains, of  $\hat{F}$  and  $j$ . This is because both  $j$  and  $\hat{F}$  square to minus one and commute with  $M$ . That is

$$M = e^{\log M} = e^{\log M + 2n\pi\hat{F}} = e^{\log M + 2m\pi j} = e^{\log M + \pi\hat{F} + \pi j} = e^{\log M + (2n+1)\pi\hat{F} + (2m+1)\pi j} \tag{59}$$

where  $n$  and  $m$  are integers, where we can add even powers of  $\pi$ . Hence  $M = e^{\log M}$  whereas  $M / = \log(e^M)$  due to the multivalued nature of the log operation.

Now, we can easily see that for  $n$  an integer that  $M^n = (e^{\log M})^n = e^{\log M} e^{\log M} \dots e^{\log M} = e^{n\log M}$ , which can be used as an alternative to Eq. (51). This leads us to define the multivector power

$$M^P = e^{\log(M)P}, \tag{60}$$

where the power  $P$  is now generalized to a multivector. This implies, for example, the power law  $(M^P)^n = M^{nP}$ , where  $n \in \mathbb{Z}$ . With this definition of power we can then define the log of a multivector  $Y$  to the multivector base  $M$  as

$$\log_M Y = \frac{1}{\log M} \log Y. \tag{61}$$

Although, if the power  $P$  does not commute with  $\log M$  then we can also define a power as  $e^{P\log(M)}$ , that has a logarithm  $\log Y / \log M$ . These expressions however need care due to the multivalued nature of the logarithm operation and the non-commutativity.

Now, using the logarithm function  $e^{0.5\log M}$

$$e^{0.5(\log|M| + \phi\hat{F})}, e^{0.5(\log|M| + \phi\hat{F} + \pi\hat{F} + \pi j)}, \tag{62}$$

we produce the two roots of  $M$  defined in Eq. (49), as required.

### Special cases

We will now consider some special cases where we do have commuting multivectors, such as the case with two multivectors  $M$  and  $Z = a + jt$ . We then have that

$\log M^Z = Z \log M + \hat{F} m \pi + j n \pi$ , where  $m, n \in \mathbb{Z}$  add possible phase terms. We can eliminate the phase terms using the exponential function and write a more explicit expression as  $e^{\log M^Z} = e^{Z \log M}$ . We also then recover the well known relations that  $e^z e^M = e^{Z+M}$  and  $\log ZM = \log Z + \log M$ .

A further special case [5] involves the product of two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , and we have from Eq. (38) that

$$\mathbf{ab} = \rho e^{\theta \hat{B}} = e^{c + \theta \hat{B}} \tag{63}$$

where  $\theta$  is the angle between the two vectors,  $\cos \theta = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ ,  $c = \log \rho$  and  $\hat{B}$  is the unit bivector of the plane defined by the vectors. We can then produce the result for two vectors  $\mathbf{a}$  and  $\mathbf{b}$  that

$$\log(\mathbf{ab}) = \log \|\mathbf{a}\| + \log \|\mathbf{b}\| + \theta \hat{B} = \frac{1}{2} \log(\mathbf{a}^2 \mathbf{b}^2) + \theta \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|} \tag{64}$$

where  $\hat{B} = \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|}$  is the unit bivector formed by  $\mathbf{a} \wedge \mathbf{b}$  and  $\theta = \arcsin \frac{|\mathbf{a} \wedge \mathbf{b}|}{\|\mathbf{a}\| \|\mathbf{b}\|}$  is the angle between the two vectors.

### Linear equations and linear functions

We define a linear function over multivector variables

$$F(M) = \sum_{m=1}^n R_m M S_m, \tag{65}$$

where  $R_m, S_m, M$  are multivectors. The series cannot in general be simplified due to non-commutativity. The case of  $n = 1$ , giving  $F(M) = RMS$  is particularly useful. For example, for the special case where  $R$  and  $S$  are vectors we have a reflection of a multivector

$$M' = -\mathbf{v} M \mathbf{v}. \tag{66}$$

When  $R$  and  $S$  lie in the even subalgebra, isomorphic to the quaternions we have a rotation operation in three dimensions

$$M' = RMR^\dagger, \tag{67}$$

where  $RR^\dagger = 1$ . The quaternions form a division algebra and so they are suitable to use as rotation operators that require an inverse. There is also a generalization to describe rotations in  $\mathfrak{R}^4$ ,

$$M' = RMS, \tag{68}$$

where  $M = xe_1 + ye_2 + ze_3 + jt$  represent a 4D Cartesian vector, with  $RR^\dagger = SS^\dagger = 1$ .

For the second case from Eq. (65) with  $n = 2$  we have  $F(M) = RMS + PMQ$ . Now, premultiplying by  $S^{-1}$  from the right and  $P^{-1}$  from the left we produce  $Y = P^{-1}F(M)S^{-1} = P^{-1}RM + MQS^{-1}$ . Setting  $A = P^{-1}R$  and  $B = QS^{-1}$  we produce

$$Y = AM + MB, \tag{69}$$

which is called Sylvester's equation [9] that can in general be solved for  $M$ . Assuming  $|A| \neq 0$  (or alternatively  $|B| \neq 0$ ) we firstly calculate

$A^{-1}Y\bar{B} + Y = M\bar{B} + A^{-1}MB\bar{B} + AM + MB = M(B + \bar{B}) + A^{-1}MB\bar{B} + AM$ . Now  $B + \bar{B}$  and

$B\bar{B}$  are commuting complex-like numbers and so we can write  $(\bar{B} + B + A^{-1}B\bar{B} + A)M = A^{-1}Y\bar{B} + Y$ , thus succeeding in isolating the unknown multivector  $M$ . Hence we have the solution

$$M = (B + \bar{B} + A^{-1}B\bar{B} + A)^{-1}(A^{-1}Y\bar{B} + Y). \tag{70}$$

This result is analogous to results using quaternions or matrices [9], though solved here for a general multivector.

Regarding polynomial equations in multivectors, the fundamental theorem of algebra tells us that the number of solutions of a complex polynomial is equal to the order of the polynomial. With multivector polynomials however, such as the simple quadratic equation  $M^2 + 1 = 0$  we can find an infinite number of solutions.

A common operation in complex number theory is the process of ‘rationalizing the denominator’ for a complex number  $\frac{1}{a+ib}$  that involves producing a single real valued denominator, given by  $\frac{a-ib}{a^2+b^2}$ . We can also duplicate this process for a multivector  $\frac{1}{a+v+jw+jt}$ . Now,  $1/M = M^{-1} = \bar{M}/(M\bar{M})$ . Notice that  $M\bar{M}$  is a complex-like number that we can now ‘rationalize’ by multiplying the numerator and the denominator by  $(M\bar{M})^\dagger$  forming  $\frac{1}{M} = \frac{\bar{M}(M\bar{M})^\dagger}{R}$ , where  $R = M\bar{M}(M\bar{M})^\dagger$  is a scalar real value, as required.

**Trigonometric functions of multivectors in 3D.** The trigonometric functions in three-dimensions are more straightforward than in two-dimensions, because the unit imaginary  $j = e_{123}$  is commuting. Using the general expressions in Eq. (10) and using  $M = Z + F$ , we can once again write these expression in a closed form

$$\begin{aligned} \cosh M &= \frac{1}{2}(e^{Z+F} + e^{-Z-F}) = \cos|F|\cosh Z + \hat{F}\sin|F|\sinh Z \\ \sinh M &= \frac{1}{2}(e^{Z+F} - e^{-Z-F}) = \cos|F|\sinh Z + \hat{F}\sin|F|\cosh Z. \end{aligned} \tag{71}$$

Using the commuting trivector  $j$ , we can now write the trig relations given in the general case in Eq. (11), as

$$\begin{aligned} \cos M &= \frac{1}{2}(e^{jM} + e^{-jM}) = \cosh|F|\cos Z - \hat{F}\sinh|F|\sin Z \\ \sin M &= \frac{1}{2j}(e^{jM} - e^{-jM}) = \cosh|F|\sin Z + \hat{F}\sinh|F|\cos Z. \end{aligned} \tag{72}$$

All the usual identities will hold such as  $\sinh jM = j\sin M$  and  $\cosh jM = \cos M$ . Also, we can see that  $\sin M$  and  $\cos M$  are commuting, and so we can define without any difficulties with non-commutativity

$$\tan M = \frac{\sin M}{\cos M} = \frac{\tan Z + \hat{F}\tanh|F|}{1 - \hat{F}\tanh|F|\tan Z}. \tag{73}$$

These expressions are easily calculated because both  $Z$  and  $|F|$  are complex-like numbers and so we can utilize the well known results from complex number theory. We can also see that, given three multivectors  $M_1, M_2, M_3$  then writing  $\sin \phi = M_1 | M_3$  and  $\cos \phi = M_2 | M_3$  we have  $\sin \phi / \cos \phi = M_1 M_3^{-1} (M_2 M_3^{-1})^{-1} = M_1 M_3^{-1} M_3 M_2^{-1} = M_1 M_2^{-1} = \tan \phi$ . That is, we can view  $M_1, M_2, M_3$  as three sides of a triangle, subject to the normal trigonometric relations.

We also have the results that for a general vector  $\mathbf{v}$  that  $\cos \mathbf{v} = \cos \sqrt{\mathbf{v}^2} = \cos ||\mathbf{v}||$  that neatly generalizes scalar values to vector values. That is, the  $\cos$  of a vector is the  $\cos$  of the length of the vector though  $\sin \mathbf{v} = \hat{\mathbf{v}} \sin ||\mathbf{v}||$ . For a field  $F = \mathbf{E} + j\mathbf{B}$  we have  $\cos F = \cos \sqrt{F^2}$ .

**Inverse trigonometric functions.** Now, using the result that  $e^M = \cosh M + \sinh M$  substituting  $M = \operatorname{arcsinh} X$  then taking the log of both sides we find

$$\operatorname{arcsinh} X = \log(\cosh(\operatorname{arcsinh} X) + X). \tag{74}$$

Re-arranging  $\cosh^2 X - \sinh^2 X = 1$  we find that  $\cosh(\operatorname{arcsinh} X) = (1 + X^2)^{1/2}$  and so

$$\operatorname{arcsinh} X = \log((1 + X^2)^{1/2} + X). \tag{75}$$

This will coincide with the power series  $\operatorname{arcsinh} X = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! X^{2n+1}}{2^{2n} (n!)^2 2n+1}$ . Similarly we find  $\operatorname{arccosh} X = \log(X + (X^2 - 1)^{1/2})$  and  $\operatorname{arctanh} X = \frac{1}{2} \log((1 + X)/(1 - X)) = \frac{1}{2} (\log(1 + X) - \log(1 - X))$  that coincides with the conventional power series  $\operatorname{arctanh} X = \sum_{n=0}^{\infty} \frac{X^{2n+1}}{2n+1}$ .

Similarly, from the definitions of  $\cos$  and  $\sin$  we know that

$$e^{jM} = \cos M + j \sin M, \tag{76}$$

and once again substituting  $M = \operatorname{arcsin} X$  and using  $\cos(\operatorname{arcsin} X) = (1 - X^2)^{1/2}$  we find

$$\operatorname{arcsin} X = -j \log((1 - X^2)^{1/2} + jX). \tag{77}$$

Similarly we have  $\operatorname{arccos} X = -j \log(X + j(1 - X^2)^{1/2})$ .

Hence we produce the result that  $\operatorname{arcsinh}(jX) = j \operatorname{arcsin} X$ ,  $\operatorname{arccosh} X = j \operatorname{arccos} X$  and  $\operatorname{arctanh}(jX) = j \operatorname{arctan} X$ . Therefore, finally

$$\operatorname{arctan} X = -\frac{j}{2} \log((1 + jX)/(1 - jX)) = -\frac{j}{2} (\log(1 + jX) - \log(1 - jX)). \tag{78}$$

As an example of solving multivector trigonometric equations, if we are asked to solve the equation  $\sinh M = 0$  then we can proceed as follows. Given  $\sinh M = \frac{1}{2}(e^M - e^{-M})$  we therefore need to solve  $e^M = e^{-M}$  or  $e^{2M} = 1$ . Now

$$e^{2M} = e^{2(a+jt)+2(v+jw)} = e^{2a} (\cos 2t + j \sin 2t) (\cos 2|F| + \hat{F} \sin 2|F|) = 1. \tag{79}$$

Hence we require  $a = 0$ ,  $t = n\pi$  and  $|F| = m\pi$  so we have a solution  $M = m\pi\hat{F} + n\pi j$ , where  $m, n \in \mathbb{Z}$ . We can also identify a second solution in which both terms in the brackets are simultaneously negative  $M = (m + \frac{1}{2})\pi\hat{F} + (n + \frac{1}{2})\pi j$ .

**Inter-relationships in  $C\ell(\mathfrak{R}^3)$ .** We have the well known result from complex number theory that  $i^i = e^{-\pi/2}$  that is duplicated with the pseudoscalar in Clifford algebra, finding  $i^i = j^j = e^{-\pi/2}$ . However with a more general multivector number now available we can also find other more general relationships. For example, for a unit vector  $\hat{v}$  with  $\hat{v}^2 = 1$ , we find that  $\hat{v}^{\hat{v}} = \hat{v}$ . That is raising a unit vector to this unit vector power produces the same unit vector. Alternatively, if we raise a unit vector to an orthogonal unit vector we find  $\hat{v}^{\hat{v}^\perp} = 1$ .

Also, consider the expression  $(\cos \theta + j \sin \theta)^v$ , where  $v = v_1 e_1 + v_2 e_2 + v_3 e_3$  is a Cartesian vector, with  $\hat{v} = v/\sqrt{v^2} = v/s$ , then we find

$$(\cos \theta + j \sin \theta)^v = e^{v \log(\cos \theta + j \sin \theta)} = e^{s \theta \hat{v}} = \cos s \theta + j \hat{v} \sin s \theta. \tag{80}$$

Now  $\hat{q} = \cos s \theta + j \hat{v} \sin s \theta$  lies in the even sub-algebra and so is isomorphic to the quaternions with  $|q| = 1$  and  $\hat{z} = \cos \theta + j \sin \theta$  is isomorphic to the complex numbers, with  $\hat{z}$  representing a unit complex number. We thus can write

$$r \hat{z}^v = q. \tag{81}$$

**Table 2. Algebraic relations in three dimensions  $Cl(\mathbb{R}^3)$ .**

Main results	Notes: $j = e_{123}, i = e_{12}, v \in \mathbb{R}^3$
$j^j = i^i = e^{-\pi/2}$	Compare with $\sqrt{-1}^{\sqrt{-1}} = e^{-\pi/2}$
$(j\hat{v})^{\hat{v}} = e^{-\pi/2}$	$\hat{v}^2 = 1$ . E.g. $(je_1)^{e_1} = (e_2e_3)^{e_2e_3} = e^{-\pi/2}$
$(j\hat{v})^{j\hat{v}^\perp} = j\hat{w}$	$\hat{v}^\perp \cdot \hat{v} = 0, \hat{w}\hat{v}^\perp\hat{v} = j$ . E.g. $(je_1)^{e_3} = je_2$
<b>Powers of vectors</b>	$\hat{v}^2 = 1$
$\hat{v}^{\hat{v}} = \hat{v}$	E.g. $e_1^{e_1} = e_1$
$\hat{v}^{\hat{v}^\perp} = 1$	E.g. $e_1^{e_2} = e_2^{e_1} = 1$
$\hat{v}^{j\hat{v}^\perp} = 1$	E.g. $e_2^{e_3} = 1$
$\mathbf{v}^{\frac{1}{2}} = \frac{1}{\sqrt{2j\sqrt{\mathbf{v}^2}}}(\mathbf{v} + j\sqrt{\mathbf{v}^2})$	E.g. $e_1^{\frac{1}{2}} = \frac{1}{\sqrt{2j}}(e_1 + j) = \frac{1}{2}(1 - j)(e_1 + j)$
<b>Trigonometric relationships</b>	
$\cos \mathbf{v} = \cos \ \mathbf{v}\ , \cos \hat{\mathbf{v}} = 1$	
$\operatorname{arcsinh} \mathbf{v} = \log(\mathbf{v} + (1 + \mathbf{v}^2)^{1/2})$	
<b>General relationships</b>	
$r\hat{z}^{\mathbf{v}} = q$	$r \in \mathbb{R}, z \in \mathbb{C},  z  = 1, q \in \mathbb{H}$ and $\mathbf{v} \in \mathbb{R}^3$
<b>Special cases</b>	
$(j\hat{v})^{\hat{v}} = j$	E.g. $(je_3)^{e_3} = (e_1e_2)^{e_3} = j$
$j^{\hat{v}} = j\hat{v}$	E.g. $j^{e_3} = je_3$

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This formula thus links real numbers  $r \in \mathbb{R}$ , complex numbers  $z \in \mathbb{C}$ , Cartesian vectors  $\mathbf{v} \in \mathbb{R}^3$  and quaternions  $q \in \mathbb{H}$  into a single relationship, a Rosetta stone for the algebra of three-dimensional space. Interpreting this formula, we can see that raising a unit complex number to a vector power produces a quaternion. A unit complex number being a rotation operator in the plane with a rotation of  $\theta$ , when raised to a unit vector power in the direction  $\hat{v}$  produces a rotation operator rotating  $2\theta$  about the axis  $\hat{v}$ . Hence raising a complex number to a vector power  $\hat{v}$  converts a planar rotation operator into a three dimensional rotation operator about an axis  $\hat{v}$ . This relates to our previous discussion on the Cayley-Dickson construction that generates quaternions from complex numbers, but illustrates an alternate construction to achieve this.

These results are summarized in [Table 2](#).

### Multivector in one and four dimensions

We can extend the sequence  $Cl(\mathbb{R}^3), Cl(\mathbb{R}^2)$  down to one dimension giving the multivector in  $Cl(\mathbb{R}^1)$

$$M = a + ve_1 \tag{82}$$

where  $a, v \in \mathbb{R}$ . We now do not have a pseudoscalar, however most functions can still be derived and a table similar to [Table 1](#) and [Table 3](#) can be constructed. It should also be noted that all multivectors are now commuting because we only have a single algebraic variable  $e_1$ .

The four dimensional case  $Cl(\mathbb{R}^4)$  is significantly harder than three dimensions, due to a larger sixteen dimensional space as well as a non-commuting pseudoscalar  $I = e_{1234}$ . We have a

**Table 3. Multivector functions in three dimensions  $Cl(\mathfrak{R}^3)$ .**

Main results	Notes ( $j = e_{123}$ commuting)
$M = a + \mathbf{v} + j\mathbf{w} + jt$	$F = \mathbf{v} + j\mathbf{w}, Z = a + jt, \mathbf{v}, \mathbf{w} \in \mathfrak{R}^3$
$\bar{M} = a - \mathbf{v} - j\mathbf{w} + jt$	Conjugation
$ M  = \sqrt{M\bar{M}} = \sqrt{a^2 - \mathbf{v}^2 + \mathbf{w}^2 - t^2 + 2j(at - \mathbf{v} \cdot \mathbf{w})}$	Amplitude
$M^{-1} = \bar{M}/(M\bar{M}):  M  \neq 0$	Inverse
$\phi = \arg M = \arctan\left(\frac{F}{Z}\right)$	Argument
$M =  M e^{j\phi} =  M (\cos\phi + \hat{F}\sin\phi):  M , F \neq 0$	Polar form, $\hat{F} = F/ F , \hat{F}^2 = -1$
$M^{\frac{1}{2}} = \frac{M \pm  M }{\sqrt{M + M \pm 2 M }}; Z \pm  M  \neq 0$	Square root
$M^x =  M ^x(\cos x\phi + \hat{F}\sin x\phi)$	Complex powers $x \in \mathfrak{R} \oplus \wedge^3 \mathfrak{R}^3$
$e^M = e^{Z+F} = e^Z(\cos F  + \hat{F}\sin F )$	Exponential If $ F  = 0$ then $e^M = e^Z(1 + F)$
$\log_e M = \log_e  M  + \phi\hat{F}$	Logarithm
$M^P = e^{\log(M)P}$ or $e^{P\log(M)}$	General powers
Hyperbolic/Trigonometric functions	
$\cosh M = \cos F \cosh Z + \hat{F}\sin F \sinh Z$	$M = Z + F = Z + \hat{F} F $ $eM = \cosh M + \sinh M$
$\sinh M = \cos F \sinh Z + \hat{F}\sin F \cosh Z$	$\cosh^2 M - \sinh^2 M = 1$
$\cos M = \cosh jM = \cosh F \cos Z - \hat{F}\sinh F \sin Z$	$e^{jM} = \cos M + j \sin M$
$\sin M = -j\sinh jM = \cosh F \sin Z + \hat{F}\sinh F \cos Z$	$\cos^2 M + \sin^2 M = 1$
$\operatorname{arcsinh} M = \log\left(M + (1 + M^2)^{\frac{1}{2}}\right)$	$\operatorname{arcsinh}(jM) = j\operatorname{arcsin} M$
$\operatorname{arccosh} M = \log\left(M + (M^2 - 1)^{\frac{1}{2}}\right)$	$\operatorname{arccosh} X = j\operatorname{arccos} X$
$\operatorname{arctanh} M = \frac{1}{2}\log\left(\frac{1 + M}{1 - M}\right)$	$\operatorname{arctanh}(jM) = j\operatorname{arctan} M$

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multivector  $M = a + \mathbf{v} + \mathbf{B} + I\mathbf{w} + It$ , where  $\mathbf{v}, \mathbf{w} \in \mathfrak{R}^4$  and the bivectors  $\mathbf{B} = \sum_{i,j=1}^4 b_{ij}e_i e_j$ . We have Clifford conjugation  $\bar{M} = a - \mathbf{v} - \mathbf{B} + I\mathbf{w} + It$  as well as a new involution  $M^\# = a - \mathbf{v} + \mathbf{B} - I\mathbf{w} - It$ . We can then find a multivector amplitude  $|M| = (M\bar{M}(M\bar{M})^\#)^{1/4}$  that allows us to find an inverse  $M^{-1} = \bar{M}(M\bar{M})^\# / |M|^4$  provided  $|M| \neq 0$ .

If we seek the next space that has a commuting pseudoscalar that squares to minus one we need to go to a seven dimensional space. This space consists of eight grades with a total of  $2^7 = 128$  elements.

## Results and Discussion

In this paper, we explore the elementary functions when generalized to act over the space of Clifford multivectors in two and three dimensions, refer [Table 1](#) and [Table 3](#). Two key points that need to be kept in mind when working with multivectors, is firstly their non-commutativity and secondly multivaluedness, as found with the log function and the square root functions, for example. According to Frobenius' theorem the only associative division algebras are the reals, complex numbers and quaternions and conveniently these form subalgebras within  $Cl(\mathfrak{R}^3)$ . For the multivector, as represented in [Eq. \(37\)](#), the reals form the scalar component  $a$ , the complex-like numbers represented by the scalar and trivector components  $a + jt$  and the quaternions by the even subalgebra  $a + j\mathbf{w}$ . The full multivector in three dimensions, on the other hand, do not form a division algebra as the inverse operation is not defined for the multivectors with zero amplitude.

We have used the symbols  $i$  and  $j$  to replace the unit imaginary in two and three dimensions respectively. The use of the bivector and trivector for this purpose allows us to duplicate imaginary quantities with real algebraic entities thus remaining within a real space. Also the quantities  $i = e_1e_2$  and  $j = e_1e_2e_3$  can be endowed with specific geometrical meaning as a unit area and a unit volume respectively.

We find that in two and four dimensions with a non-commuting pseudoscalar the elementary functions lack generality, whereas in three dimensions with a commuting pseudoscalar the functions are defined more generally, so providing a very elegant mathematical framework. For example, in three dimensions the exponential form (polar decomposition) exists for all multivectors provided  $|M|, |F| \neq 0$ , whereas in two dimensions, they only exist if  $|M|$  is real. In fact, in three dimensions with a commuting pseudoscalar allows us to identify  $Cl(\mathbb{R}^3)$  with a complex algebra, isomorphic to complexified quaternions.

We find that because the complex numbers and quaternions appear as subalgebras within the more general multivector, we can explore their mutual inter-relationships within this context. We find that we can link the three algebraic systems of vectors, quaternions and complex numbers into a single expression, finding that a complex number raised to a vector power produces a quaternion, as shown in Eq. (81). A relationship between complex numbers and quaternions is already provided by the Cayley-Dickson construction of quaternions from complex numbers, however our expression is more explicit generating quaternions from raising a complex number to the power of a Cartesian vector. Inspecting the list of functions we also identify a single formula that produces the square root, amplitude and inverse in two and three dimensions. Also vectors are given a much more versatile formulation in Clifford algebra compared to Gibbs formulation of vectors, and indeed we can explore various vector expressions, such as raising a vector to a vector power, as well as logarithms and trigonometric relationships with vectors, these and other relationships listed in Table 2. We also find that the elementary functions can be defined using a single involution of Clifford conjugation, although for convenience two other involutions of reversion and space inversion are also defined. Naturally, the elementary functions over complex numbers and quaternions can be recovered as special cases from the three dimensional case.

The two dimensional algebra has the even subalgebra isomorphic to the complex numbers and has application within planar geometry, such as planar waveguides, and the three-dimensional algebra has the obvious application to three-dimensional space and forms an elegant space due to the commuting pseudoscalar, as well as possessing complex numbers and quaternions as subalgebras that can be utilized for rotations and containing Cartesian vectors that perform reflections.

It has been found that the multivector can also provide a unifying perspective on physical phenomena in three dimensions. For example, the Schrödinger, Pauli and Dirac wave functions [10], are all subalgebras within the three-dimensional multivector  $Cl(\mathbb{R}^3)$ . This example serves to illustrate the value of detailing the functions over multivector variables as undertaken in this paper.

## Author Contributions

Conceived and designed the experiments: JC AI LG DA. Performed the experiments: JC AI LG DA. Analyzed the data: JC AI LG DA. Contributed reagents/materials/analysis tools: JC AI LG DA. Wrote the paper: JC AI LG DA.

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