Staggered and Non-Staggered
Time-Domain Meshless Radial Point
Interpolation Method in
Electromagnetics

by

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To my parents
and my husband, Ali
with all my love.
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Abstract

Meshless methods have gained attention recently as a new class of numerical methods for the solution of partial differential equations in various disciplines of computational engineering. This class of methods offers several promising features compared to mesh-based approaches. The principle of domain discretization with arbitrary node distributions allows accurate modeling of complex geometries with fine details. Moreover, an elaborate and time-consuming re-meshing in the grid-based methods can be replaced in meshless counterparts by an adaptive node refinement during the simulation. This can be exploited to enhance solution accuracy or in optimization procedures.

In this thesis, the meshless Radial Point Interpolation Method (RPIM) is investigated for application in time-domain computational electromagnetics. The numerical algorithm is based on a combination of locally defined radial and polynomial basis functions and yields a highly accurate local interpolation of field values and associated derivatives based on the values at close neighboring positions. These interpolated partial derivatives are used to solve the partial differential equations.

The thesis is firstly focused on the staggered meshless RPIM. The classical implementation of the staggered meshless RPIM in electromagnetics using the first-order Maxwell’s curl equations is described and the update equations for the staggered electric and magnetic fields are shown. To enhance the capability of the algorithm, a novel implementation of the Uniaxial Perfectly Matched Layer (UPML) is introduced. It is shown however that UPML has intrinsically a long-time instability. Therefore, to avoid this instability two loss terms are introduced, which are added to the update equations in the UPML region after almost all the energy from the computational domain is absorbed. Various capabilities of the meshless method are then validated through different numerical examples using staggered node arrangements in the staggered meshless RPIM. However, the generation of a dual node distribution can be computationally costly and restricts the freedom of node positions, which might reduce the potential advantages of the scheme.

To overcome this challenge, the thesis next proposes a novel non-staggered algorithm for the meshless RPIM based on a magnetic vector potential technique. In this method instead of solving Maxwell’s curl equations for the electric and magnetic fields, the
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wave equation for the magnetic vector potential is solved. Therefore, a single set of nodes can be used to discretize the computational domain. Importantly in the proposed implementation, solving the second-order vector potential wave equation intrinsically enforces the divergence-free property of the electric and magnetic fields and the computational effort associated with the generation of a dual node distribution is avoided. In this part of the thesis, a hybrid algorithm is further proposed to implement staggered perfectly matched layers in the non-staggered RPIM framework. The properties of the proposed non-staggered RPIM are evaluated through several numerical examples both in 2D and 3D implementations.

In the last part of the thesis, the staggered and non-staggered implementations of meshless RPIM are directly compared in terms of efficiency and accuracy. It is shown that the non-staggered meshless RPIM not only bypasses the requirement of the dual node distribution, but also suppresses the spurious solutions observed in the staggered implementation.

The results of this research show the capability of meshless RPIM for being used efficiently in time-domain computational electromagnetics.
Statement of Originality

I certify that this work contains no material, which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree.

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My endless appreciation goes to my family, especially my mother, my father, mother-in-law and father-in-law, who always endow me with infinite support, wishes, continuous love, encouragement, and patience. Your prayer for me was what sustained me thus far.

Last, but certainly not least, I wish to give my heartfelt and warmest thanks to my dear husband, Ali, whose unconditional love, patience, and continuous support of my academic endeavors enabled me to complete this journey.
Thesis Conventions

The following conventions have been adopted in this Thesis:

**Typesetting**

This document was compiled using $\LaTeX$2e. Texmaker and TeXstudio were used as text editor interfaced to $\LaTeX$2e. Inkscape was used to produce schematic diagrams and other drawings.

**Referencing**

Referencing and citation style in this thesis are based on the Institute of Electrical and Electronics Engineers (IEEE) Transaction style.

**System of units**

The units comply with the international system of units recommended in an Australian Standard: AS ISO 1000–1998 (Standards Australia Committee ME/71, Quantities, Units and Conversions 1998).

**Spelling**

American English spelling is adopted in this thesis.
Publications

Book Chapter


Journal Articles


Conference Articles

1. Z. Shaterian, T. Kaufmann, and C. Fumeaux, “On the choice of basis functions for the meshless radial point interpolation method with small local support domains,” in *International Conference on Computational Electromagnetics (iCCEM)*, Hong Kong, 2-5 February 2015.*

2. Z. Shaterian, A. K. Horestani, and C. Fumeaux, “Rotation sensing based on the symmetry properties of an open-ended microstrip line loaded with a split ring resonator,” in *German Microwave Conference (GeMiC)*, Nuremberg, Germany, 16-18 March 2015.
3. Z. Shaterian, T. Kaufmann, and C. Fumeaux, “Hybrid staggered perfectly matched layers in non-staggered meshless time-domain vector potential technique,” in International Workshop on Antenna Technology (iWAT), Sydney, Australia, 4-6 March 2014, pp. 408–411.* (Best Student Paper Award)

4. ——, “First- and second-order meshless radial point interpolation methods in electromagnetics,” in 1st Australian Microwave Symposium (AMS), Melbourne, Australia, 26-27 June 2014.* (Best Student Paper Award)

5. ——, “On the staggered and non-staggered time-domain meshless radial point interpolation method,” in 17th International Symposium on ElectroMagnetic Compatibility (CEM), Clermont-Ferrand, France, 30 June- 3 July 2014.* (Best Paper Award)


8. ——, “Impact of different node distributions on the meshless radial point interpolation method in time-domain electromagnetic simulations,” in Asia-Pacific Microwave Conference (APMC), Kaohsiung, Taiwan, 4-7 December 2012.*


Note: Articles with an asterisk (*) are directly relevant to this thesis.
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THIS introductory chapter presents a short general description of meshless methods in numerical electromagnetics and highlights their attractive features compared to the mesh-based numerical methods. This is followed by an overview of the objectives of the thesis and its original contributions. The outline of this work is sketched out and the content of each chapter is given at the end of this chapter.
1.1 Introduction

The behavior of physical systems is described mathematically by Partial Differential Equations (PDEs). In electromagnetics, Maxwell’s equations, developed in 1864, form the PDEs governing electromagnetic waves. Maxwell’s equations can be expressed as follows

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1.1a)
\]

\[
\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \sigma \mathbf{E} + \mathbf{J}, \quad (1.1b)
\]

\[
\nabla \cdot \mathbf{D} = \rho, \quad (1.1c)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (1.1d)
\]

where \( \mathbf{E} \) is the electric field, \( \mathbf{H} \) is the magnetic field, \( \mathbf{D} \) is the electric flux density, \( \mathbf{B} \) is the magnetic flux density, \( \mathbf{J} \) is the electric current density, \( \sigma \) is the electric conductivity and \( \rho \) is the charge density. In the above equations, \( \mathbf{B} = \mu \mathbf{H} \) and \( \mathbf{D} = \varepsilon \mathbf{E} \) where \( \varepsilon \) is the electrical permittivity, \( \mu \) is the magnetic permeability.

Solving Maxwell’s equations is necessary to understand and predict the behavior of electromagnetic devices and systems. There are different methods to solve Maxwell’s equations. These methods can be classified as experimental, analytical and numerical methods. Experimental methods are expensive, time consuming and not very flexible for parameter variations [18]. Before 1960, most electromagnetic problems were solved analytically. Those analytical solutions did not cover all practical problems and they required deep knowledge of mathematics and physics. From the mid-1960s, with the emergence of high speed computers, numerical methods started to be used [18]. Numerical methods simplify analytical methods and give approximated solutions for a wide range of problems. These methods are easy to apply and require less advanced knowledge about mathematics and physics. Also, these methods make the design and analysis of engineering systems easier. Indeed, numerical methods (simulations) are a primary investigation tool for engineers to reduce or replace costly experimental studies and save time and cost. For example, the behavior of a system can be predicted before the physical system is built, so the number of different measurement scenarios are reduced. Electromagnetic simulations have a wide range of applications. Figure 1.1 shows some applications of electromagnetic simulations including antennas, nano-photonics, communications, biomedical engineering, remote sensing, chip design and circuits, lasers and optoelectronics, metamaterials, Micro-Electro-Mechanical Systems (MEMS), among others.
Chapter 1

Introduction

Figure 1.1. Impact of electromagnetic simulations. Different applications of electromagnetic simulations.

Systems (MEMS) and microwave engineering, signal processing, and Electromagnetic Compatibility/Electromagnetic Interference (EMC/EMI) analysis. The scientific discipline handling the numerical approximation of Maxwell’s equations is commonly known as Computational Electromagnetics (CEM).

1.2 Why Meshless Methods

The Finite Element Method (FEM) is one of the most important numerical methods in various engineering fields and there are different commercial packages based on FEM. In this method, the domain is divided into finite elements which are connected together. The FEM is a robust method and flexible for many linear and non-linear problems.

Although FEM is a widely used numerical method, it has some inherent issues since it is dependent on predefined meshes or elements. Some of these issues are highlighted in the following:

- FEM is a mesh-based method and the generation of a discretizing mesh for a problem domain is required. However, mesh generation is costly and time consuming.

  - Discretization of complex geometries with large deformations results in considerable degradation in the accuracy of FEM.
1.3 Definition of Meshless Methods

- To have an accurate solution for a discretized problem, the quality of the mesh is very important. An adaptive analysis and re-meshing techniques are required to obtain a desired accuracy. For re-meshing techniques, the required adaptive mesh generation leads to additional computational time and cost.

These mentioned problems come from the use of meshes in the FEM. To overcome these difficulties in the mesh-based numerical methods, meshless methods have been proposed.

1.3 Definition of Meshless Methods

In meshless methods a set of scattered nodes within the problem domain and on the boundaries of the domain is used to represent the problem domain. These scattered nodes are not forming a mesh and so no priori information related to the nodes relationships is required.

Many meshless methods have found their applications in numerical methods. Although meshless methods have a great potential to become successful and powerful numerical tools, they are still in the early stage and more developments are required to solve their technical problems.

1.4 Objectives of the Thesis

The time-domain meshless Radial Point Interpolation Method (RPIM) is investigated in this thesis. The introductory parts describe general features of meshless methods in Chapter 2, and more particularly the RPIM framework in Chapter 3. In the first main part of this thesis in Chapter 4, the focus is on the time-domain staggered meshless RPIM in three-dimensional (3D) domain. Time-domain meshless methods in electromagnetics which are based on solving the first-order Maxwell’s equations have been proposed in [19, 20, 21, 22]. Using this formulation both electric and magnetic fields are simultaneously unknown and are solved in a time iteration loop. Because of the type of intrinsic coupling between electric (E) and magnetic (H) fields expressed in Maxwell’s curl equations, the magnetic field values are best sampled between the locations where the electric field values are sampled. Therefore, the computational domain needs to be discretized by staggered electric and magnetic node distributions. To this end, an initial node distribution can be generated randomly for the electric field E and then a
Figure 1.2. Voronoi diagram of 10 random nodes and the position of dual node distribution.

Voronoi diagram of 10 random nodes and dual node distribution (a) on the Voronoi vertices, and (b) on the edge center of the Voronoi diagram.

dual node distribution for the magnetic field $\mathbf{H}$ can be generated, for example using a Voronoi decomposition [23]. Figure 1.2 shows the Voronoi diagram for 10 randomly distributed node. The vertices or the edge centers of the Voronoi diagram can be used as a second (dual) node distribution (denoted $\times$ in Fig. 1.2). In this thesis the edge centers of the Voronoi diagram are used to create a dual node distribution since nodes are clearly staggered in space in this case.

Higher accuracy in solving the first-order derivatives in both space and time is the advantage of the staggered meshless RPIM. However, the requirement of a staggered dual node distribution could be seen as mesh generation step, thus defeating the purpose of meshless solution. Furthermore, to avoid spurious solutions the divergence property of the fields needs to be considered. The existence of spurious solution in time-domain meshless RPIM method has been reported in [24]. Recently, divergence-free vector basis functions have been proposed in [25] to enforce the divergence property of the fields in the RPIM. In this thesis, another method, which is a magnetic vector potential technique, is proposed to avoid spurious solutions.

In second-order formulation, one of the field types in Maxwell’s equations is eliminated and the wave equation for the electric or magnetic field is solved. This method has been applied in [26] for analyzing the time-domain wave equation based on a Sheppard approximation which is a weak-form meshless method. Although in this method second-order derivatives need to be solved, the discretization is easier since one single set of nodes is required for one field type calculation. The second field type can then be calculated in a post-processing step. However, the divergence property of the field still needs to be considered to avoid spurious solutions. As an alternative approach in this
1.5 Statement of Original Contribution

category, the solution of the wave equation for the vector potential can be considered, as demonstrated for the finite-difference time-domain method in [27]. An adaptation of this approach for the time-domain meshless RPIM is proposed in the second major part of the thesis.

In the second main part, Chapters 5 and 6 propose non-staggered meshless RPIM using the magnetic vector potential technique. The specific implementation is based on the local calculation of the magnetic vector potential in small support domains. Considering the relation for the magnetic flux density \( \mathbf{B} = \nabla \times \mathbf{A} \) and \( \mathbf{J} \) as a current source, the magnetic vector potential \( \mathbf{A} \) satisfies the inhomogeneous wave equation as

\[
\nabla^2 \mathbf{A} - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}.
\]

Using RPIM to discretize the space derivatives and finite differences to discretize the time derivatives in (1.2) lead to the discretized expression for the magnetic vector potential. In this formulation the divergence property of the fields is automatically enforced, and a single set of nodes is required in the discretized domain for computing the magnetic vector potential. Boundary conditions for the magnetic vector potential can be adapted from the boundary conditions for the electric or magnetic fields. However, applying boundary conditions for the magnetic vector potential is not straightforward since they are related to the derivatives of the magnetic vector potential components.

1.5 Statement of Original Contribution

This thesis includes several original contributions in meshless RPIM. The original contributions can be described in two major parts. The first main part of the thesis is focused on the staggered meshless RPIM. The second main part of the thesis investigates the non-staggered meshless RPIM.

1.5.1 Staggered Meshless RPIM

This section lists the original contributions of the first major part of the thesis, which is focused on the staggered meshless RPIM.

- The behavior of two different types of basis functions for the meshless RPIM is investigated. A 2D test function is interpolated through Gaussian and Wendland
basis functions and the approximation errors on the low-order derivatives of the test function are calculated. It is shown that the Gaussian basis function is more appropriate for the interpolation in small support domains, whereas Wendland basis function is more accurate for larger support domains. The results were presented at the *International Conference on Computational Electromagnetics (iCCEM)*, 2015 and is published in the proceedings under the title “On the choice of basis functions for the meshless radial point interpolation method with small local support domains” [4].

- The time-domain behavior of the Uniaxial Perfectly Matched Layer (UPML) in the 3D meshless RPIM is investigated. It is theoretically shown that the UPML will become unstable after a very long time, when the energy in the computational domain almost completely vanishes. A timed introduction of loss terms in the equations inside the UPMLs, i.e. at a time after absorption of most of the energy, can significantly delay or even remove the occurrence of this instability without compromising the accuracy of the solution. The proposed method was presented at the *Asia-Pacific Microwave Conference (APMC)*, 2013 and is published in the proceedings under the title “On the late-time instability of perfectly matched layers in the meshless radial point interpolation method” [10].

- The effect of different node distributions on the accuracy of electromagnetic simulations performed with meshless method is investigated. As a test case, a rectangular waveguide truncated with Perfectly Matched Layers (PMLs) is simulated using the 3D meshless RPIM in the time domain. For the discretization of the geometry, different strategies of node distribution are utilized, namely, a uniform grid distribution, a cylindrical node distribution, and disturbed grid distributions with random displacements amounting to 5% and 10% of the grid average node distance. All distributions are generated with a similar node density, and the results are compared in terms of phase and amplitude of the propagating wave. The application of RPIM in all cases demonstrates similar levels of error, which indicates the robustness of this meshless algorithm with respect to different node distribution strategies. The results of the study were presented at the *Asia-Pacific Microwave Conference (APMC)*, 2012 and are published in the proceedings under the title “Impact of different node distributions on the meshless radial point interpolation method in time-domain electromagnetic simulations” [11].
1.5 Statement of Original Contribution

- The capability of the meshless RPIM for implementing several arbitrarily oriented regular grids is investigated. For this purpose, a waveguide Y-junction diplexer is simulated with a 3D RPIM time-domain solver. The result of the study was published in a book chapter under the title “Conformal and Multi-Scale Time-Domain Methods: From Unstructured Meshes to Meshless Discretizations”, published by Springer-Verlag [1].

- The scattering from a perfect electrically conducting sphere is implemented using 3D meshless RPIM and the simulated far-field patterns are compared with the theoretical results. To this end, the perfect electric sphere is simulated in a 3D domain, which is truncated by the PMLs. To compute the far-field pattern, first the simulated electric and magnetic current densities are obtained on a so-called Huygens’ surface. Then using a near-field to far-field transformation, far-field scattering from a conductive sphere is obtained and compared to the theoretical Mie solution. It is shown that simulation and theoretical results are in a good agreement.

1.5.2 Non-Staggered Meshless RPIM

This section lists the original contributions of the second major part of the thesis, which is focused on the non-staggered meshless RPIM.

- A time-domain vector potential solver for the analysis of transient electromagnetic field with the meshless RPIM is proposed. Solving the second-order vector potential wave equation enforces the divergence-free property of the fields and avoids computational effort related to a dual node distribution required for solving the first-order Maxwell equations. The proposed method is validated with several examples of 2D parallel plate waveguide and filters, and the convergence is empirically demonstrated in terms of node density or size of local support domains. It is further shown that inhomogeneous node distributions can provide accelerated convergence, i.e., the same accuracy with smaller number of nodes compared to a solution for homogeneous node distribution. The proposed method is published in International Journal for Numerical methods in Engineering (IJNME) under the title “Time-domain vector potential technique for the meshless radial point interpolation method” [2].
Chapter 1

Introduction

- A hybrid algorithm for the implementation of PMLs in the meshless magnetic vector potential technique is proposed. Solving the wave equation in time-domain, the magnetic vector potential technique avoids using staggered node distributions which are needed for calculating the $E$ and $H$ fields when directly solving Maxwell’s equations. However, implementing PMLs with stretched coordinate formulation requires auxiliary variables on a staggered (dual) node distribution. To avoid defining staggered nodes in the whole computational domain, a hybrid algorithm is proposed: The algorithm keeps a single set of nodes for the magnetic vector potential $A$ inside the free space, while it uses staggered nodes for $A$ and auxiliary variables inside the PML. The hybrid algorithm is validated in a 2D rectangular waveguide and numerical reflection coefficients are compared for different thicknesses of the PML and for different orders of a polynomial conductivity profile inside the PML. A good agreement between theoretical results and converged solutions validates the approach, with best performance using a polynomial order $m = 3$. The proposed method was presented at the International Workshop on Antenna Technology (iWAT), 2014 and is published in the proceedings under the title “Hybrid staggered perfectly matched layers in non-staggered meshless time-domain vector potential technique” [6]. This paper was awarded the best student paper award of iWAT 2014.

- Two different approaches for the time-domain meshless RPIM in electromagnetic simulations are compared. These two algorithms are categorized based on the order of partial differential equations which are needed for solving electromagnetic problems. Then the advantages and issues related to those two types of formulations are discussed. The results were presented at the 1st Australian Microwave Symposium (AMS), 2014 and is published in the proceedings under the title “First- and second-order meshless radial point interpolation methods in electromagnetics” [7]. This paper was awarded the best student paper award of AMS 2014.

- The performance of the staggered and non-staggered RPIM for time-domain electromagnetics are compared. As an illustrative example, the scattering parameters of a waveguide filter are computed using both staggered and non-staggered RPIM. The accuracy of the results obtained with both methods is assessed through comparison with a reference solution. The results were presented at the 17th International Symposium on ElectroMagnetic Compatibility (CEM), 2014 and is published in the proceedings under the title “On the staggered and non-staggered
time-domain meshless radial point interpolation method” [8]. This paper was awarded the best conference paper award of CEM 2014.

1.6 Overview of the Thesis

As outlined in Fig. 1.3 the thesis encompasses five parts, including background, two major parts that describe the original contributions of the study, one comparison part, and the conclusion. The detailed description for each part of the thesis is as follows.

**Background (Chapters 1 & 2 & 3)** includes the current introductory chapter which presents a motivation for meshless method, objectives of this study and the thesis outline, as well as Chapter 2 that reviews different meshless methods, and Chapter 3 that provides the context and background information required for the rest of the thesis. This includes the history and advantages of meshless methods compared to mesh-based methods. This part also discusses the concept of RPIM. Finally, a comparison of Gaussian and Wendland basis functions is presented at the end of Chapter 3.

**Staggered Meshless RPIM (Chapter 4)** is focused on the formulation for the staggered meshless RPIM. Moreover, the UPML is introduced to absorb the energy in the boundaries and the update equations for the staggered meshless RPIM in the presence of UPML are introduced. Also, it is shown that UPML has long time instability. The long-time instability of the UPMLs is solved by introducing loss terms in the UPMLs. Moreover, the impact of different node distributions in meshless method is investigated. Also, the ability of meshless method to discretize the domain with different node arrangements is shown for a diplexer example. Finally, a conducting sphere is implemented in 3D and the far field pattern is compared with the theoretical results.

**Non-Staggered Meshless RPIM (Chapter 5)** introduces a meshless magnetic vector potential technique. The method is proposed with convergence study and several examples for validation. This chapter also introduces a hybrid algorithm to implement staggered PMLs in a non-staggered meshless RPIM framework.

**Comparison of Staggered and Non-Staggered Meshless RPIM (Chapter 6)** This chapter compares the conventional staggered meshless RPIM and the proposed non-staggered meshless RPIM.
Figure 1.3. Thesis outline. The thesis is composed of 7 chapters including background and conclusion. The original contributions are distributed over Chapter 3, Chapter 4, Chapter 5 and Chapter 6.

Conclusion (Chapter 7) summarizes the results of this thesis and recommends possible directions for future work.
THE different numerical methods for the solution of partial differential equations can be classified into two categories: mesh-based and meshless methods. This chapter reviews these two categories for the numerical solution of Maxwell’s equations. In mesh-based methods two representative methods, namely, the Finite-Difference Time-Domain (FDTD) method and the Finite Element Method (FEM), are introduced briefly. In meshless methods a historical overview of the approach is first presented. Then the solution procedure in meshless methods is described. Moreover, different meshless methods are introduced and classified in terms of formulation and approximation schemes. Finally, variations of meshless methods with different shape functions are introduced.
2.1 Introduction

There are many different numerical methods in electromagnetics. These techniques can be classified by whether they are based on integral or differential equations, whether they operate in the time or frequency domain, and whether they are mesh-based or meshless methods. This chapter describes mesh-based and meshless methods.

In the first part of this chapter mesh-based methods are reviewed. In the mesh-based category, the Finite-Difference Time-Domain (FDTD) method and the Finite Element Method (FEM) are explained and their applications in electromagnetics are provided. Then in the second part, after a brief introduction to meshless methods, main steps in meshless methods are explained. Also, different meshless methods are classified in two different ways and finally some meshless methods with various shape functions are introduced.

2.2 Mesh-based Methods

In mesh-based methods, the computational domain is discretized in a partition of numerous finite volumes, elements, or cells. Note that the mesh discretization needs to be generated in a pre-processing step or dynamically modified in adaptive meshing methods as the solution progresses. Maxwell’s equations are then approximated on the elements of the structure and a system of equations is obtained. In order to find the unknown solution, the system of equations is solved using appropriate numerical methods. Some important numerical mesh-based methods are: FDTD, Method of Moment (MoM), and FEM.

In 1966, Yee introduced the FDTD method in electromagnetic modeling [28]. He proposed a special algorithm for discretizing the space of electromagnetic problems. Since then many other finite-difference algorithms have been introduced to discretize Maxwell’s equations, but none of them have been as robust as Yee’s method [29]. This method being implemented in the time domain, the responses in a chosen frequency range can be computed in a single simulation run. Also, non-linear responses can be naturally calculated by this type of methods [29].

The MoM in electromagnetics was introduced as one of the first numerical methods for the analysis of antennas and scatterers. This method discretizes the integral equations. It is generally based on surfaces and currents, whereas FEM and FDTD are
based on volumes and fields. This means that in the classical form of MoM, only the material interfaces are discretized and solved, whereas for FEM and FDTD all volumes of interest must be discretized [30]. The general idea of MoM was introduced by Boris Grigoryevich Galerkin before 1920. In electromagnetics, MoM was first utilized by Nomura in 1952 [31] and Storm in 1953 [32] for linear antennas. Later, the mathematical foundations of the MoM were introduced in 1964 by Kantorovich, et al. [33, 34]. The MoM became popular following the work of Harrington, who proposed a systematic approach for the method in 1967-1968 [35, 36].

FEM is a more powerful method than FDTD and MoM for complex geometries and inhomogeneous media [18]. Although FEM was introduced by Courant in 1943 [37], it has not been widely used in electronic engineering until 1969, when FEM found its application in electromagnetics [38, 39]. Today, FEM is widely adopted and powerful commercial FEM software tools are available.

In this section, the FDTD method and the FEM are reviewed since some of their concepts are used in the meshless methods as well.

### 2.2.1 Finite-Difference Time-Domain (FDTD)

The FDTD method, which was introduced for electromagnetics in 1966, uses Yee’s algorithm for spatial discretization [28]. Yee’s algorithm defines a staggered grid for the electric and magnetic field components. A schematic of the algorithm for one dimensional problems is shown in Fig. 2.1 and the general staggered cell arrangement for 3D discretization is shown in Fig. 2.2. In this algorithm, the electric field is sampled at full steps in both space and time and the magnetic field is sampled at half steps in both space and time as shown in Fig. 2.1. Figure 2.2 (b) shows the arrangement of the electric and magnetic field components for one hexahedral cell as applied in 3D problems.

In the next step, the governing Partial Differential Equations (PDEs) are approximated through finite-difference technique in each cell to solve Maxwell’s equations. It is noted that the finite differences are calculated at discrete points, whose locations are determined by the cell arrangement. Then boundaries and initial conditions are introduced in the algorithm. Using the finite-difference approach, the solutions of the PDEs can be reduced to a time-stepping algorithm which take the form of explicit recursive equations, and thus can be easily implemented in a computer.
2.2 Mesh-based Methods

Figure 2.1. Yee’s algorithm in 1D. Yee’s algorithm for 1D wave propagation.

Figure 2.2. Yee’s algorithm in 3D. (a) E and H cells, (b) electric and magnetic fields in one cell for Yee’s algorithm in 3D.

Finite Difference Methods (FDMs)

There are three methods to approximate partial derivatives by finite differences: forward difference, backward difference and central difference.

**Forward Difference**  The approximation of a derivative of a function \( f \) at a point \( x_0 \) by forward finite difference is given by

\[
f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},
\]

as shown in Fig. 2.3 (a). It can be shown that this approximation is numerically instable, i.e., results in a solution growing with no bounds when applied recursively. Also, from Taylor series expansion it can be shown that this technique has first-order accuracy.
Chapter 2  
Overview of Mesh-based and Meshless Methods

Figure 2.3. Finite difference techniques. (a) Forward difference, (b) backward difference, and (c) central difference.

**Backward Difference** The backward finite-difference approximation of a derivative is shown in Fig. 2.3 (b) and is defined through

\[ f'(x_0) \approx \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}. \]  

(2.2)

This technique is always stable and it has first-order accuracy. However, its application does not lead to an explicit update equation.

**Central Difference** The central finite-difference approximation of a derivative is given by

\[ f'(x_0) \approx \frac{f(x_0 + \Delta x/2) - f(x_0 - \Delta x/2)}{\Delta x}, \]  

(2.3)

as shown in Fig. 2.3 (c). It can be shown that this technique has second-order accuracy and is conditionally stable. In other words, if \( \Delta x \) is smaller than a given stability condition, the solution will remain stable. Moreover, the central finite-difference approximation is perfectly fitted for implementation into the staggered grid in space and time. The FDTD method is based on this technique and thus has second-order accuracy. It is worth mentioning that central finite differences are used to discretize the time derivatives for meshless Radial Point Interpolation Method (RPIM) in this thesis.

**Application of FDTD in Electromagnetics**

Maxwell’s curl equations (1.1) (a) and (b) in Cartesian coordinates result in the following six equations in 3D

\[ \frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right), \]  

(2.4)
2.2 Mesh-based Methods

Figure 2.4. $H_x$ is surrounded by $E_z$ and $E_y$. $H_x$ is surrounded by circulating arrangement of $E_z$ and $E_y$ in the Yee's algorithm.

\[
\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right),
\]

(2.5)

\[
\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right),
\]

(2.6)

\[
\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x - J_x \right),
\]

(2.7)

\[
\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y - J_y \right),
\]

(2.8)

\[
\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z - J_z \right).
\]

(2.9)

Recursive update equations are obtained for electric and magnetic fields by applying central difference approximations to all space and time derivatives in Yee’s grid. For example, using central differences in equation (2.4) in a grid shown in Fig. 2.4 at time $t = n\Delta t$ and position $(i\Delta x, (j + 1/2)\Delta y, (k + 1/2)\Delta z)$ results in

\[
\frac{H_x^n|_{i,j+1/2,k+1/2} - H_x^n|_{i,j+1/2,k+1/2}}{\Delta t} = \frac{1}{\mu} \left( \frac{E_y^n|_{i,j+1/2,k+1} - E_y^n|_{i,j+1/2,k}}{\Delta z} - \frac{E_z^n|_{i,j+1,k+1/2} - E_z^n|_{i,j+1,k+1/2}}{\Delta y} \right),
\]

(2.10)
Chapter 2  Overview of Mesh-based and Meshless Methods

Figure 2.5. $E_x$ is surrounded by $H_z$ and $H_y$. $E_x$ is surrounded by a circulating arrangement of $H_z$ and $H_y$ in Yee’s algorithm.

where the superscript shows the index for time step and the subscripts are the indices for the node position. An update equation for $H_x$ is then obtained as follows

$$H_x|i,j,k+1/2 = H_x|i,j,k+1/2 + \frac{\Delta t}{\mu} \left( \frac{E_x|i,j+1/2,k+1 - E_x|i,j+1/2,k}{\Delta z} - \frac{E_z|i,j+1,k+1/2 - E_z|i,j+1,k+1/2}{\Delta y} \right).$$

(2.11)

In a similar way, applying central differences in (2.7) in a grid shown in Fig. 2.5 results in

$$\frac{E_x|i,j+1/2,k+1 - E_x|i,j+1/2,k}{\Delta t} = \frac{1}{\epsilon} \left( \frac{H_z|i+1/2,j+1/2,k - H_z|i+1/2,j-1/2,k}{\Delta y} \right) - \frac{H_y|i+1/2,j+1/2,k - H_y|i+1/2,j-1/2,k}{\Delta z} - \sigma \frac{E_x|i+1/2,j,k - E_x|i+1/2,j,k+1/2}{\Delta z} - J_x|i+1/2,j,k).$$

(2.12)

and an update equation for $E_x$ is obtained as follows

$$E_x|i,j+1/2,k+1 = \frac{\epsilon - \sigma \Delta t}{\epsilon + \sigma \Delta t} E_x|i,j+1/2,k + \Delta t \left( \frac{H_z|i+1/2,j+1/2,k - H_z|i+1/2,j-1/2,k}{\Delta y} \right) - \frac{H_y|i+1/2,j+1/2,k - H_y|i+1/2,j-1/2,k}{\Delta z} - J_x|i+1/2,j,k).$$

(2.13)

The other components for $E$ and $H$ can be found in a similar way. From (2.11) (or (2.13)), it can be easily seen that unknown field $H_x$ (or $E_x$) at time step with index
2.2 Mesh-based Methods

\( n + 1/2 \) (or \( n + 1 \)) can be found from known field values at previous time steps, with indices \( n - 1/2 \) and \( n \) (or \( n \) and \( n + 1/2 \)). Initial values or time-dependent sources are applied to the equations before the start of the iteration.

To achieve good accuracy in simulations, the spatial discretization needs to be chosen smaller than the simulated wavelength. Although decreasing the discretization steps decreases the error, it also decreases the required time step (as mentioned in next paragraph), which slows the time iteration. A good trade off between accuracy and efficiency is a discretization between \( \lambda/10 \) and \( \lambda/20 \).

To have a stable simulation, the time step \( \Delta t \) has to be bounded by a value which is determined by the space discretization. In 3D simulation, the stability condition for the time step \( \Delta t \) is determined by

\[
\Delta t \leq \frac{1}{c \cdot \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}},
\]

where \( c \) is the speed of light in free space, \( \Delta x \), \( \Delta y \) and \( \Delta z \) are the spatial steps in \( x \), \( y \) and \( z \)-directions. This condition is called the Courant-Friedrich-Levy condition [29].

2.2.2 Finite Element Method (FEM)

The FEM found its first application in electromagnetics in 1969 [38], [39]. Basic steps in FEM are as follows [40, 41]

- Domain discretization
  The whole geometry of interest is divided into a partition of elements, which are usually triangles in 2D and tetrahedrons in 3D as shown in Fig. 2.6.

- Selection of interpolation functions
  Interpolation functions provide an approximate solution within an element. They are usually polynomial functions. High-order polynomials are very accurate but their formulations are more complex than low-order polynomials.

- Formulation of equations
  The Rayleigh-Ritz and Galerkin methods can be used to formulate the system of equations. The problem is first formulated in each element. Then, interaction with other elements are considered and a global matrix for the system of equations is found.
Solution of the system of equations
As shown later in this chapter, a solution is based on a matrix inversion. Note that in time-domain solver the matrix has to be inverted at all time steps.

Post processing
The desired parameters of interest are computed, e.g., to characterize the performance of an electromagnetic device, and the results can be displayed in this step.

Since FEM uses variational methods that can be also applied in meshless methods, it is necessary to discuss this type of methods here.

Variational Problems
In mathematics and engineering, some problems described by differential equations under given boundary conditions can be solved by finding the minimum value of some integral so-called variational integral. This kind of problems are called variational or functional problems [18]. There are two methods to solve variational problems [18], namely,

- Direct method: Rayleigh-Ritz,
- Indirect method: method of weighted residuals (e.g., collocation, subdomain, Galerkin, and least square methods).

Both are described in the following.
2.2 Mesh-based Methods

**Rayleigh-Ritz Method**  In the Rayleigh-Ritz method, an approximate solution with unknown coefficients is substituted into the relevant functional equation. This approximate solution is assumed to be a linearly independent set of functions, which is defined over the domain and satisfies the boundary conditions. Unknown coefficients in the approximate solution are determined by minimizing the functional equation with respect to those coefficients. Consequently an approximated solution will be determined. The Rayleigh-Ritz method has two major limitations. These limitations are

- There is no variational equivalent for some problems,
- It is difficult to apply boundary conditions to an approximate solution in some complex geometries.

Therefore, if a suitable functional does not exist for the above method, the weighted residual method will be more appropriate.

**Weighted Residual Method**  The weighted residual method is more general, and it is not limited to variational problems. In this method, the residual, i.e., the error due to the approximate solution, is calculated. Then a weighting function is chosen such that the inner product of the residual and the weighting function is minimized. In discrete form, these equations can be converted to a matrix form. Finally, the unknown coefficients can be calculated from the obtained matrix equation.

To understand how to use the residual of the differential equation to solve the problem, we assume that we start from the equation

\[
L(u) = g, \tag{2.15}
\]

with a linear operator \(L\). In this equation \(u\) is the unknown quantity and \(g\) is the excitation (source). The solution of this equation inside each element can be approximated by the following finite series

\[
u \approx u^h = \sum_{n=1}^{N} a_n f_n, \tag{2.16}\]

Where \(a_n\) are coefficients, \(u^h\) is the approximated value for \(u\), and \(f_n\) are referred as basis or shape functions. The scalar basis function \(f_n\) is assumed to have a unitary value at node \(n\) and zero at other nodes. Accuracy and efficiency of the weighting method is dependent on the selection of the basis functions. Basis functions should satisfy the two conditions below
• Basis functions should satisfy the boundary conditions for the linear operator ($L$),

• Basis functions must result in a complete set to accurately describe the field [18].

The residual $R$ (amount of error due to this approximation) is given by

$$ R = g - \sum_{n=1}^{N} a_n L(f_n). \quad (2.17) $$

To find the best approximation for $u^h$, the residual function should be minimized at all points in the domain through the weighting function $W$. In other words, the inner product of $R$ and $W$ should be zero. An inner product of functions $u$ and $v$ defined on domain $\Omega$ is denoted here as $\langle u, v \rangle$ and can be written as

$$ \langle u, v \rangle = \int_{\Omega} uv^*d\Omega, \quad (2.18) $$

where * shows the complex conjugate. Therefore, we have

$$ \langle R, W \rangle = 0. \quad (2.19) $$

The weighting function $W$ is usually written as finite series

$$ W = \sum_{m=1}^{M} W_m. \quad (2.20) $$

Substituting (2.17) in (2.19) gives $M$ equations of the form

$$ \sum_{n=1}^{N} a_n \langle W_m, L(f_n) \rangle = \langle W_m, g \rangle. \quad (2.21) $$

Equation 2.21 can be rewritten in a matrix form as follows

$$ ZI = V, \quad (2.22) $$

where the matrix elements are as follows

$$ Z_{mn} = \langle W_m, L(f_n) \rangle, \quad (2.23a) $$

$$ I_n = a_n, \quad (2.23b) $$

$$ V_m = \langle W_m, g \rangle. \quad (2.23c) $$

Unknown coefficients $I$ in (2.22) can be found as

$$ I = Z^{-1}V. \quad (2.24) $$
Finally, by substituting the unknown coefficients (2.24) \( \mathbf{t}^T = [a_1, a_2, \ldots, a_N] \) in equation 2.16, an approximated solution will be obtained.

There are different ways to choose weighting functions which lead to: point collocation method, subdomain collocation method, Galerkin’s method, and least squares method [18]. As a widespread implementation, in the Galerkin’s method, the weighting functions \( W_m \) are same as basis functions \( f_n \).

**Vector Elements in FEM**

Scalar basis functions \( f_n \) (nodal elements) can be used to find scalar functions and they are not suitable for finding vector functions. Using nodal elements to represent vector fields causes some problems such as the inconvenience of imposing boundary conditions and the occurrence of non-physical or spurious solutions [41]. Therefore, edge or vector elements, which are related to the edges of the cells rather than to their corner nodes, were proposed for solution involving vector fields. With edge elements, applying boundary conditions is much easier for vector fields, and by selecting divergence-free vector elements, nonphysical solutions are canceled automatically.

Figure 2.7 shows an illustration of a triangular cell in FEM discretization. Vector elements use simplex coordinates, which for a point \( P \) inside the triangle are defined as

\[
\lambda_1 = \frac{\text{area} \Delta(P23)}{A}, \quad \lambda_2 = \frac{\text{area} \Delta(P13)}{A}, \quad \lambda_3 = \frac{\text{area} \Delta(P12)}{A},
\]

(2.25)

where \( A \) is the total area of the triangle. Therefore, the vector element basis function can be calculated as

\[
W_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i,
\]

(2.26)

where \( W_{ij} \) is related to the edge from node \( i \) to node \( j \), and

\[
\nabla \lambda_i = \frac{l_i}{2A} \hat{r}_i,
\]

(2.27)
Figure 2.8. Vector basis functions for a triangle in FEM. Vector basis functions (a) $W_{12}$, (b) $W_{23}$, and (c) $W_{13}$.

where $l_i$ is the length of edge $i$ and $\hat{n}_i$ is the normal to edge $i$. Three vector elements for one triangle are shown in Fig. 2.8. Each vector element has a constant tangential component only along one edge. The divergence and curl of the vector elements are as follows

$$\nabla \cdot W_{ij} = \nabla \cdot (\lambda_i \nabla \lambda_j) - \nabla \cdot (\lambda_j \nabla \lambda_i) = 0, \quad (2.28a)$$

$$\nabla \times W_{ij} = \nabla \times (\lambda_i \nabla \lambda_j) - \nabla \times (\lambda_j \nabla \lambda_i) = 2\nabla \lambda_i \times \nabla \lambda_j, \quad (2.28b)$$

In summary, vector elements are divergence free and their curl can be calculated easily. As we will discuss later, the concept of vector element is used in meshless method as well. For example, in meshless RPIM method divergence-free basis functions were introduced in [25].

Application of FEM (with Galerkin’s Method) in Electromagnetics

For the solution of the wave equation

$$\nabla \times \left( \frac{1}{\mu} \nabla \times E \right) - \omega^2 \epsilon E = -j \omega J, \quad (2.29)$$

the linear operator is

$$L = \nabla \times \left( \frac{1}{\mu} \nabla \times \right) - \omega^2 \epsilon. \quad (2.30)$$

By applying weighting residual approach, we have

$$\langle W, R \rangle = 0, \quad (2.31)$$

or

$$F(E) = \langle W, g - L(E) \rangle = 0, \quad (2.32)$$
where \( g = -j\omega J \). We assume that \( E \) is approximated by

\[
E = \sum_{i=1}^{N} e_i \hat{\phi}_i, \tag{2.33}
\]

with basis functions \( \hat{\phi}_i \), and coefficients \( e_i \), which are determined numerically. Substituting (2.30) in (2.32) gives

\[
F(E) = \iiint_\Omega W \cdot \left( \nabla \times \left( \frac{1}{\mu} \nabla \times E \right) - \omega^2 \varepsilon E \right) \, d\Omega + j\omega \iiint_\Omega W \cdot J \, d\Omega, \tag{2.34}
\]

and after some algebraic simplifications the following equation is obtained

\[
F(E) = \iiint_\Omega \left( \frac{\nabla \times W \cdot \nabla \times E}{\mu} - \omega^2 \varepsilon W \cdot E \right) \, d\Omega - \frac{1}{\mu} \iiint_\partial\Omega (W \times \nabla \times E) \cdot \hat{n} \, dS \\
+ j\omega \iiint_\Omega W \cdot J \, d\Omega. \tag{2.35}
\]

Since weighting function and basis functions are identical in Galerkin’s method the electric field \( E \) can be written as

\[
E = \sum_{i=1}^{N} e_i W_i. \tag{2.36}
\]

The matrix form of (2.35) with \( k_0 = \omega \sqrt{\varepsilon} \) is as follows

\[
A e - k_0^2 B e + C e = f_e, \tag{2.37}
\]

where

\[
e = (e_i), \tag{2.38}
\]

\[
A_{ij} = \frac{1}{\mu} \iiint_\Omega \nabla \times W_i \cdot \nabla \times W_j \, d\Omega, \tag{2.39}
\]

\[
B_{ij} = \iiint_\Omega W_i \cdot W_j \, d\Omega, \tag{2.40}
\]

\[
C_{ij} = \frac{1}{\mu} \iiint_\partial\Omega W_i \times \nabla \times W_j \, dS, \tag{2.41}
\]

\[
f_e = j\omega \iiint_\Omega W \cdot J \, d\Omega. \tag{2.42}
\]

Unknown coefficients \( e_i \) can be calculated from equation 2.37, and consequently the solution for the electric field will be found from equation 2.36.
Chapter 2

Overview of Mesh-based and Meshless Methods

2.2.3 Limitations of Mesh-based Methods

In spite of their many advantages, mesh-based methods have some issues. They often do not have enough flexibility for complex geometries, need large computer resources due to the mesh generation, and might only reach a low-accuracy solution in some problems. For example, FDTD cannot effectively and accurately simulate problems that have features at multiple scales, such as resolving small details in an electrically large structure. Moreover, the accuracy of mesh-based methods depends on the type and quality of the mesh that is used to discretize the computational domain. Moreover, it is difficult to generate and modify the mesh since the mesh cells are not independent and they have an interconnected structure. It is worth mentioning that some mesh-based methods have been improved to locally increase the accuracy. For example, in [42], a FDTD variation denoted as “zoom” FDTD has been introduced to discretize the geometry with different discretizations, e.g., a fine discretization for small dimensions and a coarse discretization for the rest of the domain. The Equivalent Circuit (EC) FDTD has also been used in [43] to discretize the geometry of interest by non-uniform meshes. However, the last mentioned issue of mesh-based methods, which is mesh generation and interconnection between meshes, is still remaining.

In order to bypass the mesh generation operation which is a complex and time consuming process, meshless methods were proposed recently as an alternative to mesh-based methods.

2.3 Meshless Methods

Meshless methods have received considerable attention in different fields of study including solid mechanics, fluid dynamics, and electromagnetics. A major advantage of meshless methods is that they rely on node distributions rather than an explicitly defined mesh. Therefore, some difficulties associated with mesh generation can be eliminated. In fact, because of the unstructured nature of the meshless methods, a comparison with a structured mesh-based method would be heavily problem-dependent. For example, for a well-behaved problem, FDTD will be the most efficient approach. However, complex multi-scale geometries can be discretized by meshless methods more efficiently, e.g., using locally higher node densities. Moreover, since meshless methods are based on arbitrary node distributions and each node can be defined independently, a costly and time-consuming re-meshing process can be replaced by a local adaptive approach.
2.3 Meshless Methods

node refinement [44]. In contrast, in mesh-based methods a re-meshing process is necessary for moving material discontinuity or refining meshes.

However, meshless methods in science and engineering especially in electromagnetics are in their early stages. Therefore, they still require more developments to find their places among more established and robust techniques based on Finite Difference (FD), Finite Element (FE), and Finite Volume (FV) algorithms. Meshless methods are also called meshfree, gridless, element-free, or cloud methods in the literature [45].

2.3.1 Historical Overview

The idea of mesh-free method comes from Gingold [46] and Lucy [47] in 1977. They introduced a Lagrangian method based on the kernel estimates method to model astrophysics problems such as exploding stars. This method, called Smoothed Particle Hydrodynamics (SPH) [48, 49, 50], was later widely extended in particular for continuum solid and fluid mechanics. A decade after the introduction of the SPH method, the mesh-free method was introduced by Nayroles and Touzot in 1992 [51]. Their method, which is based on Galerkin’s technique, is called Diffuse Element Method (DEM). In the DEM algorithm, the FEM interpolation was replaced within an element by the Moving Least Square (MLS) local interpolation [52]. In 1993, the DEM was applied to model two-dimensional static problems by Yves Maréchal [53]. Since then different meshless methods have been developed with many different names, including the wavelet Galerkin method [54], Element-Free Galerkin (EFG) [55, 56, 45], particle-in-cell method [57], Reproducing Kernel Particle Method (RKPM) [58, 59], Partition of Unity (PU) [60, 61], hp clouds [62, 63], Finite Point Method (FPM) [64, 65], free-mesh method [66], meshless local boundary integration equation method, Meshless Local Petrov-Galerkin method (MLPG) [67, 68], and multi-scale methods [69, 70, 71, 72]. These different meshless methods will be briefly introduced and classified later in this chapter.

2.3.2 Main Steps in Meshless Methods

The main steps to find the solution in meshless methods are as follows

- Domain representation and node generation
  In this step, the computational domain and its boundary are discretized by a set of scattered nodes. The density of nodes depends on the desired accuracy.
• Function approximation
In this step, the field or function can be approximated through global or local basis functions. In global approximation, the field variable $u$ at any point $x$ in the domain is approximated using the field values at the rest of nodes in the domain as follows

$$u(x) = \sum_{i=1}^{n} \phi_i(x)u_i,$$  \hspace{1cm} (2.43)

where $n$ is the total number of nodes in the domain, $\phi_i(x)$ is the shape function of the $i$th node and $u_i$ is the value of the function $u$ at the $i$th node.

To reduce the computational cost a localized formulation can be used. In local approximation, the approximation (2.43) at each node $x$ is limited to the nodes in the vicinity of that particular node. To this end, the domain is decomposed into a large number of small subdomains. Each subdomain is associated with a center node $I$ and is called support domain. A local support domain for a center node can have different number of nodes and different shapes. The most commonly used support domains are circles and rectangles in 2D as shown in Fig. 2.9, and spheres in 3D. In this method, (2.43) is applied for each support domain separately. In this case, $n$ is the number of nodes in a support domain around point $x$.

• Formulation of equations
The discrete equations are formulated using strong or weak form as introduced in the next section, and the matrix form of the global system is built.
2.3 Meshless Methods

- Solution of the equations
  The matrix form of system of equations are solved and unknown coefficients are obtained.

2.3.3 Classification of Meshless Methods

Many variations of meshless methods have been proposed over the past years. They can be classified in different ways. In this subsection we classify them in two different ways: classification based on formulation procedure and classification based on function approximation schemes.

Classification based on Formulation Procedure

Numerical methods are generally classified into two large categories for solving PDEs called direct approach (or strong form methods) and indirect approach (or weak form methods). In direct approach the PDEs are discretized and solved directly. FDMs are in this category. In indirect approach, PDEs with derivative boundary conditions are first formulated with an alternative weak-form system of equations, usually in an integral form. The approximate solution is then determined through a numerical integration in the problem domain. FEM is an example of weak form methods. Both strong and weak forms are used in meshless methods.

There are some important papers for meshless weak-form methods. Nayroles et al. [51] applied the MLS approximation to the Galerkin weak-form and formulated the DEM. Then DEM has been developed by Belytschko et al. [55] for mechanics problems. Since then different meshless weak-form methods have been developed [73] such as: the EFG method, RKPM, the MLPG [74], hp-cloud method [75], and the Partition of Unity Finite Element Method (PUFEM) [61] as shown in Fig. 2.10.

In the strong-form meshless methods, which are also called meshless collocation methods, the PDEs are directly discretized at the field nodes using simple collocation techniques. Some typical meshless strong-form methods are: General Finite Difference Method (GFDM) [76, 77, 78, 79, 80], meshless collocation methods [81, 82, 83, 84, 85, 86], and the FPM [64, 87, 88]. The meshless strong-form methods are attractive since they have a simple algorithm, high efficiency and they are truly meshless. However, solving problems with derivative boundary conditions using strong-form methods may suffer from instability. There are however some techniques to apply derivative
boundary conditions. These strategies are: introducing dummy nodes, using Hermite shape functions, or using a regular node distribution on the boundaries with derivative boundary conditions [73].

It is also worth mentioning that there are some meshless methods that are based on the combination of weak and strong-form techniques [89]. In these methods both forms are used to discretize the system of equations. For example, in [90] the weak-form formulation is used for nodes close to the boundaries with derivative boundary conditions, while the rest of the nodes use strong-form formulation. In this thesis a collocation strong form formulation is used for RPIM.

Classification based on Function Approximation Schemes

In different meshless methods, various techniques are used to construct the approximation functions. Liu [90] classified these approximation techniques into three large categories. These three major categories, which are illustrated in Fig. 2.11, are

- Integral representation methods
- Series representation methods
- Differential representation methods

Figure 2.12 shows some of the meshless methods belonging to each these three categories. In this thesis meshless RPIM, as highlighted in the shaded rectangle in Fig. 2.12, is used.
2.3 Meshless Methods

\[ f(x) = \int_{x_1}^{x_2} f(\xi)W(x - \xi)d\xi \]

\[ f(x) = a_0 + a_1p_1(x) + a_2p_2(x) + \cdots \]

\[ f(x) = f(x_0) + f'(x_0)(x - a) + \frac{1}{2}f''(x_0)(x - a)^2 + \cdots \]

Figure 2.11. Different methods for function representation. Function representation at \( x \) using the information in the vicinity of \( x \) based on (a) integral representation with weight function \( W \), (b) series representation, and (c) differential representation.

Figure 2.12. Classification of meshless methods. Classification of meshless methods based on interpolation techniques. In this thesis the meshless RPIM, as highlighted in the shaded rectangle, is used.

Recently, some of the generic meshless methods mentioned above have been adapted for time-domain electromagnetics. For example, the EFG method was used for electromagnetic scattering problems in [91]. The SPH has been reformulated for Maxwell’s equations and called smoothed particle electromagnetics in [92]. The meshless method based on Radial Basis Functions (RBFs) in 2D was applied in [20]. The MLPG method was developed for time-domain electromagnetics in [93]. And an improved meshless method was applied in [26] for analyzing the time-domain wave equation.

This thesis focuses on the staggered and non-staggered time-domain meshless RPIM in electromagnetics. RPIM belongs to the series representation category in meshless methods.
2.3.4 Meshless Methods with Different Shape Functions

A common feature in all meshless methods is a weight function. Different weight functions are used in various type of meshless methods. In fact, performance of the method is fundamentally determined by the weight function. In local formulations, the weight function is nonzero within the support domain, which is small compared to the rest of the domain, and zero outside of the support domain.

In this section, different approximations with various weight functions for meshless methods are described. To this end, the approximation of a function $u(x)$ in a domain $\Omega$ is considered. This domain includes a set of nodes $x_i, i = 1, \ldots, n_N$ and the function value at node $i$ is $u_i = u(x_i)$.

**Smooth Particle Hydrodynamics Approach (SPH)**

In the SPH method [46, 47, 48, 94, 95, 96], an integral representation of a function is used. In this method, the exact integral representation of a function $u(x)$ at point $x = (x, y, z)$, which is given by

$$ u(x) = \int_{-\infty}^{\infty} u(\xi) \delta(x - \xi) d\xi, \quad (2.44) $$

is approximated by

$$ u^h(x) = \int_{\Omega_s} u(\xi) W(x - \xi, h) d\xi, \quad (2.45) $$

where $\delta(x)$ is the Dirac delta function, $u^h(x)$ is the approximation of function $u(x)$, $W(x - \xi, h)$ is a kernel or weight or smoothing function, and $h$ is the smoothing length that controls the size of the support domain $\Omega_s$. The minimum requirements for the weight function are compactness and unity conditions, which can be mathematically described as follows

$$ W(x - \xi, h) = 0, \quad \forall \xi \notin \Omega_s, \quad \text{(Compactness)} \quad (2.46a) $$

$$ \int_{\Omega_s} W(x - \xi, h) = 1. \quad \text{(Unity)} \quad (2.46b) $$

However, additional conditions to the weight functions are often used to control stability and convergence of the solution in SPH. The required conditions for the weight function are given in [48].

The discrete form of (2.45) can be obtained using the trapezoidal rule on the right-hand side of the equation, which for 1D can be expressed as follows

$$ u^h(x) = \sum_{i \in \Omega_s} u_i W(x - x_i) \Delta x_i. \quad (2.47) $$
Table 2.1. Commonly used weight functions in SPH. Commonly used weight functions \( W(\bar{s}) \) in SPH, where \( \bar{s} = s/s_{\text{max}} \) and \( s = \| x - \xi \| \).

<table>
<thead>
<tr>
<th>Exponential</th>
<th>Cubic Spline</th>
<th>Quartic Spline</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-{(\bar{s}/s)}^2} ) for ( \bar{s} \leq 1 )</td>
<td>( \frac{2}{3} - 4s^2 + 4s^3 ) for ( \bar{s} \leq \frac{1}{2} )</td>
<td>( 1 - 6s^2 + 8s^3 - 3s^4 ) for ( \bar{s} \leq 1 )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{4}{3} - 4s + 4s^2 - \frac{4}{3}s^3 ) for ( \frac{1}{2} &lt; \bar{s} \leq 1 )</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>for ( \bar{s} &gt; 1 )</td>
<td>for ( \bar{s} &gt; 1 )</td>
</tr>
</tbody>
</table>

The discrete form for multi-dimensions is generally given by

\[
u^h(x) = \sum_{i \in \Omega_s} u_i \underbrace{W(x - x_i)\Delta V_i}_{\phi_i(x)} = \sum_{i \in \Omega_s} \phi_i(x)u_i, \quad (2.48)
\]

where \( \Delta V_i \) is some dimensional measure of the domain (length, area or volume) and \( \phi_i(x) \) is the SPH shape function. It is interesting to notice that most weight functions are bell shaped. Commonly used weight functions in SPH are the exponential, the cubic spline, and the quartic spline. These weight functions are shown in table 2.1. Since SPH was introduced for open problems, applying boundary conditions is difficult in this method. Moreover, SPH method suffers from interpolation inconsistency. Therefore, some modifications have been introduced to increase the accuracy of this method [94]. A correction function for SPH is introduced in the following.

**Reproducing the Kernel Particle Method (RKPM)**

To improve the SPH approximation near the boundaries, a correction function was introduced in [58]. The integral representation with the correction function \( C(x, \xi) \) is given by

\[
u^h(x) = \int_{\Omega_s} u(\xi)C(x, \xi)W(x - \xi, h)d\xi, \quad (2.49)
\]

which after discretization can be written as follows

\[
u^h(x) = \sum_{i \in \Omega_s} u_i \underbrace{C(x, x_i)W(x - x_i, h)\Delta V_i}_{\phi_i(x)} = \sum_{i \in \Omega_s} \phi_i(x)u_i, \quad (2.50)
\]

where \( \phi_i(x) \) is the RKPM shape function [70, 97, 98, 71, 99]. In fact, the weighting function \( W(x - x_i, h) \) in SPH is modified with a correction function \( C(x, x_i) \) to fulfill the required consistency requirements.
Finite Difference Method with Arbitrary Grids

The classical FDM is based on regular node distribution. Therefore, it is generally less accurate in treatment of curved and slanted boundary conditions compared to FEM. To overcome this difficulty the idea of irregular grids has been introduced. The FDM with arbitrary grids, which is also called the General FDM (GFDM) [100, 101, 76, 77, 78, 79, 80], belongs to the strong-form category in meshless methods. In this case, a Taylor series expansion can be used for finite-difference approximation [79]. The Taylor series expansion around a point \((x_0, y_0)\) for function \(u = u(x, y)\) is given by

\[
u = u_0 + h \frac{\partial u_0}{\partial x} + k \frac{\partial u_0}{\partial y} + \frac{h^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{k^2}{2} \frac{\partial^2 u_0}{\partial y^2} + k h \frac{\partial^2 u_0}{\partial x \partial y} + O(\Delta^3),
\]

where \(u_0 = u(x_0, y_0)\), \(h = x - x_0\), \(k = y - y_0\), \(\Delta = \sqrt{h^2 + k^2}\). Equation (2.51) for all nodes in a given domain can be written as

\[
[A]{Du} - \{u\} = \{0\},
\]

where matrix \(A\) is

\[
[A] = \begin{bmatrix}
h_1 & k_1 & h_1^2 / 2 & k_1^2 / 2 & h_1 \cdot k_1 \\
h_2 & k_2 & h_2^2 / 2 & \cdots & h_2 \cdot k_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h_m & k_m & h_m^2 / 2 & \cdots & h_m \cdot k_m,
\end{bmatrix},
\]

and the unknown derivatives are

\[
{Du}^T = \{\frac{\partial u_0}{\partial x}, \frac{\partial u_0}{\partial y}, \frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_0}{\partial y^2}, \frac{\partial^2 u_0}{\partial x \partial y}\}.
\]

The solution can be obtained by minimizing the following norm

\[
B = \sum_{i=1}^{m} \left[ u_0 - u_i + \frac{\partial u_i}{\partial x} + \cdots + \frac{\partial^3 u_i}{\partial x^3} \right] \Delta_i^3
\]

where considering \(\frac{\partial B}{\partial{Du}} = 0\) a set of five equations with five unknowns can be obtained.
Table 2.2. Commonly used polynomial basis functions in MLS. Commonly used basis functions \( p^T(x) \) in the MLS approximation.

<table>
<thead>
<tr>
<th>Name</th>
<th>1D</th>
<th>2D</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear basis</td>
<td>((1, x))</td>
<td>((1, x, y))</td>
<td>((1, x, y, z))</td>
</tr>
<tr>
<td>quadratic basis</td>
<td>((1, x, x^2))</td>
<td>((1, x, y, x^2, xy, y^2))</td>
<td>((1, x, y, z, x^2, xy, y^2, yz, xz, z^2))</td>
</tr>
</tbody>
</table>

Moving Least Squares (MLS) Approximation

The MLS method is one of the most popular meshless interpolation method. This method approximates the function \( u(x) \) at point \( x \) by series representation of the function as follows [45]

\[
u^h(x) = \sum_{j=1}^{m} p_j(x) a_j(x) = p^T(x) a(x),
\]

where \( m \) is the number of monomials, \( a(x) \) is a vector of coefficients that are functions of \( x \)

\[
a^T(x) = \{a_0(x), a_1(x), \ldots, a_m(x)\},
\]

and \( p^T(x) \) is a vector of basis functions. Commonly used basis functions \( p^T(x) \) for 1D, 2D, and 3D are shown in table 2.2.

The approximated function at \( n \) nodes in the support domain of center node \( x \) is as follows

\[
u^h(x, x_I) = p^T(x_I) a(x), \quad I = 1, 2, \ldots, n
\]

then a functional of weighted residual is given by

\[
J = \sum_{i=1}^{n} W(x - x_i) [u^h(x, x_I) - u(x_I)]^2,
\]

where \( u^h(x, x_I) - u(x_I) \) is the residual, and approximated value \( u^h(x_I) \) and nodal value \( u_I = u(x_I) \) are shown in Fig. 2.13. Equation (2.60) can be rewritten as

\[
J = (Pa - u)^T W(x)(Pa - u),
\]
where

$$u^T = (u_1, u_2, \cdots, u_n),$$  

(2.62a)

$$P = \begin{bmatrix}
p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) 
p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) 
\vdots & \vdots & \ddots & \vdots 
p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n)
\end{bmatrix},$$  

(2.62b)

and

$$W(x) = \begin{bmatrix}
W(x - x_1) & 0 & \cdots & 0 
0 & W(x - x_2) & \cdots & 0 
\vdots & \vdots & \ddots & \vdots 
0 & 0 & \cdots & W(x - x_n)
\end{bmatrix}.$$

(2.62c)

In the least squares approximation, $a(x)$ is chosen to minimize the weighted residual as follows

$$\frac{\partial J}{\partial a} = A(x)a(x) - B(x)u = 0,$$

(2.63)

where moment matrix $A$ and matrix $B$ are

$$A = P^T W(x) P,$$

(2.64a)

$$B = P^T W(x).$$

(2.64b)

The unknown coefficient $a(x)$ in (2.63) can be found as

$$a(x) = A^{-1}(x)B(x)u.$$  

(2.65)

Substituting (2.65) in (2.57) results in

$$u^h(x) = p^T(x)A^{-1}(x)B(x)u = \sum_{i=1}^{n} \phi_i^k(x)u_i,$$

(2.66)
where the shape function is

$$\phi^k = [\phi^k_1(x), \ldots, \phi^k_n(x)] = p^T(x)A^{-1}(x)B(x), \quad (2.67)$$

and $k$ is the order of the polynomial basis function. For $k = 0$ the shape function is called Shepard function [102].

To improve the efficiency of the MLS method, a developed interpolation scheme has been introduced in [103].

**Point Interpolation Method (PIM)**

The point interpolation method is in the series representation category. There are two types of point interpolation method based on different forms of basis functions

- Polynomial PIM
- Radial PIM (RPIM)

The polynomial PIM was introduced as an alternative to MLS in [104]. In the polynomial PIM, approximation function is defined by

$$u^h(x) = \sum_{i=1}^{m} p_i(x)a_i = p^T(x)a, \quad (2.68)$$

where $p_i(x)$ is the monomials basis function, $m$ is the number of monomials, and $a_i$ is the unknown coefficients as follows

$$a = \{a_1, a_2, \ldots, a_m\}^T.$$  

The coefficients $a_i$ can be found through point matching technique. The value of $u$ at $i$th node is given by

$$u_i = p^T(x_i)a, \quad i = 1, 2, \ldots, n \quad (2.69)$$

Vector $p^T(x)$ contains a polynomial basis function with order of $p$, which can be expressed for 1D as follows

$$p^T(x) = \{1, x, x^2, \ldots, x^p\}, \quad (2.70)$$

and for 2D as follows

$$p^T(x) = \{1, x, y, x^2, xy, y^2, \ldots, x^p, y^p\}. \quad (2.71)$$
Equation (2.69) results in following $n$ equations for $n$ nodes in the domain

\[ u_1 = \sum_{i=1}^{m} p_i(x_1)a_i = a_1 + a_2 p_2(x_1) + \cdots + a_m p_m(x_1), \]  
\[ u_2 = \sum_{i=1}^{m} p_i(x_2)a_i = a_1 + a_2 p_2(x_2) + \cdots + a_m p_m(x_2), \]  
\[ \vdots \]  
\[ u_n = \sum_{i=1}^{m} p_i(x_n)a_i = a_1 + a_2 p_2(x_n) + \cdots + a_m p_m(x_n), \]  

which can be expressed by the following matrix form

\[ \mathbf{U}^0 = \mathbf{P}\mathbf{a}, \]  

where $\mathbf{U}^0$ holds the values of the function at all the $n$ nodes

\[ \mathbf{U}^0 = \{u_1, u_2, \ldots, u_n\}^T, \]

and $\mathbf{P}$ is given by

\[ \mathbf{P} = \begin{bmatrix} \mathbf{p}^T(x_1) \\ \mathbf{p}^T(x_2) \\ \vdots \\ \mathbf{p}^T(x_n) \end{bmatrix}. \]  

The matrix $\mathbf{P}$ is a square matrix since $n = m$ in the PIM [73]. So, the unknown coefficients $\mathbf{a}$ can be obtained as

\[ \mathbf{a} = \mathbf{P}^{-1}\mathbf{U}^0. \]  

Substituting (2.76) in (2.69), and considering $n = m$ result in

\[ u(x) = \mathbf{p}^T(x)\mathbf{P}^{-1}\mathbf{U}^0 = \sum_{i=1}^{n} \phi_i(x)u_i = \mathbf{\Phi}^T(x)\mathbf{U}^0, \]  

where $\mathbf{\Phi}(x)$ is a vector of shape functions. Polynomial PIM suffers from singularity problem in the interpolation process. To avoid the singular matrix in polynomial PIM, the RBFs were used and RPIM was introduced. RPIM will be described in the next chapter in more detail.

### 2.4 Summary

Numerical methods can be broadly classified into mesh-based and meshless methods. This chapter has first reviewed some mesh-based methods, e.g., FDTD and FEM, in
2.4 Summary

the perspective of comparing their properties with meshless methods. The difficulties associated to mesh-based methods have been discussed briefly and meshless methods were introduced as an alternative approach. Also, different meshless methods have been introduced and categorized based on two different classification schemes. Moreover, meshless methods with various basis functions have been discussed including PIM, as applied in the rest of this thesis. The next chapter will discuss PIM based on RBFs in more detail.
Chapter 3

Radial Point Interpolation Method

This chapter reviews the concept and the fundamental theory of the Radial Point Interpolation Method (RPIM) as basic knowledge for the other chapters of this thesis. This interpolation method is introduced first for globally supported basis functions and then a localized approach using locally supported basis functions is explained. In the second part of this chapter, different local basis functions are considered, with in particular a comparison of the Gaussian and Wendland basis functions for use in RPIM. It is shown that the Gaussian basis functions are more appropriate for the interpolation in small support domains, whereas Wendland basis functions are more accurate for larger support domains.
3.1 Introduction

As shown in the previous chapter, the Point Interpolation Method (PIM) approximates a function at a certain location by interpolation of the function values in scattered nodes within a considered domain. One type of PIM is the polynomial PIM [104]. Using polynomial basis functions is simple and accurate, but may result in singular matrices in the process of interpolation [105, 85]. To overcome the singularity problem, the radial PIM which uses Radial Basis Functions (RBFs) was introduced in [85].

Interpolating with RBFs is an appropriate method for interpolating scattered data. A RBF at a particular node $x_i$ is a function of the distance between the node $x_i$ and interpolating point $x$. Using RBFs for the interpolation has many advantages [106], such as reproducing sufficiently smooth functions with high accuracy, and straightforward extension to higher-dimentional formulations. It is shown in [107] that RBF interpolation is one of the best scheme among various 2D interpolation methods.

The RBFs have been used in various applications including geodesy, geophysics, surveying and mapping, photogrammetry, remote sensing, signal processing, geography, and hydrology [108, 109]. Importantly, Kansa [110, 81] introduced a method for solving Partial Differential Equations (PDEs) using RBFs. Since then, the RBFs have been increasingly used in the numerical solution of PDEs [111, 112, 113, 114, 115, 116, 117, 118, 119]. RBFs-based methods are a viable alternative to more traditional methods, such as finite differences or finite elements, and have several attractive features, such as the flexibility in the choice of node locations and a fast convergence (exponential for some cases).

RBFs can be used in the formulation of PIM and the method is called Radial Point Interpolation Method (RPIM) [85]. The RPIM will be introduced in this chapter.

3.2 Interpolation Method

Among different types of meshless methods based on different interpolation techniques, the RPIM has been selected for investigation in this thesis. In this interpolation method, global or local basis functions can be used, as introduced in the following.
3.2.1 Global Basis Functions

In the global RPIM, the field value at a given position $\mathbf{x}$ can be interpolated based on the field values at $N$ node locations scattered in the entire computational domain. This is realized as follows using RBFs

$$u^h(\mathbf{x}) = \sum_{n=1}^{N} a_n r_n(\mathbf{x}), \quad (3.1)$$

where $a_n$ represent unknown coefficients, $N$ is the number of nodes in the domain, and $r_n(\mathbf{x})$ are the RBFs, which will be discussed later in this chapter. To make sure that the interpolation can reconstruct a linear function and increase the accuracy of the approximation [90], this interpolation can be extended with monomial basis functions $p_m(\mathbf{x})$ as follows

$$u^h(\mathbf{x}) = \sum_{n=1}^{N} a_n r_n(\mathbf{x}) + \sum_{m=1}^{M} b_m p_m(\mathbf{x}) = \mathbf{R}^T(\mathbf{x})\mathbf{a} + \mathbf{P}^T(\mathbf{x})\mathbf{b}, \quad (3.2)$$

where $\mathbf{a}$ and $\mathbf{b}$ are unknown coefficients, which can be expressed in vectorial form as

$$\mathbf{a}^T = \left( a_1, a_2, \cdots, a_N \right)_{1 \times \text{N}} \quad (3.3)$$

$$\mathbf{b}^T = \left( b_1, b_2, \cdots, b_M \right)_{1 \times \text{M}}. \quad (3.4)$$

The vector $\mathbf{P}(\mathbf{x})$ contains polynomial basis function with $M$ terms as shown in the previous chapter in table 2.2

$$\mathbf{P}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots, p_m(\mathbf{x})] = [1, x, y, \cdots], \quad (3.5)$$

and the vector $\mathbf{R}(\mathbf{x})$, which contains the RBFs $r_n(\mathbf{x})$, can be expressed as follows

$$\mathbf{R}(\mathbf{x}) = [r_1(\mathbf{x}), r_2(\mathbf{x}), ..., r_N(\mathbf{x})]. \quad (3.6)$$

As will be discussed in the next section, there are different types of RBFs that can be chosen as $r_n(\mathbf{x})$.

The unknowns coefficients $\mathbf{a}$ and $\mathbf{b}$ in (3.2) can be determined in a pre-processing step as follows: Applying (3.2) for all $N$ nodes in the computational domain results in the following expression

$$\mathbf{U}^e = \mathbf{R}_0 \mathbf{a} + \mathbf{P}_0 \mathbf{b}, \quad (3.7)$$
where \( \mathbf{U}^e = (u_1, u_2, ..., u_N)^T \) holds the field values at \( N \) node locations. The matrices \( \mathbf{R}_0 \) and \( \mathbf{P}_0 \) are then defined as follows

\[
\mathbf{R}_0 = \begin{pmatrix}
    r_1(x_1) & r_2(x_1) & \cdots & r_N(x_1) \\
    r_1(x_2) & r_2(x_2) & \cdots & r_N(x_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    r_1(x_N) & r_2(x_N) & \cdots & r_N(x_N)
\end{pmatrix} \in \mathbb{R}^{N \times N},
\]

and

\[
\mathbf{P}_0 = \begin{pmatrix}
    p_1(x_1) & p_2(x_1) & \cdots & p_M(x_1) \\
    p_1(x_2) & p_2(x_2) & \cdots & p_M(x_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    p_1(x_N) & p_2(x_N) & \cdots & p_M(x_N)
\end{pmatrix} \in \mathbb{R}^{N \times M}.
\]

The equation (3.2) has \( N + M \) variables. The additional \( M \) equations can be considered using the constraint condition \( \mathbf{P}_0^T \mathbf{a} = \mathbf{0} \) \([90, 73]\). Considering this condition and (3.7) the following matrix form can be given

\[
\begin{pmatrix}
    \mathbf{R}_0 & \mathbf{P}_0 \\
    \mathbf{P}_0^T & 0
\end{pmatrix}
\begin{pmatrix}
    \mathbf{a} \\
    \mathbf{b}
\end{pmatrix}
= \begin{pmatrix}
    \mathbf{U}_e \\
    \mathbf{0}
\end{pmatrix},
\]

where the condition \( \mathbf{P}_0^T \mathbf{a} = \mathbf{0} \) results in a square \((N + M) \times (N + M)\) matrix \( \mathbf{G} \). Therefore, the unknown coefficients can be determined as

\[
\begin{pmatrix}
    \mathbf{a} \\
    \mathbf{b}
\end{pmatrix}
= \mathbf{G}^{-1} \begin{pmatrix}
    \mathbf{U}_e \\
    \mathbf{0}
\end{pmatrix}.
\]

Substituting (3.11) in (3.2) gives

\[
u^h(x) = \left[ \mathbf{R}^T(x) \mathbf{P}^T(x) \right] \mathbf{G}^{-1} \begin{pmatrix}
    \mathbf{U}_e \\
    \mathbf{0}
\end{pmatrix} = \Phi(x) \mathbf{U}_e,
\]

where \( \Phi(x) = [\phi_1(x), \phi_2(x), ..., \phi_N(x)] \) is called shape function. Using this shape function, the interpolated value for a function, i.e., \( u(x) \) and its space derivatives can be re-written as

\[
u^h(x) = [\phi_1(x), \phi_2(x), ..., \phi_N(x)] \mathbf{U}_e,
\]

\[
\frac{\partial u^h(x)}{\partial k} = \left[ \frac{\partial \phi_1(x)}{\partial k}, \frac{\partial \phi_2(x)}{\partial k}, ..., \frac{\partial \phi_N(x)}{\partial k} \right] \mathbf{U}_e,
\]

\[
\frac{\partial^2 u^h(x)}{\partial k^2} = \left[ \frac{\partial^2 \phi_1(x)}{\partial k^2}, \frac{\partial^2 \phi_2(x)}{\partial k^2}, ..., \frac{\partial^2 \phi_N(x)}{\partial k^2} \right] \mathbf{U}_e.
\]

Note that these approximations of the space derivatives will be used for the spatial discretization of PDEs in the meshless RPIM method, as described later in this thesis.
Chapter 3  Radial Point Interpolation Method

3.2.2 Local Basis Functions

In the local RPIM, the field value at a given position $x$ is interpolated based on field values at $N_s$ node locations in a surrounding bounded influence domain, instead of all $N$ nodes in the computational domain. This can be expressed as follows

$$u^h(x) = \sum_{n=1}^{N_s} a_n r_n(x) + \sum_{m=1}^{M} b_m p_m(x) = R^T(x)a + P^T(x)b, \quad (3.14)$$

where $N_s$ is the number of nodes in the influence domain around location $x$. In the local RPIM a circular or spherical influence domain (for 2D or 3D) with radius $d_s$ is generally considered around each point. Figure 3.1 shows a rectangular computational domain in 2D with illustration of circular support domains. Note that this figure shows only selected support domains for clarity. To have an accurate approximation, a suitable size of support domain is required. For convenience, the radius $d_s$ of the support domain for node $x_I$ can be defined as

$$d_s = \alpha_0 d_c, \quad (3.15)$$

where $\alpha_0$ is a dimensionless parameter that controls the actual size of the support domain, and $d_c$ is the average nearest node distance in the computational domain. In uniform node distributions, $d_c$ is the minimum distance between two nodes. In non-uniform distributed nodes, $d_c$ can be defined as an average distance of all nodes in the domain. The simplest method to define an average node distance in 1D is as follows

$$d_c = \frac{D_s}{nD_s - 1}, \quad (3.16)$$
3.2 Interpolation Method

where $D_s$ is the dimension of the computational domain and $n_{D_s}$ is the number of nodes in the domain [73]. This definition turns to the following equations for 2D cases

\[ d_c = \frac{\sqrt{A_s}}{\sqrt{n_{A_s}} - 1}, \]  

(3.17)

and for 3D cases

\[ d_c = \frac{\sqrt[3]{V_s}}{\sqrt[3]{n_{V_s}} - 1}, \]  

(3.18)

where $A_s$ and $V_s$ are the area and the volume of the domain, respectively. Also, $n_{A_s}$ and $n_{V_s}$ are the number of nodes in the domain. After finding $d_c$, the size of the support domain $d_s$ can be determined by (3.15). The local support domain around each point can be determined using a kd-tree data structure [120]. This tree splits the space and provides an optimal search strategy to find the nearest neighbors for a point.

The procedure of the locally supported RPIM is shown in Fig. 3.2 and can be summarized as follows: First a kd-tree is built for all nodes in the computational domain. Then, the nearest $N_s$ neighbors within the support domain of each node with radius $d_s$ are found using the kd-tree in previous step. In this step, the matrix $G$ is also calculated and the local shape function $\Phi$ and its derivatives for the considered node are determined. Finally, after performing this step for all nodes and finding all local shape functions, global matrices for $\Phi$ and its derivatives are built using the local shape functions calculated in the previous step.

As described above, local basis functions lead to a large number of small matrices. These small matrices will be used to create a large matrix for the global system. Since each node uses only its $N_s$ nearest neighbors to calculate the local shape function $\Phi$, these global matrices are sparse matrices. It is noted that, although the application of global basis functions is more accurate than the use of local basis functions, it results in a large full matrix. This large matrix has to be inverted, which requires large memory to be stored and large computational time to process. Therefore, using local basis functions will reduce the computational cost and time. In this thesis the local RPIM is used. It is worth mentioning that MoM uses also a full matrix. However, since the discretization is limited to the surface of the structure rather than in the full volume, the implementation of the global full matrix is not cumbersome in the MoM.
3.3 Different Radial Basis Functions (RBFs)

A RBF is a function of Euclidean distance, which in 2D can be written as

$$\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad (3.19)$$

where $(x_0, y_0)$ is a center node position in a support domain. There are different types of RBFs, and a non-exhaustive list can be found in the table 3.1 [121]. In this section, two types of basis functions, which are Gaussian and Wendland basis functions, are studied and compared for RPIM applied in different support domain sizes.

3.3.1 Gaussian and Wendland Basis Functions

Here, Gaussian and Wendland basis functions are studied for RPIM with two different support domain sizes $d_s = 3\Delta x$ and $d_s = 5\Delta x$. The following Gaussian basis function

$$r_n(x) = \exp\left(-\alpha \left(\frac{\rho}{d_s}\right)^2\right), \quad (3.20)$$

is considered as first type of basis function. Gaussian basis functions have in principle an infinite extent, and need to be truncated for application in a compact support domain. In (3.20), $\rho = |x - x_n|$ is the Euclidean distance and $\alpha$ is called the shape
3.3 Different Radial Basis Functions (RBFs)

Table 3.1. Commonly used RBFs. Typical RBFs \( r_n(x) \) for use in the RPIM.

| Name                     | Function with \( \rho = |x - x_n| \) | Parameter |
|--------------------------|----------------------------------------|-----------|
| Gaussian                 | \( \exp\left(-\alpha \left(\frac{\rho}{d_s}\right)^2\right) \) | \( \alpha \) |
| Multi-Quadric (MQ)       | \( (\rho^2 + (ad_s)^2)^q \)            | \( \alpha, q \) |
| Thin Plate Spline (TPS)  | \( \rho^n \)                           | \( n \)    |
| Logarithmic              | \( \rho^n \log \rho \)                | \( n \)    |
| Wendland C^4             | \( \begin{cases} 
(1 - \frac{\rho}{d_s})^6 \left(35\left(\frac{\rho}{d_s}\right)^2 + 18\left(\frac{\rho}{d_s}\right) + 3\right), & 0 \leq \rho \leq d_s \\
0, & \rho > d_s 
\end{cases} \) | \( d_s \) |
| Wendland C^6             | \( \begin{cases} 
(1 - \frac{\rho}{d_s})^8 \left(32\left(\frac{\rho}{d_s}\right)^3 + 25\left(\frac{\rho}{d_s}\right)^2 + 8\left(\frac{\rho}{d_s}\right) + 1\right), & 0 \leq \rho \leq d_s \\
0, & \rho > d_s 
\end{cases} \) | \( d_s \) |

The parameter which controls the flatness of the function. Figure 3.3 shows Gaussian basis functions for different shape parameters \( \alpha \). It can be seen that a smaller value of the shape parameter results in a flatter Gaussian basis function. It has been shown in [24] that flatter basis functions, i.e., lower values of \( \alpha \), yield higher accuracy, while a too-low value leads to unstable time-domain iterations. Therefore, the optimum shape parameter can be chosen as a trade-off between accuracy and stability [122].

The second type of basis function considered is the compactly supported Wendland basis function with degree of \( \lfloor s/2 \rfloor + 3k + 1 \) and smoothness \( 2k \), where \( s \) is the space dimension and \( k \) is a positive integer number. In this study \( s = 2 \) and \( k = 2 \) are chosen. The resulting Wendland basis functions with \( C^4 \) continuity can then be expressed as follows [123]

\[
r_n(x) = \begin{cases} 
\left(1 - \frac{\rho}{d_s}\right)^6 \left(\frac{35}{3}\left(\frac{\rho}{d_s}\right)^2 + 6\left(\frac{\rho}{d_s}\right) + 1\right), & 0 \leq \rho \leq d_s, \\
0, & \rho > d_s 
\end{cases}
\]

(3.21)

where \( d_s \) is the radius of the support domain.

The Gaussian and Wendland basis functions with different support domain sizes are shown in Fig. 3.4. It is noted that here the Wendland basis function is normalized to
3.3.2 Comparing Gaussian and Wendland Basis Functions

To compare the Gaussian and Wendland basis functions, a 2D test function

\[ F(x, y) = \sin(x) \cos(y), \]  

(3.22)
is considered in a computational domain as shown in Fig. 3.5 (a) [4]. The computa-
3.3 Different Radial Basis Functions (RBFs)

Figure 3.5. Geometry and node distribution to approximate test function $F(x, y) = \sin(x) \cos(y)$. (a) Geometry and node distribution of a domain to approximate the test function using local support domains with radius of $d_s$, and (b) the exact value of the test function in the computational domain.

The computational domain is discretized by two interlaced sets of node distributions. Figure 3.5 (b) shows the exact values of the test function in the computational domain. To mimic the procedure used in RPIM, the exact values of the test function are employed in the initial node distribution to interpolate the function values in the secondary nodes as well as their low-order derivatives. The RPIM is applied using either Gaussian or Wendland basis functions. Finally, the interpolation errors for the test function and its derivatives are calculated at the location of the secondary nodes.

Figure 3.6 (a) and (b) shows the interpolation errors for the test function and its derivatives for a support domain size of $3\Delta x$ ($\Delta x = 2\pi/60$) using Gaussian and Wendland basis functions. It can be seen that the errors for the Gaussian basis function is generally one order of magnitude smaller than the errors for the Wendland basis function. Note that the error for the Gaussian basis function can even be decreased by decreasing the shape parameter $\alpha$.

When the support domain size is increased to $5\Delta x$, the interpolation accuracy increases. This is due to the higher number of considered points used for the interpolation. In this case, the Wendland basis function becomes more accurate than the Gaussian basis function as shown in Fig. 3.7. In both different support domain sizes, the accuracy of the interpolation is degraded close to the boundaries since the nodes
Figure 3.6. Interpolation error using Gaussian and Wendland basis functions for support domain size $d_s = 3\Delta x$. Interpolation error for a test function and its low-order derivatives using (a) Gaussian and (b) Wendland basis functions for support domain size $d_s = 3\Delta x$. 

(a) Gaussian  
error $F$  
error $dF/dx$  
error $dF/dy$  
error $d2F/dx2$  

(b) Wendland  
error $F$  
error $dF/dx$  
error $dF/dy$  
error $d2F/dx2$
Figure 3.7. Interpolation error using Gaussian and Wendland basis functions for support domain size $d_s = 5\Delta x$. Interpolation error for a test function and its low-order derivatives using (a) Gaussian and (b) Wendland basis functions for support domain size $d_s = 5\Delta x$. 

3.3 Different Radial Basis Functions (RBFs)
used for interpolation are limited to the computational domain. The implementation of appropriate boundary conditions would mitigate this effect.

For application in computational electromagnetics using meshless RPIM, the first-order spatial derivatives for the electric and magnetic fields in Maxwell’s curl equations have to be approximated through basis functions. Alternatively, second-order derivatives can be utilized for solving the wave equations. In both cases, choosing more accurate basis functions allows to decrease approximation errors in the computational domain.

It is worth mentioning that in time-domain algorithms, small support domains are preferred because of a higher computational efficiency of sparse matrices instead of full matrices. To this aim, truncated Gaussian basis functions appear to be better suited to get more accurate results in the small support domain.

3.4 Conclusion

The RPIM has been introduced in this chapter. In this method, a combination of radial and polynomial basis functions is used to obtain a highly accurate interpolation of a function and its associated derivatives. The variations of this interpolation method using globally and locally supported domain basis functions have been distinguished. Also, the general procedure for implementing local RPIM has been summarized. In the local RPIM only nodes in the local support domain are used for the interpolation at a particular node. Therefore, local RPIM reduces the computational cost and time.

In the second part, two different types of RBFs typically used in the meshless RPIM have been studied. The interpolation errors for a test function and its derivatives have been calculated using Gaussian and Wendland basis functions with different support domain sizes. It has been shown that the Gaussian basis functions have smaller errors for interpolation of a function and its derivatives in small support domains. However, as the size of the support domain is increased, the use of the Wendland basis functions gradually allows to achieve higher interpolation accuracy. The application of local RPIM in time-domain electromagnetics will be shown in the next chapters. Since small support domains are preferred in time-domain methods, the truncated Gaussian basis functions, which have a better accuracy in this case, will be used.
This chapter is dedicated to the staggered meshless Radial Point Interpolation Method (RPIM) in electromagnetics. In the first part of the chapter, the update equations are derived for the staggered electric and magnetic fields in the framework of the first-order meshless RPIM. Moreover, the truncation of the computational domain is introduced and the implementation of a Uniaxial Perfectly Matched Layer (UPML) is explained as the most popular technique to absorb the energy in the simulation of unbounded problems. The update equations for the staggered meshless RPIM are then modified to include a novel implementation of UPML. It is also shown that UPML suffers from late-time instability that can be solved by introducing small loss factors in the UPMLs after almost all the energy in the computational domain is absorbed. Finally, in this chapter some of the properties and capabilities of meshless methods are demonstrated through selected numerical examples.
4.1 Staggered Meshless RPIM in Electromagnetics

In the staggered meshless Radial Point Interpolation Method (RPIM), the first-order spatial derivatives in the Maxwell’s curl equations (1.1) are discretized through the RPIM. The curl operators in the Maxwell’s equations are defined as

\[ \nabla \times \mathbf{E} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix}, \tag{4.1} \]

and

\[ \nabla \times \mathbf{H} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix}. \tag{4.2} \]

Considering the curl definition, the following time-domain equations can be obtained from Maxwell’s curl equations in 3D

\[ \frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right), \tag{4.3a} \]

\[ \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right), \tag{4.3b} \]

\[ \frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right), \tag{4.3c} \]

and

\[ \frac{\partial E_x}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_x \right), \tag{4.4a} \]

\[ \frac{\partial E_y}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - J_y \right), \tag{4.4b} \]

\[ \frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z \right). \tag{4.4c} \]

It is noted that there are both time and space derivatives in the above equations. To implement the staggered meshless RPIM these equations have to be discretized. The discretization of time and space as well as the path for obtaining the update equations for the staggered meshless RPIM are described in the following subsections.

4.1.1 Time Discretization

The time derivatives in the time-domain meshless RPIM method can be approximated by central finite differences in a leapfrog time-stepping scheme similarly as commonly
Time discretization for the electric and magnetic fields in a staggered manner in time.

done in the Finite-Difference Time-Domain (FDTD) method. Figure 4.1 shows the time discretization scheme for the electric and magnetic fields. The electric and magnetic fields are staggered in time in the leapfrog scheme which means that second-order accuracy can be efficiently obtained without requiring additional cost in memory. In this method, the time is divided into intervals \( \Delta t \) and time staggered electric and magnetic fields are then considered as follows: Magnetic fields are sampled at half time steps \((n + \frac{1}{2})\Delta t\) and electric fields are sampled at full time steps \(n\Delta t\) as follows

\[
\begin{align*}
H(t) &\rightarrow H((n + \frac{1}{2})\Delta t) \overset{\text{def}}{=} H^n + \frac{1}{2}, \quad (4.5a) \\
E(t) &\rightarrow E(n\Delta t) \overset{\text{def}}{=} E^n. \quad (4.5b)
\end{align*}
\]

After time discretization, central finite differences are used to discretize time derivatives in equations (4.3) and (4.4) as follows

\[
\begin{align*}
\frac{\partial H^n}{\partial t} &= \frac{H^{n+\frac{1}{2}} - H^{n-\frac{1}{2}}}{\Delta t}, \quad (4.6a) \\
\frac{\partial E^{n+\frac{1}{2}}}{\partial t} &= \frac{E^{n+1} - E^n}{\Delta t}. \quad (4.6b)
\end{align*}
\]

This time discretization scheme will be used to find explicit update equations for the staggered meshless RPIM later in this chapter.

### 4.1.2 Space Discretization

The space derivatives in equations (4.3) and (4.4) are discretized using the RPIM explained in Chapter 3. For space discretization in the Maxwell’s equations two distinct
sets of node distributions, i.e., a primary set and a dual set, are needed for the electric and magnetic fields because of the intrinsic coupling between the electric and magnetic fields in the Maxwell’s equations. Either of these node distributions can be chosen as locations for sampling the electric or magnetic fields. However, in the commonly encountered case of electric sources (electric currents) or perfect electric boundary conditions, it is usually more convenient to assign the primary set of nodes to the electric field and the dual set of nodes to the magnetic field.

There are some techniques for generation of a staggered dual node distribution from a primary node distribution, for example, using a Voronoi tessellation [23]. Recently a new technique has been introduced based on Coulomb’s law for node generation for meshless RPIM in electromagnetics [124]. A Voronoi tessellation for an arbitrary set of nodes divides the whole geometry into several subdomains, where the boundary of each subdomain has the same distance from two nearest nodes. The resulting subdomains are polygons in 2D and polyhedrons in 3D as shown in Fig. 4.2. The second (dual) set of nodes can be generated using, for example, center edges in the Voronoi diagram for 2D or center faces in the Voronoi diagram for 3D. An example of the staggered node distributions in 2D and 3D domains is shown in Fig. 4.3. In this figure, the first node distribution is shown by black points and the dual node distribution is shown by green crossed nodes.

In summary, the staggering of nodes in space for the meshless RPIM can be achieved as follows.
Figure 4.3. Dual nodes for three arbitrary nodes in 2D and 3D using Voronoi diagram. Dual nodes for arbitrary node distributions are chosen at (a) center edges of Voronoi diagram in 2D, and (b) center faces of Voronoi diagram in 3D.

- An arbitrary node distribution is generated as a first set of nodes.
- A Voronoi tessellation is applied for the above nodes.
- The second set of nodes can be chosen using the Voronoi diagram, e.g., the edge center of each polygon of the Voronoi diagram in 2D, and the face center of polyhedrons of the Voronoi diagram in 3D structures.

It has been shown in Chapter 3 that the electric and magnetic fields at node $x_i$ can be approximated (interpolated) through the value of the fields in the surrounding nodes $(x_j, j = 1, \cdots, N_s)$ as shown in Fig. 4.4. Also, the space derivatives at node $x_i$ can be found based on the space derivatives of the shape functions at that node and on the field values in the support domain as follows

$$\frac{\partial E^n(x_i)}{\partial x} = \left[ \frac{\partial \phi_1(x_i)}{\partial x}, \frac{\partial \phi_2(x_i)}{\partial x}, \cdots, \frac{\partial \phi_{N_s}(x_i)}{\partial x} \right] \left[ E^n_1 \ E^n_2 \ \cdots \ E^n_{N_s} \right] = \sum_{j=1}^{N_s} E^n_j \frac{\partial \Phi_j(x_i)}{\partial x}. \quad (4.7)$$

After time and space discretizations, the solutions for the electric and the magnetic fields in Maxwell’s equations can be obtained through a time iteration using explicit update equations. The update equations for the electric and magnetic fields in the staggered meshless RPIM are explained in the next paragraph.
4.1 Staggered Meshless RPIM in Electromagnetics

4.1.3 Update Equations

After discretization of Maxwell’s equations using a leapfrog time-stepping scheme and RPIM approximation of the space derivatives, the resulting explicit formulation for the $x$, $y$ and $z$-components of both magnetic and electric fields can be written as follows

$$H^{n+1/2}_{x,i} = H^{n-1/2}_{x,i} + \frac{\Delta t}{\mu_0 \mu_r} \left( \sum_j E^n_{y,j} \frac{\partial \Phi_j(x_i)}{\partial z} - \sum_j E^n_{z,j} \frac{\partial \Phi_j(x_i)}{\partial y} \right), \quad (4.8a)$$

$$H^{n+1/2}_{y,i} = H^{n-1/2}_{y,i} + \frac{\Delta t}{\mu_0 \mu_r} \left( \sum_j E^n_{z,j} \frac{\partial \Phi_j(x_i)}{\partial x} - \sum_j E^n_{x,j} \frac{\partial \Phi_j(x_i)}{\partial z} \right), \quad (4.8b)$$

$$H^{n+1/2}_{z,i} = H^{n-1/2}_{z,i} + \frac{\Delta t}{\mu_0 \mu_r} \left( \sum_j E^n_{x,j} \frac{\partial \Phi_j(x_i)}{\partial y} - \sum_j E^n_{y,j} \frac{\partial \Phi_j(x_i)}{\partial x} \right), \quad (4.8c)$$

and

$$E^{n+1}_{x,i} = E^n_{x,i} + \frac{\Delta t}{\epsilon_0 \epsilon_r} \left( \sum_j H^{n+1/2}_{z,j} \frac{\partial \Phi_j(x_i)}{\partial y} - \sum_j H^{n+1/2}_{y,j} \frac{\partial \Phi_j(x_i)}{\partial z} \right), \quad (4.9a)$$

$$E^{n+1}_{y,i} = E^n_{y,i} + \frac{\Delta t}{\epsilon_0 \epsilon_r} \left( \sum_j H^{n+1/2}_{z,j} \frac{\partial \Phi_j(x_i)}{\partial x} - \sum_j H^{n+1/2}_{x,j} \frac{\partial \Phi_j(x_i)}{\partial z} \right), \quad (4.9b)$$

$$E^{n+1}_{z,i} = E^n_{z,i} + \frac{\Delta t}{\epsilon_0 \epsilon_r} \left( \sum_j H^{n+1/2}_{x,j} \frac{\partial \Phi_j(x_i)}{\partial y} - \sum_j H^{n+1/2}_{y,j} \frac{\partial \Phi_j(x_i)}{\partial x} \right), \quad (4.9c)$$

where $i$ is the current node location, $n$ is the time step, and $j$ is the node index in the local support domain. An acceptable time step $\Delta t$ for stability is derived from the Courant-Friedrich-Levy condition as $\Delta t \leq \frac{d_{\text{min}}}{c}$, where $d_{\text{min}}$ is the shortest distance between any two nodes in the domain and $c$ is the velocity of light in free space.

The application of the above update equations is shown in Fig. 4.5. Firstly, in the preprocessing step the two staggered node distributions are generated as described before. In this step the shape functions for the spatial derivatives of the electric and magnetic...
4.2 Domain Truncation

Electromagnetic problems are often open and unbounded. Therefore, radiated fields propagate outwards in the infinite free space surrounding the geometry of interest as shown in Fig. 4.6 (a). However, since computers can not store an infinite number of elements for discretization of an infinite volume, Maxwell’s equations can only be solved within a finite number of elements in volume-based numerical methods. In other words, to fit into the finite memory of the computer, the spatial domain must be
4.2 Domain Truncation

**Figure 4.6. Comparing domain set up in reality and in the simulation.** (a) Unbounded domain: source emits waves into the infinite space, and (b) truncated domain: the infinite space is truncated to a finite computational domain by an artificial computational absorbing boundary.

The solution of the problem inside the truncated domain should converge to the solution of the original problem in the unbounded domain. To this end, an artificial boundary with accurate and reliable boundary conditions is required. In fact, simulating problems with unsuitable boundary conditions can cause numerical errors. For example, waves traveling outwards in the computational domain will generate reflections at the boundaries and these reflections will travel back into the computational domain and degrade the accuracy of the solution. Therefore, all out-going waves should be absorbed without reflections.

Absorbing Boundary Conditions (ABCs) were introduced for truncation of the computational domain in volume-discretization methods and absorption of the energy on the boundaries. In fact an unbounded physical space is replaced with a finite computational domain via ABCs. Various type of locally defined ABCs have been proposed in the literature, such as Engquist-Majda’s one-way wave equation ABC [125], Mur’s ABC of first, second, and third order [126], Litva’s second order Dispersive Boundary Condition (DBC) [127], etc. However, the solution with local ABCs is an estimation and can never be perfect.

Instead of being truncated by an ABC, the physically unbounded domains can be surrounded by an artificial absorbing layer of finite thickness as shown in Fig. 4.7. The
outermost boundary is usually a Perfect Electric Conductor (PEC). The equations in the absorbing layers need to be modified such that there is no reflection at the layers interface, and the solutions decay rapidly in the layers. This approach is analogous to the anechoic chambers used for antenna testing in terms of physical treatment. Absorbing layers were introduced for Maxwell’s equations by Bérenger in 1994 [128] and called Perfectly Matched Layers (PMLs). Using this type of absorbing layers, outgoing waves are absorbed and the level of reflection back into the problem space is minimized [129]. A PML is a non-physical lossy media that is operating nearly-independently of the angle of incident and frequency. In its original work, Bérenger introduced an unphysical splitting of the wave function into tangential components. To absorb the out-going waves he added a damping loss term in each normal direction. This method is called Split-field PML (SPML).

Following Bérenger’s work, different types of PMLs have been introduced in the literature, including Uniaxial PML (UPML) [130], absorbing Lorentz material model (MPML) [131], General Theory PML (GTPML) [132], and Complex Frequency Shifted PML (CFS-PML) [133]. Recently some PMLs have been adapted to the meshless RPIM time-domain algorithm such as Bérenger’s PML [134], Convolution PML (CPML) [135, 136] and UPML in 2D [22]. In this thesis a 3D UPML in the meshless RPIM is implemented.

In the next section, UPML is introduced for absorption in one and more directions, and the first-order meshless RPIM formulation is modified in the presence of a UPML. Also, the late time instability of the UPML is studied, and a practical solution to avoid this instability is introduced.
4.3 Uniaxial Perfectly Matched Layer (UPML)

PMLs are an efficient approach for the simulation of open problems in computational electromagnetics. However, it is shown in [137] that splitting the fields in Bérenger’s PML causes a non-physical exponential growth of the fields in the solutions, which in most cases only manifests itself after very long simulation times. To avoid splitting the fields, un-split PMLs were proposed, e.g., as UPML [130].

4.3.1 Formulation of the UPML

According to the concept of UPML, the wave will be perfectly transmitted into a uniaxial medium without reflection if the uniaxial medium is an anisotropic medium

\[ \epsilon = \epsilon_0 \epsilon_r \Lambda, \quad \mu = \mu_0 \mu_r \Lambda, \]  \hspace{1cm} (4.10)

where \( \Lambda \) is a diagonal matrix appropriately defined to absorb the waves in one or more directions. This matrix is explained in more details in the following.

Absorption in One Direction

To absorb an incident wave along one direction, e.g., in \( y \)-direction, the matrix \( \Lambda \) in (4.10) is defined as follows

\[ \Lambda = \begin{pmatrix} s_y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_y \end{pmatrix}. \]  \hspace{1cm} (4.11)

where \( s_y \) must be chosen such that the waves which enter the PML region are quickly attenuated. To this end, \( s_y \) was chosen by Bérenger as follows

\[ s_y = 1 + \frac{\sigma_y}{j \omega \epsilon_0}, \]  \hspace{1cm} (4.12)

where \( \sigma_y \) is the conductivity or loss term for absorption in \( y \)-direction. To have a smooth transition into the PML medium and to minimize reflections at the interface between the PML and free space, the conductivity \( \sigma_y \) is considered with a spatial variation defined as follows [128]

\[ \sigma_y = \sigma_{\text{max}} \left| \frac{y - y_{\text{pml}}}{d_{\text{pml}}} \right|^m, \quad 0 \leq y - y_{\text{pml}} \leq d_{\text{pml}} \]  \hspace{1cm} (4.13)
where $d_{pml}$ is the thickness of the PML medium in the $y$-direction, $m$ is the order of the conductivity profile, $y_{pml}$ is the position of the start point in the PML, and $\sigma_{\text{max}}$ is the maximum conductivity at the end of PML. Figure 4.8 shows the conductivity inside the PMLs at both ends of a rectangular domain.

In practical the maximum conductivity $\sigma_{\text{max}}$ is chosen for a certain reflection coefficient at normal incidence. The theoretical reflection coefficient for an angle of incidence $\theta$ is defined by [128]

$$R(\theta) = \exp\left(-\frac{2 \cos \theta}{\varepsilon_0 c} \int_0^{d_{pml}} \sigma(\rho) d\rho\right), \quad (4.14)$$

where $c$ is the speed of light in free space. Therefore, substituting equation (4.13) in equation (4.14) ($\rho = y - y_{pml}$), the reflection coefficient at normal incidence ($\theta = 0^\circ$) is obtained as follows

$$R(0) = \exp\left(-\frac{2\sigma_{\text{max}} d_{pml}}{(m+1)\varepsilon_0 c}\right). \quad (4.15)$$

Considering a specified level of reflection at normal incidence, e.g., $R(0) = 10^{-4}$, the parameter $\sigma_{\text{max}}$ is determined as follows

$$\sigma_{\text{max}} = -\frac{(m+1)\varepsilon_0 c}{2d_{pml}} \ln(R(0)). \quad (4.16)$$

The update equations for meshless RPIM in the presence of UPML can be derived by considering equations (4.10) in the Maxwell’s equations. Let’s start with finding the
update equations for all components of the electric field in 3D. The frequency-domain version of Ampere’s law in the presence of UPML is expressed as follows

\[ \nabla \times \mathbf{H} = j\omega \varepsilon_0 \epsilon_r \Lambda \mathbf{E}, \]  

(4.17)

which turns to the following equations by considering the curl operator and \( \Lambda \) as in (4.11)

\[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega \varepsilon_0 \epsilon_r s_y \mathbf{E}_x, \]  

(4.18a)

\[ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega \varepsilon_0 \epsilon_r \frac{1}{s_y} \mathbf{E}_y, \]  

(4.18b)

\[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega \varepsilon_0 \epsilon_r s_y \mathbf{E}_z. \]  

(4.18c)

By substituting the lossy anisotropy term in \( y \)-direction as \( s_y = 1 + \sigma_y / (j\omega \varepsilon_0) \) in the equation (4.18) (a), the following equation is obtained

\[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega \varepsilon_0 \epsilon_r \left(1 + \frac{\sigma_y}{j\omega \varepsilon_0}\right) \mathbf{E}_x = j\omega \varepsilon_0 \epsilon_r \mathbf{E}_x + \epsilon_r \sigma_y \mathbf{E}_x, \]  

(4.19)

which is equivalent to the following time-domain equation

\[ \frac{1}{\varepsilon_0 \epsilon_r} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) = \frac{\partial \mathbf{E}_x}{\partial t} + \frac{\sigma_y}{\varepsilon_0} \mathbf{E}_x. \]  

(4.20)

Using RPIM to discretize the space derivatives, and using central finite differences to discretize the time derivatives give

\[ \frac{1}{\varepsilon_0 \epsilon_r} \left( \sum_j H_{z,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j H_{y,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} \right) = \frac{E_{x}^{n+1} - E_{x}^{n}}{\Delta t} + \frac{\sigma_y}{\varepsilon_0} \frac{E_{x}^{n+1} + E_{x}^{n}}{2}. \]  

(4.21)

After some simple manipulations the update equation for \( \mathbf{E}_x \) is obtained as follows

\[ E_{x,i}^{n+1} = E_{x,i}^{n} \left(1 - \frac{\sigma_y \Delta t}{\varepsilon_0 \epsilon_r 2\Delta x} \right) + \cfrac{\Delta t}{\varepsilon_0 \epsilon_r (1 + \frac{\sigma_y \Delta t}{\varepsilon_0 2\Delta x})} \left( \sum_j H_{z,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j H_{y,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} \right). \]  

(4.22)

In a very similar manner, the update equation for \( \mathbf{E}_z \) can be determined as follows

\[ E_{z,i}^{n+1} = E_{z,i}^{n} \left(1 - \frac{\sigma_y \Delta t}{\varepsilon_0 \epsilon_r 2\Delta x} \right) + \cfrac{\Delta t}{\varepsilon_0 \epsilon_r (1 + \frac{\sigma_y \Delta t}{\varepsilon_0 2\Delta x})} \left( \sum_j H_{y,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial x} - \sum_j H_{x,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial y} \right). \]  

(4.23)
Chapter 4 Staggered Meshless RPIM

The update equation for the \( y \)-component of the electric field is obtained slightly differently, as follows: An auxiliary variable \( D_y \) is introduced to split the equation (4.18) (b) into the two sub-equations [130]

\[
\frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) = j \omega D_y, \tag{4.24a}
\]

\[
\frac{E_y}{1 + \frac{\sigma_y}{j \omega \varepsilon_0}} = D_y. \tag{4.24b}
\]

Converting the above equations to the time domain and then using space and time discretizations result in the following update equations for the auxiliary variable \( D_y \) and the electric field \( E_y \)

\[
D_{y,i}^{n+1} = D_{y,i}^n + \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left( \sum_j H_{x,i}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} - \sum_j H_{z,i}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial x} \right), \tag{4.25a}
\]

\[
E_{y,i}^{n+1} = E_{y,i}^n + D_{y,i}^{n+1} (1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0}) - D_{y,i}^n (1 - \frac{\sigma_y \Delta t}{2 \varepsilon_0}), \tag{4.25b}
\]

where \( i \) is the current node location, \( n \) is the time step, and \( j \) is used for the node indices in the local support domain.

The update equations for the magnetic field can be obtained same as the electric field. Starting with Faraday’s law in the presence of UPML

\[
\nabla \times \mathbf{E} = -j \omega \mu_0 \mu_r \Lambda \mathbf{H}, \tag{4.26}
\]

and substituting the curl operator and \( \Lambda \), we obtain

\[
\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = j \omega \mu_0 \mu_r s_y H_x, \tag{4.27a}
\]

\[
\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j \omega \mu_0 \mu_r \frac{1}{s_y} H_y, \tag{4.27b}
\]

\[
\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = j \omega \mu_0 \mu_r s_y H_z. \tag{4.27c}
\]

Discretizing the (4.27) (a) and (c) is straightforward following the same procedure as shown before. The update equations for \( H_x \) is then obtained as

\[
H_{x,i}^{n+\frac{1}{2}} = H_{x,i}^{n-\frac{1}{2}} \left( 1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0} \right) + \frac{\Delta t}{\mu_0 \mu_r (1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0})} \left( \sum_j E_{y,j}^n \frac{\partial \phi_j(x_i)}{\partial z} - \sum_j E_{z,j}^n \frac{\partial \phi_j(x_i)}{\partial y} \right), \tag{4.28}
\]
and for $H_z$ as

$$H_{z,i}^{n+\frac{1}{2}} = H_{z,i}^{n-\frac{1}{2}} \left(1 - \frac{\sigma_y \Delta t}{2 \varepsilon_0}\right) + \Delta t \mu_0 \mu_r \frac{1}{\left(1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0}\right)} \left(\sum_j E_{x,j}^n \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j E_{y,j}^n \frac{\partial \phi_j(x_i)}{\partial x}\right).$$

(4.29)

To discretize (4.27) (b), an auxiliary variable $B_y$ is introduced to split this expression into the two following equations

$$\frac{1}{\mu_0 \mu_r} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = j \omega B_y,$$  

(4.30a)

$$\frac{H_y}{1 + \frac{\sigma_y}{j \omega \varepsilon_0}} = B_y,$$  

(4.30b)

and finally the update equations for the auxiliary variable $B_y$ and the magnetic field $H_y$ are obtained as follows

$$B_{y,i}^{n+\frac{1}{2}} = B_{y,i}^{n-\frac{1}{2}} + \Delta t \left(\sum_j E_{z,j}^n \frac{\partial \phi_j(x_i)}{\partial x} - \sum_j E_{x,j}^n \frac{\partial \phi_j(x_i)}{\partial z}\right),$$  

(4.31a)

$$H_{y,i}^{n+\frac{1}{2}} = H_{y,i}^{n-\frac{1}{2}} + B_{y,i}^{n+\frac{1}{2}} \left(1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0}\right) - B_{y,i}^{n-\frac{1}{2}} \left(1 - \frac{\sigma_y \Delta t}{2 \varepsilon_0}\right).$$  

(4.31b)

These new update equations are a modification of the previous update equations in (4.9) and (4.8) with introduction of UPML parameters in $y$-direction.

**Absorption in Three Directions**

To have absorption in three $x$, $y$ and $z$-directions, the matrix $\Lambda$ is defined as follows

$$\Lambda = \begin{pmatrix}
    s_y s_z & 0 & 0 \\
    s_x & 0 & 0 \\
    0 & s_x s_z & 0 \\
    0 & 0 & s_x s_y s_z
\end{pmatrix}.$$  

(4.32)

In this case, three auxiliary variables $D_x$, $D_y$ and $D_z$ are needed to split the three equations in the Ampere’s law as follows,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j \omega s_z D_x, \quad D_x = \epsilon_0 \varepsilon_r \frac{s_y}{s_x} E_y, $$  

(4.33a)

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j \omega s_x D_y, \quad D_y = \epsilon_0 \varepsilon_r \frac{s_z}{s_y} E_y, $$  

(4.33b)

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j \omega s_y D_z, \quad D_z = \epsilon_0 \varepsilon_r \frac{s_x}{s_z} E_x. $$  

(4.33c)
Substituting \( s_x, s_y \) and \( s_z \) in (4.33) (a) results in

\[
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega D_x + \frac{\sigma_z}{\varepsilon_0} D_x,
\]

and

\[
j\omega D_x + \frac{\sigma_x}{\varepsilon_0} D_x = \varepsilon_0 \varepsilon_r (j\omega E_x + \frac{\sigma_y}{\varepsilon_0} E_x).\]

Converting equations in (4.34) and (4.35) into the time domain, and using finite differences to discretize time derivatives and RPIM to discretize space derivatives result in

\[
\left( \sum_j H_{z,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j H_{y,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} \right) = \frac{D_x^{n+1} - D_x^{n-1}}{\Delta t} + \frac{\sigma_z}{\varepsilon_0} \frac{D_x^{n+1} + D_x^{n-1}}{2},
\]

\[
\frac{D_x^{n+1} - D_x^{n-1}}{\Delta t} + \frac{\sigma_x}{\varepsilon_0} \frac{D_x^{n+1} + D_x^{n-1}}{2} = \varepsilon_0 \varepsilon_r \left( \frac{E_x^{n+1} - E_x^{n-1}}{\Delta t} + \frac{\sigma_y}{\varepsilon_0} \frac{E_x^{n+1} + E_x^{n-1}}{2} \right),
\]

which can be transformed into the following update equations for the auxiliary variable \( D_x \) and the electric field \( E_x \)

\[
D_x^{n+1} = D_x^{n-1} \left( 1 - \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta x} \right) + \frac{\Delta t}{1 + \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta x}} \left( \sum_j H_{z,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j H_{y,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} \right),
\]

\[
E_x^{n+1} = E_x^{n-1} \left( 1 - \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta x} \right) + D_x^{n+1} \frac{1}{\varepsilon_0 \varepsilon_r} \left( 1 + \frac{\sigma_y \Delta t}{\varepsilon_0 \Delta y} \right) - D_x^{n-1} \frac{1}{\varepsilon_0 \varepsilon_r} \left( 1 + \frac{\sigma_y \Delta t}{\varepsilon_0 \Delta y} \right).
\]

In a similar procedure the update equations for \( D_y \) and \( E_y \) can be obtained from (4.33) (b) as follows

\[
D_y^{n+1} = D_y^{n-1} \left( 1 - \frac{\sigma_y \Delta t}{\varepsilon_0 \Delta y} \right) + \frac{\Delta t}{1 + \frac{\sigma_y \Delta t}{\varepsilon_0 \Delta y}} \left( \sum_j H_{x,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial x} - \sum_j H_{y,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} \right),
\]

\[
E_y^{n+1} = E_y^{n-1} \left( 1 - \frac{\sigma_y \Delta t}{\varepsilon_0 \Delta y} \right) + D_y^{n+1} \frac{1}{\varepsilon_0 \varepsilon_r} \left( 1 + \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta x} \right) - D_y^{n-1} \frac{1}{\varepsilon_0 \varepsilon_r} \left( 1 + \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta x} \right).
\]

Also, the update equations for the auxiliary variable \( D_z \) and the electric field \( E_z \) can be found from (4.33) (c) as follows

\[
D_z^{n+1} = D_z^{n-1} \left( 1 - \frac{\sigma_z \Delta t}{\varepsilon_0 \Delta z} \right) + \frac{\Delta t}{1 + \frac{\sigma_z \Delta t}{\varepsilon_0 \Delta z}} \left( \sum_j H_{z,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j H_{x,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial x} \right),
\]

\[
E_z^{n+1} = E_z^{n-1} \left( 1 - \frac{\sigma_z \Delta t}{\varepsilon_0 \Delta z} \right) + D_z^{n+1} \frac{1}{\varepsilon_0 \varepsilon_r} \left( 1 + \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta y} \right) - D_z^{n-1} \frac{1}{\varepsilon_0 \varepsilon_r} \left( 1 + \frac{\sigma_x \Delta t}{\varepsilon_0 \Delta y} \right).
\]
Now to find the update equations for the magnetic field components, matrix $\Lambda$ is used in Faraday’s law. Here again three auxiliary variables $B_x$, $B_y$, and $B_z$ are required to split the three equations in the Faraday’s law into the following equations

\[
\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = j\omega s_y B_x, \quad B_x = \mu_0 \mu_r \frac{s_z}{s_x} H_x, \\
\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j\omega s_z B_y, \quad B_y = \mu_0 \mu_r \frac{s_x}{s_y} H_y, \\
\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = j\omega s_x B_z, \quad B_z = \mu_0 \mu_r \frac{s_y}{s_z} H_z. 
\] (4.41a, 4.41b, 4.41c)

Again, substituting $s_x$, $s_y$, and $s_z$ in the above equations, converting them to the time domain, and finally discretizing space and time derivatives give the following update equations for $B_x$ and $H_x$

\[
B_{x}^{n+\frac{1}{2}} = B_{x}^{n-\frac{1}{2}} \left( \frac{1 - \frac{\sigma_x \Delta t}{2 \varepsilon_0}}{1 + \frac{\sigma_x \Delta t}{2 \varepsilon_0}} \right) + \frac{\Delta t}{1 + \frac{\sigma_x \Delta t}{2 \varepsilon_0}} \left( \sum_j E_{y,j}^n \frac{\partial \phi_j(x_i)}{\partial z} - \sum_j E_{z,j}^n \frac{\partial \phi_j(x_i)}{\partial y} \right), \\
H_{x}^{n+\frac{1}{2}} = H_{x}^{n-\frac{1}{2}} \left( \frac{1 - \frac{\sigma_x \Delta t}{2 \varepsilon_0}}{1 + \frac{\sigma_x \Delta t}{2 \varepsilon_0}} \right) + B_{x}^{n+\frac{1}{2}} \frac{1}{\mu_0 \mu_r} \left( \frac{1 + \frac{\sigma_x \Delta t}{2 \varepsilon_0}}{1 + \frac{\sigma_x \Delta t}{2 \varepsilon_0}} \right) - B_{x}^{n-\frac{1}{2}} \frac{1}{\mu_0 \mu_r} \left( 1 - \frac{\sigma_x \Delta t}{2 \varepsilon_0} \right), 
\] (4.42a, 4.42b)

and the update equations for the auxiliary variable $B_y$ and the magnetic field $H_y$ are as follows

\[
B_{y}^{n+\frac{1}{2}} = B_{y}^{n-\frac{1}{2}} \left( \frac{1 - \frac{\sigma_y \Delta t}{2 \varepsilon_0}}{1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0}} \right) + \frac{\Delta t}{1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0}} \left( \sum_j E_{x,j}^n \frac{\partial \phi_j(x_i)}{\partial z} - \sum_j E_{z,j}^n \frac{\partial \phi_j(x_i)}{\partial y} \right), \\
H_{y}^{n+\frac{1}{2}} = H_{y}^{n-\frac{1}{2}} \left( 1 - \frac{\sigma_y \Delta t}{2 \varepsilon_0} \right) + B_{y}^{n+\frac{1}{2}} \frac{1}{\mu_0 \mu_r} \left( 1 + \frac{\sigma_y \Delta t}{2 \varepsilon_0} \right) - B_{y}^{n-\frac{1}{2}} \frac{1}{\mu_0 \mu_r} \left( 1 - \frac{\sigma_y \Delta t}{2 \varepsilon_0} \right). 
\] (4.43a, 4.43b)

Also, the update equations for the auxiliary variable $B_z$ and the magnetic field $H_z$ can be obtained as

\[
B_{z}^{n+\frac{1}{2}} = B_{z}^{n-\frac{1}{2}} \left( \frac{1 - \frac{\sigma_z \Delta t}{2 \varepsilon_0}}{1 + \frac{\sigma_z \Delta t}{2 \varepsilon_0}} \right) + \frac{\Delta t}{1 + \frac{\sigma_z \Delta t}{2 \varepsilon_0}} \left( \sum_j E_{x,j}^n \frac{\partial \phi_j(x_i)}{\partial y} - \sum_j E_{y,j}^n \frac{\partial \phi_j(x_i)}{\partial x} \right), \\
H_{z}^{n+\frac{1}{2}} = H_{z}^{n-\frac{1}{2}} \left( 1 - \frac{\sigma_z \Delta t}{2 \varepsilon_0} \right) + B_{z}^{n+\frac{1}{2}} \frac{1}{\mu_0 \mu_r} \left( 1 + \frac{\sigma_z \Delta t}{2 \varepsilon_0} \right) - B_{z}^{n-\frac{1}{2}} \frac{1}{\mu_0 \mu_r} \left( 1 - \frac{\sigma_z \Delta t}{2 \varepsilon_0} \right). 
\] (4.44a, 4.44b)
These update equations will be used later in the numerical examples where absorption in all three directions is required. In these cases, UPMLs will be arranged to truncate the computational domain in all directions.

### 4.3.2 Late-Time Instability in UPML

The existence of a late-time linear growth of the fields in the un-split PMLs was shown in [138]. A small perturbation in the equations to avoid a singular coefficient matrix was introduced as a remedy. This method successfully suppresses the late-time instability and the cost is a loss of the perfect matching condition at the air-PML interface. In [139] it was shown that using CPML [133] results in stable solutions in the late time and also the perfectly matched property is satisfied. However, it requires a larger numerical effort and more memory compared to UPML.

In this section, we analyze the temporal behavior of the UPML in the meshless RPIM in 3D. To this end a rectangular waveguide with UPMLs at both sides is implemented as test case. For the suppression of the late-time instability the method introduced in [138] is modified for RPIM. It is shown that late-time instability can be significantly delayed to later time by introducing of small losses inside the UPML after a sufficient decay of the energy in the computational domain. This late introduction does not compromise the accuracy of the solution as the perfect matching is retained in the initial simulation phase.

**Algorithm**

Here, the late time instability of the UPML in y-direction is studied by considering equation (4.24) in more detail. Transforming equation (4.24) to the time domain yields

\[
\frac{\partial D_y}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_z}{\partial z} - \frac{\partial H_x}{\partial x} \right),
\]

(4.45a)

\[
\frac{\partial E_y}{\partial t} = \frac{\partial D_y}{\partial t} + \frac{\sigma_y}{\varepsilon_0} D_y.
\]

(4.45b)

After a long time, the fields in the absorbing layer are almost quiescent and the spatial derivatives \(\partial/\partial x, \partial/\partial z\) are nearly zero. In the limit of \(\lim_{\partial_x, \partial_z \to 0}\) the matrix form of (4.45) is

\[
\frac{\partial}{\partial t} \begin{bmatrix} D_y \\ E_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{\sigma_y}{\varepsilon_0} & 0 \end{bmatrix} \begin{bmatrix} D_y \\ E_y \end{bmatrix}.
\]

(4.46)
The coefficient matrix in (4.46) is singular with two zero eigenvalues. The solutions for \( D_y \) and \( E_y \) are as follows

\[
D_y = A_1, \\
E_y = \frac{\sigma_y}{\varepsilon_0} A_1 t,
\]

where \( A_1 \) is a constant. It can be clearly seen that the field values will increase with time and the late-time solution is unbounded, i.e., will become unstable. To avoid this singular coefficient matrix, two loss terms \( \delta_1/\varepsilon_0 D_y \) and \( \delta_2 E_y \) can be introduced in equation (4.45) as follows

\[
\begin{align*}
\frac{\partial D_y}{\partial t} &= \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) - \frac{\delta_1}{\varepsilon_0} D_y, \\
\frac{\partial E_y}{\partial t} &= \frac{\partial D_y}{\partial t} + \frac{\sigma_y}{\varepsilon_0} D_y - \delta_2 E_y.
\end{align*}
\]

In this case and by considering the condition \( \lim_{\delta_x, \delta_z \to 0} \) after a long time, the matrix (4.46) is changed to the following

\[
\frac{\partial}{\partial t} \begin{bmatrix} D_y \\ E_y \end{bmatrix} = \begin{bmatrix} -\frac{\delta_1}{\varepsilon_0} & 0 \\ \frac{\sigma_y}{\varepsilon_0} & -\delta_2 \end{bmatrix} \begin{bmatrix} D_y \\ E_y \end{bmatrix}, \tag{4.49}
\]

where both \( \delta_1, \delta_2 \) are small positive real values introduced to yield two eigenvalues with negative real parts. The late-time solutions for (4.49) are as follows

\[
\begin{align*}
D_y &= A_1 \exp \left( -\frac{\delta_1}{\varepsilon_0} t \right), \\
E_y &= B_1 \exp \left( -\frac{\delta_1}{\varepsilon_0} t \right) + C_1 \exp (-\delta_2 t),
\end{align*}
\]

where \( A_1, B_1 \) and \( C_1 \) are constant. Since the new coefficient matrix in (4.49) has two unequal eigenvalues \( (\delta_1/\varepsilon_0, \delta_2) \), the late-time solutions, which are linearly increasing with time, can be avoided.

The new update equations for the auxiliary variable \( D_y \) and the electric field \( E_y \) in the presence of two loss terms are as follows

\[
\begin{align*}
D_{y,i}^{n+1} &= D_{y,i}^n \left( 1 - \frac{\delta_1 \Delta t}{\varepsilon_0} \right) + \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left( \sum_j H_{x,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial x} - \sum_j H_{z,j}^{n+\frac{1}{2}} \frac{\partial \phi_j(x_i)}{\partial z} \right), \\
E_{y,i}^{n+1} &= E_{y,i}^n \left( 1 - \frac{\delta_2 \Delta t}{\varepsilon_0} \right) + D_{y,i}^{n+1} \left( 1 + \frac{\sigma \Delta t}{\varepsilon_0} \right) - D_{y,i}^n \left( 1 - \frac{\sigma \Delta t}{\varepsilon_0} \right).
\end{align*}
\]

\( n \) is an index for the time step and \( i \) is the index for the spatial point.
The same scenario occurs for the magnetic field in the $y$-direction. The new update equations for the auxiliary variable $B_y$ and the magnetic field $H_y$ in the presence of two loss terms are as follows

$$B_{y,i}^{n+\frac{1}{2}} = B_{y,i}^{n-\frac{1}{2}} \left( \frac{1 - \delta_1 \Delta t}{2 \epsilon_0} \right) + \frac{\Delta t}{\mu_0 \mu_r (1 + \frac{\delta_1 \Delta t}{2 \epsilon_0})} \left( \sum_j E_{z,j}^n \frac{\partial \phi_j(x_i)}{\partial x} - \sum_j E_{x,j}^n \frac{\partial \phi_j(x_i)}{\partial z} \right),$$

(4.52a)

$$H_{y,i}^{n+\frac{1}{2}} = H_{y,i}^{n-\frac{1}{2}} \left( \frac{1 - \delta_2 \Delta t}{2 \epsilon_0} \right) + B_{y,i}^{n+\frac{1}{2}} \left( \frac{1 + \delta_y \Delta t}{2 \epsilon_0} \right) - B_{y,i}^{n-\frac{1}{2}} \left( \frac{1 - \delta_y \Delta t}{2 \epsilon_0} \right).$$

(4.52b)

Note that adding $\delta_1$ and $\delta_2$ does not require additional variables in the implementation. Therefore, there is no additional memory cost in the implementation. By adding these values however, the perfectly matched properties are not satisfied anymore at the interfaces between the PMLs and the free-space domain. To prevent any undesirable effect on the numerical solutions, these two losses are added to the update equations only after the energy in the computational domain is absorbed sufficiently in the PMLs, but before the simulation gets unstable. To find the right time step, where the correction needs to be switched on, the total energy inside the computational domain can be considered. The total energy in the waveguide at time step $n$ is approximated as

$$E^n = \sum_i \epsilon_i \Delta V_i |E_i|^2 + \sum_i \mu_i \Delta V_i (|H_i|^2),$$

(4.53)

where $\Delta V_i$ is the volume surrounding node $i$. Considering the total energy in the computational domain, the additional losses can be activated in the PMLs when the energy is sufficiently absorbed in the PMLs and just before the energy starts to increase in an unphysical manner.

**Results**

A 3D rectangular waveguide with perfect conducting walls is utilized as a canonical validation tool. The width of the waveguide is chosen for a cut-off frequency of $f_c = 2$ GHz. Figure 4.9 shows the geometry of the structure. The waveguide is truncated on both ends with PMLs with thickness $12 \Delta x$, where $\Delta x = 9.4$ mm ($\lambda_{\text{min}}/8$). The conductivity $\sigma_y$ has a cubic profile ($m = 3$) and a magnitude chosen such that the theoretical reflection for normal incidence is $10^{-4}$. A modulated sinusoidal Gaussian pulse with a spectrum covering $f = [2 - 4]$ GHz is used as a plane source with a TE$_{10}$
4.3 Uniaxial Perfectly Matched Layer (UPML)

Figure 4.9. Rectangular waveguide with a plane source in the middle and PMLs in both ends. Geometry of rectangular waveguide with width \( w = 7.5 \) cm and height \( h = w/2 \).

Figure 4.10. Time-domain electric field with and without loss terms. Time function of \( E_z \) at observation point \( P \) without loss terms, i.e., \((\delta_1, \delta_2) = (0, 0)\) and with loss terms \((\delta_1, \delta_2) = (0.0015, 0.0015)\) activated at \( t=500\)ns.

mode distribution in the center of the waveguide. An observation point \( P \) is defined as shown in Fig. 4.9 to record time-domain data.

Figure 4.10 shows the time domain electric field at sensor node \( P \). Without the two loss terms, it can be seen that at time \( t = 1500 \) ns the instability becomes visible at the sensor location. However, activating \( \delta_1 \) and \( \delta_2 \) inside the PMLs at \( t_\delta = 500 \) ns (\( n = 40000 \) time steps, corresponds to 1500 periods at \( f = 3 \) GHz) shifts the instability to much later times or even remove the instability.

Figure 4.11 shows the normalized total energy (4.53) in the waveguide as a function of time. After absorption of the energy in the PMLs, the inherent instability causes the energy to rise again, as visible from time \( t = 544 \) ns, for the case when no parasitic losses \( \delta_1, \delta_2 \) are introduced. At that time, the energy in the domain has however already significantly decayed through absorption in the PMLs. Therefore, a timed introduction
of the loss terms, e.g., \((\delta_1, \delta_2) = (0.001,0.001)\), inside the PMLs can avoid the solution to grow linearly and shifts the instability to much later time. Furthermore, a parameter study shows that with loss terms \(\delta \in [0.0015, 0.5]\) instability can be suppressed completely. The result for \(\delta=0.0015\) is shown in Fig. 4.11. Further investigations are required to automate the choice of the switching time \(t_\delta\), as well as to explore the possibility of having the values of \(\delta_1\) and \(\delta_2\) as spatial functions in the PML.

### 4.4 Numerical Examples for the Staggered Meshless RPIM

In this section some capabilities of meshless methods are shown in several examples using the staggered meshless RPIM. The update equations for the staggered meshless RPIM in the presence of UPML are used to find the electric and magnetic fields in each case.

#### 4.4.1 Impact of Different Node Distributions

Commonly, meshless methods are based on an arbitrary (unstructured) node distribution that allows the simulation of complex problems with fine geometrical details. Moreover, the flexibility of this node-based approach allows to increase the accuracy by moving or adding nodes [44].

In most previous publications describing simulations with the RPIM scheme in electromagnetics, canonical node distributions have generally been applied, e.g. in [20] and
4.4 Numerical Examples for the Staggered Meshless RPIM

[22]. Here, the impact of non-ideal node distributions is considered, as this is a crucial aspect for the robustness of the method. In a 3D waveguide example, a time-domain simulation is performed for a regular node distribution and compared to the results when the regular node distribution is randomly disturbed. A further comparison is performed with a cylindrical node distribution that represents a case of a non-intuitive distribution and can potentially introduce errors in the simulation. The results are obtained in an otherwise identical setup.

Node Distribution

The spatial discretization of the RPIM scheme can be realized through arbitrarily placed node distributions. As mentioned, for the solution of the first-order Maxwell’s equations, because of the coupling nature of electric and magnetic field components, two separate sets of electric and magnetic field nodes (E- and H-nodes) are chosen to be staggered in space. Similar to the FDTD method, this is best combined with a staggered leapfrog time-stepping which results in second-order accuracy. Since meshless methods are node based, complex geometries can be approximated by sophisticated node distributions to conformally resolve the geometry. So far, the question of optimal node distributions for a given geometry have only been partially answered, e.g., through node adaptivity. It can be generally assumed though that in complex geometries, an initial node distribution will be based on the (known) geometry, but will not be optimal in terms of the (unknown) electromagnetic field distribution. Hence, the RPIM scheme is required to be robust for arbitrary node distributions.

In order to investigate the impact of non-ideal node placement, a test case of a rectangular waveguide is solved with different node distributions. The first strategy considers a uniform node distribution: E-nodes are defined on a regular 3D grid. The Voronoi algorithm [140] is then applied to the E-nodes, and H-nodes are placed in the middle faces of each Voronoi cell. Figure 4.12 (a) shows a regular electric node distribution with equal spacing of $\Delta x$. Due to the rectangular geometry of the waveguide structure with field components predominantly along the coordinate axes, this can be considered as a “good” node distribution. To test the effect of adding randomness in the node distribution, i.e., by making the distribution of the nodes non-ideal, this regular distribution is disturbed by introducing random node displacements. Random displacements corresponding to a maximum of 5% and 10% of the grid period $\Delta x$ have been generated, and an example is shown in Fig. 4.12 (b). As another example of a non-ideal node arrangement, a cylindrical node distribution as shown in Fig. 4.12 (c) has
Figure 4.12. Different node distributions. Electric node distribution for (a) regular, (b) 10\% random, and (c) cylindrical node arrangements.

Figure 4.13. Geometry of a rectangular waveguide to test different node distributions. Geometry of a rectangular waveguide with width $w = 11.25$ cm and height $h = w/2$.

been chosen. This further adds randomness to this particular problem, which would be best solved with a rectangular node distribution.

Numerical Results

To illustrate the impact of different node distributions in the meshless RPIM scheme, a waveguide with width $w = 11.25$ cm and height $h = w/2$ is used, as shown in Fig. 4.13, corresponding to a cut-off frequency of $f_c = 1.33$ GHz. The waveguide is excited by a modulated Gaussian pulse with a bandwidth within $f = [1 - 3]$ GHz. The excitation is introduced on a port plane with a TE$_{10}$ mode distribution on one end of the waveguide, which is truncated at $y_{pml} = 32\Delta x$ with a UPML with thickness $d = 12\Delta x$, where $\Delta x = 9.4$ mm ($\lambda_{center}/16$). A UPML [130] with a cubic conductivity profile and a theoretical reflection coefficient of $10^{-4}$ is used in this example. Four observation points $P_1, P_2, P_3$ and $P_4$ are located at distance $l = [8\Delta x, 10\Delta x, 12\Delta x, 14\Delta x]$
from the source plane in the center of the waveguide cross section. The locations of these observation points are chosen close to the first end of the waveguide to minimize parasitic reflections from the PML termination. The spatial domain is discretized as described in Section 4.4.1 and as shown in Fig. 4.12. The support domain size, i.e., the domain of influence of each point in the interpolation scheme is $d_s = 1.7d_c$, where $d_c$ is the average node distance.

The time-domain electric field recorded at the observation point $P_1$ for different node distributions is plotted in Fig. 4.14. Small deviations between the different distributions are observed. In the following these deviations are quantified considering the regular distribution as a reference, as it is the most adapted to the rectangular waveguide geometry. To this aim, the frequency-domain quantities obtained by Fourier transform at observation points $P_1$ to $P_4$ are plotted in Fig. 4.15. As observed from this graph, the largest phase deviation from the regular node distribution is arising in the cylindrical node distribution. Maximum relative phase deviations over all frequencies for all observation points are shown in table 4.1. In this table $\phi_{rr}$, $\phi_{5\%}$, $\phi_{10\%}$, and $\phi_{cy}$ are the phase of the electric fields in regular, 5% random, 10% random, and cylindrical node distributions, respectively. Simulation results show that maximum phase deviation from the regular node distribution over all observation points are 0.0014% for 5% random node distributions.
Figure 4.15. Phase difference from regular node distribution for different node distributions.

Phase difference at observation points $P_1$ to $P_4$ for different node distributions.

Table 4.1. Maximum phase deviations for different node distributions in a rectangular waveguide. Maximum relative phase deviations from the regular node distribution for different 5% random, 10% random, and cylindrical node distributions at four observation points in a rectangular waveguide.

| Observation Point | $\max (\left| \phi_n - \phi_{5\%} \right| / \phi_n) \times 100$ | $\max (\left| \phi_n - \phi_{10\%} \right| / \phi_n) \times 100$ | $\max (\left| \phi_n - \phi_{cy} \right| / \phi_n) \times 100$ |
|-------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| $P_1$             | 0.0007                                          | 0.0029                                          | 0.1021                                          |
| $P_2$             | 0.0014                                          | 0.0042                                          | 0.5054                                          |
| $P_3$             | 0.0001                                          | 0.0003                                          | 0.0809                                          |
| $P_4$             | 0.0013                                          | 0.0006                                          | 0.0031                                          |

placement, 0.0042% for 10% random node placement and 0.5054% for cylindrical node distribution.

The amplitudes of the Fourier transformed signals at different node positions at $f_1 = 1.8$ GHz and $f_2 = 2.2$ GHz are shown in Fig. 4.16. The variations in amplitude are very local (numerical noise), and are also influenced by the imperfections of the absorbing boundary conditions. This can be manifested as an amplitude slightly larger than one in the Fourier transformed data. The relative amplitude deviations from the regular node distribution for different node distributions at $f_1 = 1.8$ GHz and $f_2 = 2.2$ GHz are shown in tables 4.2 and 4.3, respectively. In these tables $E_n$, $E_{5\%}$, $E_{10\%}$, and $E_{cy}$ are
4.4 Numerical Examples for the Staggered Meshless RPIM

Figure 4.16. Normalized amplitude at different observation points for different node distributions. Amplitude at observation points $P_1$ to $P_4$ for $f_1 = 1.8$ GHz and $f_2 = 2.2$ GHz, normalized to the source amplitude.

Table 4.2. Amplitude deviations for different node distributions at $f_1 = 1.8$ GHz in a rectangular waveguide. Relative amplitude deviations (in %) from the regular node distribution for different 5% random, 10% random, and cylindrical node distributions at four observation points in a rectangular waveguide at $f_1 = 1.8$ GHz.

| Observation Point | $\epsilon_1 = \frac{|E_n - E_{5\%}|}{E_n} \times 100$ | $\epsilon_2 = \frac{|E_n - E_{10\%}|}{E_n} \times 100$ | $\epsilon_3 = \frac{|E_n - E_{cyl}|}{E_n} \times 100$ |
|-------------------|---------------------------------|---------------------------------|---------------------------------|
| $P_1$             | 0.64                            | 1.4                             | 0.95                             |
| $P_2$             | 0.14                            | 1.28                            | 1                                |
| $P_3$             | 0.25                            | 1.71                            | 1.83                             |
| $P_4$             | 0.67                            | 2.37                            | 2.26                             |

the amplitude of the electric fields in regular, 5% random, 10% random, and cylindrical node distributions, respectively. Maximum relative amplitude deviations ($\epsilon$) at $f_1 = 1.8$ GHz are $\epsilon_1 = 0.67\%$ for 5% random node placement, $\epsilon_2 = 2.37\%$ for 10% random node placement and $\epsilon_3 = 2.26\%$ for cylindrical node distribution, and those at $f_2 = 2.2$ GHz are $\epsilon_1 = 1.7\%$, $\epsilon_2 = 2.1\%$ and $\epsilon_3 = 2.16\%$. These values lie within acceptable limits for this fineness of discretization (around $\lambda/16$), considering the very different node distributions.
Table 4.3. Amplitude deviations for different node distributions at $f_2 = 2.2$ GHz in a rectangular waveguide. Relative amplitude deviations (in %) from the regular node distribution for different 5% random, 10% random, and cylindrical node distributions at four observation points in a rectangular waveguide at $f_2 = 2.2$ GHz.

| Observation Point | $\epsilon_1 = \frac{|E_n - E_{5\%}|}{E_n} \times 100$ | $\epsilon_2 = \frac{|E_n - E_{10\%}|}{E_n} \times 100$ | $\epsilon_3 = \frac{|E_n - E_{cy}|}{E_n} \times 100$ |
|-------------------|---------------------------------|---------------------------------|---------------------------------|
| $P_1$             | 1.72                            | 2.1                             | 1.49                            |
| $P_2$             | 0.47                            | 1.41                            | 0.94                            |
| $P_3$             | 0.1                             | 1.3                             | 1.48                            |
| $P_4$             | 0.65                            | 1.88                            | 2.16                            |

4.4.2 Diplexer

In this example, a 3D waveguide Y-junction diplexer is solved. In the diplexer, two bandpass filters with different passbands are connected through a Y-junction waveguide. The design process and dimensions of the diplexer are described in [141]. The geometry of the diplexer and the node distribution for the electric field in the center height of the waveguide is shown in Fig. 4.17. In this test case, three regular node distributions with different node orientations are used. As can be seen in the figure, the feeding part is discretized by a regular node distribution with 45-degree node alignment, and then it is connected to the two iris filters in two arms of the diplexer. The node distributions in two iris filters are horizontal and vertical regular node distributions. The node density around the smallest iris in the upper arm of the diplexer is doubled to increase the accuracy. This local node refinement is shown in Fig. 4.17.

To excite the diplexer, ramped sinusoidal signals at frequencies $f_1$ and $f_2$ are used as a plane source with a TE$_{10}$ mode distribution in the $\lambda/4$ distance from the beginning of the waveguide. Magnitude of the electric field for excitation at $f_1 = 12.65$ GHz at time $t = 5.23$ ns is shown in Fig. 4.18. As shown in the figure, the wave propagates through the right filter and it is rejected in the other filter. Figure 4.19 shows magnitude of the electric field at $f_2 = 14.1$ GHz at time $t = 5.04$ ns. At this frequency the wave propagates through the upper filter, while it is rejected in the right filter. Note that the wave is absorbed in the PMLs in the end of each filter and reflected in the beginning of the feeding part since there is no PML there.
### 4.4 Numerical Examples for the Staggered Meshless RPIM

[Figure 4.17. Geometry of the waveguide Y-junction diplexer. Electric nodes for the diplexer. Magnetic nodes are not shown for clarity.]

[Figure 4.18. Propagation of the electric field in the diplexer for frequency $f_1 = 12.65$ GHz. Magnitude of the electric field in the diplexer for frequency $f_1 = 12.65$ GHz at time $t = 5.23$ ns.]
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Figure 4.19. Propagation of the electric field in the diplexer for frequency \( f_2 = 14.1 \text{ GHz} \).

Magnitude of the electric field in the diplexer for frequency \( f_2 = 14.1 \text{ GHz} \) at time \( t = 5.04 \text{ ns} \).

This example illustrates how different node distributions can be connected in meshless methods. This potential is very helpful for discretization of complex geometries in different applications.

4.4.3 Scattering From a Conducting Sphere

Scattering by a sphere or cylinder of arbitrary radius has an exact theoretical solution and can be used to verify the implementation of the meshless RPIM. The scattering from a perfectly conducting cylinder has been implemented for 2D meshless RPIM in [142]. In this section, a 3D conducting sphere is implemented by a 3D meshless RPIM and the simulated scattered electric field in the far-field region is compared with the theoretical result (Mie scattering solution) using a near-field to far-field transformation.

Node Distribution

Figure 4.20 shows the geometry and node distribution of a sphere inside a cubic computational domain. The radius of the metallic sphere is chosen as \( r = \lambda/2 \) and the dimensions of the computational domain are \( 4\lambda \times 4\lambda \times 4\lambda \). The sphere is discretized by a
4.4 Numerical Examples for the Staggered Meshless RPIM

Figure 4.20. Geometry and node distribution of a conducting sphere in a cubic computational domain. The electric node distribution for a conducting sphere with radius \( r = \lambda / 2 \) in a cubic domain with dimensions \( 4\lambda \times 4\lambda \times 4\lambda \).

set of regular node distribution in spherical coordinate system for the electric nodes as shown in Fig. 4.20, and rest of the domain is discretized by regular node distributions for the electric and magnetic nodes with \( \Delta x = \lambda / 12 \) at \( f = 3 \) GHz. Figure 4.20 shows only the electric node distribution for clarity. The domain is truncated by UPMLs in all directions using layers with total thickness of \( \lambda / 2 \).

This example is solved to find the scattered electric field instead of the total electric field in the far-field region. The scattered field formulation and also the near-field to far-field transformation for finding the far-field scattered electric field are described in the following.

**Scattered Field Formulation**

Since Maxwell’s equations are linear, the formulation for the total electric and magnetic fields can be converted to a formulation for the scattered electric and magnetic fields in the presence of known incident fields [29]. The total electric field \( \mathbf{E}_{\text{total}} \) is as follows

\[
\mathbf{E}_{\text{total}} = \mathbf{E}_{\text{scattered}} + \mathbf{E}_{\text{inc}}
\]

(4.54)

where \( \mathbf{E}_{\text{scattered}} \) is the scattered electric field and \( \mathbf{E}_{\text{inc}} \) is the incident electric field. The incident wave is a modulated Gaussian pulse with a bandwidth within \( f = [1.5 - 4.5] \) GHz. The \( x \)-polarized incident plane wave propagating in \( z \)-direction is expressed
as follows

\[ E_{\text{inc}} = \exp \left( -\left( \frac{t - \frac{(z - z_0)}{c_0} - 4\sigma_s}{\sqrt{2}\sigma_s} \right)^2 \right) \sin \left( 2\pi f \left( t - \frac{(z - z_0)}{c_0} - 4\sigma_s \right) \right) \], \quad (4.55) 

where \( \sigma_s \) determines the pulse width and \( z_0 \) is a spatial offset. Since the incident electric field is known in each point and each time step as shown in the equation (4.55), the problem can be solved just for the unknown scattered electric field. To this end, the scattered field formulation with its own boundary condition needs to be used. The boundary condition for the total electric field on a conducting sphere is given by

\[ E_{\text{total},t} = E_{\text{scattered},t} + E_{\text{inc},t} = 0, \quad (4.56) \]

where subscript \( t \) shows the tangential component of the fields. From equation (4.56) the following boundary condition for the scattered field on the conducting sphere can be obtained

\[ E_{\text{scattered},t} = -E_{\text{inc},t}. \quad (4.57) \]

As a result, instead of solving the total electric field with condition \( E_{\text{total},t} = 0 \) on the sphere, the problem can be solved for the scattered electric field with applying the source on the sphere as a boundary condition as shown in equation (4.57). Note that the solution for the total electric field can be determined by adding the incident field to the scattered field solution in the post processing.

**Near-Field to Far-Field Transformation**

The far-field radiation pattern is one of the main characteristics that needs to be determined for a radiating object. The far-field region is typically assumed to be at distances larger than \( 2D^2 / \lambda \) from the radiating object, where \( D \) is the dimension of the object and \( \lambda \) is the wavelength. On the other hand, to save computational cost the computational domain can not be extended to the far-field area. Therefore, usually the computational domain is limited to the near-field region and a near-field to far-field transformation is performed based on fields sampled on a closed Huygens’ surface around the object.

The near-field to far-field transformation is based on the equivalence theorem [143]. According to this theorem, if the tangential components of the fields are completely known over a closed surface, the fields outside of that closed surface within the source-free region can be determined.
4.4 Numerical Examples for the Staggered Meshless RPIM

Figure 4.21. Equivalence principle. (a) Original problem with actual sources $J_1$ and $M_1$, and (b) equivalent problem with sources on the Huygens’ surface.

Figure 4.21 illustrates the equivalence principle. Figure 4.21 (a) shows the actual radiating source that can be represented by currents $J_1$ and $M_1$. These sources produce the fields $E_1$ and $H_1$ everywhere. The field outside an arbitrary closed surface enclosing the source can be determined by the equivalence principle. To this end the original problem shown in Fig. 4.21 (a) can be replaced by equivalent problem shown in Fig. 4.21 (b). In the equivalent model, actual sources are removed and equivalent current sources $J$ and $M$ on the surface enclosing the original source are considered. The same electric and magnetic field in region $R_2$ can be obtained by these equivalent sources.

The electric and magnetic current densities on the Huygens’ surface for having $E_1$ and $H_1$ in region $R_2$ and zero fields in region $R_1$ as shown in Fig. 4.21 (b) are defined as follows

\[
J = n \times H_1, \quad (4.58)
\]

\[
M = -n \times E_1, \quad (4.59)
\]

where $E_1$ and $H_1$ are the simulated scattered electric and magnetic field (in the scattered field formulation) on the Huygens’ surface and $n$ is the unit normal vector to the Huygens’ surface.
Figure 4.22. Coordinate system for finding the far fields. Coordinate system for analysis the far fields. The source is shown with primed coordinates and unprimed coordinates show the observation point.

Figure 4.22 shows the spherical coordinate system used for far-field calculations. The magnetic and electric vector potentials $A$ and $F$ for electric and magnetic current densities $J$ and $M$ located in the source point $r' = (x', y', z')$ are given by

$$A = \frac{\mu}{4\pi} \int_S J(x', y', z') e^{-jkR} \frac{e^{-jkR}}{R} dS', \quad (4.60)$$

$$F = \frac{\varepsilon}{4\pi} \int_S M(x', y', z') e^{-jkR} \frac{e^{-jkR}}{R} dS', \quad (4.61)$$

where $R$ is the distance between the source point $r'$ and the observation point $r$, and $k = 2\pi/\lambda$ is the wavenumber. For far-field calculations, the observation point is very far from the source and it can be assumed that vector $R$ is in parallel with vector $r$ in Fig. 4.22. In this condition, $R$ in equations (4.60) and (4.61) can be approximated by $R \approx r$ for amplitude variations and $R \approx r - r' \cos \psi$ ($\psi$ is the angle between $r$ and $r'$) for phase variations. Considering these approximations equations (4.60) and (4.61) can be written as follows in the far field

$$A \approx \frac{\mu e^{-jkr}}{4\pi r} \int_S J(x', y', z') e^{jkr' \cos \psi} dS' = \frac{\mu e^{-jkr}}{4\pi r} N, \quad (4.62)$$

$$F \approx \frac{\varepsilon e^{-jkr}}{4\pi r} \int_S M(x', y', z') e^{jkr' \cos \psi} dS' = \frac{\varepsilon e^{-jkr}}{4\pi r} L, \quad (4.63)$$
4.4 Numerical Examples for the Staggered Meshless RPIM

where \( N \) and \( L \) are as follows

\[
N = \iint_S J(x', y', z') e^{jkr' \cos \psi'} dS',
\]
(4.64)

and

\[
L = \iint_S M(x', y', z') e^{jkr' \cos \psi'} dS'.
\]
(4.65)

Since spherical coordinates are more convenient for far-field calculations than the Cartesian coordinates, the following conversion matrix can be used to convert \( J(x', y', z') \) and \( M(x', y', z') \) to spherical coordinates \((r, \theta, \phi)\)

\[
\begin{bmatrix}
J_r \\
J_\theta \\
J_\phi 
\end{bmatrix} =
\begin{bmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{bmatrix}
\begin{bmatrix}
J_x \\
J_y \\
J_z
\end{bmatrix}.
\]
(4.66)

In this case, the integrals \( N \) and \( L \) in equations (4.64) and (4.65) in spherical coordinates are written as

\[
N_\theta = \iint_S J_\theta e^{jkr' \cos \psi'} dS' = \iint_S [J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta] e^{jkr' \cos \psi'} dS',
\]
(4.67a)

\[
N_\phi = \iint_S J_\phi e^{jkr' \cos \psi'} dS' = \iint_S [-J_x \sin \phi + J_y \cos \phi] e^{jkr' \cos \psi'} dS',
\]
(4.67b)

and

\[
L_\theta = \iint_S M_\theta e^{jkr' \cos \psi'} dS' = \iint_S [M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta] e^{jkr' \cos \psi'} dS',
\]
(4.68a)

\[
L_\phi = \iint_S M_\phi e^{jkr' \cos \psi'} dS' = \iint_S [-M_x \sin \phi + M_y \cos \phi] e^{jkr' \cos \psi'} dS'.
\]
(4.68b)

Finally, the approximated electric field at the far field can be obtained as

\[
E_r \simeq 0,
\]
(4.69a)

\[
E_\theta \simeq -\frac{jke^{-jk}}{4\pi r}(L_\phi + \eta N_\theta),
\]
(4.69b)

\[
E_\phi \simeq \frac{jke^{-jk}}{4\pi r}(L_\theta - \eta N_\phi),
\]
(4.69c)
and the approximated magnetic field at the far field can be given as

\[
H_r \simeq 0, \quad (4.70a)
\]

\[
H_\theta \simeq + \frac{jke^{-jk_r}}{4\pi r} (N_\phi - \frac{L_\theta}{\eta}), \quad (4.70b)
\]

\[
H_\phi \simeq - \frac{jke^{-jk_r}}{4\pi r} (N_\theta + \frac{L_\phi}{\eta}). \quad (4.70c)
\]

where \(\eta\) is the intrinsic impedance. Therefore, using the simulated electric and magnetic fields on the Huygens’ surface, the equivalent sources \(J\) and \(M\) are obtained as equations (4.58) and (4.59). The electric and magnetic fields in the far field can then be obtained by equations (4.69) and (4.70).

**Results**

In this section, the theoretical solution of the scattered electric field for a conducting sphere is compared with the numerical solution obtained using RPIM.

**Theoretical Results (Mie Solution)** The theoretical solution for the scattering from a conducting sphere, the Mie solution, is a summation of an infinite number of Hankel and Bessel functions that can be expressed as follows [144, 145]

\[
E_s = \sum_{n=1}^{\infty} E_n (ja_n N_{e1n} - b_n M_{o1n}), \quad (4.71)
\]

where \(M_{o1n}\) and \(N_{e1n}\) are the spherical vector harmonics, and scattering coefficients \(a_n\) and \(b_n\) are a combination of Hankel and Bessel functions, which are given in Appendix A in more details.

**Numerical Results** Figure 4.23 shows the Huygens’ surface around the conducting sphere. Using tangential electric and magnetic fields on the Huygens’ surface, electric and magnetic fields on the far-field observation points are calculated based on the equivalence principle. Figure 4.24 shows the propagation of the simulated scattering electric fields at different time steps in a cross section plane. Figures 4.25 and 4.26 show comparison of the numerical and theoretical scattered electric field in the far field in two orthogonal planes \(\phi = 0^\circ\) and \(\phi = 90^\circ\). The red solid line shows the Mie solution and the blue starred marks show the simulation results. It can be seen that there is a good agreement between simulation and theoretical solutions.
In the first part of this chapter, the meshless RPIM has been described for first-order electromagnetic simulations and the update equations in a 3D domain have been obtained. In the second part of this chapter, the domain truncation necessary to handle open problems has been introduced. In particular, the UPML has been explained and applied for the meshless method. The temporal behavior of UPMLs in the meshless RPIM has been studied in a 3D rectangular waveguide. It has been shown that after absorption of the fields in the UPMLs the solutions will intrinsically become unstable. This instability is caused by near zero spatial derivatives that causes the coefficient matrix to become singular with double zero eigenvalues. Adding small loss terms inside the UPMLs, after the initial absorption of most of the energy, can significantly shift the instability to later time without compromising the accuracy of the numerical solution. Finally, in the last part of this chapter, several numerical examples have illustrated several features of the method:

- In the first example, different node distributions have been investigated: regular, regular with 5% and 10% random node displacement, and cylindrical node distribution. This provides information about the impact of non-ideal node distributions. Results for all node distributions are in a good agreement with each other. Taking the regular node distribution as reference, the maximum phase deviation is 0.5% for the cylindrical node distribution and the maximum amplitude...
Figure 4.24. Scattering from the conducting sphere. Scattering from the conducting sphere, shown in Fig. 4.20, after (a) 50, (b) 90, (c) 170, and (d) 220 time steps. The incident wave is a $x$-polarized plane wave propagating in $z$-direction.
4.5 Conclusion

Figure 4.25. Comparing theoretical and numerical solutions for scattering from the sphere shown in Fig. 4.20 at plane $\phi = 0^\circ$. Comparing theoretical and numerical solutions for scattering from the sphere at plane $\phi = 0^\circ$.

Figure 4.26. Comparing theoretical and numerical solutions for scattering from the sphere shown in Fig. 4.20 at plane $\phi = 90^\circ$. Comparing theoretical and numerical solutions for scattering from the sphere at plane $\phi = 90^\circ$. 
deviation is 2.37% for 10% random node displacement. The results show that the
meshless RPIM scheme is robust for various node distributions, i.e., it yields very
close results for very different node arrangements. This increases confidence that
the method is suitable in versatile applications using complex geometries.

• In the second example, different parts of a diplexer have been discretized with
vertical, horizontal and a 45-degree oriented node distributions. The simulated
results show the wave propagation in the diplexer for two different frequencies.
This example shows the capability of meshless method to connect several do-
 mains with various orientation of regular node distributions. This capability can
be useful in discretization of complex geometries.

• In the last example, the RPIM has been implemented to simulate scattering from
a perfect conducting sphere. A scattering formulation has been introduced and
the far-field scattered electric field has been obtained using a near-field to far-
field transformation. Finally, the far-field scattering simulation results have been
compared with the theoretical solutions. It has been shown that the simulation
results are in a good agreement with the theoretical results.
In the first part of this chapter a time-domain meshless algorithm based on vector potentials is introduced for the analysis of transient electromagnetic fields. The proposed numerical algorithm is a modification of the Radial Point Interpolation Method (RPIM), where radial basis functions are used for local interpolation of the vector potentials and their derivatives. In the proposed implementation, solving the second-order vector potential wave equation intrinsically enforces the divergence-free property of the electric and magnetic fields. Furthermore, the computational effort associated with the generation of a dual node distribution (as required for solving the first-order Maxwell’s equations) is avoided. In the second part of this chapter an implementation of perfectly matched layer for the magnetic vector potential method is proposed. In the final part of this chapter, the proposed magnetic vector potential method is validated with several examples of 2D waveguides and filters, and the convergence is empirically demonstrated in terms of node density or size of local support domains. It is further shown that inhomogeneous node distributions can provide increased convergence rates, i.e., the same accuracy with smaller number of nodes compared to a solution for homogeneous node distribution.
5.1 Introduction

A time-domain implementation of meshless Radial Point Interpolation Method (RPIM) for the Maxwell’s equations was introduced in the previous chapter relying on staggered node distributions for the electric and magnetic fields (Fig. 5.1 (a)). Such staggered node arrangements have been proven to be robust in conjunction with most time-domain numerical methods. However, in the framework of meshless methods, the generation of an unstructured dual node distribution can be computationally costly and restricts the freedom of node positions, which might reduce the potential advantages of the scheme.

To overcome this challenge, a meshless vector potential technique is proposed in this chapter to solve electromagnetic problems in a non-staggered node distribution as shown in Fig. 5.1 (b). In this approach, on the one hand the order of the solved partial differential equations is increased to second order, compared to the first-order Maxwell’s equations solved in the staggered meshless RPIM. On the other hand, since the vector potentials are smoother functions of space than the field vectors near sources and sharp discontinuities such as corners and edges, the numerical errors for a given space discretization can be decreased [27]. Moreover, in this formalism, fields can be described using three scalar wave equations, which has been shown in [27] to increase the efficiency of a finite-difference time-domain solution and to save CPU time compared to the classical staggered approach. Finally, and most importantly, the vector potential technique preserves divergence property of the electric and magnetic fields and thus prevents the appearance of spurious solutions [24, 25].

The vector potential technique was introduced to solve time-domain electromagnetic fields with finite differences in [27, 146]. Various meshless implementations have been later reported [147, 26, 148, 149]. In this chapter, we propose an efficient implementation of the time-domain meshless RPIM based on the local calculation of the magnetic vector potential (A) in small support domains. The proposed method solves the wave equation using Gaussian radial basis functions and a simplified explicit time integration, allowing for efficient high-accuracy simulations on small stencil sizes. This implementation relies on the intrinsic capability of the RPIM method for accurate local interpolations of higher-order derivatives, both in homogeneous and inhomogeneous node distributions [150]. For simplicity, the method is firstly introduced in a 2D framework. The convergence of the method is then validated using parallel plate waveguide
and iris filter examples discretized with homogeneous and inhomogeneous node distributions. Moreover, a first extension to 3D is illustrated with the example of a square current loop antenna radiating in free space.

### 5.2 Maxwell’s Equations: The Vector and Scalar Potentials

The field values in terms of magnetic vector potential can be obtained using the definition of magnetic vector potential for the Maxwell’s equations. The Maxwell’s equations are repeated here for the reader’s convenience

\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad (5.1a)
\]

\[
\nabla \times H = J + \frac{\partial D}{\partial t}, \quad (5.1b)
\]

\[
\nabla \cdot D = \rho, \quad (5.1c)
\]

\[
\nabla \cdot B = 0. \quad (5.1d)
\]

Since the magnetic flux density \( B \) in equation (5.1) (d) is divergence free, it can be defined as the curl of a vector as follows [143]

\[
\mathbf{B} = \nabla \times \mathbf{A}, \quad (5.2)
\]

where \( \mathbf{A} \) is called magnetic vector potential. Substituting (5.2) in (5.1) (a) leads to

\[
\nabla \times E = -\frac{\partial (\nabla \times \mathbf{A})}{\partial t}, \quad (5.3)
\]

or

\[
\nabla \times (E + \frac{\partial \mathbf{A}}{\partial t}) = 0. \quad (5.4)
\]
5.3 Magnetic Vector Potential

Since the curl of the vector in the parenthesis in (5.4) is equal to zero, this vector can be expressed as the gradient of a scalar function $V$ as follows

$$E + \frac{\partial A}{\partial t} = -\nabla V,$$  \hfill (5.5)

where $V$ is called electric scalar potential. From equation (5.5) the electric field can be expressed as

$$E = -\frac{\partial A}{\partial t} - \nabla V.$$ \hfill (5.6)

Substituting (5.2) and (5.6) in equation (5.1) (b) gives

$$\nabla \times \left( \frac{1}{\mu} \nabla \times A \right) = J + \epsilon \frac{\partial}{\partial t} \left( -\frac{\partial A}{\partial t} - \nabla V \right),$$ \hfill (5.7)

where using the identity $\nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A$ and rearranging the terms in equation (5.7) result in

$$\nabla^2 A - \mu \epsilon \frac{\partial^2 A}{\partial t^2} - \nabla (\nabla \cdot A + \mu \epsilon \frac{\partial V}{\partial t}) = -\mu \mathbf{J}.$$ \hfill (5.8)

Finally, substituting (5.2) in (5.1) (c) gives

$$\nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla V \right) = -\frac{\rho}{\epsilon'},$$ \hfill (5.9)

or

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot A) = -\frac{\rho}{\epsilon}.$$ \hfill (5.10)

Note that the definition of $A$ in (5.2) automatically enforces the divergence-free property of the magnetic flux density $B$. Therefore, equations (5.2), (5.6), (5.8), and (5.10) are Maxwell’s equations in potential formulation.

5.3 Magnetic Vector Potential

Equations (5.2) and (5.6) do not uniquely define the potentials. In other words, if $A$ is changed to $A' = A + \nabla \phi$, the equation (5.2) is still satisfied since $\nabla \times (\nabla \phi) = 0$. However, the new $A$ will change the $E$ in (5.6) to $-\frac{\partial A}{\partial t} - \nabla (V + \frac{\partial \phi}{\partial t})$. Therefore, if $A$ and $V$ are simultaneously changed to $A' = A + \nabla \phi$ and $V' = V - \frac{\partial \phi}{\partial t}$, then both $E$ and $B$ in (5.2) and (5.6) are consistently changed to satisfy Maxwell’s equations. Such changes in $A$ and $V$ are called gauge transformation.
There are different gauges in the literature, e.g., the Lorenz gauge is written as

$$\nabla \cdot \mathbf{A} = -\mu \varepsilon \frac{\partial V}{\partial t}. \quad (5.11)$$

Using this gauge the magnetic vector potential $\mathbf{A}$ and the electric scalar potential $V$ in (5.8) and (5.10) satisfy the inhomogeneous wave equations as

$$\nabla^2 \mathbf{A} - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}, \quad (5.12)$$

and

$$\nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}. \quad (5.13)$$

The electric field can be calculated using (5.6) and (5.11) as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{\mu \varepsilon} \int \nabla \left( \nabla \cdot \mathbf{A} \right) dt, \quad (5.14)$$

and the magnetic field using (5.2) is

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}. \quad (5.15)$$

From (5.12), it is seen that in the presence of a current source $\mathbf{J}$, the magnetic vector potential $\mathbf{A}$ satisfies the inhomogeneous wave equation. In this case instead of discretizing the domain for $\mathbf{E}$ and $\mathbf{H}$ and solving Maxwell’s equations using staggered node distributions, a single set of nodes for $\mathbf{A}$ need only to be considered. Using the Lorenz gauge condition, equation (5.12) is first solved and in a post-processing step, $\mathbf{E}$ and $\mathbf{H}$ can be calculated through equations (5.14) and (5.15).

The vector potential technique automatically preserves the divergence property of the electric field as well as of the magnetic field. From equation (5.14) the electric field can be written in the frequency domain as follows

$$\mathbf{E} = -j\omega \mathbf{A} + \frac{1}{j\omega \mu \varepsilon} \nabla \left( \nabla \cdot \mathbf{A} \right). \quad (5.16)$$

Taking the divergence of both sides of the above equation and multiplying by $j\omega \mu \varepsilon$ result in

$$j\omega \mu \varepsilon \nabla \cdot \mathbf{E} = -\mu \varepsilon (j\omega)^2 \left( \nabla \cdot \mathbf{A} \right) + \nabla \cdot \nabla \left( \nabla \cdot \mathbf{A} \right) = -\mu \varepsilon (j\omega)^2 \left( \nabla \cdot \mathbf{A} \right) + \nabla^2 \left( \nabla \cdot \mathbf{A} \right). \quad (5.17)$$

Using (5.12), the right hand side of the above equation can be rewritten to yield

$$j\omega \mu \varepsilon \nabla \cdot \mathbf{E} = -\mu \nabla \cdot \mathbf{J}, \quad (5.18)$$
5.4 Update Equations for $A$

or

$$j\omega(\epsilon \nabla \cdot E) = -\nabla \cdot J.$$  \hspace{1cm} (5.19)

It can be concluded that in the source-free region ($J = 0$), $\nabla \cdot E$ is a constant in time. Considering zero initial values for the electric field $E$ in a time-domain solver it can be concluded that the electric field is divergence free in the source-free region.

It is worth mentioning that in the presence of magnetic sources, the electric vector potential $F$ needs to be solved [143]. Note that this could be made in parallel to the $A$, and on the same node distribution. Finally the total electric and magnetic fields can be calculated by superposition of the fields obtained by $A$ and $F$.

5.4 Update Equations for $A$

Using the RPIM algorithm to discretize space derivatives and finite differences to discretize time derivatives in (5.12) lead to a discretized expression for the magnetic vector potential. In the non-staggered meshless RPIM the boundary conditions are related to the derivative of the fields. Therefore, a local coordinate system is required to implement accurate derivative boundary conditions. The update equations for the magnetic vector potential in Cartesian coordinate system is first introduced. To handle curved boundaries, the update equations for the magnetic vector potential in polar coordinate system is then introduced, which will be used for the PEC curved boundary nodes in the next section.

5.4.1 Cartesian Coordinates

The explicit update equations for the magnetic vector potential can be obtained in Cartesian coordinates by discretization of (5.12) through RPIM and finite difference methods. In a 2D TE$_z$ framework, i.e., with the magnetic vector potential and consequently the electric field perpendicular to the $xy$-plane and constant in $z$-direction, the update equation for the $z$-component of the magnetic vector potential at location $x = (x, y)$ is derived as

$$A_{z}^{n+1} = 2A_{z}^{n} - A_{z}^{n-1} + \frac{(\Delta t)^2}{\mu\epsilon} \left( \frac{\partial^2 \phi(x)}{\partial x^2} A_{z}^{n} + \frac{\partial^2 \phi(x)}{\partial y^2} A_{z}^{n} \right) + \frac{(\Delta t)^2}{\epsilon} J_{z}^{n}. \hspace{1cm} (5.20)$$

Expanding to 3D Cartesian coordinates, the Laplacian operator is given as

$$\nabla^2 A = (\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z), \hspace{1cm} (5.21)$$
Chapter 5 Non-Staggered Meshless RPIM: Vector Potential Technique

and the update equations for different components of the magnetic vector potential can be obtained separately in a similar way starting from (5.12).

5.4.2 Polar Coordinates

The polar coordinate system with components denoted as \((\rho, \Phi)\) can be used as a local coordinate system to increase the accuracy of simulations for curved boundaries. In order to write the vector Laplacian operator in the polar coordinates, it is best to start from the equivalent expression

\[
\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A) .
\] (5.22)

Using the curl and divergence in the polar coordinate system, as explicitly given in Appendix B, and after some manipulations, the following expression is obtained for the Laplacian operator

\[
\nabla^2 A = (\nabla^2 A_{\rho} - \frac{A_{\rho}}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_{\Phi}}{\partial \Phi}) \hat{\rho} + \left(\nabla^2 A_{\Phi} - \frac{A_{\Phi}}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_{\rho}}{\partial \Phi}\right) \hat{\Phi},
\] (5.23)

where

\[
\nabla^2 A_{\rho} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial (A_{\rho})}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 A_{\rho}}{\partial \Phi^2},
\] (5.24)

and

\[
\nabla^2 A_{\Phi} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial (A_{\Phi})}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 A_{\Phi}}{\partial \Phi^2}.
\] (5.25)

Using the expression (5.23) in (5.12), each component of \(A\) in the polar coordinate system satisfies the wave equation for the magnetic vector potential as follows

\[
\left(\nabla^2 A\right)_{\rho} - \mu \epsilon \frac{\partial^2 A_{\rho}}{\partial t^2} = -\mu J_{\rho},
\] (5.26)

and

\[
\left(\nabla^2 A\right)_{\Phi} - \mu \epsilon \frac{\partial^2 A_{\Phi}}{\partial t^2} = -\mu J_{\Phi}.
\] (5.27)

The update equations for each component of the magnetic vector potential in the polar coordinate system can be obtained by discretizing these two equations. However, since the polar coordinate system is only used for the curved boundary nodes in this thesis, the update equations for boundary nodes can be simplified using the boundary conditions as shown in the next section.
5.5 Perfect Electric Boundary Conditions for A

Boundary conditions for the magnetic vector potential can be adapted from boundary conditions for the electric or magnetic fields in the Maxwell’s equations.

Perfect electric boundary conditions for the magnetic vector potential in Cartesian and polar coordinate systems are introduced in the following.

5.5.1 Cartesian Coordinates

Boundary conditions for the Perfect Electric Conductor (PEC) in the magnetic vector potential method can be obtained by enforcing the tangential component of the electric field to zero \( E_\tau = 0 \) in (5.14). These boundary conditions can be derived as follows

\[
A_\tau = 0, \quad \frac{\partial A_n}{\partial n} = 0, \quad (5.28)
\]

where \( n \) indicates the component normal to the boundary, \( A_\tau \) and \( A_n \) are the tangential and the normal components of \( A \), respectively.

5.5.2 Polar Coordinates

As mentioned before, a polar coordinate system can be used as a local coordinate system to handle curved boundaries. Generally, an arbitrary curved boundary can be approximated by connecting different arcs of various circles. For simplicity an arc of a circle centered at the origin is considered here. In this case, perfect electric boundary conditions for the magnetic vector potential in the polar coordinate system can be obtained by enforcing \( E_\Phi = 0 \) in (5.14). In this coordinate system, the term \( \nabla(\nabla \cdot A) \) in (5.14) becomes

\[
\nabla(\nabla \cdot A) = \frac{\partial}{\partial \rho} \left( \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\Phi}{\partial \Phi} \right) \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \Phi} \left( \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\Phi}{\partial \Phi} \right) \hat{\Phi}. \quad (5.29)
\]

The identities to find this expression are given in Appendix B. Imposing the condition \( E_\Phi = 0 \) in (5.14) leads to

\[
E_\Phi = -\frac{\partial A_\Phi}{\partial t} + \frac{1}{\mu \epsilon} \int \frac{1}{\rho} \frac{\partial}{\partial \Phi} \left( \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\Phi}{\partial \Phi} \right) \, dt = 0. \quad (5.30)
\]

Considering that each term of the sum in (5.30) must be equal to zero results in the following conditions

\[
A_\Phi = 0, \quad (5.31)
\]
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Figure 5.2. Boundary and dummy nodes in polar coordinate system. Boundary and dummy nodes in polar coordinate system for applying derivative boundary conditions.

and

\[ \frac{1}{\mu \varepsilon} \int \frac{1}{\rho} \frac{\partial}{\partial \Phi} \left( \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\Phi}{\partial \Phi} \right) dt = 0. \]  \hspace{1cm} (5.32)

The condition in (5.31) is straightforward to apply. To implement derivative boundary conditions in (5.32), a row of dummy nodes is introduced outside the computational domain. For simplicity, a structure of only four neighbors are considered in a support domain of each boundary node as shown in Fig. 5.2. This limitation is achieved by adjusting the radius of the support domain. Note that in this case the normal derivative \( \frac{\partial A_\rho}{\partial \rho} \) is only related to the two neighbors in normal direction whereas the tangential derivative \( \frac{\partial A_\Phi}{\partial \Phi} \) is related to two neighbors on the boundary. In this case, the first condition \( A_\Phi = 0 \) on the boundary nodes results in \( \frac{\partial A_\Phi}{\partial \Phi} = 0 \). Therefore, the second condition in (5.32) can be simplified to

\[ \frac{1}{\mu \varepsilon} \int \frac{1}{\rho} \frac{\partial}{\partial \Phi} \left( \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} \right) dt = 0, \]  \hspace{1cm} (5.33)

or

\[ \frac{\partial A_\rho}{\partial \rho} = - \frac{A_\rho}{\rho}. \]  \hspace{1cm} (5.34)

The finite-difference approximation is used to discretize this boundary condition. The discretization of condition in (5.34) for a node in a concave boundary (upper boundary in Fig. 5.2) is as follows

\[ \frac{A_\rho^t - A_\rho^b}{2 \Delta \rho} = - \frac{A_\rho^c}{\rho}, \]  \hspace{1cm} (5.35)

where \( A_\rho^c \) is the normal component of the magnetic vector potential at the boundary node (center node), \( A_\rho^t \) is the normal component of the magnetic vector potential at
5.6 Perfectly Matched Layer (PML) for $A$

dummy node (top node) and $A^b_\rho$ is the normal component of the magnetic vector potential at node inside the domain (bottom node) as shown in Fig. 5.2. So, the normal component of the magnetic vector potential at dummy nodes in a concave boundary can be expressed as

$$A^t_\rho = A^b_\rho - \frac{2\Delta\rho}{\rho}A^c_\rho,$$  \hspace{1cm} (5.36)

and similarly the normal component of the magnetic vector potential at dummy nodes in a convex boundary (lower boundary in Fig. 5.2) is as

$$A^b_\rho = A^t_\rho + \frac{2\Delta\rho}{\rho}A^c_\rho.$$  \hspace{1cm} (5.37)

Therefore, by updating the dummy nodes based on (5.36) and (5.37) the boundary condition in (5.34) is satisfied.

Using the boundary conditions in (5.31) and (5.34) for PEC curved surface centered at the origin, the update equations for boundary nodes in (5.26) and (5.27) can be simplified. Considering the condition in (5.31), the tangential component of the magnetic vector potential at boundary nodes is zero. Therefore the update equation for $A_\Phi$, which can be found from (5.27) is trivial ($A_\Phi = 0$). The Laplacian operator for the $\hat{\rho}$-component is as follows

$$(\nabla^2 A)_{\hat{\rho}} = \frac{\partial^2 A_\rho}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 A_\rho}{\partial \Phi^2} + \frac{1}{\rho} \frac{\partial A_\rho}{\partial \rho} - \frac{2}{\rho^2} \frac{\partial A_\Phi}{\partial \Phi} - \frac{A_\rho}{\rho^2}. \hspace{1cm} (5.38)$$

Therefore, according to (5.26) the update equation for boundary nodes is as follows

$$A^{n+1}_\rho = 2A^n_\rho - A^{n-1}_\rho + \Delta t^2 (\nabla^2 A)^n_{\hat{\rho}}$$

$$= 2A^n_\rho - A^{n-1}_\rho + \Delta t^2 \left( \frac{\partial^2 A^n_\rho}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 A^n_\rho}{\partial \Phi^2} + \frac{1}{\rho} \frac{\partial A^n_\rho}{\partial \rho} - \frac{A^n_\rho}{\rho^2} \right), \hspace{1cm} (5.39)$$

where using boundary condition in (5.34), the update equation for boundary nodes is simplified to

$$A^{n+1}_\rho = 2A^n_\rho - A^{n-1}_\rho + \Delta t^2 \left( \frac{\partial^2 A^n_\rho}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 A^n_\rho}{\partial \Phi^2} - \frac{2A^n_\rho}{\rho^2} \right). \hspace{1cm} (5.40)$$

The boundary conditions and update equations in the Cartesian and polar coordinate systems will be used to implement the numerical examples later in this chapter.

5.6 Perfectly Matched Layer (PML) for $A$

In this section an implementation of PML is introduced to truncate the computational domain in the magnetic vector potential technique. A popular formulation for PML
is a stretched coordinate approach [151] that can be implemented by introducing auxiliary variables. Therefore a second set of (staggered) nodes for the auxiliary variables is required in the PML regions, where a structured (grid-like) node distribution is generally in use and dual grid generation is straightforward. However, the formulation in the main computational domain remains unchanged, and only a single set of nodes is required for the solution of the second-order magnetic vector potential equation. This section introduces an algorithm for implementing hybrid staggered stretched-coordinate PML equations in the framework of the non-staggered magnetic vector potential solution domain, and it is validated in a 2D rectangular waveguide in the next section.

The PML parameters can be introduced by using a stretched coordinate approach for the Laplacian in (5.12) as follows [151]

\[
\nabla_s^2 = \frac{1}{s_x} \frac{\partial}{\partial x} \left( \frac{1}{s_x} \frac{\partial}{\partial x} \right) + \frac{1}{s_y} \frac{\partial}{\partial y} \left( \frac{1}{s_y} \frac{\partial}{\partial y} \right),
\]

where

\[s_\xi = 1 + \frac{\sigma_\xi}{j\omega \epsilon}, \quad \xi = x, y.\]  

(5.42)

The PML conductivity is defined with the following conductivity profile

\[
\sigma_\xi(\rho) = \sigma_{\text{max}} \left( \frac{\rho}{d_\xi} \right)^m,
\]

where \(d_\xi\) is the thickness of PML in \(\xi\)-direction, \(\rho\) is the position in the PML, and \(m\) is the polynomial order of the conductivity profile. The variable \(\sigma_{\text{max}}\) is the maximum conductivity at the end of the PML, which can be defined for a desired theoretical reflection coefficient at normal incident \(R_0\) as follows [128]

\[
\sigma_{\text{max}} = -\ln(R_0) \epsilon \sigma_0 (m + 1) \]

\[2d\]  

(5.44)

Therefore, substituting (5.41) into the wave equation (5.12) gives the following expression for the PML region

\[
\frac{1}{s_x} \frac{\partial}{\partial x} \left( \frac{1}{s_x} \frac{\partial A}{\partial x} \right) + \frac{1}{s_y} \frac{\partial}{\partial y} \left( \frac{1}{s_y} \frac{\partial A}{\partial y} \right) - \mu \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J.
\]

(5.45)

For the numerical solution of this equation, two auxiliary variables are defined for each dimension similarly as in [152]. In frequency domain, these auxiliary variables in
5.6 Perfectly Matched Layer (PML) for $A$

$x$-direction are as follows

$$j\omega X_1 = \frac{1}{s_x} \frac{\partial A}{\partial x},$$  
(5.46a)

$$j\omega X_2 = \frac{1}{s_x} \frac{\partial (j\omega X_1)}{\partial x}.  
(5.46b)

Auxiliary variables $Y_1, Y_2$ for the $y$-direction are similarly defined as follows

$$j\omega Y_1 = \frac{1}{s_y} \frac{\partial A}{\partial y},$$  
(5.47a)

$$j\omega Y_2 = \frac{1}{s_y} \frac{\partial (j\omega Y_1)}{\partial y}.  
(5.47b)

Substituting the auxiliary variables $X_2, Y_2$ into (5.45) in frequency domain leads to

$$j\omega X_2 + j\omega Y_2 - \mu \epsilon (j\omega)^2 A = -\mu J.$$  
(5.48)

The corresponding time-domain equations for (5.46) are as follows

$$\frac{\partial X_1}{\partial t} + \frac{\sigma_x}{\epsilon} X_1 = \frac{\partial A}{\partial x},$$  
(5.49a)

$$\frac{\partial X_2}{\partial t} + \frac{\sigma_x}{\epsilon} X_2 = \frac{\partial^2 X_1}{\partial x^2 t},$$  
(5.49b)

and the corresponding time-domain equations for (5.47) are as follows

$$\frac{\partial Y_1}{\partial t} + \frac{\sigma_y}{\epsilon} Y_1 = \frac{\partial A}{\partial y},$$  
(5.50a)

$$\frac{\partial Y_2}{\partial t} + \frac{\sigma_y}{\epsilon} Y_2 = \frac{\partial^2 Y_1}{\partial y^2 t}. 
(5.50b)

Finally, the time-domain equation for (5.48) is given by

$$\frac{\partial X_2}{\partial t} + \frac{\partial Y_2}{\partial t} - \mu \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J.$$  
(5.51)

Using finite differences and RPIM discretizations for time and spatial derivatives in (5.49) result in

$$X_1^{n+\frac{1}{2}} = \left( \frac{1 - r_x}{1 + r_x} \right) X_1^{n-\frac{1}{2}} + \frac{\Delta t}{1 + r_x} \frac{\partial \phi (x)}{\partial x} A^n, 
(5.52a)$$

$$X_2^{n+\frac{1}{2}} = \left( \frac{1 - r_x}{1 + r_x} \right) X_2^{n-\frac{1}{2}} + \frac{1}{1 + r_x} \left( \frac{\partial \phi (x)}{\partial x} X_1^{n+\frac{1}{2}} - \frac{\partial \phi (x)}{\partial x} X_1^{n-\frac{1}{2}} \right), 
(5.52b)$$

where $r_x = \sigma_x \Delta x / 2 \epsilon$. The discretized form of (5.50) gives

$$Y_1^{n+\frac{1}{2}} = \left( \frac{1 - r_y}{1 + r_y} \right) Y_1^{n-\frac{1}{2}} + \frac{\Delta t}{1 + r_y} \frac{\partial \phi (y)}{\partial y} A^n, 
(5.53a)$$

$$Y_2^{n+\frac{1}{2}} = \left( \frac{1 - r_y}{1 + r_y} \right) Y_2^{n-\frac{1}{2}} + \frac{1}{1 + r_y} \left( \frac{\partial \phi (y)}{\partial y} Y_1^{n+\frac{1}{2}} - \frac{\partial \phi (y)}{\partial y} Y_1^{n-\frac{1}{2}} \right), 
(5.53b)$$
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where \( r_y = \sigma_y \Delta x / 2\epsilon \). The discretized form of (5.51) is as follows

\[
\left( \frac{X_2^{n+\frac{1}{2}} - X_2^{n-\frac{1}{2}}}{\Delta t} + \frac{Y_2^{n+\frac{1}{2}} - Y_2^{n-\frac{1}{2}}}{\Delta t} \right) - \mu \epsilon \left( \frac{D_t A^{n+\frac{1}{2}} - D_t A^{n-\frac{1}{2}}}{\Delta t} \right) = -\mu J, \tag{5.54}
\]

where \( D_t A = \frac{\partial A}{\partial t} \). To find the update equation for \( A \), the variable \( D_t A \) is obtained from (5.54) as

\[
D_t A^{n+\frac{1}{2}} = D_t A^{n-\frac{1}{2}} + \frac{\Delta t}{\mu \epsilon} \left( \frac{X_2^{n+\frac{1}{2}} - X_2^{n-\frac{1}{2}}}{\Delta t} + \frac{Y_2^{n+\frac{1}{2}} - Y_2^{n-\frac{1}{2}}}{\Delta t} + \mu J \right). \tag{5.55}
\]

The magnetic vector potential \( A \) can then be updated staggered in time as follows

\[
A^{n+1} = A^n + \Delta t (D_t A^{n+\frac{1}{2}}). \tag{5.56}
\]

The algorithm to update the magnetic vector potential can be summarized as follows

**Step 1:** Update the auxiliary variables \( X_1 \) and \( Y_1 \) using (5.52) (a) and (5.53) (a),

**Step 2:** Update the auxiliary variables \( X_2 \) and \( Y_2 \) using (5.52) (b) and (5.53) (b),

**Step 3:** Update the variable \( D_t A \) according to (5.55),

**Step 4:** Update the magnetic vector potential \( A \) according to (5.56)

In the next section, this algorithm will be modified to implement a staggered PML formulation in the non-staggered framework of magnetic vector potential.

### 5.6.1 Hybridization of Staggered and Non-Staggered Meshless RPIM

In this section a hybrid algorithm for the implementation of PMLs in the meshless magnetic vector potential technique is proposed. Solving the wave equation in time-domain, the magnetic vector potential technique avoids using staggered node distributions that are needed for calculating the \( E \) and \( H \) fields when directly solving Maxwell’s equations. However, as demonstrated above, implementing PMLs with stretched coordinate formulation requires auxiliary variables on a staggered (dual) node distribution. To avoid defining staggered nodes in the whole computational domain, a hybrid algorithm is proposed in this section: The algorithm keeps a single set of nodes for
5.6 Perfectly Matched Layer (PML) for $\mathbf{A}$

![Figure 5.3. 2D rectangular waveguide with staggered and non-staggered node distributions.](image)

Node distribution for 2D rectangular waveguide with width $w = 7.49$ cm, i.e., cut-off frequency of 2 GHz. The free space is discretized with $\mathbf{A}$ nodes and PMLs are discretized with staggered nodes for $\mathbf{A}$ and auxiliary variables $X_1, X_2$.

the magnetic vector potential $\mathbf{A}$ inside the main computational domain, while it uses staggered nodes for $\mathbf{A}$ and auxiliary variables inside the PML. The hybrid algorithm is validated in a 2D rectangular waveguide and numerical reflection coefficients are compared for different thicknesses of the PML and for different orders of a polynomial conductivity profile inside the PML. A good agreement between theoretical results and converged solutions validates the approach, with best performance using a polynomial order $m = 3$ for the conductivity profile (5.43).

As described in the previous section in order to implement PML using the stretched coordinate approach, auxiliary variables need to be defined. For the described time-staggered update equations, the best approach is to discretize the computational domain into a staggered node distribution of $\mathbf{A}$ and auxiliary variables. To avoid staggered nodes in the whole domain, the introduction of the auxiliary nodes for $X_1, X_2$ can be limited to the PML region. Figure 5.3 shows a regular node distribution for a 2D rectangular waveguide, which is truncated with PMLs at both ends in the $x$-direction. The dot points show $\mathbf{A}$ nodes in free space, and $\mathbf{A}$ and $X_2$ nodes inside the PMLs. Furthermore, the crosses in Fig. 5.3 show $X_1$ nodes inside the PML and at the interface of the two regions. Since the auxiliary variables only exist in the PML region, the update equation for free space (5.20) can be directly applied in this region [6]. Hence, the free-space region is updated based on (5.20) and the PML region is updated through equations (5.52) to (5.56). Note that at the PML interface there is an overlap, where each region uses nodes that are updated in the other region. Thus, the communication at the interface requires regions overlap and a careful implementation of the explicit algorithm to maintain causality and consistency.

Figure 5.4 shows an example node distribution around the interface between free-space and the PML. The two regions of PML and free space are shown with different colors. Figure 5.4 illustrates the five steps of the updating process for the two regions. In each
Step 1: The auxiliary variable $X_1$ in the PML (cross points) is updated through vector potentials $A$ using equation (5.52) (a). As shown in Fig. 5.4.1, the support domain around each node $X_1$ at the interface includes $A$ nodes in the free-space domain. The auxiliary variable $Y_1$ is similarly updated through equation (5.53) (a).

Step 2: $X_2$ in the PML is updated using $X_1$ as in (5.52) (b). $Y_2$ is updated similarly using (5.53) (b).

Step 3: The first-order finite-difference variable $D_t A$ in the dot points inside the PML is calculated using auxiliary variables from the previous step. These values will be used in Step 5 for updating $A$ in the PML.

Step 4: The magnetic vector potential $A$ in free space is updated through (5.20). As shown in Fig. 5.4.4, the support domain of each $A$ node in the interface includes both $A$ nodes from the PML and free-space region.

Step 5: The magnetic vector potential in the PML region is calculated through $D_t A$ which was calculated in Step 3.

This procedure yields a stable hybrid algorithm to link the second-order magnetic vector potential method and the staggered PML formulation.

5.6.2 Numerical Results

To validate the hybridization algorithm for the staggered and the non-staggered magnetic vector potentials in the presence of a PML, the 2D rectangular waveguide structure shown in Fig. 5.3 is considered. The fundamental TE$_{10}$ mode distribution is excited on a source plane as a modulated sinusoidal Gaussian pulse with a spectrum covering $f = [2 - 4]$ GHz. The structure is discretized with $\Delta x = \lambda_{\text{min}}/16$ and PMLs are used to truncate the domain on both sides. The theoretical reflection coefficient in (5.44) is set to $R_0 = 10^{-4}$. To extract the reflection coefficient from the PML, an output port is defined along the $y$-axis towards the end of the waveguide as shown in Fig. 5.3. The S-parameters over the port plane are calculated as in [153].

Figure 5.5 shows the reflection coefficient from the PML with a thickness of $d = 10\Delta x$ and different polynomial orders of conductivity profile (5.43) from $m = 1$ to $m = 5$.
Figure 5.4. Updating steps for hybridization of staggered PML formulation and non-staggered magnetic vector potential method. Five steps are required for updating magnetic vector potential $\mathbf{A}$. The nodes that are at the interface of PML and free space are shown inside the dash lines. In steps 1, 2, and 4, darker red points show the position of the variable which is updated through variables in the darker black or blue point positions.
inside the PML. The blue solid line shows the theoretical reflection coefficient in the waveguide [128]. It can be seen that for a polynomial order of $m = 3$, best results are achieved. Generally, this observation has been made across a wide range of PML thicknesses (not shown here).

Figure 5.6 shows the reflection coefficient from the PML for an order of $m = 3$ and different thicknesses of the PML. It can be clearly seen that increasing the thickness of the PML improves its reflection coefficient, illustrating convergence of the discretized PML toward the ideal (continuous) PML. The numerical reflection coefficient for $m = 3$ and $d = 24\Delta x$ is at the same level as the theoretical result.

### 5.7 Numerical Examples for the Non-Staggered Meshless RPIM

In this section, different aspects of the proposed meshless vector potential method are numerically validated through several examples. The method is first validated with the canonical example of a 2D parallel plate waveguide. The dispersion characteristics of the parallel plate waveguide are obtained and compared with theoretical results and a convergence study is performed. Then, as a second example, scattering parameters of an iris filter computed with the proposed method are compared with mode-matching
5.7 Numerical Examples for the Non-Staggered Meshless RPIM

Figure 5.6. Reflection coefficient from PML for different PML thicknesses. Comparison of reflection coefficient from PML for different PML thicknesses when the order of polynomial conductivity profile is \( m = 3 \). The blue solid line shows the expected theoretical reflection coefficient in the waveguide.

results. Moreover, to evaluate the application in the orthogonal polarization (TM \( z \) in 2D) the S-parameters of a tilted parallel plate waveguide and a bend parallel plate waveguide with TEM mode are obtained. Finally, the simulated far-field radiation pattern for a square loop antenna in 3D is compared with the theoretical results.

5.7.1 Parallel Plate Waveguide, TE\(_1\) Mode

Figure 5.7 (a) shows the geometry of a 2D parallel plate waveguide with a regular node distribution for the magnetic vector potential \( A \). Note that compared to previous implementations of the time-domain RPIM method (Section 4.1), no staggered electric and magnetic field node distributions are required. The separation distance of two plates \( a \) is chosen for a cut-off frequency \( f_c = 2 \) GHz. A modulated sinusoidal Gaussian pulse with spectrum covering \( f = [2 - 4] \) GHz is used as pulsed current excitation \( J_z \). The current source is located at distance \( D = \lambda_g / 4 \) (\( \lambda_g \) at center frequency \( f_0 = 3 \) GHz) from a shorted end, and the parallel plate waveguide is truncated with a PML at the other end as shown in Section 5.6 [6]. Two observation points are defined at distances \( d_1 \) and \( d_2 \) from the first end to record time-domain data. Figure 5.7 (b) illustrates the propagation of the computed magnetic vector potential in the parallel plate waveguide. Figures 5.7 (c) and (d) are the magnetic and electric field distributions, which can be found in a post-processing step from equations (5.14) and (5.15).
Figure 5.7. Parallel plate waveguide. (a) Geometry and distribution of nodes for a 2D parallel plate waveguide with separation distance $a = 7.49$ cm, i.e., cut-off frequency of 2 GHz, (b) instantaneous magnetic vector potential $\mathbf{A}$ for a wideband pulse propagation at $t = 1.53$ ns, (c) magnetic field, and (d) electric field calculated from the magnetic vector potential $\mathbf{A}$.

The dispersion characteristics of the parallel plate waveguide are obtained from Fourier-transformed time-domain data recorded at $d_1$ and $d_2$. To this end, the phase constant from the simulation ($\beta_{\text{sim}}$) can be found by computing the phase differences at two observation points divided by the distance $(d_2 - d_1)$. A comparison of the theoretical phase constant $\beta_{\text{th}}$ with the simulation results for $\Delta x = \lambda/8$ with a shape parameter $\alpha = 0.03$ and $N = 9$ neighbors in the local support domain is presented in Fig. 5.8. To quantify the accuracy of the method, a comparison of the theoretical phase constant $\beta_{\text{th}}$ with the simulation results is performed and the relative numerical error is calculated for different node discretizations. Figure 5.9 shows the error as a function of frequency considering the same number of neighbors ($N = 9$) and shape parameter $\alpha = 0.001$ for different discretizations. Note that the physical size of the influence domain is reduced for finer resolutions. It is observed that the error decreases as the discretization gets finer. Since the simulation has to be stopped at a certain finite time, the simulation...
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Figure 5.8. Comparing simulated and theoretical phase constant in the parallel plate waveguide. Theoretical and simulated phase constant for $\Delta x = \lambda/8$, number of neighbors $N = 9$ and shape parameter $\alpha = 0.03$.

Figure 5.9. Relative numerical error. Error for different discretizations, considering support domains with $N = 9$ neighbors and shape parameter $\alpha = 0.001$. 

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Figure 5.10. Relative numerical error. Error for different numbers of neighbors for $\Delta x = \lambda/16$ and shape parameter $\alpha = 0.001$.

accuracy is reduced close to the cut-off frequency, where the group velocity tends to zero, and thus the relative error diverges. Also, it can be seen that the error generally increases towards higher frequencies since the relative discretization becomes coarser.

Figure 5.10 shows the comparison of the error for different numbers of neighbors $N$, as calculated using a discretization $\Delta x = \lambda/16$ and shape parameter $\alpha = 0.001$. A decrease of the error for a larger number of neighbors is observed. The results of Fig. 5.9 and Fig. 5.10 empirically demonstrate convergence of the method both in terms of increasing node density and increasing number of neighboring nodes.

5.7.2 Iris Filter

The second numerical example to validate our proposed vector potential RPIM technique is an iris filter. The scattering parameters of a bandpass waveguide filter, with two identical irises separated by a distance of $d = 20$ mm [154], are obtained through simulation with the proposed method and compared with results from mode-matching method [155].

Figure 5.11 represents the geometry and dimensions of the filter consisting of two cascaded irises. The fundamental TE$_{10}$ mode is excited in the waveguide on the source plane via a Gaussian pulse with spectrum covering $[8 - 12]$ GHz. The structure is discretized with $\Delta x = \lambda_{\text{min}}/16$ and PMLs are used to truncate the domain on both
5.7 Numerical Examples for the Non-Staggered Meshless RPIM

Figure 5.11. Iris filter. Geometry of an iris filter with two cascading irises. All dimensions are in millimeter.

To extract the scattering parameters from the iris waveguide filter, two ports are defined along the x-axis on both ends (input and output) of the filter as shown in Fig. 5.11. The mode voltage $V_m$ and current $I_m$ over the port plane $m$ are calculated through the magnetic vector potential as follows

\[ V_m(\omega) = \sum_{i} n_p \Delta x_i \sin\left(\frac{\pi}{a} x_i\right) \left( -j\omega \right) A_{m,z}(\omega) \], (5.57a)

\[ I_m(\omega) = \sum_{i} n_p \Delta x_i \sin\left(\frac{\pi}{a} x_i\right) \left( \frac{1}{\mu_0} \frac{\partial A_{m,z}(\omega)}{\partial y} \right) \], (5.57b)

where $n_p$ is the number of port nodes, and $\sin(\pi x_i/a)$ corresponds to the TE$_{10}$ mode at node location $(x_i, y_i)$. The incident wave $a_m(\omega)$ and reflected wave $b_m(\omega)$ amplitudes at port $m$ can be then calculated using $V_m(\omega)$ and $I_m(\omega)$ as follows [153]

\[ a_m(\omega) = \frac{V_m(\omega) + Z_m(\omega) I_m(\omega)}{2\sqrt{Z_m(\omega)}}, \]

and

\[ b_m(\omega) = \frac{V_m(\omega) - Z_m(\omega) I_m(\omega)}{2\sqrt{Z_m(\omega)}}, \]

where the mode impedance $Z_m(\omega)$ is

\[ Z_m(\omega) = \sqrt{\frac{V_m(\omega) V'_m(\omega)}{I_m(\omega) I'_m(\omega)}}. \]

In the above equation $V'_m(\omega)$ and $I'_m(\omega)$ are the spatial derivatives of the voltage and current along the normal to the port plane $m$. Finally, the scattering parameters for input reflection ($S_{11}$) and transmission coefficient ($S_{21}$) are obtained from $a_m(\omega)$ and $b_m(\omega)$ as

\[ S_{11}(\omega) = \frac{b_1(\omega)}{a_1(\omega)}, \]

and

\[ S_{21}(\omega) = \frac{a_2(\omega)}{a_1(\omega)}. \]
Figure 5.12. S-parameters for iris filter. Input reflection ($S_{11}$) and transmission coefficient ($S_{21}$) for an iris filter with dimensions given in Fig. 5.11, and comparison with a mode-matching technique solution.

Figure 5.12 shows a good agreement between S-parameters for vector potential RPIM and mode-matching solutions. To investigate the convergence of the proposed vector potential technique, the obtained resonance frequency for different node distributions is compared to the mode-matching result.

Figure 5.13 shows the resonance frequency error in terms of the number of nodes in the domain for homogeneous and inhomogeneous node distributions. The blue line shows the resonance frequency error for a homogeneous node distribution with different discretizations $\Delta x = \lambda_{\text{min}}/10, 12, 18, 20, 25, 32$ and 40. The error is less than 1% for discretization finer than $\Delta x = \lambda_{\text{min}}/10$, considering the resonance frequency of the mode-matching solution as a reference ($f_{\text{MM}} = 9.134$ GHz). Moreover, as the number of nodes is increased, the results converge to the results obtained with the mode-matching technique. It is worth mentioning that since the iris filter is sensitive to the exact dimensions of the irises, inhomogeneous node distributions can be used to reduce the number of nodes in the computational domain while maintaining accuracy of the results. Figure 5.14 shows one example of inhomogeneous node distributions. Since the field variations in the irises and central cavity are stronger than in the rest of the domain, a higher node density is chosen for this inner cavity, with a gradually decreasing node density towards the PMLs. Note that an adaptive node refinement algorithm as introduced in [156] can also be applied to obtain the optimum node distribution for this structure. This inhomogeneous node distribution accelerates the convergence: The resonance frequency error for distributions similar to the
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Figure 5.13. Resonance frequency error for homogeneous and inhomogeneous nodes. Error of computed resonance frequency for homogeneous regular, inhomogeneous irregular, and inhomogeneous regular nodes for different number of nodes, considering the mode-matching resonance frequency of \( f_{\text{MM}} = 9.134 \text{ GHz} \) as a reference.

Figure 5.14. Inhomogeneous node distribution. Iris filter with inhomogeneous node distribution.

one shown in Fig. 5.14 is shown as a black line with star markers in Fig. 5.13. The resonance frequency can be computed with similar or better accuracy than in a homogeneous node distribution, however with a significantly smaller number of nodes. The nodes in the inhomogeneous regular distributions are then randomly moved by \( \pm 10\% \Delta x_{\text{max}} \). The resonance frequency error for the resulting inhomogeneous irregular node distributions is shown as a red dotted line in Fig. 5.13. As can be expected, an irregular distribution slightly increases the error, but generally shows a similar convergence behavior. Note that for inhomogeneous node distributions, discretizing the time derivatives by finite difference can cause the simulation to get unstable. To alleviate this problem, backward differentiation time integrators (BDS) are used [157]. This discretization by the BDS is described in Appendix C.
5.7.3 Tilted Parallel Plate Waveguide with TEM Mode

To verify the magnetic vector potential technique for the polarization in the \( xy \)-plane, the S-parameters for the 2D parallel plate waveguide previously shown in Fig. 5.7 is simulated when excited by an in-plane current source \( J_x \), with a Gaussian pulse time-dependence covering the spectrum \([2 - 4]\) GHz. With this excitation, the TEM mode is excited in the parallel plate waveguide. For generality, the parallel plate waveguide is tilted by 45 degree as shown in Fig. 5.15 (a). Since the boundary conditions in equation (5.28) are related to \( A_n \) and \( A_\tau \), a local coordinate system is defined for boundary nodes with axes in normal (\( n \)) and tangential (\( \tau \)) directions as shown in Fig. 5.15 (a). The spatial derivatives at each boundary node can be obtained by rotating their values in the \( xy \)-coordinate to the new local coordinate system. To satisfy the second boundary condition in (5.28), a single row of dummy nodes is added on the exterior of the boundary. These dummy nodes are obtained by appropriately mirroring the last row in the inside of the waveguide.

The structure is discretized by a regular node distribution with distance \( \lambda_{\text{min}} / 16 \), and solved for the magnetic vector potential. The electric and magnetic fields are then calculated based on equations (5.14) and (5.15). Figure 5.15 (a) illustrates the absolute value of the electric field at time \( t = 0.9 \) ns and Fig. 5.15 (b) shows the simulated scattering parameters. There is a complete transmission, demonstrating energy conservation, and a reflection coefficient in the order of \(-140\) dB, which corresponds to numerical noise. This validates the propagation of the wave for the orthogonal polarization.
5.7 Numerical Examples for the Non-Staggered Meshless RPIM

Figure 5.16. Parallel plate waveguide bend. Geometry and node distribution for a parallel plate waveguide bend. The width of the waveguide is \( W = 3\lambda/4 \) and the radius of the bend is \( R = \lambda \) at \( f = 4\)GHz.

5.7.4 Parallel Plate Waveguide Bend with TEM Mode

To verify the implementation of the boundary conditions for curved boundaries in the magnetic vector potential technique, the S-parameters for a 2D parallel plate waveguide bend shown in Fig. 5.16 are simulated. The excitation source is an in-plane current source \( J_y \), with a Gaussian pulse time-dependence covering the spectrum \([2 – 4]\) GHz. With this excitation, the TEM mode is excited in the parallel plate waveguide. As in the previous example, a local coordinate system is defined to update the boundary nodes, with axes in normal \((n)\) and tangential \((\tau)\) directions as shown in Fig. 5.16. However, to improve the accuracy of the results, polar coordinates are used to describe the boundary nodes in the bend part of the structure. Boundary conditions for straight parts are obtained through (5.28) similar to the previous example, and boundary conditions for the bend part are obtained through calculations in the polar coordinate system as described in Section 5.5.2. The structure shown in Fig. 5.16 is discretized using a node distribution with distance \( \lambda_{\text{min}}/64 \), and solved for the magnetic vector potential. The electric and magnetic fields are then calculated based on equations (5.14) and (5.15). Figures 5.17, 5.18 and 5.19 show the absolute value of the magnetic vector.
Figure 5.17. Magnetic vector potential in a parallel plate waveguide bend. Magnitude of the magnetic vector potential in a parallel plate waveguide bend at time $t=1$ ns.

Figure 5.18. Magnetic vector potential in a parallel plate waveguide bend. Magnitude of the magnetic vector potential in a parallel plate waveguide bend at time $t=1.8$ ns.
potential at $t = 1$ ns, $t = 1.8$ ns and $t = 2.8$ ns, respectively. Figure 5.20 shows the simulated scattering parameters compared to HFSS and CST results. There is a very good agreement between magnetic vector potential results and the reference solutions. This example shows how conformal structure can be discretized by conformal node distributions, and also validates the PEC curved boundary conditions for the magnetic vector potential technique.

### 5.7.5 Square Loop Antenna in 3D

To verify the magnetic vector potential technique in a 3D implementation, a small square electric current loop with dimension $L = 2\Delta x$ ($\Delta x = \lambda/50$ at $f = 3$ GHz) is simulated. Figure 5.21 shows the propagation of the electric field in two orthogonal planes ($\phi = 0^\circ$ and $\phi = 90^\circ$). A near-field to far-field transformation, as shown in Section 4.4.3 [143], using the simulated $E$ and $H$ on a Huygens’ surface surrounding the square loop antenna is performed to retrieve the radiation pattern of the loop. Figure 5.22 shows a good agreement between simulation and theoretical results [143] for the electric field patterns in both planes. This is a preliminary implementation validating free-space wave propagation in 3D and a comprehensive 3D study is a concept of our future work.
5.8 Conclusion

In the first part of this chapter, the meshless RPIM has been introduced for the time-domain vector potential wave equation in electromagnetic simulations. The technique exploits the ability of the interpolation-based meshless techniques to accurately compute derivatives, and avoids staggered dual node distributions. Moreover, the vector potential technique automatically enforces the divergence-free property of the electric and magnetic fields, which avoids spurious solutions.

In the second part of this chapter, a hybrid algorithm for solving the magnetic vector potential in meshless RPIM in presence of a PML has been introduced. The time-domain magnetic vector potential approach is based on solving the second-order wave equation and does not require a staggered node distribution for $E$ and $H$ fields. However, implementing PMLs based on a stretched coordinate technique requires some auxiliary variables, which are best computed on a staggered node distribution in the PML domain. In the proposed implementation staggered nodes are only added inside the PML using the hybrid algorithm proposed in this section. The introduced algorithm has been validated in a 2D waveguide and numerical reflection coefficients have been compared for different thicknesses of PML as well as for different order of PMLs. The convergence of the reflection coefficients has been empirically demonstrated by comparison of simulated and theoretical reflection coefficients when the order of PML
5.8 Conclusion

Figure 5.21. Electric field for a small current loop in 3D in two planes. Magnitude of the electric field for a small current loop in 3D in both $H$ and $E$ planes at $t = 0.492$ ns.

Figure 5.22. Comparison of simulated and theoretical far-field pattern for a small current loop in 3D. Far-field pattern for the radiated $E$ field compared to theoretical results in two planes.

or the thickness of PML is increased. It has been shown that for PMLs with order $m = 3$ and thickness $d = 24\Delta x$, the numerical reflection coefficient is very close to the theoretical reflection coefficient.

In the last part of this chapter, the magnetic vector potential method has been validated by several examples. Firstly, a 2D parallel plate waveguide example has been provided as first canonical validation of the approach. The theoretical dispersion relation has been compared to the simulation results and the convergence of the results has been investigated. The results show that the error decreases as the discretization gets finer or as the number of neighbor nodes used for interpolation is increased. Secondly, the scattering parameters of a bandpass waveguide filter with two cascaded
irises have been extracted through the simulation and a good agreement with the results of a mode-matching computation has been demonstrated. It has been also shown that discretizing the domain using an inhomogeneous node distribution improves the accuracy of the results with a smaller number of nodes. Finally, the method has been validated for orthogonal polarization and a 3D implementation.
Chapter 6

Comparing Staggered and Non-Staggered Meshless RPIM

This chapter directly compares the staggered and non-staggered algorithms of meshless methods presented in the previous parts of this thesis. In the first part, the salient features of both algorithms are briefly recapitulated. This starts with the Radial Point Interpolation Method (RPIM) to solve the $E$ and $H$ fields in a staggered node arrangements. Then, the solver for the second-order vector potential wave equation is summarized. The second algorithm avoids ambiguity and effort related to a dual node distribution required for solving the first-order Maxwell’s equations. In the second part of the chapter, the actual direct comparison between the staggered and non-staggered RPIM for time-domain electromagnetics is performed. As illustrative example, the scattering parameters of a waveguide filter are computed using both staggered and non-staggered RPIM. The accuracy of the results obtained with both methods is assessed through comparison with a reference solution.

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6.1 Introduction

Time-domain meshless methods in electromagnetics have been introduced for field computations in staggered node distributions for electric and magnetic fields, i.e., for the first-order Maxwell’s equations in [20, 21, 22, 158] and an implementation of 3D staggered Radial Point Interpolation Method (RPIM) has been described in Chapter 4. However, generating unstructured dual node distributions for the electric and magnetic fields increase the computational load and can decrease the accuracy of the results. Moreover, the need of a staggered node distribution restricts the freedom of node positions in meshless methods and might reduce the potential advantages of the method.

To overcome this problem, a meshless vector potential technique can be used to solve electromagnetic problems. Although the order of the partial differential equations is increased to second order compared to the first-order Maxwell’s equations, the numerical errors for a given space discretization can be decreased in vector potential methods. Moreover, in this formalism fields can be described using three scalar wave equations, and it has been shown in [27] that less CPU time is needed for the vector potential FDTD method compared to the classical staggered approach. An implementation of the time-domain meshless RPIM based on the local calculation of the magnetic vector potential (A) in small support domains has been introduced in Chapter 5.

In this chapter, staggered and non-staggered meshless RPIM are directly compared. To this end, the main features of staggered and non-staggered meshless RPIM are first recapitulated briefly, and then these two methods are compared using illustrative numerical results.

As shown in Chapter 3, in RPIM the approximation of the field value \( u(x) \) and its derivatives at position \( x \) is given by

\[
\begin{align*}
  u^h(x) &= [\phi_1(x), \phi_2(x), ..., \phi_N(x)]U^e, \\
  \frac{\partial u^h(x)}{\partial k} &= \left[ \frac{\partial \phi_1(x)}{\partial k}, \frac{\partial \phi_2(x)}{\partial k}, ..., \frac{\partial \phi_N(x)}{\partial k} \right]U^e, \\
  \frac{\partial^2 u^h(x)}{\partial k^2} &= \left[ \frac{\partial^2 \phi_1(x)}{\partial k^2}, \frac{\partial^2 \phi_2(x)}{\partial k^2}, ..., \frac{\partial^2 \phi_N(x)}{\partial k^2} \right]U^e,
\end{align*}
\]

where \( U^e \) is the field value at \( N \) surrounding nodes and \( \phi_i(x) \) are the shape functions, which can be computed using radial basis functions in a pre-processing step.

These approximations are used in the following to summarize the implementation of the staggered and non-staggered meshless RPIM.
Chapter 6 Comparing Staggered and Non-Staggered Meshless RPIM

Figure 6.1. Staggered and non-staggered node distributions. Discretization of a computational domain by (a) staggered, and (b) non-staggered node distributions. \( d_s \) is the radius of the support domain, \( d_c \) is the average node distance and \( d_{\text{min}} \) is the minimum node distance in the domain.

6.2 Staggered Meshless RPIM

In the staggered meshless RPIM, the computational domain is discretized into two set of nodes as shown in Fig. 6.1 (a). To obtain explicit update equations for the electric and magnetic fields from the first-order Maxwell’s equations, we consider that \( u(\mathbf{x}) \) in the equation (6.1) (a) can be any component of \( \mathbf{E} \) and \( \mathbf{H} \). Then, the first-order spatial derivatives of those field components in the curl operator can be obtained by (6.1) (b). The time derivatives in time-domain meshless methods can be approximated by central finite differences in a leapfrog time-stepping scheme similarly to the FDTD method. The resulting explicit formulation is given here for the \( y \)-component of \( \mathbf{E} \) and \( \mathbf{H} \) with similar expressions for the other components:

\[
E_{y,i}^{n+1} = E_{y,i}^n + \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left( \sum_j H_{x,j}^{n+1/2} \frac{\partial \phi_j(x_i)}{\partial z} - \sum_j H_{z,j}^{n+1/2} \frac{\partial \phi_j(x_i)}{\partial x} \right), \quad (6.2a)
\]
\[
H_{y,i}^{n+1/2} = H_{y,i}^{n-1/2} + \frac{\Delta t}{\mu_0 \mu_r} \left( \sum_j E_{z,j}^n \frac{\partial \phi_j(x_i)}{\partial x} - \sum_j E_{x,j}^n \frac{\partial \phi_j(x_i)}{\partial z} \right), \quad (6.2b)
\]

where \( i \) is the index of the considered node location \( \mathbf{x} \), \( n \) is the time step and \( j \) denotes the node indices in the local support domain. An acceptable time step \( \Delta t \) is derived from the Courant-Friedrich-Levy stability condition as \( \Delta t \leq \frac{d_{\text{min}}}{c} \), where \( d_{\text{min}} \) is the shortest distance between any two nodes in the domain and \( c \) is the velocity of light in free space.
In the magnetic vector potential technique, the computational domain is discretized into one set of nodes for the vector potential $A$ as shown in Fig. 6.1 (b). The discretized second-order wave equation is numerically solved on that node distribution. The magnetic vector potential is defined by

$$B = \nabla \times A,$$  \hspace{1cm} (6.3)

and the electric and magnetic fields from Maxwell’s equations are related to $A$ as

$$E = -\frac{\partial A}{\partial t} + \frac{1}{\mu \varepsilon} \int \nabla \nabla \cdot A \, dt,$$ \hspace{1cm} (6.4)

and

$$H = \frac{1}{\mu} (\nabla \times A).$$ \hspace{1cm} (6.5)

The electric field in (6.4) is obtained through the Lorenz gauge condition. In the presence of a current source $J$, the magnetic vector potential satisfies the inhomogeneous wave equation as

$$\nabla^2 A - \mu \varepsilon \frac{\partial^2 A}{\partial t^2} = -\mu J.$$ \hspace{1cm} (6.6)

Note that in the presence of magnetic sources, the wave equation for the electric vector potential $F$ can be obtained in a similar way as the magnetic vector potential $A$. As mentioned before, the simulation for the electric vector potential $F$ could be made in parallel to the $A$, and on the same node distribution.

Using RPIM to discretize space derivatives and finite differences to discretize time derivatives in (6.6) lead to the discretized expression for the magnetic vector potential. In other words, $u(x)$ can be considered as magnetic vector potential and the second spatial derivatives of each magnetic vector potential components can be obtained by (6.1) (c). The second-order time-derivatives in the time-domain wave equation can be approximated by conditionally stable second-order central differences. In this chapter, a 2D framework with the magnetic vector potential perpendicular to the $xy$-plane and constant in the $z$-direction is considered. According to (6.5) and (6.4), the corresponding magnetic field is in the $xy$-plane, and the corresponding electric field is along $z$. The update equations for the magnetic vector potential in this case can be derived as

$$A_{z}^{n+1} = 2A_z^n - A_z^{n-1} + \frac{(\Delta t)^2}{\mu \varepsilon} \left( \frac{\partial^2 \phi(x)}{\partial x^2} A_z^n + \frac{\partial^2 \phi(x)}{\partial y^2} A_z^n \right) + \frac{(\Delta t)^2}{\varepsilon} j_z^n.$$ \hspace{1cm} (6.7)
Perfectly conducting boundary conditions for the magnetic vector potential can be applied in the explicit update equations and are derived as shown in Chapter 5 [27]

\[ A_\tau = 0, \quad \frac{\partial A_n}{\partial n} = 0, \quad (6.8) \]

where \( n \) is the normal vector to the boundary, \( A_\tau \) and \( A_n \) are the tangential and the normal components of \( A \), respectively.

### 6.4 Numerical Results

To compare and validate staggered and non-staggered RPIM, a 2D metal insert waveguide filter is simulated. The obtained scattering parameters are also compared to a reference solution obtained with Ansys HFSS.

Fig. 6.2 represents the geometry and dimensions of the waveguide filter consisting of two identical metal insertions centered in the waveguide and separated by a distance that determines the resonance frequency. The structure is first discretized with the staggered \( E \) and \( H \) nodes according to Section 6.2 and as shown in Fig. 6.2. The fundamental \( TE_{10} \) mode is excited in the waveguide on the source plane via a Gaussian pulse with spectrum covering \([2 - 6]\) GHz. The structure is discretized with \( \Delta x = \lambda_{\text{min}}/64 \) and PMLs are used to truncate the domain on both sides. Fig. 6.3 shows the comparison of simulated scattering parameters for staggered RPIM and HFSS results. A close general agreement between staggered RPIM and finite element solution can be observed. However, spurious solutions are observed as spikes in the staggered RPIM that can result in instability in the time-domain simulations [24]. Using divergence-free basis functions [159] in this method would be a viable path to suppress these spurious solutions.

In a second step, the meshless magnetic vector potential technique, described in Section 6.3 [2], is used to avoid the requirement of a dual node distribution. This technique uses only one set of nodes for solving the wave equation and can bypass the ambiguity of the generation of dual nodes. Moreover, in non-staggered magnetic vector potential technique the numerical effort is reduced compared to the staggered RPIM since only computation of \( A \) is required instead of both \( E \) and \( H \). However, the implementation of PMLs in a stretched coordinate approach [151] needs introduction of auxiliary variables [152, 146]. Therefore a second set of (staggered) nodes for the auxiliary...
6.4 Numerical Results

Figure 6.2. A metal insert waveguide filter with E and H nodes. Geometry and dimensions of a metal insert waveguide filter with two identical metal insertions. All dimensions are in millimeter. The domain is discretized with staggered E (dot) and H (cross) nodes.

Figure 6.3. S-parameters for the metal insert waveguide filter with staggered nodes. Input reflection ($S_{11}$) and transmission coefficient ($S_{21}$) for the metal insert waveguide filter with dimensions given in Fig. 6.2, and comparison with a finite element solution in HFSS.

variables is required. To avoid introducing staggered nodes in the whole domain, an algorithm for implementing hybrid staggered stretched-coordinate PML equations in the framework of the non-staggered magnetic vector potential solution domain is used as described in detail in Section 5.6 [6]. As a result, a single set of nodes is required for the solution of the second-order magnetic vector potential equation in the free-space region and a staggered node distribution is required only in the PML regions.

Fig. 6.4 shows the previously introduced metal-insert waveguide filter with the structure now discretized with $\Delta x = \lambda_{\text{min}}/32$ by non-staggered magnetic vector potential ($A$) nodes inside the computational domain. In the PMLs on each side, a staggered distribution for $A$ and the auxiliary variables are applied as visible in the figure. The
Figure 6.4. A metal insert waveguide filter with non-staggered nodes in the free-space region and staggered nodes in the PML regions. Geometry and dimensions of a metal insert waveguide filter with two identical metal insertions. All dimensions are in millimeter. The domain is discretized with non-staggered $A$ nodes in the free-space region and staggered nodes for $A$ and auxiliary variables $X_1$ and $X_2$ in the PML regions.

Figure 6.5. $S$-parameters for the metal insert waveguide filter with staggered and non-staggered nodes. Input reflection ($S_{11}$) and transmission coefficient ($S_{21}$) for a metal insert waveguide filter with dimensions and discretization given in Fig. 6.4, and comparison with the finite element solution in HFSS.

computed scattering parameters for non-staggered RPIM are shown in Fig. 6.5 and compared with HFSS results. An excellent agreement between staggered RPIM and finite element solution can be observed even with larger $\Delta x$ compared to the staggered RPIM. Moreover, the spurious solutions are removed in the non-staggered RPIM.

As another investigation for comparing the staggered and non-staggered meshless RPIM, the convergence of the resonance frequency error in the metal-insert waveguide filter is studied for different discretizations. Figure 6.6 shows the resonance frequency error in terms of discretization for the staggered and non-staggered meshless
Figure 6.6. Convergence study of the resonance frequency error for the staggered and non-staggered meshless RPIM. Resonance frequency error in terms of discretization for the staggered and non-staggered meshless RPIM.

The resonance frequency of these two methods for different discretizations of $\Delta x = \lambda/11, \lambda/17, \lambda/32, \text{ and } \lambda/64$ are compared with the resonance frequency obtained in HFSS, which is $f_{\text{ref}} = 4.348 \text{ GHz}$, as a reference solution. As can be seen from the figure the resonance frequency error for both staggered and non-staggered meshless RPIM are decreasing for finer discretizations, empirically indicating convergence. However, the non-staggered meshless RPIM is more accurate than the staggered meshless RPIM.

6.5 Conclusion

The staggered and non-staggered meshless RPIM for time-domain electromagnetic simulations have been summarized. It has been shown on one hand that in the staggered RPIM, a dual node distribution is required for solving the coupled first-order Maxwell’s equations. On the other hand, solving the second-order wave equation in a vector potential formulation needs only one set of nodes and avoids the staggered dual node distributions. The S-parameters of a metal insert waveguide filter have been computed using both RPIM approaches. The results show that the non-staggered RPIM algorithm is free of the spurious modes observed in the staggered RPIM. Moreover, the non-staggered RPIM algorithm is preferable since it needs less computational effort compared to the staggered RPIM and bypasses the requirement of a dual node
distribution. Finally, a convergence study of the resonance frequency error in terms of discretization has been conducted for the staggered and non-staggered meshless RPIM demonstrating a faster convergence for the non-staggered meshless RPIM, with more accurate results especially for a coarse discretization.
Chapter 7

Conclusion and Future Work

The research presented in this thesis has been described in two major parts related to meshless Radial Point Interpolation Method (RPIM): The first part of the thesis has placed the focus on the staggered meshless RPIM. The second part of the thesis has investigated the non-staggered meshless RPIM. This chapter concludes the thesis by summarizing the original contributions, results and findings of the thesis. The chapter also considers possible future research on staggered and non-staggered meshless RPIM method.
7.1 Part I: Staggered Meshless RPIM

Meshless methods were introduced to eliminate the costs and difficulties associated with the generation of meshes in unstructured numerical methods. Compared to mesh-based numerical techniques, the concept of meshless discretization suggests an efficient numerical treatment for complex geometries and multi-scale structures since partial differential equations are solved on a cloud of arbitrarily distributed nodes. Moreover, the straightforward addition of nodes to efficiently refine an initial coarse node distribution can adaptively lead to more accurate results.

This chapter provides a summary of the original contributions and results of the two major parts of the thesis along with a number of possible future directions of research related to each part.

7.1 Part I: Staggered Meshless RPIM

The first major part of the thesis mainly encompasses Chapter 4 that focused on staggered meshless Radial Point Interpolation Method (RPIM). This section gives a summary of the achieved results highlighting the original contributions described in the last part of Chapter 3 and in Chapter 4. The section ends with suggestions for related possible future work.

7.1.1 Summary of Original Contributions

- The Gaussian and Wendland basis functions have been compared for interpolations in small local support domains. It has been shown that Gaussian basis functions have a smaller interpolation error when applied on local support domain with small sizes. As the size of support domain is increased the interpolation error for the Wendland basis function becomes progressively smaller than the interpolation error for the Gaussian basis function. This results show that truncated Gaussian basis functions are a more appropriate choice compared to Wendland basis functions in terms of accuracy when used in a time-domain meshless solver where the small local support domains are preferable to achieve high computational efficiency [4].
• The staggered meshless RPIM for the first-order Maxwell’s equations has been implemented for 3D structures. In this method the space derivatives are discretized by RPIM approximations and the time derivatives are discretized by central finite differences. The update equations have been provided for 3D staggered meshless RPIM and have been validated through several numerical examples.

• In order to avoid spurious reflections caused by the truncation of the computational domains, absorbing boundaries are needed in numerical techniques to absorb the energy of radiated waves. There are numerous techniques to simulate infinite space in numerical methods using absorbing boundary conditions. Among them, the Perfectly Matched Layers (PMLs) are considered as one of the most efficient techniques as they work for a wide range of incident angles and frequencies. Therefore, variations of PML absorbing boundary conditions have been developed for most volume-discretizing numerical methods in time and frequency domain. In this thesis, the Uniaxial Perfectly Matched Layer (UPML) has been adopted for 3D meshless RPIM and the long-time behavior of the UPML has been studied. It has been shown that the UPML has long time instability that can be avoided by introducing small losses in the UPML after most of the energy in the computational domain has been absorbed in the UPML [10].

• The impact of different node distributions has been studied in a waveguide example. A regular node distribution, a regular node distribution with 5% and 10% random node displacement, and a cylindrical node distribution have been compared at some observation points in terms of amplitude and phase of the electric fields. The results have been shown to be very close to each other for different node distributions, which demonstrates the robustness of the meshless RPIM for different node distributions [11].

• Different node arrangements have been used to discretize a diplexer. This example shows that meshless method can be used to discretize complex geometries with in particular the ability to connect regular node arrangements with arbitrary axes orientations [1].

• The scattering from a conducting sphere has been used for further validation of the 3D RPIM method. In this example instead of solving the total electric field, the scattered electric field is computed since the incident field is known. To this end, the scattered field formulation has been described. Moreover, a near-field to
far-field transformation has been implemented to compute the scattered electric field in the far field. The simulation results have been compared to the theoretical solutions demonstrating a very good agreement.

### 7.1.2 Future Work

- The face center of the Voronoi diagram has been chosen as dual node distributions in the 3D staggered meshless RPIM. Choosing Voronoi vertices as a dual node distributions in 3D might results in chess-board propagation pattern. A further study is required for finding the optimum dual node positions.

- One recurring issue of RPIM has been that small values of the shape parameter result in accurate but unstable results, whereas the large values of the shape parameter lead to less accurate but stable results. At present, a compromise between stability and accuracy has been chosen. However, additional studies are required to automatically determine an optimum shape parameter to have stable and accurate results.

- The Courant-Friedrich-Levy stability condition, which is based on the smallest distance between nodes in the computational domain, has been used in this thesis. In the case of complex geometries with very different densities of nodes, the local time-step scheme which was introduced in [160] for the Finite Volume Time Domain (FVTD) method can be adapted for the meshless RPIM.

- Several numerical examples have been implemented and the results have been compared with the solutions in HFSS or CST. More investigations are required to improve the computational efficiency by optimizing the algorithm.

### 7.2 Part II: Non-Staggered Meshless RPIM

This section presents original contributions and possible future work related to the second part of the thesis, which focused on the non-staggered meshless method.

#### 7.2.1 Summary of Original Contributions

- Most time-domain meshless methods are solving Maxwell’s equations in a dual node distribution with staggered nodes for the electric and magnetic fields. To
bypass the requirement of a dual node distribution, a meshless magnetic vector potential formulation has been introduced with only one set of nodes for solving the wave equation. As additional advantage, this technique can reduce the numerical effort since it solves three equations instead of six. This method has been implemented first in a 2D parallel plate waveguide and a convergence study has been performed, demonstrating convergence of the algorithm for different discretizations and also different numbers of neighbors included in the support domain. Moreover, the simulated scattering parameters of an iris filter simulated using the proposed meshless magnetic vector potential method have been compared to a mode-matching solution. A good agreement between simulation and the mode-matching technique has provided further validation of the method. To extend the considerations to all components, the magnetic vector potential technique has been further validated for in-plane polarization. Finally, a 3D implementation has also been shown for a square loop antenna and the radiation pattern has been compared with the analytical solutions [2].

- The formulation for the first and second-order meshless RPIM have been introduced and these two methods have been compared. Solving the first order Maxwell’s equations requires a dual staggered node distribution in space, whereas solving the second-order wave equation needs only a single set of nodes but can exploit the ability of meshless methods to accurately estimate second-order derivatives. Moreover, in the first-order discretization to avoid unphysical solutions of the fields, the divergence property of the fields needs to be considered. In contrast, in a formulation based on a second-order vector potential method the divergence property of the fields are automatically enforced [7].

- Implementing magnetic vector potential needs a single set of nodes for $A$. On the other hand, to implement PML with a stretched coordinate approach, a staggered node distributions for $A$ and auxiliary variables is required. To avoid having staggered nodes in the whole computational domain, a hybrid algorithm has been introduced to implement staggered PML in a non-staggered meshless RPIM formulation. In this case staggered node distribution is limited to the PML region where a structured node distribution is generally in use and dual node generation is straightforward [6].

- In the last part, staggered and non-staggered meshless RPIM have been compared. The results from a metal insert waveguide filter indicate that the solution
of the staggered meshless RPIM has spurious solutions. However, non-staggered meshless RPIM not only does not require a dual node distribution but also avoids the spurious modes in the solutions. This clearly demonstrates the superiority of the non-staggered vector potential algorithm [8].

### 7.2.2 Future Work

- The magnetic vector potential technique has been introduced and validated for several examples in 2D. This method has also been validated for the preliminary example of a square loop antenna in 3D. However, the applicable range of the 3D magnetic vector potential needs to be extended further.

- One of the main advantages of the magnetic vector potential method is to bypass the requirement for the dual node distribution. Moreover, the spurious solutions can be avoided using this technique. However, implementing derivative boundary conditions is not straightforward in this method. There are some techniques to implement the derivative boundary conditions, e.g., using dummy nodes or using Hermite interpolation. In this thesis dummy nodes have been used to apply the derivative boundary conditions. Using other techniques, such as Hermite interpolation, to apply Neumann boundary conditions can be seen as another future work.

- The stretched coordinate system has been used to implement the PML in the non-staggered meshless RPIM. This method needs staggered nodes for the auxiliary variables and the magnetic vector potential. Further research to use a different approach that can be implemented without introducing staggered auxiliary variables seems highly promising.

- In a staggered meshless RPIM, using divergence-free vector basis functions can avoid spurious solutions. In a non-staggered meshless RPIM the magnetic vector potential technique avoids unphysical solutions. Further research is required to find a mathematical relation between divergence-free vector basis functions and the basis functions in the magnetic vector potential technique.

- Parallelization of the algorithm can be considered as another future work to speed up the algorithm.
In conclusion, meshless methods can be seen as a type of generalized FDTD with flexible node distributions. The results in this thesis emphasize the attractive features of the meshless RPIM to discretize the complex geometries and to obtain highly accurate results.
Appendix A

Theoretical Solution for the Scattering of Electromagnetic Waves from a Conducting Sphere

In this section, the theoretical solution (Mie solution) for the scattering from a conducting sphere in the far field is summarized. The scattered electric field can be expressed as follows [144]

\[ E_s = \sum_{n=1}^{\infty} E_n(j a_n N e_1 n - b_n M o_1 n), \]  
(A.1)

where \( E_n \) is

\[ E_n = j^n \frac{2n + 1}{n(n + 1)}. \]  
(A.2)

Also, \( M_01 n \) and \( N_e 1n \) are the spherical vector harmonics (which will be explained later), and the scattering coefficients \( a_n \) and \( b_n \) are combination of Hankel and Bessel functions [144]. Using boundary condition for the conducting sphere, the coefficients \( a_n \) and \( b_n \) can be found as follows [161]

\[ a_n = -\frac{[x j_n(x)]'}{[x h_n(x)]'}, \]  
(A.3)

\[ b_n = -\frac{j_n(x)}{h_n(x)}, \]  
(A.4)

where \( x = kr \) at the observation point \( r \), \( k = 2\pi / \lambda \), \( j_n(x) \) is the spherical Bessel function of first kind, and \( h_n(x) \) is the spherical Hankel function of first kind. The spherical Bessel functions of first and second kind are as follows

\[ j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+0.5}(x), \]  
(A.5a)

\[ y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+0.5}(x), \]  
(A.5b)
where $J_{n+0.5}(x)$ and $Y_{n+0.5}(x)$ are the Bessel functions of first and second kind that can be computed using the commands “besselj $(n + 0.5, x)$” and “bessely $(n + 0.5, x)$” in Matlab. Also, the spherical Hankel function of the first kind in (A.3) and (A.4) can be written in terms of the spherical Bessel functions as follows

$$h_n(x) = j_n(x) + jy_n(x), \quad (A.6)$$

and

$$[xj_n(x)]' = xj_{n-1}(x) - nj_n(x), \quad (A.7a)$$

$$[xh_n(x)]' = xh_{n-1}(x) - nh_n(x). \quad (A.7b)$$

In equation (A.1), $M_{01n}$ and $N_{c1n}$ are as follows

$$N_{c1n} = \frac{h_n(x)}{x} \cos \phi n(n + 1) \sin \theta \pi_n \epsilon_r' + \cos \phi \tau_n \frac{[xh_n(x)]'}{x} \epsilon_\theta' - \sin \phi \pi_n \frac{[xh_n(x)]'}{x} \epsilon_\phi', \quad (A.8a)$$

$$M_{01n} = \cos \phi \pi_n h_n(x) \epsilon_\theta' - \sin \phi \tau_n h_n(x) \epsilon_\phi'. \quad (A.8b)$$

In above equations $\pi_n$ and $\tau_n$ are defined as

$$\pi_n = \frac{2n - 1}{n} \mu \pi_{n-1} - \frac{n}{n - 1} \pi_{n-2}, \quad (A.9a)$$

$$\tau_n = n \mu \pi_n - (n + 1) \pi_{n-1}, \quad (A.9b)$$

where $\mu = \cos \theta$, and the two first terms for $\pi_n$ and $\tau_n$ are given by

$$\pi_1 = 1, \quad \pi_2 = 3 \cos(\theta), \quad (A.10a)$$

$$\tau_1 = \cos(\theta), \quad \tau_2 = 3 \cos(2\theta). \quad (A.10b)$$

The infinite series in (A.1) can be approximated by considering $n_{\text{max}}$ terms. The proposed number for $n_{\text{max}}$ in [144] is given by

$$n_{\text{max}} = x + 4x^{\frac{1}{2}} + 2. \quad (A.11)$$

The value of $n_{\text{max}}$ for $r = \lambda/2$ is 11. Finally, the scattered electric field with the truncated series can be approximated as follows

$$E_{s\theta} \simeq -\frac{e^{jkr}}{-jkr} \cos \phi S_2(\cos \theta), \quad (A.12a)$$

$$E_{s\phi} \simeq -\frac{e^{jkr}}{-jkr} \sin \phi S_1(\cos \theta), \quad (A.12b)$$
where

\[ S_1 = \sum_{n=1}^{n_{\text{max}}} \frac{2n + 1}{n(n + 1)} (a_n \pi_n + b_n \tau_n), \quad (A.13a) \]

\[ S_2 = \sum_{n=1}^{n_{\text{max}}} \frac{2n + 1}{n(n + 1)} (a_n \tau_n + b_n \pi_n). \quad (A.13b) \]
Some useful identities in cylindrical coordinate system (or polar coordinate system by neglecting z-component) are summarized in this appendix:

\[ \nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \Phi} \hat{\Phi} + \frac{\partial f}{\partial z} \hat{z} \]  \hspace{1cm} (B.1)  

\[ \nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\Phi}{\partial \Phi} + \frac{\partial A_z}{\partial z} \]  \hspace{1cm} (B.2)  

\[ \nabla \times \mathbf{A} = \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \Phi} - \frac{\partial A_\Phi}{\partial z} \right) \hat{\rho} + \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \Phi} \right) \hat{\Phi} + \frac{1}{\rho} \left( \frac{\partial (\rho A_\Phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \Phi} \right) \hat{z} \]  \hspace{1cm} (B.3)  

\[ \nabla^2 \mathbf{A} = \begin{bmatrix} \frac{\partial^2 A_\rho}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 A_\rho}{\partial \Phi^2} + \frac{\partial^2 A_\rho}{\partial z^2} + \frac{1}{\rho} \frac{\partial A_\rho}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial A_\Phi}{\partial \Phi} - \frac{2}{\rho^2} \frac{\partial A_\Phi}{\partial \Phi} - \frac{A_\rho}{\rho^2} \\ \frac{\partial^2 A_\Phi}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 A_\Phi}{\partial \Phi^2} + \frac{\partial^2 A_\Phi}{\partial z^2} + \frac{1}{\rho} \frac{\partial A_\Phi}{\partial \rho} + \frac{2}{\rho^2} \frac{\partial A_\Phi}{\partial \Phi} + \frac{A_\Phi}{\rho^2} \\ \frac{\partial^2 A_z}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \Phi^2} + \frac{\partial^2 A_z}{\partial z^2} + \frac{1}{\rho} \frac{\partial A_z}{\partial \rho} \end{bmatrix} \]  \hspace{1cm} (B.4)
Staggered Backward-Differentiation Time Integrators

In order to increase the stability and accuracy in differentiation methods, staggered grids in time and space can be used. In [157] some staggered ordinary differential equations including staggered backward-differentiation time integrators (BDSp) have been introduced, where \((p+1)\) is the order of the normalized local truncation error. The explicit Taylor equation for BDS3 is defined as follows

\[
y(t + \Delta t) = \frac{21}{23}y(t) + \frac{2}{23}y(t - \Delta t) - \frac{1}{23}y(t - 2\Delta t) + \frac{24\Delta t}{23}y'(t + \frac{\Delta t}{2}) + O(\Delta t^4), \tag{C.1}
\]

or

\[
y'(t + \frac{\Delta t}{2}) \simeq \frac{23}{24\Delta t} \left(y(t + \Delta t) - \frac{21}{23}y(t) - \frac{2}{23}y(t - \Delta t) + \frac{1}{23}y(t - 2\Delta t)\right). \tag{C.2}
\]

In this section, the discretization of the wave equation for magnetic vector potential using BDS3 is introduced. The wave equation is repeated here for convenience as

\[
\nabla^2 A - \mu\varepsilon \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} \right) = -\mu J. \tag{C.3}
\]

An auxiliary variable \(DA\) is then defined for discretization as

\[
DA = \frac{\partial A}{\partial t}. \tag{C.4}
\]

Therefore, equation (C.3) using the auxiliary variable \(DA\) at time step \(n\) is discretized as follows

\[
\left( \frac{\partial^2 A^n}{\partial x^2} + \frac{\partial^2 A^n}{\partial y^2} \right) - \mu\varepsilon \frac{\partial DA^n}{\partial t} = -\mu J^n. \tag{C.5}
\]
After some manipulations, (C.5) is written as

\[
\frac{1}{\mu \epsilon} \left( \frac{\partial^2 A^n}{\partial x^2} + \frac{\partial^2 A^n}{\partial y^2} \right) + \frac{J^n}{\epsilon} = \frac{\partial DA^n}{\partial t}.
\]  
(C.6)

Using RPIM and BDS3 to discretize the space and time derivatives in (C.3) gives

\[
\frac{1}{\mu \epsilon} \left( \frac{\partial^2 \phi(x)}{\partial x^2} A^n + \frac{\partial^2 \phi(x)}{\partial y^2} A^n \right) + \frac{J^n}{\epsilon} = \frac{23}{24 \Delta t} \left( DA^{n+\frac{1}{2}} - \frac{21}{23} DA^{n-\frac{1}{2}} - \frac{3}{23} DA^{n-\frac{3}{2}} + \frac{1}{23} DA^{n-\frac{5}{2}} \right),
\]  
(C.7)

which results in the following update equation for the axillary variable \(DA\)

\[
DA^{n+\frac{1}{2}} = \frac{24 \Delta t}{23} \frac{1}{\mu \epsilon} \left( \frac{\partial^2 A^n}{\partial x^2} + \frac{\partial^2 A^n}{\partial y^2} \right) + \frac{24 \Delta t J^n}{23 \epsilon} + \frac{21}{23} DA^{n-\frac{1}{2}} + \frac{3}{23} DA^{n-\frac{3}{2}} - \frac{1}{23} DA^{n-\frac{5}{2}}.
\]  
(C.8)

Also, using the BDS3 to discretize (C.4) gives

\[
DA^{n+\frac{1}{2}} = \frac{\partial A^{n+\frac{1}{2}}}{\partial t} = \frac{23}{24 \Delta t} (A^{n+1} - \frac{21}{23} A^n - \frac{3}{23} A^{n-1} + \frac{1}{23} A^{n-2}),
\]  
(C.9)

which gives the following update equation for \(A\) after some manipulations

\[
A^{n+1} = \frac{24 \Delta t}{23} DA^{n+\frac{1}{2}} + \frac{21}{23} A^n + \frac{3}{23} A^{n-1} - \frac{1}{23} A^{n-2}.
\]  
(C.10)

This update equation is an explicit equation in time that depends on the magnetic vector potential \(A\) and \(DA\) from the past time steps. Note that \(DA\) can be updated based on update equation in (C.8).
Bibliography


Bibliography


## Acronyms

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<td>Diffuse Element Method</td>
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<td>EC FDTD</td>
<td>Equivalent Circuit Finite Difference Time Domain</td>
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Biography

Zahra Shaterian was born in Kashan, Iran in 1982. She received her B.Eng. and M.Eng. degrees in Electrical and Electronics Engineering from The Amirkabir University of Technology and K.N. Toosi University of Technology, Iran, in 2004 and 2008, respectively. She was working as a lecturer at Feiz University of Kashan from 2008 to 2010, and as a researcher in the Iran Telecommunication Research Center (ITRC) in 2010. In 2011, she joined the School of Electrical and Electronic Engineering at The University of Adelaide to study towards her PhD in the area of computational electromagnetics under the supervision of Prof. Christophe Fumeaux and Dr Thomas Kaufmann.

During her candidature, she received an IEEE SA Section Student Travel Award in 2012 to attend the IEEE Asia-Pacific Microwave Conference (APMC) in Taiwan, an IEEE Australian MTT/AP travel grant in 2013 to attend the IEEE Asia-Pacific Microwave Conference (APMC) in South Korea, and the German Microwave Conference student travel grant to attend GeMiC 2015 in Germany. She was the recipient of the best paper award at the International Symposium on ElectroMagnetic Compatibility (CEM) 2014. Also, she was the recipient of 2 best student paper awards at the International Workshop on Antenna Technology (iWAT) 2014, and Australian Microwave Symposium (AMS) 2014. She was the people’s choice winner in the Three Minute Thesis (3MT) University Final in 2014, and also the recipient of the 2014 Simon Rockliff Award for outstanding postgraduate mentorship from the School of Electrical and Electronic Engineering and DSTO. Her research interests include computational electromagnetics, time-domain meshless methods, finite-difference time-domain analysis, metamaterials and their applications in planar circuits.

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