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Fossil fuels, alternative energy and economic growth

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We present a theoretical framework that incorporates energy within an endogenous growth model. The model explicitly allows for the interaction and substitution between fossil fuels, defined as a non-renewable resource derived from some fixed initial stock, and alternative energy, defined as renewable resource whose production requires capital input. The dynamics of the model depict a unique balance growth to a saddle point. The consumption path temporarily peaks, when fossil fuels are plentiful and cheap, followed by a fall, as fossil fuel become more scarce and alternative energy production has yet to take over, until finally the steady state is reached where alternative energy production fuels the entire economy.

The model depicts a sort of energy rich heyday when fossil fuels are plentiful and cheap. As oil stocks fall, alternative energy sources become increasingly more viable until a time when alternative energy has almost completed replaced oil. Graphically, the model generates a hump in the growth path of consumption such that a short run "peak oil" heyday may be compared to the long run renewable energy dependent steady state.

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A. Introduction

The modern economy requires energy to produce its goods. Although alternatives exist, fossil fuels are still its cheapest source. Unfortunately, fossil fuels are non-renewable and can therefore run out. More colourful extensions of this line of thought gave birth to "peak oil" within popular vernacular. In general, "peak oil" refers to the period after 50% of the planet's oil endowment is exhausted causing oil production to drop as costs rise. Although the significance of the 50% mark is questionable, the more general idea is that "peak oil" represents a sort of golden age of cheap energy that in effect fuels our consumption. As oil depletes without viable substitute, its price will rise and welfare will decline, or so the story goes.

We consider a growing economy that is energy dependent. Energy can be either extracted from the ground at a fixed depletion rate or alternatively produced at some capital cost. We develop a two sector model that explicitly considers the dynamic trade-off between a non-renewable energy source, whose flow is determined by its extraction rate, and an alternative renewable energy source, whose flow is determined by a capital intensive production process. The two forms of energy drive the production of final goods such that the representative economy can operate with either energy or with both simultaneously. Their relative quantities are endogenously determined by the marginal product of each energy source in terms final goods. The analytical framework applies the work of Solow (1956) as well as the endogenous growth literature pioneered by Ramsey (1928), Cass (1965), Koopmans (1965) and Lucas (1988) to a representative economy that requires energy flows to produce goods.

We contribute to the literature that follows the concerns first expressed by The Club of Rome (Meadows, et al, 1972) pertaining to the hypothetical collapse of any economy that is solely dependent upon a non-renewable energy source that exists without a viable substitute or backstop. Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974a and 1974b) consider the conditions under which per capita consumption in the long run may avoid collapse. The consumption effects of non-renewable's steady decline can by mitigated by substitution, resource augmenting technological progress and or increasing returns to scale.[1]

We consider a methodology of endogenous substitution and eventual replacement of renewable for non-renewable resources as a means to counter-balance consumption's decline. For ease of exposition, we refer to the non-renewable resource as oil and the renewable resource as alternative energy. Our model allows both oil and alternative energy to coexist and simultaneously contribute to the production process.

We consider a world where the relative factor productivity of alternative energy rises over time. Our experiment addresses the stylized generalization that alternative energy and fossil fuels are imperfect substitutes whose productivity differential is narrowing over time. In other words, the quality as well as the quantity of alternative energy relative to oil is increasing. Analytically, the productivity differential is captured by assuming the productivity of oil is fixed, while the productivity of alternative energy, through some process of technological diffusion, improves over time to eventually catches up to that of oil.

Our representative economy follows an endogenously determined growth path across three dimensions, consumption, capital and energy, to a steady state that is analogous to a modified golden rule.

The quantitative results are obtained from simulations of a general equilibrium endogenous growth model whose consumption path necessarily converges to a steady

[1] Aznar-Marquez and Ruiz-Tamarit (2005) consider a Lucas (1988) type endogenous growth model with increasing returns to scale in renewable resource production.

state equilibrium growth rate. By construction, a stable endogenous growth model will converge to the steady state equilibrium irrespective of its initial condition. For example, consider an initial labor and capital stock of K_0=1 and L_0=1, which we loosely refer to as the 'beginning of time.' At $t = 0$, oil is more productive, plentiful and therefore cheaper than alternative energy. Our concern is with the dynamic substitution away from oil as the factor productivity of alternative energy increases and thereby approaches that of oil.

The model has implications on both the price of energy and its relation to aggregate welfare. The model predicts rising per effective capita consumption while oil is still relatively plentiful. As oil stocks dwindle and society depends increasingly upon alternative energy, per capita consumption declines. The falling consumption path is eventually curtailed by the expanding ability to produce alternative energy flows, which eventually replace oil altogether. The transitions follow three phases. Initially oil energy exists almost entirely alone, followed by oil plus alternative energies used simultaneously, and finally alternative energy exists almost entirely alone. The transitions result from the narrowing productivity differentials between energy sources and the changing rate of alternative energy's productivity catch-up. The greater the productivity gap between oil and its alternative, the longer it will take for the alternative energy sector to catch up and the greater will be the impact on consumption of the shift away from oil. The negative trend in consumption turns around as the productivity of alternative energy improves and sufficient alternative energy flows can be created to offset oil's depletion. Analytically, the result is a hump in the saddle path of consumption to the steady state. The larger the productivity differential between oil and its alternative, the more profound is the impact on the path of per effective capita consumption - the greater is the difference between consumption's peak and trough and the longer is the transition from trough to steady state.

The results are insightful given the current state of oil versus alternative energy. There are several types of alternative energies. They are all plagued by the same basic problem. Wide scale exploitation of alternative energy sources are each very costly in terms of the capital and energy needed for their production. For example, wind farms, solar energy, tidal energy and hydroelectric power all require extensive capital outlays, significant maintenance costs, long time horizons to initiate and carry serious environmental costs. Their outputs are not storable or reliable in the sense that when the wind doesn't blow or the sun doesn't shine, there is no power. To produce biomass energy, it must first be grown, collected, dried and burned. These steps require resources and infrastructure. The inescapable conclusion is that large scale alternative energy sources, albeit improving, are presently more costly and less efficient that traditional fossil fuel based technology.[2]

B.1. Baseline model 1: depleting oil

As a basis for discussion, consider a standard Ramsey type growth model, which requires a flow of energy for production.[3] Suppose that this flow of energy, for example oil, is fixed in total initial stock and is depleted at a fixed rate.[4] The model may be summarized as follows.

Social welfare results from the combined result of individual agents that each faces an identical utility function.

$$Max\ W = \int_{t=0}^{\infty} U\left(\frac{C_t}{L_t}\right) \cdot L_t e^{-\rho t} dt\ ,\ 0 < \rho < 1 \qquad (1.1)$$

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[2] See Ghenai and Janajreh (2013) for a operational parameters comparing fossil fuels and renewable energy sources.

[3] The baseline model is drawn from Sinclair (2006)

[4] We are not explicitly concerned with optimal extraction of oil, the discovery new oil stocks or changing extraction technology. So long as the resource is finite, a change in the extraction rate may extend the life of the resource and thereby delay, but does not change, the inevitable collapse. Optimal extraction, growing resource base and changing extraction technology are posed as extensions of the model at the conclusion of the paper.

subject to:

$$U\left(\frac{C_t}{L_t}\right)=\frac{\left(\frac{C_t}{L_t}\right)^{1-\theta}}{1-\theta}=\frac{\left(c_t A_t\right)^{1-\theta}}{1-\theta},\ 0<\theta<1 \qquad (1.2)$$

$$L_t = L_0 e^{nt},\ A_t = A_0 e^{gt} \qquad (1.3)$$

$$S_t = S_0 e^{-\chi t} \qquad (1.4)$$

$$Y_t = C_t + \dot{K}_t \qquad (1.5)$$

$$\begin{aligned} Y_t &= F\left(K_t, \dot{S}_t, A_t L_t\right) \\ &= K_t^{\alpha}\left(-\dot{S}_t\right)^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta},\ 0<\alpha<1,\ 0<\beta<1 \end{aligned} \qquad (1.6)$$

Upper case letters represent levels and lower case letters represent per effective capita. C_t is consumption, ρ is the discount rate and θ is the coefficient of relative risk aversion. L_t and A_t are labor and technology which each grow at some exogenous rate n and g respectively.[5] Output, Y_t, requires capital, K_t, effective labor, A_tL_t, and energy flow, $\dot{S}_t$. The energy flow is drawn from a fixed stock, S_t, at some fixed rate, χ. For simplicity assume that energy is not replenished such that the extraction rate, χ, is also the depletion rate. Agents may consume or save their incomes. Savings results trivially in capital accumulation. The Euler equation, derived from the Hamiltonian yields the per effective capita growth rate, ξ_t.[6]

$$H = U\left(c_t\right) - \lambda \dot{k}_t - \upsilon \frac{\ddot{S}_t}{A_t L_t} \qquad (1.7)$$

---

[5] Although labor does not play a fundamental role in the analysis, its inclusion is necessary for convergence to a steady state. The inclusion of labor augmenting technology adds richness to the model without significant increase in complexity.

[6] Note that the third term in equation (1.7) is the second derivative with respect to time. Again, this reflects final production's need for oil flows as opposed to oil stock. We adopt the simplifying assumption of a constant extraction rate such that $\frac{\ddot{S}_t}{A_t L_t}=0$. See appendix 1 for the full specification of the model B.1, including transversality conditions.

$$\xi_t = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(\frac{\alpha y_t}{k_t} - \rho - \theta g\right) = \frac{1}{\theta}\left[\alpha k_t^{\alpha-1}\left(\frac{-\dot{S}_t}{A_t L_t}\right)^{\beta} - \rho - \theta g\right] \tag{1.8}$$

Albeit interesting, the model's solution is not inspiring. Since oil is constantly depleting and by assumption, no additional stocks are ever added, it will eventually run out, given $\lim_{t\to\infty} S_t = 0$. Without the constant reintroduction of new oil stocks at a rate of at least χ or the constant improvement in energy technology, the model always collapses to zero such that there is no true steady state.

Proposition 1: Without addition of new stock, an ever depleting resource that is necessary in the production process will result in a consumption path that collapses to zero.

Proof of proposition 1: If $\lim_{t\to\infty} S_t = 0$, then $\lim_{t\to\infty} Y_t = 0$ and $\lim_{t\to\infty} \xi_t = 0$.

Stability implies convergence to a modified golden rule irrespective of the initial conditions of the state variables- labor, capital, technology and oil. In figure 1, we consider the simulated growth path from the initial conditions L_0=1, K_0=1, A_0=1 and S_0=1000. The thin line is the $\dot{k}_t = 0$ locus and the thick line is the unique consumption path. Optimizing the extraction rate extends the positive growth period to higher levels of per effective capital and slows down the transition back to the origin. But so long as energy is finite and depleting without substitute, the null steady state result is always the same.

B.2. Baseline model 2: renewable alternative energy

Consider the above same model replacing depleting oil flows, $\dot{S}_t$, within the production function, with renewable alternative energy flows, $\dot{Q}_t$. Equations (1.4) and (1.6) are replaced by equations (1.9) and (1.10).

$$
\begin{aligned}
Y_t &= F\left(K_{Yt}, \dot{Q}_{Yt}, A_t L_t\right) \\
&= K_{Yt}{}^{\alpha}\left(-\dot{Q}_{Yt}\right)^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta}
\end{aligned} \quad (1.9)^7
$$

$$
\begin{aligned}
\dot{Q}_t &= G\left(K_{Qt}, \dot{Q}_{Qt}, B_t\right) \\
&= B_t K_{Qt}{}^{\pi} \dot{Q}_{Qt}{}^{1-\pi}
\end{aligned} \quad (1.10)^8
$$

Equation (1.10) represents the production of alternative energy which requires capital, K_{Qt}, energy flows, $\dot{Q}_{Qt}$, and technology, B_t.[9] Implicit in (1.9) and (1.10) is that alternative energy cannot be stored.[10] Alternative energy production requires both capital and energy such that the economy wide income constraints (1.11) and (1.12) are always binding.

$$K_{Yt} + K_{Qt} = K_t \quad (1.11)$$

$$\dot{Q}_{Yt} + \dot{Q}_{Qt} = \dot{Q}_t \quad (1.12)$$

The Hamiltonian and Lagrangian with non-negativity constraints are defined as follows.

$$H = U\left(c_t\right) - \lambda \dot{k}_t - \upsilon \dot{q} \quad (1.13)$$

$$L = H + \omega\left(K_t - K_{Yt} - K_{Qt}\right) + \varpi\left(\dot{Q}_t - \dot{Q}_{Yt} + \dot{Q}_{Qt}\right) \quad (1.14)$$

In equation (1.13), note that $\dot{q}_t = \dfrac{\partial\left(\dfrac{\dot{Q}_t}{A_t L_t}\right)}{\partial t}$ which reflects the flow of energy per effective capita. By construction, energy and capital distribute themselves competitively across the two sector, Y_t and $\dot{Q}_t$, such that the proportions of K_{Yt} to K_{Qt} and $\dot{Q}_{Yt}$ to $\dot{Q}_{Qt}$ are defined as follows.

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[7] $0 < \alpha < 1$, $0 < \beta < 1$

[8] $0 < \pi < 1$

[9] Alternative specifications of renewable resource production can be found in Chambers and Guo (2009) as well as Aznar-Marquez and Ruis-Tamarit (2005). The former considers renewable resource production as a proxy for environmental quality problem and the latter considers renewable resource production as an extraction problem. In either case, the renewable resource is the economy's only source of energy.

[10] Although there exists an analytical solution to the stock of alternative energy, Q_t, it is irrelevant for our purposes. Unlike oil, alternative energy cannot be stored and thereby depleted. Although arguably of interest to consider oil as an energy source for the production of alternative energy, we have nevertheless limited the production of alternative energy flows to exclusively require alternative energy.

$$\frac{k_{Yt}}{k_{Qt}} = \frac{\alpha}{\beta} \tag{1.15}$$

$$\frac{q_{Yt}}{q_{Qt}} = \frac{\frac{\dot{Q}_{Yt}}{A_t L_t}}{\frac{\dot{Q}_{Qt}}{A_t L_t}} = \frac{\pi}{1-\pi} \tag{1.16}$$

Proposition 2: Renewable alternative energy within the production function results in a unique growth path of per capita consumption across three dimensions, *c*, *q* and *k* that converge to a three dimensional modified golden rule.

Proof of proposition 2: Equation (1.10) may be reduced to $\dot{Q}_t = g(K_{Qt})$ which implies that equation (1.9) may in turn be reduced to $Y_t = f(K_{Yt}, K_{Qt})$. Since the proportions of capital across the two sectors remain fixed, the dynamics of the model are defined by $\xi_t = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\frac{y_t}{k_t}(\alpha+\beta) - \rho - \theta g\right]$, $\frac{\dot{k}_t}{k_t} = \frac{y_t - c_t}{k_t} - (n+g)$ and $\frac{\dot{q}_t}{q_t} = \frac{h}{\pi} + \frac{\dot{k}_t}{k_t}$, where *h* is the growth rate of alternative energy technology, B_t. In the steady state equilibrium, per capita consumption, capital and alternative energy converge to $k_\infty = k|(\dot{c}=0)$, $q_\infty = q|(\dot{c}=0)$ and $c_\infty = c(k_\infty)$ as depicted in figure 2.[11]

It is useful to make a direct comparison of the model with only depleting oil versus the one with only renewable alternative energy.[12] Using the same coefficient and starting point values as the simulation described in Section B.1, figure 3 depicts the unique saddle paths of consumption and the $\dot{k} = 0$ loci in two dimensions. Note the dashed line, the alternative energy simulation, converges to the steady state from either the left or

[11] The steady state analytic solution to k_∞, c_∞ and q_∞ are reported in equations (1.25), (1.26) and (1.27) in Section B.3

[12] Figure 3 illustrates how the marriage of the two baseline models form the switching model described in section B.3. It is important to stress that optimal extraction of oil would necessarily extend the life of non-renewable oil and make the transition from oil to renewable alternative energy less dramatic. Nevertheless, the switching mechanism between oil to alternative energy would remain unchanged as would the fundamental humped shaped growth path as depicted in figure 4.

the right. Irrespective of whether we initiate the model from the left, given arbitrary starting values of $K_0 = 1$, $L_0 = 1$ and $A_0 = 1$, or from the right, given arbitrary starting values of $K_0 = 100$, $L_0 = 1$ and $A_0 = 1$, stability of the model insures convergence to the unique steady state values of $k_\infty = 34.79$, $q_\infty = 6.23$ and $c_\infty = 2.25$. These numbers, although meaningless in absolute value, are comparable. Given comparable coefficient values across the two model specifications, in the oil only economy, consumption peaks at 1.93 versus steady state 2.25 in the alternative energy world. In other words, assuming oil and alternative energy are perfect substitutes, we would be about 15% better off in the steady state world of renewable energy than the very best we ever were in the non-renewable world.

B.3. Model with Depleting Oil and an Alternative Renewable energy Source

Consider a growing economy that is initially dependent upon depleting oil but is capable of producing a renewable alternative energy. We assume oil and alternative energy are nested within a CES production function such that one energy source may substitute for the other as in equation (1.17) below. The specification also allows alternative energy to augment oil such that the two factors may be used in tandem. Eventually as oil's scarcity increases, alternative energy will replace oil all together as the source of economy wide energy. Furthermore, the price and quantity of alternative energy are endogenously determined by both the demand for energy and the relative productivity differential between oil and its alternative.

Our model contributes to the recent growth literature with backstop technology primarily through the manner in which energy enters the production function. We allow for both oil and the alternative to be used simultaneously in final goods production.[13] Our specification also considers less than perfect substitution between oil and the alternative.[14] Society effectively undertakes a dynamic switching process between oil and alternative energy. The process is driven by the technological diffusion from fossil fuel technology to alternative energy technology.

The representative agent again maximizes welfare defined by equations (1.1) and (1.2) subject to oil extraction technology (1.4), final goods production technology (1.17) and alternative energy production technology (1.10).

$$\begin{aligned} Y_t &= F\left(K_{Yt}, Total.Energy_t, A_t L_t\right) \\ &= K_{Yt}{}^{\alpha}\left[\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}} + \dot{Q}_{Yt}{}^{\frac{\mu_t}{\beta}}\right]^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta} \end{aligned} \qquad (1.17)^{15}$$

K_{Yt} represents capital and $\dot{Q}_{Yt}$ represents the flow of alternative energy to final production; together they make the total energy flows. η and μ_t represent productivities of oil and the alternative energy respectively. The specifications describe an economy whose depleting oil stocks are gradually replaced by alternative fuels. Oil eventually runs out and is ultimately replaced by alternative energy. Mobility of capital ensures that the production of alternative energy flows, $\dot{Q}_t$ is determined by the marginal product of capital in final production, given the exogenous flow of oil such that

$P_{Energy} \cdot MPK_{Qt} = MPK_{Yt}\left|\left(-\dot{S}_t\right)\right.$ where $\dfrac{\partial MPK_{Yt}}{\partial\left(-\dot{S}_t\right)} > 0$. From the beginning of time, both alternative energy and oil are therefore used simultaneously. As oil is depleted, the marginal product of capital falls and alternative energy production rises.

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[13] Valente (2011) presents a technology switching model to find the optimal time in which the backstop technology replaces the depleting resource technology. But the two technologies never operate simultaneously.

[14] van der Ploeg & Withagen (2011) and Schumacher (2011) allow simultaneous usage of both renewable and non-renewable but with perfect substitution.

[15] $\{\eta, \mu_t\} > 0$, $0 < \{\alpha, \beta\} < 1$

Energy flows into final production from oil and alternative energy may be considered them in terms of energy shares. The alternative energy share is defined by the cumulative distribution of alternative energy. [16]

$$Sh\left[Alt.Energy_t\right] = \frac{\dot{Q}_{Yt}^{\frac{\mu_t}{\beta}}}{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}} + \dot{Q}_{Yt}^{\frac{\mu_t}{\beta}}} \qquad (1.18)$$

As the transition takes place, two questions become increasingly important. What is the role of the productivity of alternative energy versus oil and how much will alternative energy cost to produce?

Substitutability of energy types within the final goods production function, equation (1.17), insures all energy share a common price.

$$p_{Energy.Flow} = p_{\dot{S}t} = p_{\dot{Q}t}$$

$$\frac{\beta Y_t}{Total.Energy_t} = \frac{\eta Y_t}{\dot{S}_t} \cdot sh\left[Oil.Energy_t\right] = \frac{\mu_t Y_t}{(1-\phi)\dot{Q}_t} \cdot sh\left[Alt.Energy_t\right] \qquad (1.19)$$

There are effectively two ways to analytically consider the productivities of oil and alternative energy. The simpler is to assume oil and alternative energy are perfectly substitutable. In other words, energy is energy irrespective of its source.[17] Alternatively, oil and alternative energy may be imperfect substitutes. Under this assumption, we would wish to consider the mechanism that determines the factor productivity differential as well as the implications of improving productivity.

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[16] $sh\left[Alt.energy_t\right]$ and $sh\left[Oil.Energy_t\right]$ represent energy shares of as a proportion of total energy used in final production. They are derived by observing that Equations (1.17) and (1.18) may be expressed as follows, $Y_t = K_{Yt}{}^{\alpha}\left(Total.Energy_t\right)^{\beta}\left(A_tL_t\right)^{1-\alpha-\beta}$,

$$= K_{Yt}{}^{\alpha}\left[\left(sh\left[Alt.Energy_t\right] + sh\left[Oil.Energy_t\right]\right)\cdot Total.Energy_t\right]^{\beta}\left(A_tL_t\right)^{1-\alpha-\beta}$$

$Sh\left[Alt.Energy_t\right] = \frac{\dot{Q}_{Yt}^{\frac{\mu_t}{\beta}}}{Total.Energy_t}$ and $Sh\left[Oil.Energy_t\right] = \frac{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}}}{Total.Energy_t}$.

[17] See van der ploeg & Withagen (2011) as well as Schumacher (2011) for examples.

What if the productivities of oil and alternative energy are equal and static, thereby perfect substitutes? Although alternative energy will be produced and used in conjunction with oil from the outset, alternative energy flows will necessarily initially be small as they are determined by the marginal product of capital which is initially very high. Oil flows diminish as oil stocks deplete, all the while being substituted by greater alternative energy flows. Eventually, alternative energy takes over as the economy's primary fuel source. The representative agent will begin on a growth path depicted in figure 1 and end up on the growth path as well as the steady state depicted by where the dashed lines cross in figure 3. The process is driven by the endogenous distribution of capital across alternative energy production and final goods production. Recall in the baseline model with only alternative energy, the ratio of capital across the two sectors, defined by equation (1.15), is static. In the presence of oil, the alternative energy flow price depends on its relative abundance such that the ratio of capital across final production and alternative energy production becomes dynamic.

Lemma 1: The mobility of capital condition, $MPK_Y = P_{Energy} \cdot MPK_Q$, given the price of energy expressed as $p_{\dot{Q}t} = \frac{\partial Y_t}{\partial \dot{Q}_t} = \frac{\mu_t Y_t}{(1-\phi)\dot{Q}_t} \cdot sh\left[Alt.Energy_t\right]$ defines the distribution of capital across the two sectors as a function of energy shares, $\frac{K_{Yt}}{K_{Qt}} = \frac{\alpha}{\mu_t \cdot sh\left[Alt.energy_t\right]}$.

The proportion of K_{Yt} to K_{Qt} is determined by equation (1.20).[18]

$$\frac{K_{Yt}}{K_{Qt}} = \frac{\alpha}{\mu_t \cdot sh\left[Alt.energy_t\right]} \tag{1.20}$$

Equation (1.20) implies that a portion of capital slowly migrates from final goods production to alternative energy production until the share of alternative energy is one in

[18] To avoid circularity, equation (1.20) may be reasonably approximated by defining the proportion of alternative energy capital as a function of the lagged alternative energy share, such that $\gamma_t = \frac{\mu \cdot sh\left[Alt.energy_{t-1}\right]}{\alpha + \mu \cdot sh\left[Alt.energy_{t-1}\right]}$.

the steady state. The steady state proportion of capital in alternative energy production to total capital is therefore defined by (1.21).

$$\gamma_\infty = \frac{K_{Q\infty}}{K_\infty} = \frac{\mu_\infty}{\alpha + \mu_\infty} \tag{1.21}$$

What if the productivity of alternative energy is not equal to that of oil? Suppose at the beginning of time, alternative energy is in fact far less productive than oil but evolves through time via some process of technological diffusion. We assume that technological diffusion is endogenously determined by alternative energy's share in final goods production.[19]

$$\mu_t = \beta \cdot sh\left[Alt.Energy_{t-1}\right] \tag{1.22}$$

The productivity of alternative energy, μ_t is time variant in order to allow for technological diffusion to alternative energy production.[20] The quality of alternative energy is lowest at the beginning of time and improves directly proportional to its share in final production. The productivity of oil is static, exogenous and serves as the upper bound for alternative energy productivity. It is important to note that our goal is not to consider the nature of technological diffusion in the production of alternative energy. We simply assume there exists a diffusion process and contrive it in such a way to allow for ultimate stability of the steady state.[21]

Lemma 2: The productivity of alternative energy in final production have initial and terminal limits defined as $\lim_{t\to 0}\mu_t = 0$ and $\lim_{t\to\infty}\mu_t = \beta$.

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[19] Anecdotal precedent to this type of diffusion may be attributed to Swanson's Law, named after Richard Swanson, founder of SunPower Corporation. He observed that the price of photovoltaic modules, the technological cornerstone of solar power, tend to drop 20% for every doubling of cumulative shipped volume. See Swanson (2006).

[20] Although certainly of interest to consider other mechanisms of technological diffusion, for the sake of brevity, we assume alternative energy technology diffuses at the same rate that alternative energy replaces oil. There exists a rich literature on technological diffusion. A notable example is Comin and Hobijn (2010).

[21] Equation (1.22) defines the cumulative distribution of alternative energy and thereby may serve to represent the diffusion of technology to the alternative energy sector. Equation (1.22) defines the "S" that is typical of technology diffusion models. A literature survey of these is available in Geroski (2000). Figure 7 depicts the endogenous diffusion curve.

Proposition 3: Stability of the steady state growth rate requires $\mu_\infty = \beta$.

Proof of proposition 3: $\frac{\partial k^*}{\partial t} = \frac{\partial q^*}{\partial t} = \frac{\partial y^*}{\partial t} = \frac{\partial c^*}{\partial t}$ iff $\beta = \mu$ where * denotes steady state equilibrium value. See technical appendix for further details.

Proposition 3 implies that equation (1.21) may be expanded as follows.

$$\gamma_\infty = \frac{\mu_\infty \cdot sh[Alt.energy_\infty]}{\alpha + \mu_\infty \cdot sh[Alt.energy_\infty]} = \frac{\mu_\infty}{\alpha + \mu_\infty} = \frac{\beta}{\alpha + \beta} \tag{1.23}$$

Proposition 3 also implies that if factor productivities are equal, stability of the steady state requires that $\beta = \mu = \eta$.

Capital is produced trivially from saving. As before, labor and technology both grow at constant exogenous rates *n* and *g*. The dynamics of the model are driven the Euler condition.[22]

$$\xi_t = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\frac{y_t}{k_t}\left(\alpha + \mu_t \cdot sh[Alt.Energy_t]\right) - \rho - \theta g\right] \tag{1.24}$$

To solve for the steady state, we observe the terminal condition, $sh[Alt.Energy_\infty] = 1$.

Recall the existence condition of Proposition 3, $\mu_\infty=\beta$. This is reasonable restriction at the steady state, when oil stocks have run dry and the productivity of total energy is really the productivity of alternative energy. But prior to then, particularly at the beginning of time when there is effectively no alternative energy, this restriction is flawed. We therefore assume that the productivity of total energy, β is equal to the productivity of oil, η, while the productivity of alternative energy is initially zero, $\mu_0 \cong 0$ and improves over time.[23]

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[22] The derivation of Equation (1.26) results from solving of the present value Hamiltonian. In more general terms, Equation (1.26) may be expressed as follows, $\xi_t = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(\frac{\partial y_t}{\partial k_t} - \rho - \theta g\right)$. The complete derivation is presented in the technical appendix.

[23] The assumption that $\eta=\beta$, although intuitively justifiable, is not necessary for convergence. If $\eta>\beta$ and $\mu_\infty=\beta$, then the productivity of alternative energy never reaches that of oil and the steady state per capita

This is tantamount to treating the productivity of oil as a numeraire and considering the productivity of alternative energy in relative terms to that of oil.

Central to the model is the productivity of oil, η, versus the productivity of alternative energy, μ_t. In the beginning of time, society is reliant on oil. In addition, by assumption society effectively lacks the technology with which to create alternative energy flow. To model these stylized facts, we assume $\eta=\beta$, $\mu_0 \cong 0$ and $\mu_\infty=\beta$. At the outset, alternative energy is hopelessly inefficient relative to oil. But as time passes and incentives increase via higher oil prices, society devotes more resources to alternative energy production which increases its flow. By construction, this process also hastens the diffusion of technology toward alternative energy production, represented by its relative productivity, μ_t. Alternative energy's productivity eventually approaches that of oil and society settles on a steady state distribution of capital between final production and alternative energy production.[24]

When oil is plentiful and cheap, society benefits; as time passes, oil depletes and the economy slowly adjusts to alternative energy. Eventually the productivity of alternative energy catches up to that of oil. So long as total energy productivity is greater than or equal to oil's productivity, $\beta \geq \eta$, and the terminal condition over alternative energy productivity is met, $\mu_\infty=\beta$, then the economy in steady state is ultimately always better off with renewable energy than it ever was with oil. In other words, so long as the productivity of alternative energy eventually catches up to that of oil, the fact that flows of alternative energy are ever increasing insures society's long run relative prosperity. In

consumption will necessarily be less than the consumption peak during the transition. Furthermore, $\eta>\beta$ implies a larger productivity differential and a longer, more dramatic transition to the steady state.

[24] It is possible to add dynamics to the productivity of oil. This would lengthen the transition to the steady state as alternative energy productivity has a greater differential to make up. Nevertheless, the results and conclusions of the model would remain unchanged.

fact, if alternative energy's productivity were to grow beyond that of oil, the model predicts the diminishing relative significance of the "peak oil" consumption hump

Figure 4, using the same parameter values, depicts the unique growth path to the steady state, starting at a capital per effective capita k_0=1. The backward bending consumption path results from the transition from relatively productive oil to initially unproductive alternative fuel. Once the productivity differential between fuels falls sufficiently and society has begun to devote sufficient resources to alternative energy production, a more traditional growth path to a steady state emerges. It is the transition as society substitutes away from fossil fuels that causes the temporary backward bend in the consumption path. Eventually society only has alternative fuels whose productivity is approaching that of total energy, $\lim_{t\to\infty}\mu_t = \beta$, and the growth path follows the traditional upward trajectory.

The backward bending shape of the consumption path in figures 4 reflects that both consumption per capita and capital per capita rise, fall and rise again to finally rest at the steady state. A more clear exposition of the saddle path in Figure 5 shows the consumption path relative to time. The peak and trough occur in periods 29 and 92 respectively, which are marked in both Figure 4 and 5.

Oil's initial relative abundance, its falling stock and its eventual replacement create a hump in the growth path of consumption, followed by a fall and finally an improvement.

Figure 6 compares three representative economies. $\mu = \eta = \beta$ represents the one where energy sources are perfect substitutes. $\mu \le \eta = \beta$ represents where oil and alternative energy are initially imperfect substitutes but alternative energy productivity improves and eventually catches up to that of oil. $\mu \le \beta < \eta$ represents where they are imperfect substitutes and alternative energy productivity improves over time but never actually catches up to that of oil. Notice that both the hump in consumption and length of transition away from oil are exacerbated by imperfect substitutability. This is because when energy sources are imperfect substitutes, in addition to having to wait for sufficient alternative energy flows, society must also allow the productivity gap to narrow. If oil's productivity is always and forever higher than that of alternative energy – the productivity gap never closes – the consumption peak will be higher, the transition to alternative energy is longer and the steady state consumption will be lower than the temporary consumption peak.

The productivity differential between oil and alternative energy ultimately drives the results. If alternative energy is potentially only as productive as oil then the steady state alternative energy only world is only as good as good as the peak of the oil dependent world. If alternative energy is never able to catch up to oil and always remains an inferior fuel source, then the peak of the oil driven economy is higher than the eventual alternative energy steady state. This latter case also implies that any further improvement in alternative energy productivity, irrespective of when it occurs, will push the consumption path to a higher steady state equilibrium. Finally, if alternative energy's productivity is potentially higher than that of oil, the resulting steady state would necessarily be higher than the consumption peak during oil dependence.

B.4. Analytical Note

The steady state occurs graphically when the saddle path of consumption passes through the $\dot{k} = 0$ locus. The analytic solution to the steady state can be found by imposing endpoint conditions on the motion of the state variables, $\frac{\dot{c}_\infty}{c_\infty} = 0$, $\frac{\dot{k}_\infty}{k_\infty} = 0$ and $\frac{\dot{q}_\infty}{q_\infty} = 0$, where $q_t = \frac{\dot{Q}_t}{A_t L_t} = B_t^{\frac{1}{\pi}} \phi^{\frac{1-\pi}{\pi}} \gamma_t k_t$ to yield the following.

$$k_T* = \left[\alpha^\alpha \beta^\beta (1-\phi)^\beta \phi^{\frac{1-\pi}{\pi}\cdot\beta} \cdot \frac{\alpha+\beta}{\rho+\theta(g+\bar{\xi})} \cdot B_T^{\frac{\beta}{\pi}} \right]^{\frac{1}{1-\alpha-\beta}} \quad (1.25)$$

$$c_T* = \left(\frac{\rho+\theta g+\frac{\beta h}{\pi}}{\alpha+\beta} - n - g \right) k_T* \quad (1.26)$$

$$q_T* = \left[\alpha^\alpha \beta^{1-\alpha} (1-\phi)^{1-\alpha} \phi^{\frac{1-\pi}{\pi}\cdot(1-\alpha)} \cdot \frac{\alpha+\beta}{\rho+\theta(g+\bar{\xi})} \cdot B_T^{\frac{1-\alpha}{\pi}} \right]^{\frac{1}{1-\alpha-\beta}} \quad (1.27)$$

$$y_T* = \left\{ \alpha^\alpha \beta^\beta (1-\phi)^\beta \phi^{\frac{1-\pi}{\pi}\cdot\beta} \left[\frac{\alpha+\beta}{\rho+\theta(g+\bar{\xi})} \right]^{\alpha+\beta} B_T^{\frac{\beta}{\pi}} \right\}^{\frac{1}{(1-\alpha-\beta)}} \quad (1.28)$$

In this numeric simulation, the steady state per capita values are $k^* \cong 34.71$, $c^* = 2.24$, $q^* \cong 6.22$ and $y^* \cong 3.81$.

The technology diffusion is defined by Equation (1.22). Figure 7 depicts the diffusion curve of the above simulation. Figure 8 shows the price of energy flows. The model predicts that the price of oil flows will rise to a peak that coincides with maximum consumption. As oil stocks continue to fall, the diffusion of technology to alternative energy production slowly picks up momentum. Since alternative energy is renewable and the economy progressively gets better at its production, the price limit of alternative energy is zero.

C.1. The Production of Alternative Energy

The above analysis' assumes the technology associated with the production of alternative energy, B_t remained static. A more realistic approach would be to consider not only the improvement in the quality of alternative fuel, μ_t, but also the improvement in our ability to create alternative fuels. The appropriate specification of the change in alternative fuel production technology is beyond the scope of this paper. Nevertheless, we can readily examine exogenous change in the technology parameter, B_t.

The analytic result would be a steady state growth rate greater than zero. In particular, we find that the balanced growth rate in the steady state is defined as follows.

$$\bar{\xi} = \frac{\dot{k}^*}{k^*} = \frac{\dot{c}^*}{c^*} = \frac{\dot{y}^*}{y^*} = \frac{\beta}{1-\alpha-\beta} \cdot \frac{h}{\pi} \tag{1.29}$$

$$\frac{\dot{q}^*}{q^*} = \frac{1-\alpha}{1-\alpha-\beta} \cdot \frac{h}{\pi} \tag{1.30}$$

where h is the exogenous growth rate of alternative energy production technology, B_t. The steady state growth rates of consumption, output and capital per capita all depend on the productivity of total energy, β, while the steady state growth rate of energy flows per capita depends on the combined productivity of energy and of technology augmented labor, $(1-\alpha)$.

C.2. Popular Attention to Alternative Energy

Rising oil prices have spurred popular attention to both oil's depletion and alternative energy's viability as a fossil fuel substitute. The term "peak oil" entered popular vernacular to describe both the possible heyday of cheap oil as well as the societal impact of depleting oil. Technologies that might have been considered too expensive in the past become more affordable as the opportunity cost of oil rises.

We address the "peak oil" phenomenon in three ways. First; a greater proportion of society's capital may be devoted to the production of alternative energy as represented by γ_t. Second; the quality of alternative fuel may improve over time in the form of alternative energy's productivity, μ_t. Third; the technology to produce alternative energy, B_t may improve.

Greater resources devoted to alternative energy, γ_t, and improvements to alternative energy's productivity, μ_t, both hasten the transition to the steady state and mitigate the fall in consumption associated with the transition from oil to alternative energy. Improvements to the rate of technological change in alternative energy production directly impact both the transition and the steady state. The transition is improved by lowering the price of alternative energy. The steady state values and growth rates of *c*, *k* and *y* all rise with better technology in alternative energy production.

Irrespective of how we choose to model the impact of greater attention to alternative energy production, the result is somewhat similar. Since society must ultimately depend on alternative energies, greater capital devoted to its production and or improved productivity can only have positive impacts on consumption and output. However, so long as there exists a productivity differential between oil and the alternative, society will necessarily suffer during the transition to alternative fuels. Only through the investment in the quality and consequent relative productivity of alternative energy can the fall in consumption during the transition be minimized.

Conclusion

We describe an energy dependent economy where oil is initially cheap and plentiful, but non-renewable. We model the dynamic substitution of oil by an alternative renewable energy source that may be produced at some capital cost. The substitution is achieved through nesting oil and its alternative within the final goods production function. Even if alternative energy is hopelessly inefficient and only a fraction as productive as fossil fuels, so long as it can substitute for its non-renewable counterpart, there will not occur the economic collapse associated with complete energy depletion. That said, the place where the economy ultimately resides – the steady state – is determined by productivity differential between oil energy and alternative energy. If society is capable of improving the productivity of alternative energy to a level at least that of oil, then the future will be at least a bright as it was at the peak of the economy's oil dependence. If instead, alternative energy always remains oil's weaker cousin, then the eventual result is a world that is at best nostalgic of the heydays of cheap oil.

We find the greater is the productivity difference between oil and its alternative, the greater will be downturn and cost to society as it adapts to alternative energy technology. As long as alternative energy is less productive than oil, we will suffer a falling growth, possibly for a prolonged period, as we are forced to switch to the less efficient alternative. But the transition to alternative energy will be temporary. Eventually, society's growth path will renew its rise and a society will again return to its slow rise toward prosperity. How that prosperity compares to the past will depend on the eventual productive efficiency of alternative energy versus that to oil.

The model describes society's eventual transition from oil to alternative energy. As the economy depletes its fixed stock of oil, initially both consumption and oil prices are simultaneously at their global highs. But as oil's scarcity rises, consumption and welfare fall sharply until alternative energy production is sufficient to effectively replace oil and the economy resumes its upward trajectory to the steady state.

The model implications and possible extensions are clear. First, the long run steady state equilibrium, although potentially above the temporary consumption high, may take a relatively significant amount of time to reach. Second, of greater policy concern is the transition to the steady state which is marked by temporarily high consumption followed by prolonged falling consumption that eventually turns back up. Greater investment in alternative energy production mitigates the length of time necessary to reach the steady state. Improvements in the quality of alternative energy, as measured by its productivity relative to oil, offsets the negative impact of the transition away from oil and ultimately defines the level of long run consumption. Whether the long run is better or worse that the temporary hump depends on society's success in improving the productive efficiency of alternative energy.

Extensions of the model include a more robust treatment of oil stocks and extraction technology. Although world oil stock is fixed in the most literal sense, a more realistic analysis would consider known versus unknown oil stocks such that supply shocks could be examined. Tied to this is the need to include optimal extraction rates of oil given changing stocks and better extraction technologies. Another important extension includes the explicit concern for technological diffusion in alternative energy production.

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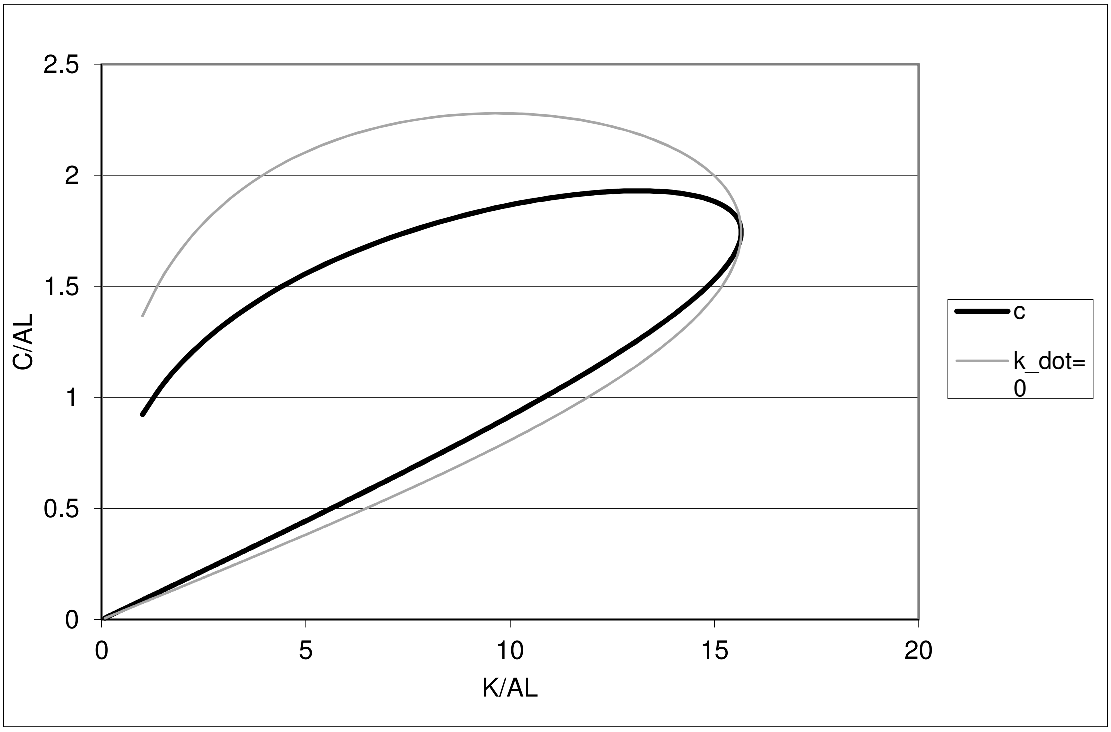
Figure 1: Growth path of consumption per capita with depleting oil



ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Figure 2: Three Dimensional modified golden

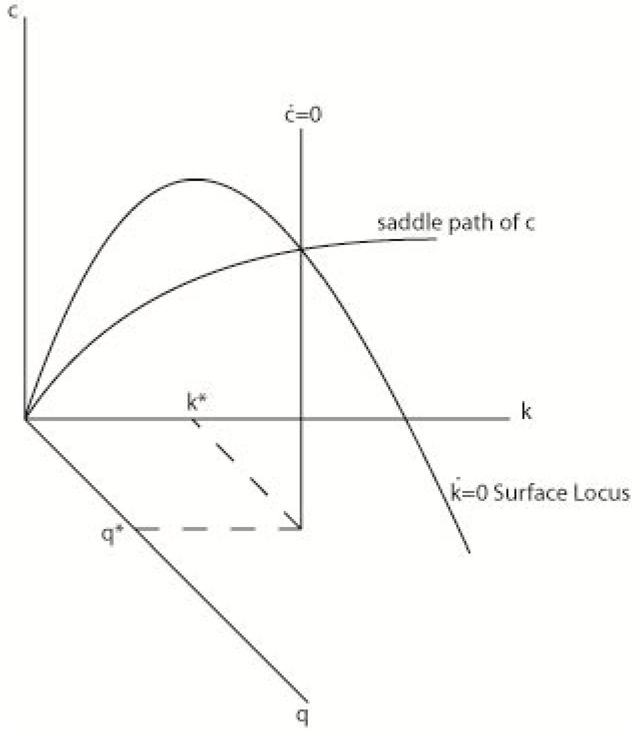
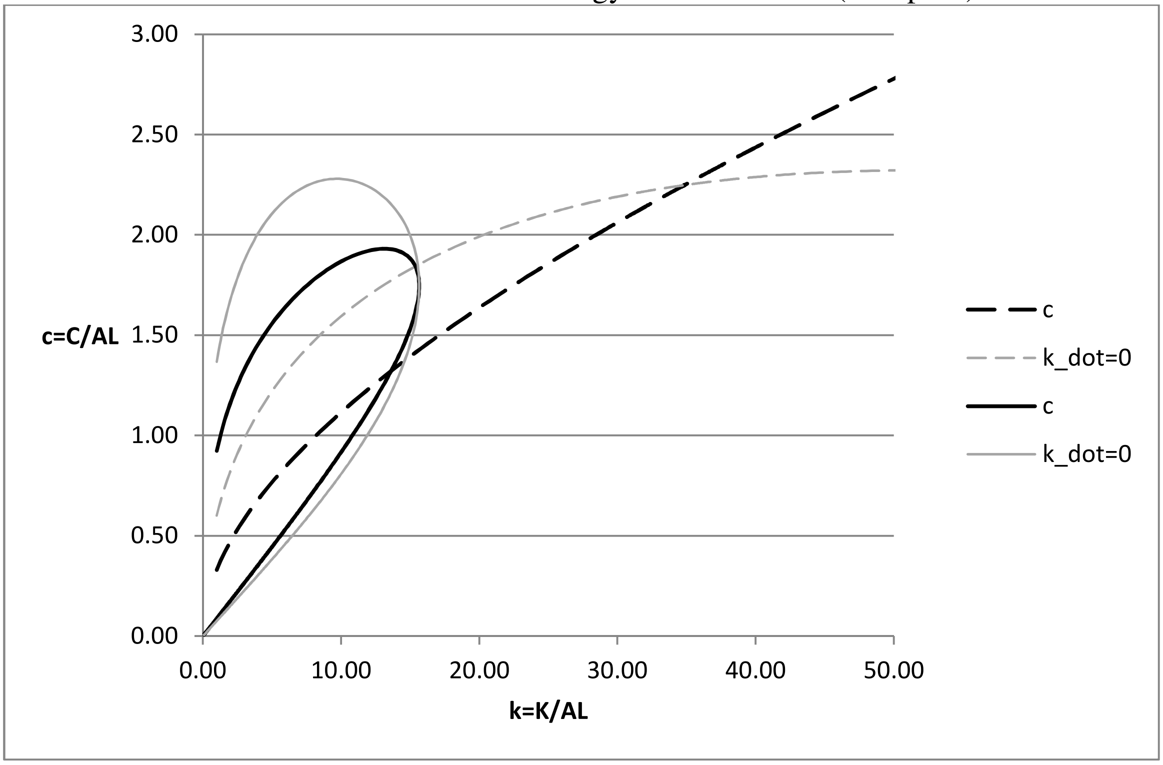
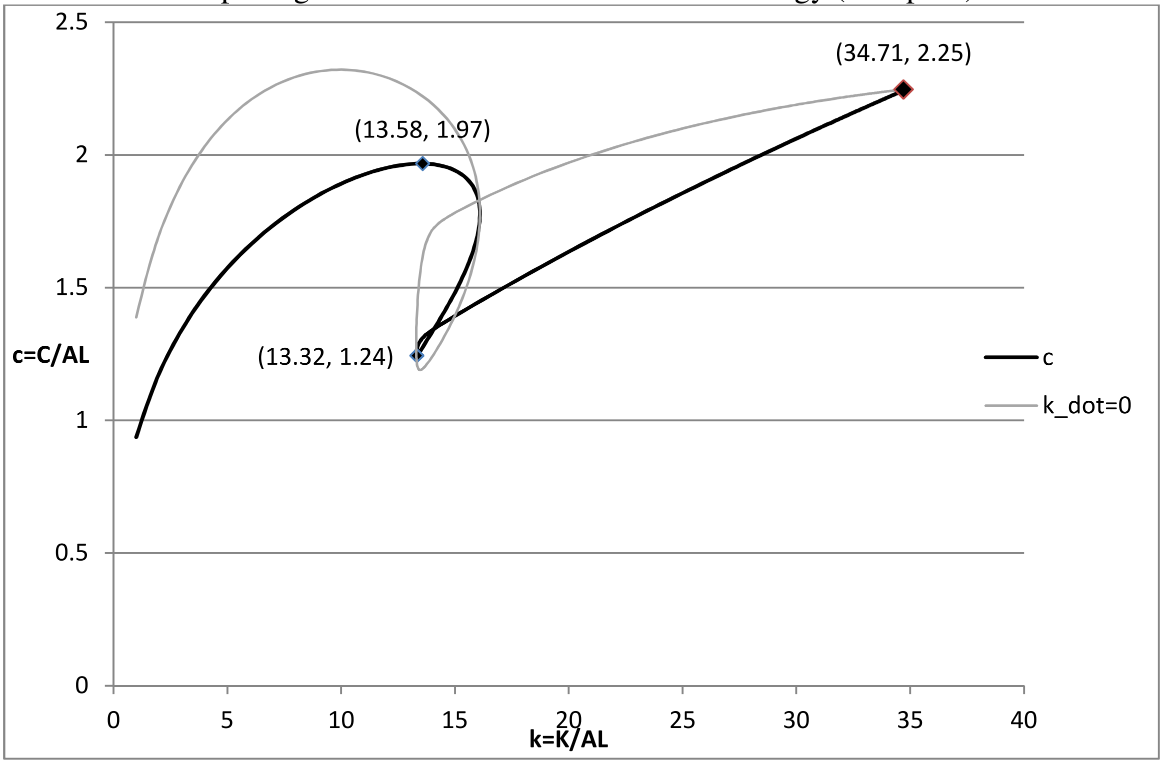


Figure 3: Saddle paths of consumption per capita with depleting oil - solid lines - versus with renewable alternative energy - hatched lines. (c-k space)



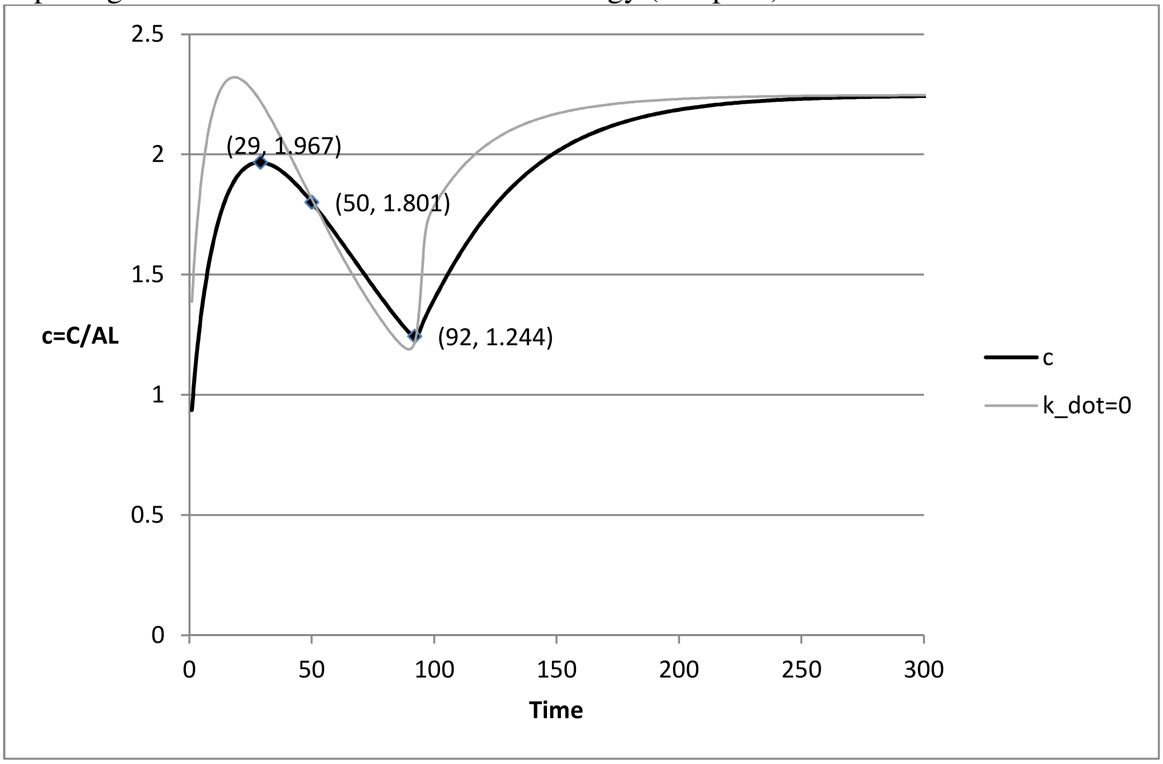
ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, π=0.7, η=0.15, μ=0.15, ϕ=0.3, B_t=1, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Figure 4: Saddle path of consumption per capita as society substitutes away from depleting oil toward renewable alternative energy (c-k space)



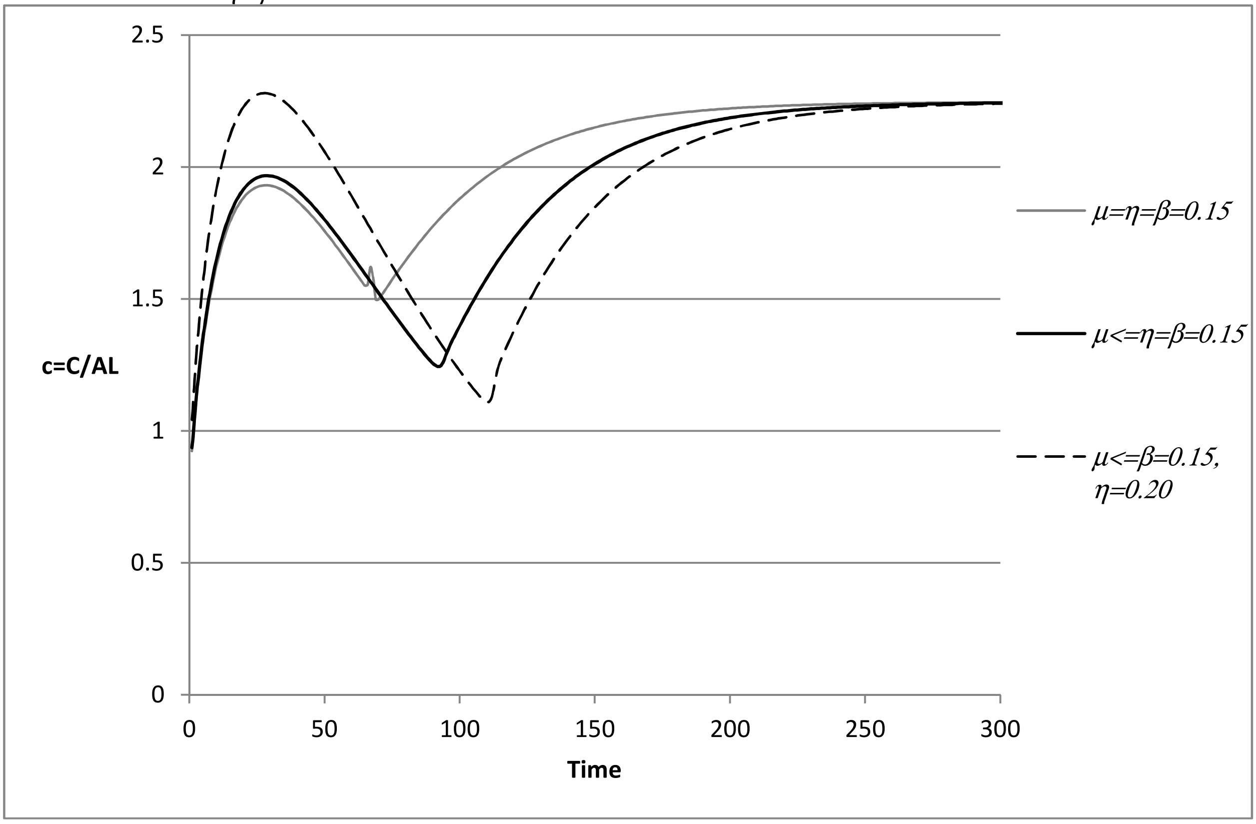
ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, π=0.7, η=0.15, ϕ=0.1, B_t=1, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Figure 5: Saddle path of consumption per capita as society substitutes away from depleting oil toward renewable alternative energy (c-t space)



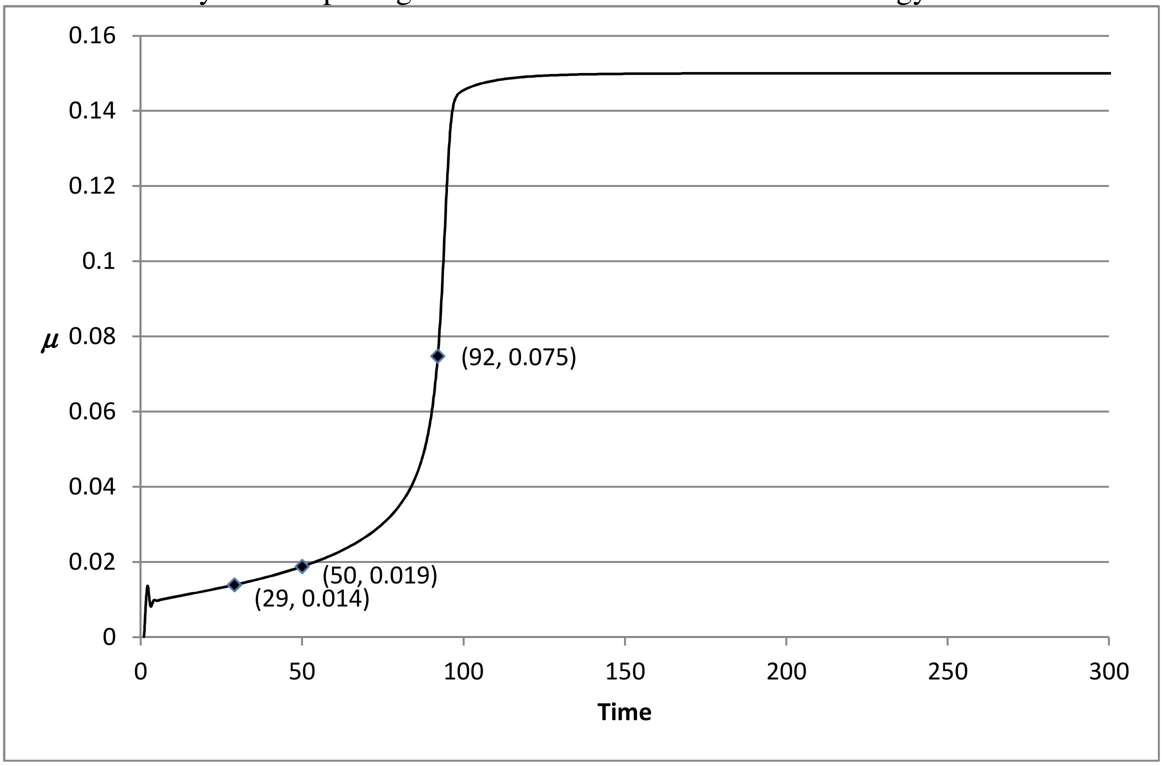
ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, π=0.7, η=0.15, ϕ=0.1, B_t=1, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Figure 6: Saddle path of consumption per capita when energy sources are perfect substitutes (ie. $\mu = \eta = \beta$) versus imperfect substitutes when $\eta=\beta$ versus imperfect substitutes when $\eta>\beta$



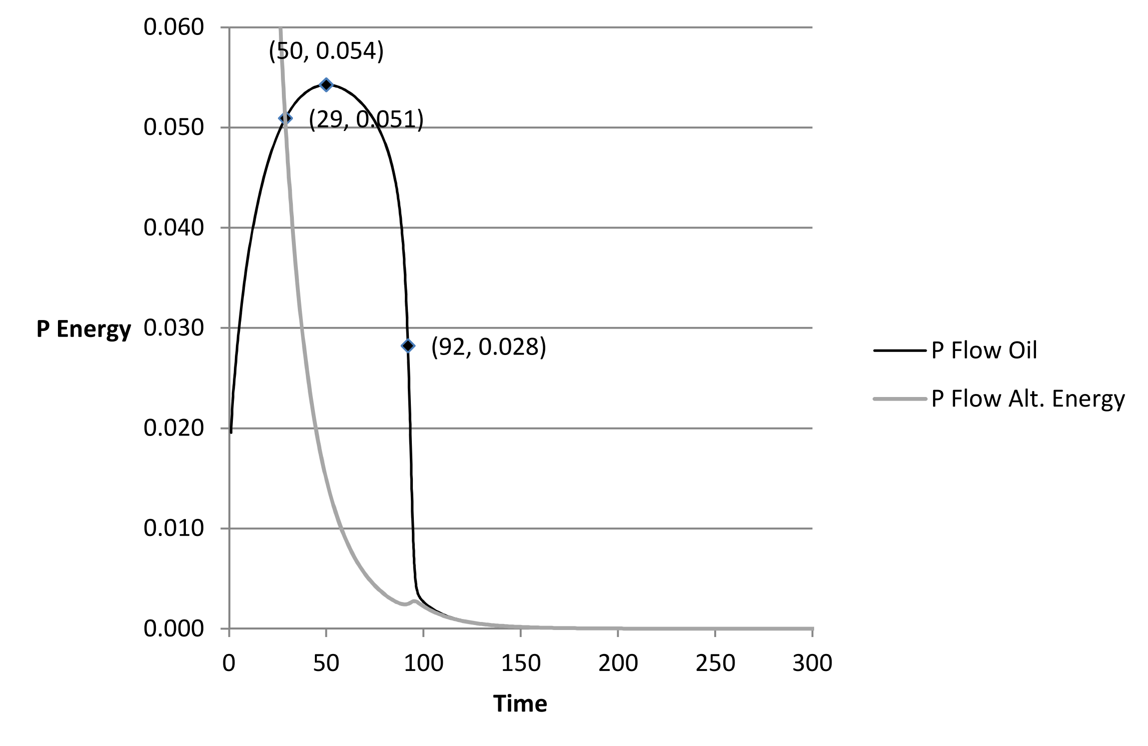
ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, π=0.7, ϕ=0.1, B_t=1, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Figure 7: Technology diffusion curve of alternative energy productivity as society substitutes away from depleting oil toward renewable alternative energy



ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, π=0.7, η=0.15, ϕ=0.1, B_t=1, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Figure 8: The price of energy flows as society substitutes away from depleting oil toward renewable alternative energy



ρ=0.03, θ=0.99, n=0.02, g=0.025, χ=-0.01, α=0.35, β=0.15, π=0.7, η=0.15, ϕ=0.1, B_t=1, K_0=1, A_0=1, L_0=1 & S_0=1,000.

Fossil fuels, alternative energy and economic growth

Technical Appendix

$$Max\ W = \int_{t=0}^{\infty} U\left(\frac{C_t}{L_t}\right) \cdot L_t e^{-\rho t} dt$$

$$c_t = \frac{C_t}{A_t L_t} \quad \Rightarrow \quad c_t A_t = \frac{C_t}{L_t}$$

$$Max\ W = \int_{t=0}^{\infty} \frac{c_t^{1-\theta}}{1-\theta} \cdot A_0^{1-\theta} L_0 e^{\left[n+(1-\theta)g-\rho\right]t} dt$$ where $\left[n+(1-\theta)g-\rho\right]$ is the effective discount rate.

subject to

Fossil fuels production:

$$-\dot{S}_t = S_t \chi_t$$

Alternative energy production:

$\dot{Q}_t = B_t K_{Qt}{}^{\pi} \dot{Q}_{Qt}{}^{1-\pi}$ given $\dot{Q}_t = \dot{Q}_{Qt} + \dot{Q}_{Yt}$, such that $\phi = \frac{\dot{Q}_{Qt}}{\dot{Q}_t}$ ➔

$$\dot{Q}_t = B_t K_{Qt}{}^{\pi} \left(\phi \dot{Q}_t\right)^{1-\pi}$$

$$= B_t^{\frac{1}{\pi}} \phi^{\frac{1-\pi}{\pi}} K_{Qt}$$

$$\frac{\dot{Q}_t}{A_t L_t} = B_t^{\frac{1}{\pi}} \phi^{\frac{1-\pi}{\pi}} k_{Qt} \quad \Rightarrow \quad let\ q_t = \frac{\dot{Q}_t}{A_t L_t} = B_t^{\frac{1}{\pi}} \phi^{\frac{1-\pi}{\pi}} k_{Qt}$$

$$\therefore \boxed{\frac{\dot{q}_t}{q_t} = \frac{h}{\pi} + \frac{\dot{k}_{Qt}}{k_{Qt}}}$$

$$\Pi_Q = p_{Qt} \dot{Q}_t - p_{Qt} \dot{Q}_{Qt} - r_t K_{Qt}$$

$$= p_{Qt} (1-\phi) \dot{Q}_t - r_t K_{Qt}$$

$$= \left[p_{Qt} (1-\phi) q_t - r_t k_{Qt} \right] A_t L_t$$

$$\Pi_Q = p_{Qt} \dot{Q}_t - p_{Qt} \dot{Q}_{Qt} - r_t K_{Qt} = 0$$

$$\frac{\partial \Pi_Q}{\partial \dot{Q}_{Qt}} = p_{Qt} \frac{\partial \dot{Q}_t}{\partial \dot{Q}_{Qt}} - p_{Qt} = 0$$

$$p_{Qt} \left(\frac{\partial \dot{Q}_t}{\partial \dot{Q}_{Qt}} - 1 \right) = p_{Qt} \left[(1-\pi) \frac{\dot{Q}_t}{\dot{Q}_{Qt}} - 1 \right] = 0$$

$$\because \boxed{\frac{\dot{Q}_{Qt}}{\dot{Q}_t} = 1 - \pi = \phi}$$

$$\Pi_q = p_{Qt} \pi q_t - r_t k_{Qt} = 0$$

$$r = p_{Qt}\pi \frac{\partial q_t}{\partial k_{Qt}} = p_{Qt}\pi \frac{q_t}{k_{Qt}}$$

$$\therefore \boxed{r_Q = p_{Qt}\pi \frac{q_t}{k_{Qt}}}$$

Final goods production:

$$Y_t = K_{Yt}{}^{\alpha}\left[Total.Energy_t\right]^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta}$$

$$= K_{Yt}{}^{\alpha}\left[\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}} + \dot{Q}_{Yt}{}^{\frac{\mu_t}{\beta}}\right]^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta}$$

$$= K_{Yt}{}^{\alpha}\left[\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}} + \left(\pi\dot{Q}_t\right)^{\frac{\mu_t}{\beta}}\right]^{\beta}\left(A_t L_t\right)^{1-\alpha-\beta}$$

$$y_t = k_{Yt}{}^{\alpha}\left[\frac{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}}}{A_t L_t} + \frac{\left(\pi\dot{Q}_t\right)^{\frac{\mu_t}{\beta}}}{A_t L_t}\right]^{\beta} \text{ given that}$$

$$Sh\left[Alt.Energy_t\right] = \frac{\left(\pi\dot{Q}_t\right)^{\frac{\mu_t}{\beta}}}{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}} + \left(\pi\dot{Q}_t\right)^{\frac{\mu_t}{\beta}}}$$

$$\frac{\partial y_t}{\partial k_{Yt}} = \frac{\alpha y_t}{k_{Yt}}$$

$$\frac{\partial Y_t}{\partial \dot{Q}_{Yt}} = \frac{\mu_t Y_t}{\dot{Q}_{Yt}} \cdot sh\left[Alt.energy_t\right] = \frac{\mu_t y_t}{\pi q_t} \cdot sh\left[Alt.Energy_t\right]$$

$$\frac{\partial y_t}{\partial S_t} = \frac{\eta y_t}{S_t} \cdot sh\left[Oil.Energy_t\right]$$

$$\frac{\partial Y_t}{\partial \dot{S}_t} = \frac{\eta Y_t}{\dot{S}_t} \cdot sh\left[Oil.Energy_t\right]$$

$$\Pi_Y = Y_t - r_t K_{Yt} - p_{St}\dot{S}_t - p_{Qt}\dot{Q}_{Yt} - w_t L_t$$

$$= \left(y_t - r_t k_{Yt} - p_{St}\frac{\dot{S}_t}{A_t L_t} - p_{Qt}\pi q_t - \frac{w_t}{A_t}\right)A_t L_t$$

$$\Pi_y = y_t - r_t k_{Yt} - p_{St}\frac{\dot{S}_t}{A_t L_t} - p_{Qt}\pi q_t - \frac{w_t}{A_t} = 0$$

$$r_t = \frac{\partial y_t}{\partial k_{Yt}} = \frac{\alpha y_t}{k_{Yt}}$$

$$p_{St} = \frac{\partial Y_t}{\partial \dot{S}_t} = \frac{\eta Y_t}{\dot{S}_t} \cdot sh\left[Oil.Energy_t\right]$$

$$p_{Qt} = \frac{\partial Y_t}{\partial \dot{Q}_{Yt}} = \frac{\mu_t Y_t}{\dot{Q}_{Yt}} \cdot sh\left[Alt.energy_t\right] = \frac{\mu_t y_t}{\pi q_t} \cdot sh\left[Alt.energy_t\right]$$

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{(1-\alpha-\beta)Y_t}{L_t} = (1-\alpha-\beta)A_t y_t = \left(\frac{1-\alpha-\beta}{\alpha}\right)A_t r_t k_{Yt}$$

Equilibrium & capital shares in capital market:

$$MPK_Y = MPK_Q$$

$$r_t = p_{Qt}\pi\frac{q_t}{k_{Qt}} = \frac{\alpha y_t}{k_{Yt}} \text{ given } p_{Qt} = \frac{\partial Y_t}{\partial \dot{Q}_{Yt}} = \frac{\mu_t y_t}{\pi q_t}\cdot sh[Alt.energy_t]$$

$$\frac{\mu_t y_t}{\pi q_t}\cdot sh[Alt.energy_t]\frac{\pi q_t}{k_{Qt}} = \frac{\alpha y_t}{k_{Yt}}$$

$$\therefore \boxed{\frac{k_{Yt}}{k_{Qt}} = \frac{\alpha}{\mu_t \cdot sh[Alt.energy_t]}}$$

Let $\gamma_{t+1} = \frac{k_{Qt}}{k_t}$, given that $\frac{k_{Yt}}{k_{Qt}} = \frac{\alpha}{\mu_t \cdot sh[Alt.energy_t]}$ and $k_{Qt} + k_{Yt} = k_t$ implies $\gamma_t = \frac{\mu_t \cdot sh[Alt.energy_{t-1}]}{\alpha + \mu_t \cdot sh[Alt.energy_{t-1}]}$.

Technological diffusion:

$$\mu_t = \beta \cdot sh[Alt.Energy_{t-1}]$$

Capital accumulation:

$$Y = C_t + \dot{K}_t \qquad L_t = L_0 e^{nt} \qquad A_t = A_0 e^{gt} \qquad B_t = B_0 e^{ht}$$

$$\dot{K}_t = Y_t - C_t$$

$$\frac{\dot{K}_t}{K_t} = \frac{Y_t - C_t}{K_t} = \frac{y_t}{k_t} - \frac{c_t}{k_t}$$

$$\frac{\dot{K}_t}{K_t} = \frac{(\xi_t + g)\theta + \rho}{\alpha + \mu_t \cdot sh[Alt.Energy_t]} - \frac{A_t L_t c_t}{K_t}$$

$$= \frac{(\xi_t + g)\theta + \rho}{\alpha + \mu_t \cdot sh[Alt.Energy_t]} - \frac{c_t}{k_t}$$

$$\dot{\left(\frac{K_t}{A_t L_t}\right)} = \left(\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} - \frac{\dot{A}_t}{A_t}\right)\frac{K_t}{A_t L_t} \quad \Rightarrow \quad \dot{k} = \left[\frac{y_t}{k_t} - \frac{c_t}{k_t} - n - g\right]k_t$$

$$= y_t - c_t - (n+g)k_t$$

$$\frac{\dot{Y}_t}{Y_t}=\alpha\frac{\dot{K}_t}{K_t}+\beta\left[\frac{\frac{\eta}{\beta}\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}}\chi+\frac{\mu_t}{\beta}\pi^{\frac{\mu_t}{\beta}}\dot{Q}_t^{\frac{\mu_t}{\beta}}\left(\frac{1}{\pi}\frac{\dot{B}_t}{B_t}+\frac{\dot{K}_t}{K_t}\right)}{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}}+\pi^{\frac{\mu_t}{\beta}}\dot{Q}_t^{\frac{\mu_t}{\beta}}}\right]+(1-\alpha-\beta)(g+n)$$

$$=\alpha\xi_t+(1-\beta)(g+n)+\eta\chi sh\left[Oil.Energy_t\right]+\mu_t\left(\frac{h}{\pi}+g+n+\xi_t\right)sh\left[alt.Energy_t\right]$$

Steady State given by $\lim_{t\to\infty}Sh\left[Oil.Energy_t\right]=0$, $\lim_{t\to\infty}Sh\left[Alt.Energy_t\right]=1$, $\lim_{t\to\infty}\mu_t=\beta$ & $\lim_{t\to\infty}\xi_t=\overline{\xi}$

$$\frac{\partial\left(\frac{Y_\infty}{K_\infty}\right)}{\partial t}=0 \quad \Rightarrow \quad \frac{\partial\left(\frac{A_tL_tc_t}{K_t}\right)}{\partial t}=\frac{A_tL_tc_t}{K_t}\left(g+n+\xi_t-\frac{\dot{K}_t}{K_t}\right)=0$$

$$\therefore \boxed{\frac{\dot{K}_t}{K_t}=\kappa_t=g+n+\xi_t}$$

$$\frac{\dot{Y}_\infty}{Y_\infty}=g+n+\frac{\beta h}{\pi}+(\alpha+\beta)\overline{\xi}$$

$$\frac{\dot{Y}_t}{Y_t}=\frac{\dot{Y}_t}{Y_t} \quad \Rightarrow \quad g+n+\frac{\beta h}{\pi}+(\alpha+\beta)\overline{\xi}=\frac{\dot{y}_\infty}{y_\infty}+g+n$$

$$\therefore \boxed{\frac{\dot{y}_\infty}{y_\infty}=\frac{\beta h}{\pi}+(\alpha+\beta)\overline{\xi}}$$

$$S_t=Y_t-C_t=sY_t$$

$$s=\frac{Y_t-C_t}{Y_t}=\frac{\dot{K}_t}{C_t+\dot{K}_t}$$

$$\therefore \boxed{s_t=\frac{\left(\alpha+\mu_t\cdot sh\left[Alt.Energy_t\right]\right)\left(\xi_t+g+n\right)}{\left(\xi_t+g\right)\theta+\rho}}$$

Steady state:

$$H=U(c_t)-\lambda\dot{k}_t$$
$$=U(c_t)-\lambda\left[y_t-c_t-(n+g)k_t\right]$$

$$\frac{\partial H}{\partial c_t} = u'(c_t) - \lambda_t \le 0$$

$$u'(c_t) = c_t^{-\theta} = \lambda_t$$

$$u''(c_t) \cdot \dot{c}_t = -\theta c_t^{-\theta-1} \cdot \dot{c}_t = \dot{\lambda}_t$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{u''(c_t)}{u'(c_t)} \cdot \dot{c}_t = \frac{u''(c_t)}{u'(c_t)} \cdot c_t \cdot \frac{\dot{c}_t}{c_t} = -\theta \cdot \frac{\dot{c}_t}{c_t}$$

$$\frac{\dot{c}_t}{c_t} = -\frac{1}{\theta} \cdot \frac{\dot{\lambda}_t}{\lambda_t}$$

$$\frac{\partial H}{\partial k_t} = \lambda_t \left(\frac{\partial y_t}{\partial k_t} - n - g \right) = -\lambda_t \left[\rho + n + (1-\theta) g \right] - \dot{\lambda}_t$$

$$\frac{y_t}{k_t}\left(\alpha + \mu_t \cdot sh\left[Alt.Energy_t\right]\right) = \rho + \theta g - \frac{\dot{\lambda}_t}{\lambda_t}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\frac{\partial y_t}{\partial k_t} - \rho - \theta g \right]$$

$$= \frac{1}{\theta}\left[\frac{y_t}{k_t}\left(\alpha + \mu_t \cdot sh\left[Alt.Energy_t\right]\right) - \rho - \theta g \right]$$

Dynamics adjustment of consumption and capital:

$$\xi_t = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\frac{y_t}{k_t}\left(\alpha + \mu_t \cdot sh\left[Alt.Energy_t\right]\right) - \rho - \theta g \right], \quad \frac{\dot{k}_t}{k_t} = \frac{y_t - c_t}{k_t} - (n+g) \text{ and}$$

$$\frac{\dot{q}_t}{q_t} = \frac{h}{\pi} + \frac{\dot{k}_{Qt}}{k_{Qt}}$$

Steady state defined by $\frac{\partial \xi_\infty}{\partial t} = 0$:

$$\frac{\partial \xi_t}{\partial t} = \left(\frac{\dot{y}_t}{y_t} - \frac{\dot{k}_t}{k_t} \right) \xi_t \qquad \Rightarrow \qquad \therefore \frac{\dot{y}_\infty}{y_\infty} = \frac{\dot{k}_\infty}{k_\infty}$$

$$\frac{y_t}{k_t}\left(\alpha + \mu_t \cdot sh\left[Alt.Energy_t\right]\right) = \rho + \theta\left(g + \xi_t\right)$$

$$(1-\gamma_t)^\alpha k_t^{\alpha-1} \left[\frac{(\chi S_t)^{\frac{\eta}{\beta}}}{A_t L_t} + B_t^{\frac{\mu_t}{\beta\pi}} \pi^{\frac{\mu_t}{\beta}} \phi^{\frac{1-\pi}{\pi}\cdot\frac{\mu_t}{\beta}} (\gamma_t k_t)^{\frac{\mu_t}{\beta}} (A_t L_t)^{\frac{\mu_t}{\beta}-1} \right]^\beta = \frac{\rho + \theta\left(g + \xi_t\right)}{\alpha + \mu_t \cdot sh\left[Alt.Energy_t\right]}$$

Terminal values & the steady state:

In S.S., $Sh\left[Oil.Energy_{\infty}\right]=0$, $Sh\left[Alt.Energy_{\infty}\right]=1$, $\mu_{\infty}=\beta$ & $\xi_{\infty}=\overline{\xi}$:

$$\left(1-\gamma_{\infty}\right)^{\alpha} k_{\infty}{}^{\alpha-1} B_T^{\frac{\beta}{\pi}} \left(\gamma_{\infty} k_{\infty}\right)^{\beta} = \frac{\rho+\theta\left(g+\overline{\xi}\right)}{\alpha+\mu_{\infty}}$$

$$k_T{}^* \cong \left(1-\gamma_{\infty}\right)^{\frac{\alpha}{1-\alpha-\beta}} \gamma_{\infty}{}^{\frac{\beta}{1-\alpha-\beta}} \pi^{\frac{\beta}{1-\alpha-\beta}} \left(1-\pi\right)^{\frac{1-\pi}{\pi}\cdot\frac{\beta}{1-\alpha-\beta}} \left[\frac{\alpha+\beta}{\rho+\theta\left(g+\overline{\xi}\right)}\right]^{\frac{1}{1-\alpha-\beta}} B_T{}^{\frac{\beta}{\pi(1-\alpha-\beta)}}$$

$$\cong \left(\alpha+\beta\right)^{\frac{-(\alpha+\beta)}{1-\alpha-\beta}} \alpha^{\frac{\alpha}{1-\alpha-\beta}} \beta^{\frac{\beta}{1-\alpha-\beta}} \pi^{\frac{\beta}{1-\alpha-\beta}} \left(1-\pi\right)^{\frac{1-\pi}{\pi}\cdot\frac{\beta}{1-\alpha-\beta}} \left[\frac{\alpha+\beta}{\rho+\theta\left(g+\overline{\xi}\right)}\right]^{\frac{1}{1-\alpha-\beta}} B_T{}^{\frac{\beta}{\pi(1-\alpha-\beta)}}$$

$$\cong \left(\alpha+\beta\right) \alpha^{\frac{\alpha}{1-\alpha-\beta}} \beta^{\frac{\beta}{1-\alpha-\beta}} \pi^{\frac{\beta}{1-\alpha-\beta}} \left(1-\pi\right)^{\frac{1-\pi}{\pi}\cdot\frac{\beta}{1-\alpha-\beta}} \left[\frac{1}{\rho+\theta\left(g+\overline{\xi}\right)}\right]^{\frac{1}{1-\alpha-\beta}} B_T{}^{\frac{\beta}{\pi(1-\alpha-\beta)}}$$

$$\therefore \boxed{k_T{}^* \cong \left(\alpha+\beta\right)\left[\frac{B_T{}^{\frac{\beta}{\pi}} \alpha^{\alpha} \beta^{\beta} \pi^{\beta} \left(1-\pi\right)^{\frac{1-\pi}{\pi}\cdot\beta}}{\rho+\theta\left(g+\overline{\xi}\right)}\right]^{\frac{1}{1-\alpha-\beta}}}$$

$$q_T{}^* \cong B_T^{\frac{1}{\pi}} \left(1-\pi\right)^{\frac{1-\pi}{\pi}} \gamma_{\infty} k_{\infty}$$

$$\cong B_T^{\frac{1}{\pi}} \left(1-\pi\right)^{\frac{1-\pi}{\pi}} \gamma_{\infty} \left(\alpha+\beta\right)\left[\frac{B_T{}^{\frac{\beta}{\pi}} \alpha^{\alpha} \beta^{\beta} \pi^{\beta} \left(1-\pi\right)^{\frac{1-\pi}{\pi}\cdot\beta}}{\rho+\theta\left(g+\overline{\xi}\right)}\right]^{\frac{1}{1-\alpha-\beta}}$$

$$\therefore \boxed{q_T{}^* \cong \left[\frac{B_T{}^{\frac{1-\alpha}{\pi}} \alpha^{\alpha} \beta^{1-\alpha} \pi^{\beta} \left(1-\pi\right)^{\frac{1-\pi}{\pi}\cdot(1-\alpha)}}{\rho+\theta\left(g+\overline{\xi}\right)}\right]^{\frac{1}{1-\alpha-\beta}}}$$

$$\frac{\partial k^*}{\partial t} = \frac{\beta}{\pi\left(1-\alpha-\beta\right)} \cdot \frac{\dot{B}}{B} k^*$$

$$\therefore \boxed{\frac{\dot{k}^*}{k^*} = \frac{\beta}{1-\alpha-\beta} \cdot \frac{h}{\pi}}$$

$$\frac{\dot{q}_{\infty}}{q_{\infty}} = \frac{h}{\pi} + \frac{\dot{k}^*}{k^*}$$

$$= \frac{h}{\pi}\left(1+\frac{\beta}{1-\alpha-\beta}\right)$$

$$\therefore \boxed{\frac{\dot{q}^*}{q^*} = \frac{1-\alpha}{1-\alpha-\beta} \cdot \frac{h}{\pi}}$$

$$\frac{\dot{k}^*}{k^*} = \frac{\beta}{(1-\alpha-\beta)} \cdot \frac{h}{\pi} \quad \rightarrow \quad \frac{y_\infty - c_\infty}{k_\infty} = \frac{\dot{k}^*}{k^*} + n + g$$

$$= \frac{\beta}{(1-\alpha-\beta)} \cdot \frac{h}{\pi} + n + g$$

$$\therefore c_t^* = y^* - \left[\frac{\beta}{(1-\alpha-\beta)} \cdot \frac{h}{\pi} + n + g \right] k^*$$

$$y_\infty^* \cong k_{Y\infty}{}^{\alpha} (\pi q_\infty)^{\beta}$$

$$\cong (1-\gamma_\infty)^{\alpha} k_\infty{}^{\alpha} B_T{}^{\frac{\beta}{\pi}} (1-\pi)^{\frac{1-\pi}{\pi}\cdot\beta} \gamma_\infty{}^{\beta} k_\infty{}^{\beta}$$

$$\cong \left(\frac{\beta}{\alpha+\beta}\right)^{\alpha} \left(\frac{\alpha}{\alpha+\beta}\right)^{\beta} B_T{}^{\frac{\beta}{\pi}} (1-\pi)^{\frac{1-\pi}{\pi}\cdot\beta} k_\infty{}^{\alpha+\beta}$$

$$\cong \left(\frac{\beta}{\alpha+\beta}\right)^{\alpha} \left(\frac{\alpha}{\alpha+\beta}\right)^{\beta} B_T{}^{\frac{\beta}{\pi}} (1-\pi)^{\frac{1-\pi}{\pi}\cdot\beta} (\alpha+\beta)^{\alpha+\beta} \left[\frac{B_T{}^{\frac{\beta}{\pi}} \alpha^{\alpha} \beta^{\beta} \pi^{\beta} (1-\pi)^{\frac{1-\pi}{\pi}\cdot\beta}}{\rho + \theta(g+\bar{\xi})} \right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

$$\therefore \boxed{y_T^* \cong \left\{ \frac{B_T{}^{\frac{\beta}{\pi}} \alpha^{\alpha} \beta^{\beta} \pi^{\beta} (1-\pi)^{\frac{1-\pi}{\pi}\cdot\beta}}{\left[\rho + \theta(g+\bar{\xi})\right]^{\alpha+\beta}} \right\}^{\frac{1}{1-\alpha-\beta}}}$$

$$\frac{c_\infty^*}{k_\infty^*} = \frac{y_\infty^*}{k_\infty^*} - (\bar{\xi} + n + g)$$

$$= \frac{\rho + \theta(g+\bar{\xi})}{\alpha+\beta} - (\bar{\xi} + n + g)$$

$$= \frac{\rho + \theta g + \bar{\xi}(1-\alpha-\beta)}{\alpha+\beta} - n - g$$

$$\therefore \boxed{\frac{c_\infty^*}{k_\infty^*} = \left(\frac{\rho + \theta g + \frac{\beta h}{\pi}}{\alpha+\beta} - n - g \right)}$$

$$\therefore \boxed{\bar{\xi} = \frac{\dot{k}^*}{k^*} = \frac{\dot{c}^*}{c^*} = \frac{\dot{y}^*}{y^*} = \frac{\beta}{1-\alpha-\beta} \cdot \frac{h}{\pi}}$$

Dynamic adjustment to the Steady State:

$$\frac{\dot{Y}_t}{Y_t} = \frac{\beta}{1-\alpha-\beta}\frac{h}{\pi} + g + n$$

1) *Stability of k*, q*, y* and c*:*

$$\therefore \frac{\partial k^*}{\partial t} = \frac{\partial q^*}{\partial t} = \frac{\partial y^*}{\partial t} = \frac{\partial c^*}{\partial t} \text{ iff } \beta = \mu \,.\otimes$$

2) *3 Dimensional Modified Golden Rule Reference surface:* $c\left|\dot{k}=0 \;\&\; q\right|\dot{k}=0$

$$c_t \left| \left(\dot{k}=0\right) = y_t - \left(n+g\right)k_t\right.$$

$$q_t \left| \left(\dot{k}=0\right) = \left[\frac{c_t + \left(n+g\right)k_t}{\left(1-\gamma\right)^{\alpha} k_t^{\alpha}}\right]^{\frac{1}{\beta}} - \frac{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}}}{A_t L_t}\right.$$

3) *Energy Prices:*

$$P_{Stock.Oil} = \frac{\partial y_t}{\partial S_t} = \frac{\eta y_t}{S_t} \cdot sh\left[Oil.Energy_t\right]$$

$$P_{Stock.Alt.Energy} = \frac{\partial y_t}{\partial \dot{Q}_t} \cdot \frac{\partial \dot{Q}_t}{\partial Q_t} = \frac{\mu_t y_t}{\dot{Q}_t} \cdot sh\left[Alt.Energy_t\right] \cdot \left[\frac{2\left(1+\pi\right)}{2\pi B_t + \left(1+\pi\right)K_t}\right]$$

$$\int_{t=0}^{\infty} \dot{Q}_t dt = Q_t + \overline{C}_Q$$

$$Q_t = \int_{t=0}^{\infty} B_t^{\frac{1}{\pi}} \gamma K_t dt - \overline{C}_Q$$

$$= \left(\frac{1}{\pi}+1\right)^{-1} B_t^{\frac{1}{\pi}+1} \gamma K_t + \overline{C}_B + \frac{1}{2} B_t^{\frac{1}{\pi}} \gamma K_t^2 + \overline{C}_K - \overline{C}_Q$$

$$= \dot{Q}_t \left[\frac{2\pi B_t + \left(1+\pi\right)K_t}{2\left(1+\pi\right)}\right] + \overline{C}_B + \overline{C}_K - \overline{C}_Q$$

$$\therefore \boxed{\frac{\partial \dot{Q}_t}{\partial Q_t} = \frac{2\left(1+\pi\right)}{2\pi B_t + \left(1+\pi\right)K_t}}$$

$$\hat{P}_{Stock.Oil} = \frac{P_{Stock.Oil}}{P_{Stock.Alt.Energy}}$$

$$= \frac{2\pi B_t + \left(1+\pi\right)K_t}{2\left(1+\pi\right)} \cdot \frac{\eta}{\mu_t} \cdot \frac{\left(-\dot{S}_t\right)^{\frac{\eta}{\beta}}}{\dot{Q}_t^{\frac{\mu}{\beta}}} \cdot \frac{\dot{Q}_t}{S_t}$$

$$= \frac{2\pi B_t + \left(1+\pi\right)K_t}{2\left(1+\pi\right)} \cdot \frac{\eta}{\mu_t} \cdot \frac{S_t^{\frac{\eta}{\beta}-1} \chi^{\frac{\eta}{\beta}}}{\dot{Q}_t^{\frac{\mu}{\beta}-1}}$$

$$\therefore \lim_{t\to\infty} \hat{P}_{Stock.Oil} \cong \frac{2\pi B_t + (1+\pi) K_T}{2(1+\pi)} \cdot \dot{Q}_T = \infty$$

$$P_{Flow.Alt.Energy} = \frac{\partial y_t}{\partial \dot{Q}_t} = \frac{\mu y_t}{\dot{Q}_t} \cdot sh[Alt.Energy_t]$$

$$\therefore \lim_{t\to\infty} P_{Flow.Alt.Energy} = \frac{\mu_\infty y_\infty}{\dot{Q}_\infty} \cong 0$$

$$P_{Flow.Oil} = \frac{\partial y_t}{\partial \dot{S}_t} = \frac{\eta y_t}{\dot{S}_t} \cdot sh[Oil.Energy_t]$$

$$\therefore \lim_{t\to\infty} P_{Flow.Oil} \cong 0$$