Abstract

The detection and tracking of underwater targets with active sonar is a challenging problem because of high acoustic clutter, fluctuating target returns and a relatively low measurement update rate. In this thesis, a Bayesian framework for the detection and tracking of underwater targets using active sonar is formulated. In general, Bayesian tracking algorithms are built on two statistical models: the target dynamics model and the measurement model. The target dynamics model describes the evolution of the target state with time and is almost always assumed to be a Markov process. The typical measurement model approximates the sensor image with a collection of discrete points at each frame and allows point measurement tracking to be performed. This thesis investigates alternative target and measurement models and considers their application to active sonar tracking.

The Markov process commonly used for target modelling assumes that the state evolves without knowledge of its future destination. Random realisations of a Markov process can also display a large amount of variability and do not, in general, resemble realistic target trajectories. An alternative is the reciprocal process, which assumes conditioning on a known destination state. The first key contribution is the derivation and implementation of a Maximum Likelihood Sequence Estimator (MLSE) for a Hidden Reciprocal Process (HRP). The performance of the proposed algorithm is demonstrated in simulated scenarios and shown to give improved state estimation performance over Markov processes for scenarios featuring reciprocal targets.

In point measurement tracking, reducing the sensor data to point detections results in the loss of valuable information. This method is generally sufficient for tracking high Signal-to-Noise Ratio (SNR) targets but can fail in the case of low SNR targets. The alternative to point measurement tracking is to provide the sensor intensity map, an image, as an input into the tracker. This paradigm is referred to as Track-Before-Detect (TkBD). This thesis will focus on a particular TkBD algorithm based on Expectation-Maximisation (EM) data association called the Histogram-Probabilistic Multi-Hypothesis Tracker (H-PMHT) as it handles multiple targets with low complexity. In the second key contribution, we demonstrate a Viterbi implementation of the H-PMHT algorithm, and show that it outperforms the Kalman Filter in the linear non-
Gaussian case.

A problem with H-PMHT is that it fails to model fluctuating target amplitude, which can degrade performance in realistic sensing conditions. The third key contribution addresses this by replacing the multinomial measurement model with a Poisson mixture process. The new Poisson mixture is shown to be consistent with the original H-PMHT modelling assumptions but it now allows for a randomly evolving mean target amplitude state with instantaneous fluctuations. This new TkBD algorithm is referred to as the Poisson H-PMHT. The Bayesian prior on the target state is also modified to ensure more robust performance.

The fourth contribution is a novel TkBD algorithm based on the application of EM data association to a new measurement model that directly describes continuous valued intensity maps and avoids using an intermediate quantisation stage like the H-PMHT. This model is referred to as the Interpolated Poisson measurement model and is integrated into the Probabilistic Multi-Hypothesis Tracker (PMHT) framework to derive a TkBD algorithm for continuous data called the Interpolated Poisson-PMHT (IP-PMHT). The performance of the Poisson H-PMHT and IP-PMHT algorithms are verified through simulations and are shown to outperform the standard H-PMHT in terms of SNR estimation, particularly for scenarios featuring targets with highly fluctuating amplitude.

The final key contribution is the application of several TkBD algorithms based on EM data association to the active sonar problem through a comparative study using trial data from an active towed array sonar. The TkBD algorithms are modified to incorporate changes in target appearance with received array bearing, and are shown to give improved SNR and state estimation performance compared with a conventional point measurement tracking algorithm. The thesis concludes by discussing the limitations of the proposed algorithms and possible avenues for future work.
Publications


Declaration

I, Han Xuan Vu, certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree. I give consent to this copy of my thesis when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

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Signature. ..........................    Date. ............
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List of Acronyms

EM  Expectation-Maximisation
EKF  Extended Kalman Filter
ER  Echo-Repeater
GNN  Global Nearest Neighbour
HMM  Hidden Markov Model
HMC  Hidden Markov Chain
HRC  Hidden Reciprocal Chain
HRP  Hidden Reciprocal Process
H-PMHT  Histogram-Probabilistic Multi-Hypothesis Tracker
iid  independent and identically distributed
IPDA  Integrated Probabilistic Data Association
IP-PMHT  Interpolated Poisson - Probabilistic Multi-Hypothesis Tracker
MAP  Maximum a Posteriori
MLSE  Maximum Likelihood Sequence Estimator
MMSE  Minimum Mean Square Error
ML  Maximum Likelihood
NN  Nearest Neighbour
KF  Kalman Filter
pdf  probability density function
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>psf</td>
<td>point spread function</td>
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<tr>
<td>PDA</td>
<td>Probabilistic Data Association</td>
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<tr>
<td>PF</td>
<td>Particle Filter</td>
</tr>
<tr>
<td>PMHT</td>
<td>Probabilistic Multi-Hypothesis Tracker</td>
</tr>
<tr>
<td>rhs</td>
<td>right hand side</td>
</tr>
<tr>
<td>RMS</td>
<td>Root Mean Square</td>
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<tr>
<td>RC</td>
<td>Reciprocal Chain</td>
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<tr>
<td>RP</td>
<td>Reciprocal Process</td>
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<tr>
<td>SMC</td>
<td>Sequential Monte Carlo</td>
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<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
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<tr>
<td>TkBD</td>
<td>Track-Before-Detect</td>
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<tr>
<td>UKF</td>
<td>Unscented Kalman Filter</td>
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</tbody>
</table>
List of Principal Symbols

! Factorial operator

$[\cdot]^T$ Transpose operator

$[\cdot]'$ Indicates that the variable is dependent on estimates from the previous EM iteration

$\eta$ Forgetting factor

$\lambda$ Poisson mixing rate

$\lambda_m^t$ Poisson mixing rate for component $m$ at time $t$

$\tilde{\lambda}$ Quantised Poisson mixing rate

$\tilde{\lambda}_t$ Quantised Poisson mixing rate at time $t$

$\tilde{\lambda}_m^t$ Quantised Poisson mixing rate for component $m$ at time $t$

$\tilde{\lambda}_{im}^t$ Quantised Poisson mixing rate for component $m$ in pixel $i$ at time $t$

$\Lambda$ Collection of Poisson mixing rate terms for all times and components

$\Lambda'$ Collection of Poisson mixing rate terms at the previous EM iteration for all time and components

$\pi$ Multinomial mixing proportion term

$\pi_{im}^t$ Multinomial mixing proportion term for component $m$ in pixel $i$ at time $t$

$\Pi$ Collection of multinomial mixing proportion terms for all times and components

$\Pi'$ Collection of multinomial mixing proportion terms at the previous EM iteration for all times and components

$\mu_{im}^t$ Proportion of power from component $m$ in pixel $i$ at time $t$

$\mathbf{Z}_{im}^t$ Energy from component $m$ in pixel $i$ at time $t$

$B^k_{i,j}$ Markov bridge transitions
Quantisation level

$f_t$ H-PMHT probability density function at time $t$

$f^i_t$ H-PMHT per-pixel probability density at time $t$

$f_t$ Poisson H-PMHT intensity function at time $t$

$g^i_t$ Poisson H-PMHT per-pixel intensity at time $t$

$F$ Linear target transition matrix

$F_t$ Estimated total power in the image at time $t$ as calculated by the H-PMHT

$F^O_t$ Estimated total power in the observed image pixels at time $t$ as calculated by the H-PMHT

$F_t$ Estimated total power in the observed image pixels at time $t$ as calculated by the Poisson H-PMHT

$G_0(\cdot)$ Clutter distribution

$h(\cdot)$ Point spread function

$h^i(x^m_t)$ Probability of a shot due to a target $x^m_t$ falls in pixel $i$

$h^i(\emptyset)$ Probability of a shot due to clutter falls in pixel $i$

$H$ Linear measurement function

$i$ Measurement pixel index

$I$ Total number observed pixels

$K^i_t$ Set of components associated with shots in pixel $i$ at time $t$

$K^i_{tr}$ Index of the component associated with the $r^{th}$ shot in pixel $i$ at time $t$

$K$ Collection of the assignments of measurements to components for all times and components

$L$ Collection of the precise locations of the shots inside its pixel for all time and components

$m_t$ Number of point detections at time $t$

$m$ Component index

$M$ Number of components

$n^i_t$ Quantised measurements (counts) in pixel $i$ at time $t$

$n^i_{tm}$ Quantised measurements (counts) from component $m$ in pixel $i$ at time $t$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<td>( n_i^t )</td>
<td>Quantised unobserved measurements (counts) in pixel ( i ) at time ( t )</td>
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<td>( N )</td>
<td>Collection of quantised observed measurements for all times and components</td>
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<td>( N^c )</td>
<td>Collection of quantised unobserved measurements for all times and components</td>
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<tr>
<td>( O )</td>
<td>Observer</td>
</tr>
<tr>
<td>( Q )</td>
<td>Process noise covariance</td>
</tr>
<tr>
<td>( Q^{(H)} )</td>
<td>H-PMHT auxiliary function</td>
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<tr>
<td>( Q^{(P)} )</td>
<td>Poisson H-PMHT auxiliary function</td>
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<tr>
<td>( Q^{(IP)} )</td>
<td>IP-PMHT auxiliary function</td>
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<tr>
<td>( Q_{ijl} )</td>
<td>Reciprocal three point transition function</td>
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<tr>
<td>( r )</td>
<td>Measurement shot index</td>
</tr>
<tr>
<td>( R )</td>
<td>Measurement covariance matrix</td>
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<tr>
<td>( S )</td>
<td>Total number of image pixels (unobserved and observed)</td>
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<tr>
<td>( t )</td>
<td>Time index</td>
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<tr>
<td>( T )</td>
<td>Number of time scans</td>
</tr>
<tr>
<td>( x_t^m )</td>
<td>State of component ( m ) at time ( t )</td>
</tr>
<tr>
<td>( X_t )</td>
<td>State in Markov Model</td>
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<tr>
<td>( X )</td>
<td>Collection of components for all times and components</td>
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<tr>
<td>( z_t )</td>
<td>An observed measurement at time ( t )</td>
</tr>
<tr>
<td>( z_i^t )</td>
<td>Observed energy in pixel ( i ) at time ( t )</td>
</tr>
<tr>
<td>( z_t^m )</td>
<td>Observed energy from component ( m ) at time ( t )</td>
</tr>
<tr>
<td>( Z )</td>
<td>Collection of observed measurements for all times and components</td>
</tr>
<tr>
<td>( Z_t )</td>
<td>Observed measurements at time ( t )</td>
</tr>
<tr>
<td>( Z_t )</td>
<td>Collection of observed measurements up until time ( t )</td>
</tr>
</tbody>
</table>
$Z$: Collection of unobserved measurements for all times

$Z_t$: Collection of unobserved measurements up to time $t$

$||Z_t||$: Total observed energy in image at time $t$