Einstein and conformally Einstein bi-invariant semi-Riemannian metrics

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Abstract

This thesis considers the geometric properties of bi-invariant metrics on Lie groups. On simple Lie groups, we show that there is always an Einstein bi-invariant metric; that when the Lie algebra is of complex type, there is another metric on a simple Lie group that is Bach-flat but not conformally Einstein and that when the metric is a linear combination of these aforementioned metrics, that the metric is not Bach-flat. This result can be used to describe all bi-invariant metrics on reductive Lie groups.

The thesis then considers bi-invariant metrics on Lie groups when the Lie algebra is created through a double extension procedure, as described initially by Medina [25]. We show two examples of bi-invariant metrics on non-reductive Lie groups that are Bach-flat but not conformally Einstein, however, we show that all Lorentzian bi-invariant metrics are conformally Einstein.
Dedication

To C.B.,

This will give us plenty to talk about.
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