Multi-channel Techniques for 3D ISAR

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April 2015
Declaration of Authorship

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“A person who never made a mistake never tried anything new.”

Albert Einstein
Abstract

Doctor of Philosophy

Multi-channel Techniques for 3D ISAR

by Federica Salvetti

This thesis deals with the challenge of forming 3D target reconstruction by using spatial multi-channel ISAR configurations. The standard output of an ISAR imaging system is a 2D projection of the true three-dimensional target reflectivity onto an image plane. The orientation of the image plane cannot be predicted a priori as it strongly depends on the radar-target geometry and on the target motion, which is typically unknown. This leads to a difficult interpretation of the ISAR images. In this scenario, this thesis aim to give possible solutions to such problems by proposing three 3D processing based on interferometry, beamforming techniques and MIMO InISAR systems. The CLEAN method for scattering centres extraction is extended to multichannel ISAR systems and a multistatic 3D target reconstruction that is based on an incoherent technique is suggested.
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<tr>
<td>2D</td>
<td>2 Dimensional</td>
</tr>
<tr>
<td>3D</td>
<td>3 Dimensional</td>
</tr>
<tr>
<td>DSTO</td>
<td>Defence Science Technology Organization</td>
</tr>
<tr>
<td>EM</td>
<td>ElectroMagnetic</td>
</tr>
<tr>
<td>FMCW</td>
<td>Frequency Modulated Continuous Wave</td>
</tr>
<tr>
<td>FT</td>
<td>Fourier Transform</td>
</tr>
<tr>
<td>IC</td>
<td>Image Contrast</td>
</tr>
<tr>
<td>ICBA</td>
<td>Image Contrast Based Autofocus</td>
</tr>
<tr>
<td>IFT</td>
<td>Inverse Fourier Transform</td>
</tr>
<tr>
<td>InISAR</td>
<td>Interferometric Inverse Synthetic Aperture Radar</td>
</tr>
<tr>
<td>ISAR</td>
<td>Inverse Synthetic Aperture Radar</td>
</tr>
<tr>
<td>IPP</td>
<td>Image Projection Plane</td>
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<tr>
<td>LoS</td>
<td>Line of Sight</td>
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<td>MC-CLEAN</td>
<td>MultiChannel CLEAN</td>
</tr>
<tr>
<td>M-ICBA</td>
<td>Multichannel Image Contrast Based Autofocus</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multiple Input Multiple Output</td>
</tr>
<tr>
<td>MF</td>
<td>Matched Filter</td>
</tr>
<tr>
<td>PLS</td>
<td>Probabilistic Least Square</td>
</tr>
<tr>
<td>Pol-CLEAN</td>
<td>Polarimetric CLEAN</td>
</tr>
<tr>
<td>PSF</td>
<td>Point Spread Function</td>
</tr>
<tr>
<td>PRF</td>
<td>Pulse Repetition Frequency</td>
</tr>
<tr>
<td>PRI</td>
<td>Pulse Repetition Interval</td>
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<tr>
<td>RADAR</td>
<td>RAdio Detection And Ranging</td>
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<tr>
<td>RCS</td>
<td>Radar Cross Section</td>
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<tr>
<td>Abbreviation</td>
<td>Full Form</td>
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<tr>
<td>RD</td>
<td>Range Doppler</td>
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<td>RT</td>
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<td>SNR</td>
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Symbols

\begin{itemize}
\item \( A \): received signal amplitude of the \( i^{th} \) spatial channel
\item \( A^i \): focusing parameter of the ICBA algorithm corresponding to the acceleration
\item \( B \): signal bandwidth
\item \( c \): speed of light
\item \( C_m \): model coordinates
\item \( C_{rt} \): coordinates of the model rotated along the trajectory
\item \( d_H \): horizontal baseline length
\item \( d_V \): vertical baseline length
\item \( D_{cr} \): size of the antenna along a given cross-range direction
\item \( E_{g,*} \): energy of a time-delay section in the \( i^{th} \) spatial channel
\item \( f_0 \): carrier frequency
\item \( f_d \): Doppler frequency
\item \( F \): percentage of initial energy of the ISAR image
\item \( h_H \): scatterer’s height along the horizontal baseline
\item \( h_V \): scatterer’s height along the vertical baseline
\end{itemize}
\( \mathbf{I} \)
- LoS unit vector

\( i_{\text{LoS}} \)
- ISAR image in the range-Doppler domain

\( J(\alpha) \)
- cost function of the soft assignment

\( \mathbf{K} \)
- wavenumber

\( k_0 \)
- rotation matrix

\( \mathbf{M} \)
- mean matching distance

\( mmd_k(n) \)
- centre of the reference system

\( O \)
- position at time \( t = 0 \) for a generic scatterer in the local reference system \( T_j \)

\( P_j \)
- \( m^{th} \) tx coordinates in the local system of reference

\( p_{rm} \)
- \( n^{th} \) rx coordinates in the local system of reference

\( Q \)
- number of sensor in the multistatic network

\( \mathbf{R} \)
- radar-target distance

\( R_0 \)
- yaw matrix

\( R_\mu \)
- pitch matrix

\( R_\nu \)
- roll matrix

\( R_\xi \)
- spectrum of the time-varying spatial multichannel received signal

\( S_R(f,t) \)
- transmitted pulse duration

\( T_i \)
- reference system embedded in the radar

\( T_\xi \)
- transmitted pulse duration
Symbols

$T_{\text{obs}}$  
observation time

$T_R$  
pulse repetition interval

$T_x$  
time-varying reference system embedded in the target

$T_y$  
reference system $T_x$ at $t = 0$

$U$  

$V$

$v_r$  
radial velocity

$W$

$w(\tau, \nu)$  
Point Spread Function

$W(f, t)$  
domain where the 2D FT of the reflectivity function is defined

$X$  

$Y$

$y_{ij}$  
$i^{th}$ y-coordinate in the local reference system $T_j$

$Z$

Greek Symbols

$\alpha$

azimuth angle

$\alpha_{cr}$  
angular resolution of a rectangular antenna

$\alpha$  
soft assignment matrix

$\alpha_{i,k}$  
assignment probability between the $k^{th}$ and the $i^{th}$ scatterer

$\beta$

$\gamma$

dominant parameter to adjust the threshold $\Lambda$

$\delta$

cross-range resolution

$\delta_r$  
range resolution

$\delta_r$  
pulse duration at the output of the MF

$\Delta_\nu$  
Doppler resolution

$\Delta_r$  
time delay resolution

$\Delta_{y_1}$  
spatial resolution along $y_1$
$\Delta_{y_2}$  spatial resolution along $y_2$

$\epsilon$  euclidean distance between $k^{th}$ and the $i^{th}$ scatterer

$\epsilon_h$  height error

$\epsilon_\phi$  $\phi$ error

$\epsilon_\Omega$  effective rotation angle error

$\zeta$  compression factor

$\theta$  elevation angle

$\lambda$  wavelength

$\Lambda$  threshold to identify unreliable assignments

$\mu$  chirp rate

$\nu$  Doppler frequency

$\rho$  reflectivity function

$\sigma$  standard deviation of $\epsilon_h$

$\tau$  time delay

$\phi$  rotation angle between $T_\xi$ and $T_x$

$\varphi$  angle between the scatterer's trace and the abscissa axis

$\chi$  function to be minimised for the estimation of $\Omega_{eff}$ and $\phi$

$\Psi(a, b)$  function to be minimised for the estimation of $\Omega_{eff}$ and $\phi$
Symbols

\( \Omega_{\text{eff}} \)  effective rotation vector
\( \Omega_T \)  total angular rotation vector

Math Operators

\( A\{\} \)  expectation
\( E\{\} \)  expectation
\( \delta_{i,j} \)  Dirac delta function
\( T \)  transpose operator