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We apply nonperturbative variational techniques to a relativistic scalar field theory in which heavy bosons ("nucleons") interact with light scalar mesons via a Yukawa coupling. Integrating out the meson field and neglecting the nucleon vacuum polarization, one obtains an effective action in terms of the heavy particle coordinates which is nonlocal in the proper time. As in Feynman's polaron approach, we approximate this action by a retarded quadratic action whose parameters are to be determined variationally on the pole of the two-point function. Several *Ansätze* for the retardation function are studied and for the most general case we derive a system of coupled variational equations. An approximate analytic solution displays the instability of the system for coupling constants beyond a critical value.

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I. INTRODUCTION

Variational methods have a long history and are still widely used in physics to obtain approximate nonperturbative solutions. For a very wide class of problems specified by a given set of equations, it is indeed always possible to construct a variational principle which will give an estimate of the quantity of interest correct to first order if the quantities appearing in the variational principle are known to zeroth order [1]. In quantum mechanics the best-known variational principle is the Rayleigh-Ritz variational principle for the energy which is applied extensively in molecular, atomic, and nuclear physics.

In contrast, the applications of variational principles in quantum field theory are rather limited (for a review see Ref. [2]). Within the Hamiltonian formalism, several studies exist (see, e.g., [3, 4]). The best-known covariant example is also a Rayleigh-Ritz variational principle which has been formulated in the functional Schrödinger representation [5]. It leads to the Hartree (-Fock) approximation when a Gaussian wave functional is used. Unfortunately, the latter is the only trial functional which can be used for practical purposes, which drastically restricts the power of the variational principle. In addition, in quantum field theory it is not the energy of the ground-state (vacuum) one is interested in but the energy (mass) of excitations. Already in ordinary quantum mechanics this is much harder to obtain. The need for renormalization and the infinitely many degrees of freedom get added to the "difficulties in applying the variational principle to quantum field theory" so that Feynman expressed a rather pessimistic view at a workshop devoted to that topic [6].

It is remarkable that the variational principle works very well in a nonrelativistic field-theoretical problem, the polaron (for reviews see [7–10]), but only after the infinitely many degrees of freedom for the phonons are integrated out exactly. This gives rise to a nonlocal effective action which Feynman approximated variationally by a retarded quadratic action [11]. Recent exact Monte Carlo calculations [12] have again demonstrated that the Feynman polaron approximation is the best analytical approximation which works for small as well as large coupling constants. Taking the known strong-coupling expansions as a yardstick the ground-state energy deviates less than 2.2% and the effective mass (which determines the lowest excitations) less than 12% from the exact values. This success can be attributed to the reduction in the number of variables and the explicit allowance of retardation in the quadratic trial action. Feynman used a specific parametrization for the retardation function but the most general form gives only a very small improvement in the ground-state energy [13, 14].

Although the Feynman variational principle (or Jensen's inequality in mathematical language) has sometimes been used in field theory (see, e.g., [15]), it was never used in the context which made it so successful in the polaron problem: namely, approximating a nonlocal action expressed in terms of particle coordinates by a retarded quadratic one. We will do so in the present work which is the first in a planned series about variational approximations employing the *particle representation* of field theory. The concept of using particle trajectories as dynamical variables in a relativistic quantum theory is an old one: it dates back to the 1937 paper by Fock [16] who investigated the role of proper time in relativistic equations. In the early 50's, Nambu [17], Feynman [18], and Schwinger [19] made much use of it, but canonical ("second") quantization later on took over and dominated, in particular in the text books (an exception is, of course, Ref. [20]). Only a few works [21–23] have employed this approach in the following years. The renewed interest in the particle representation (see also [24, 25]) is because of superstring-inspired techniques for efficient calculation of one-loop diagrams which have been shown to be connected to the ("first-") quantized form of field theory [26].

For the moment, we want to restrict our discussion to *scalar* field theories. This avoids the complications of spin in a path integral, for which there is extensive discussion in the literature (see, for example, [27, 28]). Also, having in mind applications in few-body physics, we take the simplest field theory where a light scalar particle (the "pion") has a Yukawa coupling to a heavy scalar (the "nucleon"). This is the Wick-Cutkosky model [29, 30] which usually is considered as a simple model for relativistic bound-state problems treated in the ladder approximation to the Bethe-Salpeter equation (see, e.g., Ref. [31], Chap. 10-2). Recently it also has become a popular playing field for light-cone techniques [32–34].

To be specific, we consider the Lagrangian in Euclidean space-time,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 + \frac{1}{2} M_0^2 \Phi^2 + \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{1}{2} m^2 \varphi^2 - g\Phi^2\varphi, \tag{1}$$

where M_0 is the bare mass of the heavy particle (which we shall call, for brevity, the "nucleon"), m is the mass of the light particle (the "meson"), and g is the (dimensionful) coupling constant of the Yukawa interaction between the two particles. It is well known [35] that such a coupling is equivalent to a Φ^3 theory and therefore the ground state of the theory is unstable. This is best seen in Fig. 1 which shows a contour plot of the classical "potential":

$$\begin{aligned} V^{(0)}(\Phi,\varphi) &= \frac{1}{2} M_0^2 \Phi^2 + \frac{1}{2} m^2 \varphi^2 - g\Phi^2 \varphi \\ &= \frac{1}{2} M_0^2 \Phi^2 - \frac{g^2}{2m^2}\Phi^4 + \frac{1}{2} m^2 \left(\varphi - \frac{g}{m^2}\Phi^2\right)^2 . \end{aligned} \tag{2}$$

The superscript zero reminds us that this is the potential in zeroth order in an expansion in powers of $\hbar$. One-loop quantum corrections modify the behavior shown in Fig. 1 somewhat, but no qualitative change occurs.

Clearly, the minimum at $\Phi = \varphi = 0$ is only a *local* minimum. For positive φ and nonzero Φ, the "potential" decreases indefinitely. Therefore the "ground state" sitting near $\Phi = \varphi = 0$ is only *metastable*, at least in a classical description. From semiclassical descriptions of tunneling [36, 37], we expect the lifetime to depend on the minimum height and thickness of the barrier for a given coupling constant.

Differentiating Eq. (2) with respect to φ, we obtain

$$\varphi_{\min} = \frac{g}{m^2}\Phi^2 \tag{3}$$

and the "potential" along this path

$$V^{(0)}(\Phi,\varphi_{\min}) = \frac{1}{2} M_0^2 \Phi^2 - \frac{g^2}{2m^2}\Phi^4 \tag{4}$$

is an *inverted* double well as shown in Fig. 2.

We are, however, not genuinely interested in this instability of the "ground state" in the Wick-Cutkosky model. Rather, we want to use it as a field-theoretical toy model for the dressing of *physical* nucleons by mesons. The arguments showing the instability of the "ground state" for a scalar "nucleon" do not apply for the case where the nucleons have spin [35]. In other words, the instability is an unwanted side effect of the simplified model considered here and we shall ignore it whenever possible. Operationally, we can do this as long as we restrict the parameters of the model such that the width of the ground state is small compared to its mass. From the above arguments it is clear that this corresponds to sufficiently small couplings. Indeed, it will turn out, quite reasonably, that the variational equations we shall derive cease to have real solutions once the coupling becomes too large; i.e., the formalism itself tells us in which region it remains applicable.

We will study the dressing of a single "nucleon" in the quenched approximation, i.e., neglecting pair creation of heavy particles which should be a good approximation in

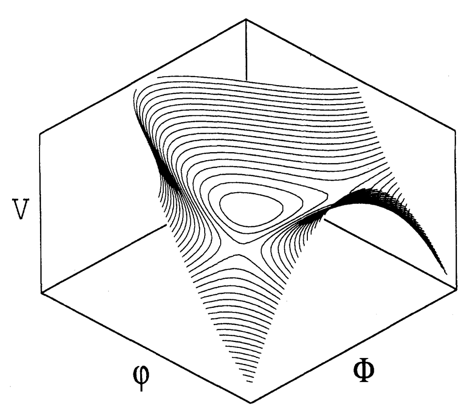

FIG. 1. Contour plot of the classical "potential" (2).

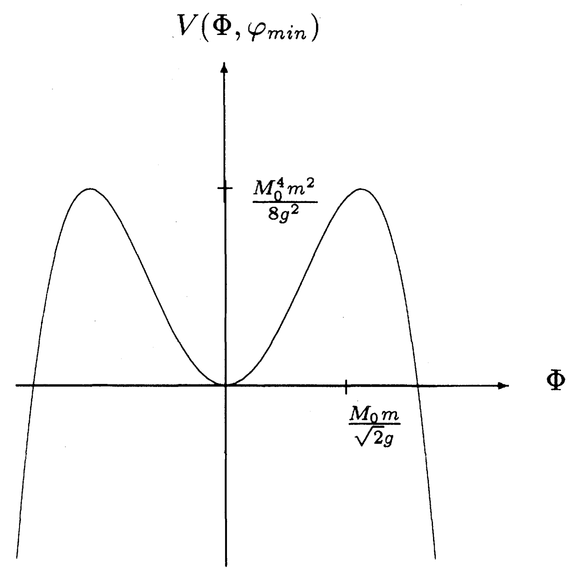

FIG. 2. Cut of the classical potential along the line (3).

low-energy processes. In this approximation it is possible to integrate out the mesons exactly and to obtain an effective nonlocal action which is a covariant functional of the particle four-coordinates with the proper time as parameter. This effective action bears a surprising similarity to the polaron action so that we could even call the dressed particle a "relativistic polaron." We then perform a variational calculation with a quadratic trial action in complete analogy to the polaron case, except that we use a covariant description and have to renormalize the mass of the heavy particle. Recently, Simonov and collaborators [38, 39] have also studied the Wick-Cutkosky model in the quenched approximation and in the particle representation. However, their aim was to solve the relativistic bound-state problem beyond the ladder approximation and they neglected all self-energy and vertex corrections. Consequently, there is no need for renormalization and no sign of any instability in their work.

This paper is organized as follows: In Secs. II and III we, respectively, derive the effective action in the particle representation of the Wick-Cutkosky model and perform the variational approximation in the manner of Feynman. The latter is done at the pole of the two-point function. In Sec. IV we discuss different variational *Ansätze* for the retardation function and we set up the coupled system of equations which arises when no assumptions are made about the form of the retardation function. We study a simple approximate solution of these variational equations which displays the instability of the ground state. The main results of this work are summarized in Sec. V, whereas some technical details are relegated to the Appendix.

II. EFFECTIVE ACTION IN THE PARTICLE REPRESENTATION

We begin with the generating functional for the Green functions of the theory:

$$Z[J,j] = \int \mathcal{D}\Phi\, \mathcal{D}\varphi\, \exp\{-S[\Phi,\varphi] + (J,\Phi) + (j,\varphi)\}\,. \tag{5}$$

Here

$$S[\Phi,\varphi] = \int d^4x\, \mathcal{L}(\Phi(x),\varphi(x)) \tag{6}$$

denotes the action and we use

$$(J,\Phi) \equiv \int d^4x\, J(x)\Phi(x), \quad \text{etc.}, \tag{7}$$

as convenient abbreviations for the source terms.

Our aim will be to integrate out the mesonic degrees of freedom in order to get an effective action for the heavy particles. Indeed, as the meson field φ appears at most quadratically in the path integral, one could do so immediately, using

$$\int \mathcal{D}\varphi \exp[-\tfrac{1}{2}(\varphi, D\,\varphi) + (j,\varphi)] = \frac{\text{const}}{(\det D)^{1/2}} \times \exp[\,\tfrac{1}{2}\,(j, D^{-1}j)\,]\,. \tag{8}$$

Considering, for simplicity, the case $j = 0$, we would obtain

$$\int \mathcal{D}\varphi\, \exp[-\tfrac{1}{2}(\varphi,(-\Box+m^2)\varphi) + g(\phi^2,\varphi)\,] = \frac{\text{const}}{(\det D_m)^{1/2}}\, \exp\left[\frac{g^2}{2}\,(\Phi^2,\, D_m^{-1}, \Phi^2)\right]\,, \tag{9}$$

where

$$D_m \equiv -\Box + m^2 \tag{10}$$

is the inverse meson propagator. In Eq. (9) the prefactor arising from the Gaussian integration is independent of the field Φ and the sources and can be absorbed in the (irrelevant) normalization factor for the path integral. Therefore, the effective action for the heavy field would be given by

$$S_{\text{eff}}[\Phi] = \frac{1}{2}(\Phi,(-\Box+M_0^2)\Phi) - \frac{g^2}{2}\left(\Phi^2, \frac{1}{-\Box+m^2}\Phi^2\right)\,. \tag{11}$$

This is a nonlocal Φ^4 theory whose interaction term has the wrong sign; i.e., this action is not bounded from below. This leads to the vacuum instability discussed above for the classical limit. To solve the model completely, one now would still have to perform a functional integration over the heavy field Φ. Because of the non-Gaussian nature of the resulting path integral, this is impossible to do analytically and one has to resort to approximative methods.

Given that we want to apply a variational approach, it turns out (as we shall see later) that it is actually advantageous to first integrate out the heavy field before doing the same for the light field. Although this sounds paradoxical in view of the stated aim, we will reintroduce the heavy particle *coordinate* at a later stage. Applying Eq. (8), we obtain

$$\int \mathcal{D}\Phi\, \exp[-\tfrac{1}{2}(\Phi,(-\Box+M_0^2-2g\varphi)\Phi) + (J,\Phi)] = \frac{\text{const}}{[\,\det(-\Box+M_0^2-2g\varphi)\,]^{1/2}}\, \exp(-I\,[\Phi,J]\,) \tag{12}$$

with

$$I\,[\Phi,J] = -\frac{1}{2}\left(J,\, \frac{1}{-\Box+M_0^2-2g\varphi}\,J\right)\,. \tag{13}$$

In contrast to Eq. (9), the prefactor now explicitly depends on the meson field φ over which one has to integrate finally. As the determinant is a highly nonlinear and nonlocal object, this makes an analytical evaluation impossible. However, it is well known that the prefactor describes pair production which is greatly suppressed if the mass of these particles is large:

$$\frac{\det(-\Box+M_0^2-2g\varphi)}{\text{const}} = \frac{\det(-\Box+M_0^2-2g\varphi)}{\det(-\Box+M_0^2)} = \det\left(1-2g\frac{1}{-\Box+M_0^2}\varphi\right) \overset{M_0\to\infty}{\longrightarrow} 1\,. \tag{14}$$

In the following we will adopt this "quenched approximation" and concentrate on the *two-point* function for one nucleon with an arbitrary number of mesons. For this object, we then have the generating functional

$$Z'[j,x] \equiv \left.\frac{\delta^2 Z[J,j]}{\delta J(x)\,\delta J(0)}\right|_{J=0}$$
$$= \int \mathcal{D}\varphi \left\langle x \left| \frac{1}{-\Box + M_0^2 - 2g\varphi} \right| y=0 \right\rangle$$
$$\times \exp[-\tfrac{1}{2}(\varphi, D_m\varphi) + (j,\varphi)]\,. \tag{15}$$

This obviously describes the propagation of a "nucleon" in the presence of an external field $g\,\varphi(x)$ over which one has to integrate functionally with a given weighting function. To perform this integration we use a trick from Schwinger and exponentiate the nucleon propagator:

$$\frac{1}{\hat{p}^2 + M_0^2 - 2g\varphi(x)} = \frac{1}{2}\int_0^\infty d\beta\, \exp\{-\tfrac{1}{2}\beta\,[\hat{p}^2 + M_0^2 - 2g\varphi(x)]\,\}, \tag{16}$$

where $\hat{p}_\mu = i\partial_\mu$ is the four-momentum operator. The integration variable β usually is called "fifth parameter" or "proper time" (Refs. [16–19]). Actually Eq. (16) only holds if the corresponding operator is positive definite which, in general, is *not* the case since the meson field $\varphi(x)$ can take any value when integrated over functionally. This means that the meson fluctuations can become so large that the nucleon locally becomes massless or even tachyonic. The correct way to exponentiate, therefore, would be

$$\frac{1}{\hat{p}^2 + M_0^2 - 2g\varphi(x) - i\epsilon} = \frac{i}{2}\int_0^\infty dT\, \exp\left\{-\frac{i}{2}T\,[\hat{p}^2 + M_0^2 - 2g\varphi(x) - i\epsilon]\right\}, \tag{17}$$

i.e., to introduce Minkowski proper time instead of the Euclidean one as in Eq. (16). We recall from the Introduction [see Eqs. (3) and (4)] that large meson fields can carry one over the barrier and induce the instability of the ground state. Since we want to disregard this instability as much as possible and since numerical calculations are much easier in Euclidean proper time, we will nevertheless use Eq. (16) in the following. However, we should expect a breakdown of this description for coupling constants large enough to induce fluctuations over the barrier.

Even with the proper time representation (16) for the nucleon propagator, we cannot perform the φ integration since the operator $\hat{p}^2$ does not commute with the external potential $g\varphi(x)$. However, formally

$$U(x,\beta;0,0) = \left\langle x \left| \exp\left[-\beta\left(\frac{\hat{p}^2}{2} - g\varphi(x)\right)\right] \right| y=0 \right\rangle \tag{18}$$

is the matrix element of the Euclidean time evolution operator of a nonrelativistic particle of unit mass[1] in the potential $g\varphi(x)$. Therefore, we can express it as a path integral over the *coordinate* $x(\tau)$ of the particle beginning at $x(0)=0$ and ending at $x(\beta)=x$ [20, 40]:

$$U(x,\beta;0,0) = \int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x(\tau) \times \exp\left(-\int_0^\beta d\tau\,[\,\tfrac{1}{2}\dot{x}^2 - g\varphi(x(\tau))\,]\right). \tag{19}$$

As all quantities in the path integral (19) are c numbers, the Gaussian φ integration,

$$\int \mathcal{D}\varphi\, \exp[-\tfrac{1}{2}(\varphi, D_m\,\varphi) + (h,\varphi)\,]\,;$$
$$h(y) = j(y) + g\int_0^\beta d\tau\, \delta(\,y - x(\tau)\,), \tag{20}$$

can now be performed with the help of Eq. (8). The result is

$$Z'[j,x] = \text{const} \times \int_0^\infty d\beta\, \exp\left(-\frac{\beta}{2}M_0^2\right) \int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x(\tau) \times \exp\{-S_{\text{eff}}\,[x(\tau),j]\,\}\,, \tag{21}$$

where the effective action is given by

$$S_{\text{eff}}\,[x(\tau),j] = \int_0^\beta d\tau\,\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}(h,\,D_m^{-1}\,h)\,. \tag{22}$$

It is convenient to write it in the form

$$S_{\text{eff}}[x(\tau),j] = S_0[x(\tau)] + S_1[x(\tau)] + S_2[x(\tau),j] + S_3[j] \tag{23}$$

with

$$S_0\,[x(\tau)] = \int_0^\beta d\tau\,\tfrac{1}{2}\dot{x}^2, \tag{24}$$

$$S_1\,[x(\tau)] = -\frac{g^2}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2\,\langle x(\tau_1)|\,D_m^{-1}\,|x(\tau_2)\rangle, \tag{25}$$

$$S_2\,[x(\tau),j] = -g\int d^4y\, j(y) \int_0^\beta d\tau\,\langle y|\,D_m^{-1}\,|x(\tau)\rangle, \tag{26}$$

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[1] A different value should not change *physical* observables since it only corresponds to a different parametrization of the particle path. It can be shown that such a "reparametrization" invariance holds in our variational approximation. The present choice is called the "proper-time gauge" [27].

$$S_3\,[j] = -\tfrac{1}{2}\int d^4y_1\, d^4y_2\; j(y_1)\,\langle y_1|\, D_m^{-1}\,|y_2\rangle\, j(y_2)\,. \quad (27)$$

Note that the last term $S_3\,[j]$ in the action does not depend on the trajectory $x(\tau)$ of the nucleon and therefore the external meson lines which are generated by differentiating with respect to the meson source j, are not attached to the nucleon line. Thus the generating functional for *connected* Green functions $G_{2,n}$ simply is

$$Z'_{\rm conn}\,[j,x] = Z'\,[j,x]\,\Big|_{S_3=0}\,. \quad (28)$$

Compared to the usual procedure via a Legendre transform, this simple identification is just one of many advantages of field theory in the "particle representation." Another one is the big reduction in degrees of freedom: although in Eq. (21) one still has to do a functional integration, it is over four functions of one variable (the proper time), whereas the previous field theoretical path integral (9) is over one function of four variables (namely, the space-time coordinates). It is for this reason that one might expect a variational approach based on particle coordinates to be superior to the one based on field variables, given that in both cases only quadratic trial actions can be used in practical calculations.

Equations (24) and (25) are the relativistic generalizations of the retarded polaron actions which Feynman [11] derived when integrating out the phonons from the polaron Hamiltonian. The meson propagator may be written as

$$\langle x|\, D_m^{-1}\,|y\rangle = \int \frac{d^4q}{(2\pi)^4}\,\frac{e^{\,iq\cdot(x-y)}}{q^2+m^2}\quad , \quad (29)$$

and so Eq. (25) becomes

$$S_1[x(\tau)] = -\frac{g^2}{2}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m^2}\times e^{\,iq\cdot(\,x(\tau_1)-x(\tau_2)\,)}\,. \quad (30)$$

Comparing with the polaron action [12]

$$S_1^{\rm polaron}[x(\tau)] = -\frac{\alpha}{2\sqrt{2}}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\int\frac{d^3q}{2\pi^2}\,\frac{e^{-|\tau_1-\tau_2|}}{\mathbf{q}^2}\times e^{\,i\mathbf{q}\cdot(\,\mathbf{x}(\tau_1)-\mathbf{x}(\tau_2)\,)}, \quad (31)$$

one observes a striking similarity. This is even more pronounced when we perform the q_0 integration in Eq. (30) which gives

$$S_1[x(\tau)] = -\frac{g^2}{16\pi}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\int\frac{d^3q}{2\pi^2}\times\frac{e^{-\omega_q|x_0(\tau_1)-x_0(\tau_2)|}}{\omega_q}\,e^{\,i\mathbf{q}\cdot(\,\mathbf{x}(\tau_1)-\mathbf{x}(\tau_2)\,)} \quad (32)$$

with $\omega_q = (\mathbf{q}^2+m^2)^{1/2}$. However, there are also some differences which should be noted: (i) All coordinates and momenta in S_1 in Eq. (30), as opposed to $S_1^{\rm polaron}$, are four dimensional and therefore Lorentz invariance is explicit; (ii) a massive meson propagator enters into the effective action of the Wick-Cutkosky model instead of the Coulomb propagator in the polaron problem; (iii) the explicitly Lorentz-invariant expression for S_1 [Eq. (30)] does not contain a retardation factor in the proper time, whereas the polaron effective action does because of the (normal) time it takes to exchange optical phonons of unit frequency. The three-dimensional version of S_1 [Eq. (32)] does contain a retardation; however, it is not just proportional to the proper time difference.

To maintain explicit covariance we will not use the form (32). It is of course also possible to fully perform the four-dimensional q integration and to obtain

$$S_1[x(\tau)] = -\frac{g^2}{8\pi^2}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\frac{m}{y(\tau_1,\tau_2)}K_1(my(\tau_1,\tau_2)), \quad (33)$$

where $K_1(x)$ is a modified Bessel function [41] and

$$y(\tau_1,\tau_2) = \sqrt{[\,x(\tau_1)\;-\;x(\tau_2)\,]^2}\,. \quad (34)$$

For small relative times, Eq. (33) exhibits a stronger divergence ($1/y^2$) than that in the polaron case $(1/y)$ and requires the usual renormalizations of relativistic field theory. As the Bessel function is difficult to handle, we will not use this explicit form in the following but rather stick to the integral representation in Eq. (30).

From the derivation presented above, it should be clear how the particle representation is generalized to N nucleons (the case $N=2$ has been considered in Ref. [39], neglecting self-energy and vertex corrections): to each heavy particle there corresponds just *one* trajectory. This is because of the quenched approximation which neglects production of heavy pairs. Therefore the nucleon number is conserved and no splitting of heavy particle trajectories can occur.

III. VARIATIONAL APPROXIMATION ON THE POLE OF THE TWO-POINT FUNCTION

In this section, we only consider the case where no external mesons are present, which corresponds to simply setting the meson sources $j(x)$ to zero. The exact two-point function (or propagator) is then given by

$$G_2(x) = {\rm const}\times\int_0^\infty d\beta\,\exp\left(-\frac{\beta}{2}M_0^2\right)\int_{x(0)=0}^{x(\beta)=x}\mathcal{D}x(\tau)\times\exp\{-S_0\,[x(\tau)] - S_1\,[x(\tau)]\,\}\,. \quad (35)$$

The normalization constant can be determined by switching off the interaction. In this case, we know [31]

$$G_2(p)\Big|_{S_1=0} = \int d^4x\,\exp(\,ip\cdot x)\,G_2(x)\Big|_{S_1=0} = \frac{1}{p^2+M_0^2}\,. \quad (36)$$

The correct normalization of Eq. (35), therefore, is

$$G_2(x) = \frac{1}{8\pi^2}\int_0^\infty d\beta\,\frac{1}{\beta^2}\,\exp\left[-\frac{\beta}{2}M_0^2 - \frac{x^2}{2\beta}\right]\times\frac{\int\mathcal{D}x\,\exp(-S_0\,-S_1\,)}{\int\mathcal{D}x\,\exp(-S_0\,)}, \quad (37)$$

where the paths are subject to the boundary conditions

$$x(0) = 0\,, \quad x(\beta) = x\,. \tag{38}$$

Similarly, in momentum space we can write

$$G_2(p) = \frac{1}{2}\int_0^\infty d\beta\, \exp\left[-\frac{\beta}{2}(p^2 + M_0^2)\right] \times \frac{\int d^4x\, \exp(ip\cdot x)\, \int \mathcal{D}x\, \exp(-S_0 - S_1)}{\int d^4x\, \exp(ip\cdot x)\, \int \mathcal{D}x\, \exp(-S_0)}\,. \tag{39}$$

Because of the nonlinear dependence of the action (30) on the paths $x(\tau)$, it is, of course, impossible to do the path integrations (37), (39) exactly. However, following Feynman [11], it is possible to find a variational approximation for the effective action starting from a solvable trial action. This variational treatment is based on the decomposition

$$S = S_t \,+\, S \,-\, S_t \,=\, S_t \,+ \Delta S \tag{40}$$

and on Jensen's inequality

$$\left\langle e^{-\Delta S}\right\rangle \;\geq\; e^{-\langle \Delta S\rangle} \tag{41}$$

which holds for averages with normalized positive weighting functions. If the weighting function is not positive (or even complex), or ΔS is complex, the inequality in Eq. (41) is replaced by a stationarity with respect to variations:

$$\left\langle e^{-\Delta S}\right\rangle \;\stackrel{\rm stat}{\simeq}\; e^{-\langle \Delta S\rangle}\,. \tag{42}$$

Obviously, Minkowski proper time and/or Minkowski space-time only allows the weaker form (42) to be used. In addition to the choice of the trial action S_t, we also have the freedom how we define the averaging, i.e., which coordinates we treat exactly and which only approximately via the Jensen stationarity. To be more precise, one can define

$$\langle \Delta S\rangle_{S_t} \;\equiv\; \frac{\int \mathcal{D}x(\tau)\, \Delta S[x(\tau)]\;\; \exp\{-S_t[x(\tau)]\}}{\int \mathcal{D}x(\tau)\, \exp\{-S_t[x(\tau)]\}} \tag{43}$$

or

$$\langle\!\langle \Delta S\rangle\!\rangle_{S_t} \;\equiv\; \frac{\int d^4x\, \exp(ip\cdot x) \int \mathcal{D}x(\tau)\, \Delta S[x(\tau)]\;\; \exp\{-S_t[x(\tau)]\}}{\int d^4x\, \exp(ip\cdot x) \int \mathcal{D}x(\tau)\, \exp\{-S_t[x(\tau)]\}}\,. \tag{44}$$

In the first case, which we will call "coordinate averaging," one has to do the Fourier transform with respect to the end point x after the averaging to get the approximate two-point function in momentum space whereas in the latter ("momentum averaging"), only the integration over the proper time β still has to be performed. This is reminiscent of the "partial averaging" procedure proposed by Doll *et al.* [42] and employed in the Monte Carlo calculations of Ref. [12]. It is clear that coordinate averaging usually is more accurate and that (with Euclidean proper time) Jensen's *inequality* (41) can be used. On the other hand, momentum averaging gives more directly the two-point function in momentum space. We will see that with suitable trial actions, both averaging procedures lead to identical results on the nucleon pole.

A. Coordinate averaging

Equation (37) may be written in the form

$$G_2(x) = \frac{1}{8\pi^2}\int_0^\infty d\beta\, \frac{1}{\beta^2}\, \exp\left(-\frac{\beta}{2}M_0^2 - \frac{x^2}{2\beta}\right) \times \left\langle\, e^{-S_1}\right\rangle_{S_0}\,, \tag{45}$$

where the averaging is performed with respect to the weighting function $\exp(-S_0)$:

$$\left\langle\, e^{-S_1}\right\rangle_{S_0} \equiv \frac{\int \mathcal{D}x\, \exp(-S_0)\, \exp(-S_1)}{\int \mathcal{D}x\, \exp(-S_0)} = \langle \exp(S_t - S)\,\rangle_{S_t}\, \frac{\int \mathcal{D}x\, \exp(-S_t)}{\int \mathcal{D}x\, \exp(-S_0)}\,. \tag{46}$$

Here, S is the sum of S_0 and S_1. Applying Jensen's inequality [Eq. (41)], we find

$$\langle\, e^{-S_1}\,\rangle_{S_0} \;\geq\; \exp(-\langle \Delta S\rangle_{S_t}\,)\, \frac{\int \mathcal{D}x \exp(-S_t)}{\int \mathcal{D}x \exp(-S_0)}\,. \tag{47}$$

The various path integrals may be easily calculated in Fourier space by parametrizing the paths as

$$x(\tau) = x\,\frac{\tau}{\beta} + \sum_{k=1}^{\infty} \frac{2\sqrt{\beta}}{k\pi}\, b_k\, \sin\left(\frac{k\pi\tau}{\beta}\right)\,. \tag{48}$$

This obviously fulfills the boundary conditions (38). As only the ratio of path integrals appears in Eq. (46), the Jacobian from the transformation to Fourier space cancels out and the path integrals are now infinite-dimensional integrals over the Fourier coefficients b_k for $k = 1, \ldots, \infty$. If one writes the end point coordinate as

$$x = \sqrt{2\beta}\, b_0, \tag{49}$$

then the free action is simply

$$S_0 = \sum_{k=0}^{\infty} b_k^2\,. \tag{50}$$

The most general trial action with which one can proceed analytically is one where the b_k's appear at most quadratically. We shall use

$$S_t = \sum_{k=0}^{\infty} A_k\, b_k^2\quad, \tag{51}$$

with coefficients $A_k > 0$ parametrized in various forms (see below) or left free as variational parameters. A term such as $b_k \cdot b_0$ may also be introduced with only minor complications, while off-diagonal terms like $b_k \cdot b_{k'}$ would require the calculation of infinite-dimensional determinants.

By this choice, all path integrals are simple Gaussian integrals and can easily be performed. We obtain

$$\frac{\int \mathcal{D}b \, \exp(-S_t)}{\int \mathcal{D}b \, \exp(-S_0)} = e^{-(A_0-1)\, b_0^2} \prod_{k=1}^{\infty} \left(\frac{1}{A_k^2} \right) , \tag{52}$$

$$\langle \, S_0 - S_t \, \rangle_{S_t} = (1 - A_0)\, b_0^2 + 2 \sum_{k=1}^{\infty} \left(\frac{1}{A_k} - 1 \right) , \tag{53}$$

and

$$\langle \, S_1 \, \rangle_{S_t} = -\frac{g^2}{2} \int_0^\beta d\tau_1 d\tau_2 \int \frac{d^4 q}{(2\pi)^4} \times \frac{1}{q^2 + m^2} \, \langle \, \exp [\, iq \cdot (x(\tau_1) - x(\tau_2)) \,] \, \rangle_{S_t} . \tag{54}$$

The last average also involves a (shifted) Gaussian integration and is given by

$$\langle \, \exp [\, iq \cdot (x(\tau_1) - x(\tau_2)) \,] \, \rangle_{S_t} = \exp \left(i \frac{\tau_1 - \tau_2}{\beta} \, q \cdot x \, - \, \frac{1}{2} \, \mu^2(\tau_1, \tau_2) \, q^2 \right) , \tag{55}$$

where we have defined

$$\mu^2(\tau_1, \tau_2) = \beta \sum_{k=1}^{\infty} \frac{\lambda_k^2(\tau_1, \tau_2)}{A_k} \tag{56}$$

and

$$\lambda_k(\tau_1, \tau_2) = \frac{\sqrt{2}}{k\pi} \left[\sin \left(\frac{k\pi\tau_1}{\beta} \right) - \sin \left(\frac{k\pi\tau_2}{\beta} \right) \right] . \tag{57}$$

We shall postpone a discussion of the meaning of the quantity μ^2, which plays a crucial role in what follows, until later. Finally, the q integration in Eq. (54) can be performed by using the representation

$$\frac{1}{q^2 + m^2} = \frac{1}{2} \int_0^\infty du \, \exp \left[-\frac{u}{2} \, (q^2 + m^2) \right] . \tag{58}$$

This gives

$$\langle \, S_1 \, \rangle_{S_t} = -\frac{g^2}{8\pi^2} \int_0^\beta d\sigma \int_{\sigma/2}^{\beta - \sigma/2} dT \times \int_0^\infty du \, \frac{1}{[u + \mu^2(\sigma, T)\,]^2} \times \exp \left[-\frac{u}{2} m^2 \, - \, \frac{x^2}{2\beta^2} \frac{\sigma^2}{u + \mu^2(\sigma, T)} \right] , \tag{59}$$

where we have used the symmetry of the integrand to restrict the proper-time integrations to $\tau_2 \leq \tau_1$ and introduced relative and total times

$$\sigma = \tau_1 - \tau_2 \, , \quad T = \tfrac{1}{2}(\tau_1 + \tau_2) \, . \tag{60}$$

The interaction term can be brought into simpler form by the transformation $u \to \mu^2/(u + \mu^2)$ which leads to

$$\langle \, S_1 \, \rangle_{S_t} = -\frac{g^2}{8\pi^2} \int_0^\beta d\sigma \int_{\sigma/2}^{\beta - \sigma/2} dT \times \frac{1}{\mu^2(\sigma, T)} \int_0^1 du \times e \left(m\mu(\sigma, T), \, \frac{x\sigma}{\beta\mu(\sigma, T)} \, , \, u \right) . \tag{61}$$

Here, the function $e(s, t, u)$ is defined as

$$e(s, t, u) = \exp \left(-\frac{s^2}{2} \frac{1 - u}{u} \, - \, \frac{t^2}{2} \, u \right) . \tag{62}$$

In principle, the u integral can be expressed in terms of a particular plasma dispersion function, the so-called Shkarofsky function [43], but there is no advantage of using this representation.

Hence, using Jensen's inequality and the trial action (51), the Green function in coordinate space is bounded by

$$G_2(x) \geq \frac{1}{8\pi^2} \int_0^\infty d\beta \, \frac{1}{\beta^2} \, \exp \left(-\frac{\beta}{2} M_0^2 \, - \, \frac{x^2}{2\beta} \right) \times \exp [- \, \beta \, \Omega(\beta) \, - \, \langle S_1 \rangle_{S_t}] \tag{63}$$

where

$$\Omega(\beta) = \frac{2}{\beta} \sum_{k=1}^{\infty} \left[\ln A_k + \frac{1}{A_k} - 1 \right] . \tag{64}$$

B. Renormalization

Actually, as it stands, Eq. (61) does not exist, since for small relative times (as we shall see later)

$$\mu^2(\sigma, T) \overset{\sigma \to 0}{\simeq} \sigma \, , \tag{65}$$

causing a logarithmic divergence in the σ integration.[2] This is, of course, one of the expected divergences of field theory which requires renormalization. In the present case, renormalization is particularly easy, since only a *mass renormalization* for the heavy particle is needed. In fact, the theory is super renormalizable in the quenched approximation, only the second-order self-energy diagram of the nucleon introduces a divergence. We regulate this with a Pauli-Villars regularization. This amounts to subtracting a term with the meson mass replaced by a cutoff mass Λ (which will eventually tend to infinity), thus removing the small σ singularity. To be specific, we subtract

$$\frac{1}{\sigma} \, e \, (\, \Lambda\sqrt{\sigma}, \sqrt{\sigma}\mu_0, u) \tag{66}$$

from $\langle S_1 \rangle$ where μ_0 is an arbitrary mass (renormalization

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[2]In D dimensions, the integrand behaves like $\sigma^{D/2-1}$ which in $D = 3$ leads to the integrable singularity $1/\sqrt{\sigma}$ of the polaron problem.

point). Since

$$\frac{\partial}{\partial\mu_0^2}\frac{1}{\sigma}\, e\left(\Lambda\sqrt{\sigma},\sqrt{\sigma}\mu_0,u\right) \;=\; -\,\frac{u}{2}\, e\left(\Lambda\sqrt{\sigma},\sqrt{\sigma}\mu_0,u\right) \tag{67}$$

is finite at $\sigma = 0$ and vanishes for $\Lambda \to \infty$, the averaged action will be independent of μ_0. We will assume a nonzero meson mass m in most of the following and, therefore, the most convenient choice for us is $\mu_0 = 0$. As shown in the Appendix, one then obtains

$$\langle\, S_1\,\rangle_{S_t} = -\frac{g^2}{8\pi^2}\,\beta\,\ln\frac{\Lambda^2}{m^2} \;+\; \langle\, S_1\,\rangle^{\rm fin} \;+\; \langle\, S_1\,\rangle^{\rm reg}\,, \tag{68}$$

where $\langle\, S_1\,\rangle^{\rm fin}$ is the finite part resulting from the subtraction (66) and is given in Eq. (A9). The regular part reads

$$\langle\, S_1\,\rangle^{\rm reg} = -\frac{g^2}{8\pi^2}\int_0^\beta d\sigma \int_{\sigma/2}^{\beta-\sigma/2} dT \int_0^1 du \times\left[\frac{1}{\mu^2(\sigma,T)}\, e\left(m\mu(\sigma,T),\, \frac{x\sigma}{\beta\mu(\sigma,T)}\,,\, u\right) -\frac{1}{\sigma}\, e\left(m\sqrt{\sigma},0,u\right)\right]. \tag{69}$$

From Eqs. (68) and (63), it is evident that the divergent part of the averaged action can be absorbed into a new mass parameter

$$M_1^2 = M_0^2 \;-\; \frac{g^2}{4\pi^2}\,\ln\frac{\Lambda^2}{m^2} \tag{70}$$

which will be found to be *finite*. After the bare mass has been replaced by M_1, all quantities are now well defined. Note that the renormalization (70) is, in fact, the same as in the lowest order perturbation theory, even though the calculation has been done in a nonperturbative way. Note also that M_1 is, in general, not yet the physical mass of the nucleon but an intermediate mass scale with no direct physical meaning. Again, the finite shift from M_1 to $M_{\rm phys}$ will be done in a nonperturbative way.

C. On-mass-shell limit

The physical mass is determined from the requirement that in momentum space, the two-point function develops a pole when approaching $p^2 = -M_{\rm phys}^2$:

$$G_2(p) \longrightarrow \;\frac{Z}{p^2 + M_{\rm phys}^2}\,. \tag{71}$$

Here $0 \le Z \le 1$ is the residue at the pole. How is it possible that

$$G_2(p) = \int d^4x\; e^{ip\cdot x}\; G_2(x) = \frac{4\pi^2}{p}\int_0^\infty dx\; x^2\; J_1(px)\; G_2(x) \tag{72}$$

diverges at $p = iM_{\rm phys}$? Obviously, this can only be the case if the large-x behavior of $G_2(x)$ (which is only a function of x^2) is not able to overcome the exponential growth of the Bessel function [41]

$$J_1\left(iM_{\rm phys}\, x\right) = i\, I_1\left(M_{\rm phys}x\right) \;\overset{x\to\infty}{\sim}\; i\,\frac{e^{M_{\rm phys}\, x}}{\sqrt{2\pi M_{\rm phys}\, x}}\,. \tag{73}$$

Therefore, the physical mass is given by

$$M_{\rm phys} = -\lim_{x\to\infty}\frac{1}{x}\,\ln\left(G_2(x)\right)\;. \tag{74}$$

This is similar to the way the ground-state energy is obtained from the partition function in nonrelativistic physics or the mass of hadrons in lattice calculations.

However, the explicit expression for $G_2(x)$ (63) contains a term $\exp(-x^2/2\beta)$ which would decay like a Gaussian unless the proper time β is proportional to x and also tends to infinity. These heuristic arguments suggest that we have to study the limit $x, \beta \to \infty$ but keep

$$\lambda = \frac{1}{M_{\rm phys}}\,\frac{x}{\beta} \tag{75}$$

fixed. In Eq. (75), the extra factor $M_{\rm phys}^{-1}$ has been introduced to obtain a dimensionless quantity.[3] From Eq. (63), we then obtain

$$G_2(x) \ge \frac{1}{8\pi^2}\,\frac{M_{\rm phys}}{x}\int_0^\infty d\lambda\; e^{-x\, F(x,\lambda)} \tag{76}$$

where

$$F(x,\lambda) = \frac{M_1^2}{2\lambda M_{\rm phys}} + \frac{\lambda}{2}M_{\rm phys} + \frac{1}{\lambda M_{\rm phys}}\,\Omega +\frac{1}{x}\left(\langle\, S_1\,\rangle^{\rm fin} \;+\; \langle\, S_1\,\rangle^{\rm reg}\right)\,. \tag{77}$$

In the limit $x \to \infty$, Laplace's method [44, 45] tells us that Eq. (76) behaves like

$$G_2(x) \;\overset{x\to\infty}{\gtrsim}\; \frac{\rm const}{x^{3/2}}\; e^{-x\, F(\lambda_0)} \tag{78}$$

where $F(\lambda_0)$ is the minimum of $F(x \to \infty, \lambda)$. Inserting this result into Eq. (74), we obtain

$$M_{\rm phys} \le F(\lambda_0)\,. \tag{79}$$

We have to study the large-x and the large-β limits of the averaged action. First, we note from Eq. (A9) that, for $\mu_0 = 0$,

$$\lim_{\beta\to\infty}\frac{1}{\beta}\langle\, S_1\,\rangle^{\rm fin} \;=\; 0. \tag{80}$$

Then we assume that

$$\lim_{\beta\to\infty}\;\mu^2(\sigma,T) = \mu^2(\sigma) \tag{81}$$

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[3] Recall from Eq. (16) that our proper time has dimension $(\text{mass})^{-2}$.

which holds in all parametrizations which we will study. Therefore

$$V \equiv \lim_{\beta\to\infty} \frac{1}{\beta}\langle S_1\rangle^{\rm reg}$$
$$= -\frac{g^2}{8\pi^2}\int_0^\infty d\sigma \int_0^1 du \left[\frac{1}{\mu^2(\sigma)} \times e\left(m\mu(\sigma), \frac{\lambda M_{\rm phys}\sigma}{\mu(\sigma)}, u\right) - \frac{1}{\sigma}\, e\left(m\sqrt{\sigma}, 0, u\right)\right] \quad (82)$$

has a well-defined limit. We will also assume (and later verify) that $\Omega(\beta)$, defined in Eq. (64), has a large-β limit:

$$\Omega = \lim_{\beta\to\infty} \Omega(\beta)\,. \quad (83)$$

Suppressing the subscript zero for λ, we finally arrive at the following inequality for the physical mass:

$$M_{\rm phys}^2 \le \frac{M_1^2}{2\lambda} + \frac{\lambda}{2} M_{\rm phys}^2 + \frac{1}{\lambda}\,(\Omega + V)\,. \quad (84)$$

Equation (84) is the main result of this section. Since $M_{\rm phys}$ is fixed, we can turn it around and use

$$M_1^2 \ge (2\lambda - \lambda^2)\, M_{\rm phys}^2 - 2\,(\Omega + V) \quad (85)$$

to maximize the right-hand side (RHS) with respect to λ and all the parameters in the trial action. In the following, we will call Ω the "kinetic" term because it has no explicit coupling constant dependence and V the "potential" term because it has. In addition, Eq. (84) looks like a variational equation for the energy in nonrelativistic quantum mechanics.

Without variation, the equality sign in Eq. (85) gives the perturbative result from the one-loop graph shown in Fig. 3(a). This can be seen as follows: while we expect $\lambda = 1 + O(g^2)$, the combination $2\lambda - \lambda^2$ is $1 + O(g^4)$. Similarly, from $A_k = 1 + O(g^2)$, we deduce $\Omega = O(g^4)$ [see Eq. (64)] and $\mu^2(\sigma) = \sigma + O(g^2)$. Therefore, to lowest order in g^2, we obtain

$$M_1^2 = M_{\rm phys}^2 - 2\,V\Big|_{\mu^2(\sigma)=\sigma}^{\lambda=1} + O(g^4)\,, \quad (86)$$

or

$$M_{\rm phys}^2 = M_1^2 + \frac{g^2}{4\pi^2}\int_0^1 du\,\ln\left[1 + \frac{M_{\rm phys}^2}{m^2}\frac{u^2}{1-u}\right], \quad (87)$$

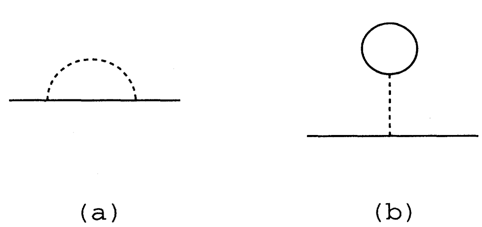

FIG. 3. Second-order graphs for the two-point function: (a) self-energy graph, (b): tadpole graph. In the quenched approximation, the tadpole graph is neglected.

after performing the σ integration. The same result is obtained from the direct calculation of the self-energy diagram in Fig. 3(a):

$$\Sigma(p^2) = -\frac{g^2}{4\pi^2}\ln\frac{\Lambda^2}{m^2} + \frac{g^2}{4\pi^2}\int_0^1 du\,\ln\left[1 + \frac{p^2}{m^2}u + \frac{M_0^2}{m^2}\frac{u}{1-u}\right]. \quad (88)$$

The pole position is determined by $M_{\rm phys}^2 = M_0^2 + \Sigma(-M_{\rm phys}^2)$ from which we obtain Eq. (87) in lowest order after renormalizing the mass [see Eq. (70)].

D. Momentum averaging

In coordinate averaging, the determination of the physical mass was a rather involved procedure. This is avoided in "momentum averaging," where we also average over the end point coordinate x with the additional weight $\exp(ip\cdot x)$. This extra weight can be formally absorbed in a modified (complex) free action

$$\tilde{S}_0 = S_0 - ip\cdot x\,. \quad (89)$$

In other words, we write Eq. (39) as

$$G_2(p) = \frac{1}{2}\int_0^\infty d\beta\,\exp\left[-\frac{\beta}{2}(p^2 + M_0^2)\right]\,\langle\!\langle\, e^{-S_1}\,\rangle\!\rangle_{\tilde{S}_0} \quad (90)$$

where

$$\langle\!\langle\, e^{-S_1}\,\rangle\!\rangle_{\tilde{S}_0} \equiv \frac{\int\mathcal{D}\tilde{x}\,\exp(-\tilde{S}_0)\,\exp(-S_1)}{\int\mathcal{D}\tilde{x}\,\exp(-\tilde{S}_0)}$$
$$= \langle\!\langle \exp(\tilde{S}_t - \tilde{S})\,\rangle\!\rangle_{\tilde{S}_t}\,\frac{\int\mathcal{D}\tilde{x}\,\exp(-\tilde{S}_t)}{\int\mathcal{D}\tilde{x}\,\exp(-\tilde{S}_0)}\,. \quad (91)$$

Here, we have defined

$$\int\mathcal{D}\tilde{x}\,\cdots = \int d^4x\int\mathcal{D}x\,\cdots\,. \quad (92)$$

Because the weight function is now complex, we can only apply Jensen's stationarity relation (42):

$$\langle\!\langle\, e^{-S_1}\,\rangle\!\rangle_{\tilde{S}_0} \simeq \exp\left(-\langle\!\langle S - \tilde{S}_t\rangle\!\rangle_{\tilde{S}_t}\right) \times\frac{\int\mathcal{D}\tilde{x}\,\exp(-\tilde{S}_t)}{\int\mathcal{D}\tilde{x}\,\exp(-\tilde{S}_0)}\,. \quad (93)$$

As trial action, we take

$$\tilde{S}_t = \sum_{k=0}^{\infty} A_k\, b_k^2 - i\,\tilde{\lambda}\,p\cdot x \quad (94)$$

where $\tilde{\lambda}$ is an additional variational parameter which rescales the momentum.

As the evaluation of the various path integrals closely follows the one in Sec. III A, we can be brief and just state the results,

$$\frac{\int \mathcal{D}\tilde{x}\ \exp(-\tilde{S}_t)}{\int \mathcal{D}\tilde{x}\ \exp(-\tilde{S}_0)} = \exp\left[-\frac{\beta}{2}p^2\left(\frac{\tilde{\lambda}^2}{A_0}-1\right)\right]\prod_{k=0}^{\infty}\left(\frac{1}{A_k^2}\right), \tag{95}$$

$$\langle\langle\, \tilde{S}_0 - \tilde{S}_t \,\rangle\rangle_{\tilde{S}_t} = 2\sum_{k=0}^{\infty}\left(\frac{1}{A_k}-1\right) - \frac{\beta}{2}p^2\,\frac{\tilde{\lambda}}{A_0^2} \times(\tilde{\lambda}+\tilde{\lambda}A_0-2A_0)\,, \tag{96}$$

and this time the interaction term is

$$\langle\langle\, S_1 \,\rangle\rangle_{\tilde{S}_t} = -\frac{g^2}{8\pi^2}\int_0^{\beta} d\sigma \int_{\sigma/2}^{\beta-\sigma/2} dT \times \frac{1}{\tilde{\mu}^2(\sigma,T)}\int_0^1 du \times e\left(m\tilde{\mu}(\sigma,T),\ \frac{-i\tilde{\lambda}p\sigma}{A_0\tilde{\mu}(\sigma,T)}\,,\ u\right). \tag{97}$$

Here

$$\tilde{\mu}^2(\sigma,T) = \frac{\sigma^2}{A_0\beta} + \mu^2(\sigma,T)\,. \tag{98}$$

Renormalization of the averaged action is along the same lines as in the Appendix. Combining all terms, we obtain the propagator in momentum space

$$G_2(p) \simeq \frac{1}{2}\int_0^{\infty} d\beta\ \exp\left[-\frac{\beta}{2}(p^2+M_1^2)+\frac{\beta}{2}p^2\left(1-\frac{\tilde{\lambda}}{A_0}\right)^2\right]\exp\Big(-\beta\,\tilde{\Omega}(\beta) - \langle\langle S_1\rangle\rangle^{\rm reg} - \langle S_1\,\rangle^{\rm fin}\Big) \tag{99}$$

where

$$\tilde{\Omega}(\beta) = \frac{2}{\beta}\sum_{k=0}^{\infty}\left[\ln A_k + \frac{1}{A_k} - 1\right] \tag{100}$$

and

$$\langle\langle\, S_1 \,\rangle\rangle^{\rm reg} = -\frac{g^2}{8\pi^2}\int_0^{\beta} d\sigma \int_{\sigma/2}^{\beta-\sigma/2} dT \int_0^1 du \times\left[\frac{1}{\tilde{\mu}^2(\sigma,T)}\, e\left(m\tilde{\mu}(\sigma,T),\ \frac{-i\tilde{\lambda}p\sigma}{A_0\tilde{\mu}(\sigma,T)}\,,\ u\right) - \frac{1}{\sigma}\, e\left(m\sqrt{\sigma},0,u\right)\right]. \tag{101}$$

Because the small σ behavior of $\tilde{\mu}^2(\sigma,T)$ is the same as that of $\mu^2(\sigma,T)$ [see Eq. (98)], we have subtracted the same term (66) as before. This explains why the finite part $\langle S_1\rangle^{\rm fin}$ of the averaged action is unchanged.

The on-shell limit of Eq. (99) is now particularly easy: a pole develops if in

$$G_2(p) \simeq \frac{1}{2}\int_0^{\infty} d\beta\ \exp\left[-\frac{\beta}{2}F(\beta,p^2)\right] \tag{102}$$

the function $F\left(\beta\to\infty, p^2 = -M_{\rm phys}^2\right)$ vanishes. This leads to

$$M_{\rm phys}^2 = M_1^2 + M_{\rm phys}^2\left(1-\frac{\tilde{\lambda}}{A_0}\right)^2 + 2\lim_{\beta\to\infty}\left[\tilde{\Omega}(\beta)+\frac{1}{\beta}\langle\langle S_1\rangle\rangle^{\rm reg}\Big|_{p=iM_{\rm phys}}\right]. \tag{103}$$

For any sensible parametrization, A_0 is finite in the large-β limit. Therefore, the tilde can be dropped from $\tilde{\mu}^2(\sigma)$ and $\tilde{\Omega}$ for large β [see Eqs. (98) and (100)], and Eq. (103) is completely equivalent to Eq. (84) if we identify

$$\tilde{\lambda} = A_0\,\lambda\,. \tag{104}$$

Because of the use of a complex trial action, momentum averaging only tells us that the RHS of Eq. (103) is an extremum (and not necessarily a minimum) under variations. Since the intermediate mass scale M_1 does not show up in any observable, this has no direct physical consequence. Of course, a minimum principle has the extra advantage that the minimal value gives a clear measure of the quality of the variational ansatz.

IV. VARIATIONAL *ANSÄTZE*

Having developed the general formalism for the variational calculation in the last two sections, we now need to turn our attention to the specific form of the trial action (51). We shall first consider two specific parametrizations of the Fourier coefficients A_k of this action, followed by the best possible parametrization (within the quadratic ansatz) where the actual functional form of the A_k's is determined by the variational principle. Before we do this, however, it is useful to discuss some general features of the trial action. We begin by writing down the general quadratic two-time action in coordinate space

$$S_t[x] = \int_0^{\beta} d\tau\ \frac{1}{2}\dot{x}^2 + \int_0^{\beta} d\tau_1 \int_0^{\tau_1} d\tau_2\ f(\tau_1-\tau_2) \times[\,x(\tau_1) - x(\tau_2)\,]^2 \tag{105}$$

where $f(\tau_1-\tau_2)$ is an undetermined *retardation function.* Inserting the Fourier parametrization (48) of the paths, we obtain the following expressions for the Fourier coefficients A_k:

$$A_0(\beta) = 1 + 2\int_0^\beta d\sigma\, f(\sigma)\,\sigma^2 \left(1 - \frac{\sigma}{\beta}\right), \tag{106}$$

$$A_k(\beta) = 1 + \frac{8\beta^2}{k^2\pi^2}\int_0^\beta d\sigma\, f(\sigma)\left(1 - \frac{\sigma}{\beta} - \frac{1}{k\pi} \times \sin\frac{k\pi\sigma}{\beta}\right)\sin^2\frac{k\pi\sigma}{2\beta}, \quad k = 1, 2 \ldots . \tag{107}$$

Here we have neglected cross terms of the form $b_k \cdot b_{k'}, k, k' = 0, 1\ldots$ which are suppressed for large β [12]. It is therefore consistent to also take the large-β limit of Eqs. (106, 107). This gives

$$A_k(\beta) = 1 + \frac{8\beta^2}{k^2\pi^2}\int_0^\infty d\sigma\, f(\sigma)\sin^2\frac{k\pi\sigma}{2\beta}, \quad k = 0, 1 \ldots . \tag{108}$$

In the following, we will use only this form. Note that in this expression, the dependence on β and the number k of the Fourier mode only come in via the combination

$$E = \frac{k\pi}{\beta}. \tag{109}$$

Writing $A_k \equiv A(k\pi/\beta)$, in particular $A_0 = A(0)$, we therefore have

$$A(E) = 1 + \frac{8}{E^2}\int_0^\infty d\sigma\, f(\sigma)\sin^2\frac{E\sigma}{2}. \tag{110}$$

Clearly, $A(E)$ is even,

$$A(-E) = A(E), \tag{111}$$

and tends to unity for large E:

$$A(E) \stackrel{E\to\infty}{\sim} 1. \tag{112}$$

The way how this limit is approached depends on the small-σ behavior of the retardation function $f(\sigma)$. We should emphasize that the trial action which we use is given by

$$S_t = \sum_{k=0}^\infty A\left(\frac{k\pi}{\beta}\right) b_k^2 \tag{113}$$

in Fourier space and not by Eq. (105) in x space. However, since one usually has more intuition in coordinate space, it is useful to deduce general properties and special parametrizations for the "profile function" $A(E)$ from the x-space formulation.

We are now in a position to express the quantities $\mu^2(\sigma)$ and Ω in terms of $A(E)$. The tool to perform the sums over Fourier modes in Eqs. (56) and (83) is Poisson's summation formula [46, 47]

$$\sum_{k=-\infty}^{+\infty} F(k) = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, F(x)\, e^{2i\pi n x} \tag{114}$$

which, for an even function $F(k\pi/\beta)$, leads to

$$\sum_{k=1}^\infty F\left(\frac{k\pi}{\beta}\right) = \frac{\beta}{\pi}\int_0^\infty dE\, F(E) - \tfrac{1}{2}F(0) + \frac{2\beta}{\pi}\sum_{n=1}^\infty\int_0^\infty dE\, F(E)\cos(2n\beta E). \tag{115}$$

This is an exact form which is much more useful for our purposes than, for example, the Euler-MacLaurin summation formula [41]. The usefulness of Eq. (115) comes from the fact that for ordinary functions, the asymptotic behavior of the Fourier cosine transformation [46] is given by

$$\int_0^\infty dx\, F(x)\cos(2xy) \sim -\frac{F'(0)}{(2y)^2} + \frac{F'''(0)}{(2y)^4} - \ldots . \tag{116}$$

Since $A(E)$ is even, all odd derivatives at $E = 0$ will vanish, unless $F(x)$ is singular at $x = 0$. Therefore, the asymptotic falloff of the last term in Eq. (115) with increasing β will not be powerlike but, in most cases, at least exponential. For brevity, such terms will be denoted by

$$\mathrm{Ex}_i(\beta) \equiv \frac{2\beta}{\pi}\sum_{n=1}^\infty\int_0^\infty dE\, F(E)\cos(2n\beta E), \tag{117}$$

where i is an index with which we label the various functions F which occur. Let us first apply Poisson's summation formula (115) to the sum in Eq. (56). Recalling the definitions (57) and (60), we obtain

$$\mu^2(\sigma, T) = 8\beta\sum_{k=1}^\infty \frac{1}{A_k}\frac{1}{k^2\pi^2}\sin^2\frac{k\pi\sigma}{2\beta}\cos^2\frac{k\pi T}{\beta} = \frac{8}{\pi}\int_0^\infty dE\,\frac{1}{A(E)}\frac{1}{E^2}\sin^2\frac{E\sigma}{2}\cos^2 ET - \frac{\sigma^2}{\beta A(0)} + \mathrm{Ex}_1(\beta). \tag{118}$$

The trigonometric identity $\cos^2 ET = (1 + \cos 2ET)/2$ allows us to simplify Eq. (118) further: again the cosine term only contributes to exponentially small terms[4] so that

$$\mu^2(\sigma, T) = \frac{4}{\pi}\int_0^\infty dE\,\frac{1}{A(E)}\frac{\sin^2(E\sigma/2)}{E^2} - \frac{\sigma^2}{\beta A(0)} + \mathrm{Ex}_2(\beta). \tag{119}$$

In this form, the limit $\beta \to \infty$ is trivial and given by the simple formula

$$\mu^2(\sigma) \equiv \lim_{\beta\to\infty}\mu^2(\sigma, T) = \frac{4}{\pi}\int_0^\infty dE\,\frac{1}{A(E)}\frac{\sin^2(E\sigma/2)}{E^2}. \tag{120}$$

We further note that because of Eq. (112), the small-σ

[4]Strictly speaking, these terms are exponentially small in T, not β. In order to obtain sensible asymptotic behavior for the theory, however, it is necessary for the trial action (105) to receive its main contribution for $\tau_{1,2}$ not too close to the end points of the path. Hence, $T = (\tau_1 + \tau_2)/2$ must grow like β.

limit of μ^2 is

$$\lim_{\sigma \to 0} \mu^2(\sigma) = \frac{4}{\pi} \int_0^\infty dE \, \frac{\sin^2(E\sigma/2)}{E^2} = \sigma \qquad (121)$$

which is what we have used for discussion of the divergences in the averaged action [see Eq. (65)]. The large-σ limit is given by

$$\lim_{\sigma \to \infty} \mu^2(\sigma) = \frac{4}{\pi} \frac{1}{A(0)} \int_0^\infty dE \frac{\sin^2(E\sigma/2)}{E^2} = \frac{\sigma}{A(0)} \quad . \qquad (122)$$

Because both the small- and large-σ limits of $\mu^2(\sigma)$ are proportional to σ, we shall call it "pseudotime."

We now turn to the sum over Fourier modes in Eq. (64). By applying Eq. (115), one easily obtains

$$\Omega(\beta) = \frac{2}{\pi} \int_0^\infty dE \left[\ln A(E) + \frac{1}{A(E)} - 1 \right] - \frac{1}{\beta} \left[\ln A(0) + \frac{1}{A(0)} - 1 \right] + \mathrm{Ex}_3(\beta) \qquad (123)$$

so that

$$\Omega = \lim_{\beta \to \infty} \Omega(\beta) = \frac{2}{\pi} \int_0^\infty dE \left[\ln A(E) + \frac{1}{A(E)} - 1 \right] . \qquad (124)$$

For convergence of the integral, $A(E)$ has to approach unity faster than $1/\sqrt{E}$ for large E.

A. Feynman parametrization

In his famous polaron paper, Feynman [11] chose the retardation function

$$f(\sigma) \equiv f_F(\sigma) = C\, e^{-w\sigma} , \qquad (125)$$

with C and w as variational parameters. This was motivated by the exact polaron effective action (31), which has an exponential retardation function because of the time it takes for phonons to be emitted and reabsorbed by the electron. Furthermore, it may be argued [20] that the exponential suppression at large relative times suppresses, at least partially, the increase of the quadratic trial action (105) for large $x(\tau_1) - x(\tau_2)$. (The exact action obviously goes to zero in this limit.) For this reason, we will still adopt Eq. (125) for the variational approximation to the meson-nucleon action [Eq. (30)] in a first try in this subsection, even though now, of course, there is no explicit retardation function in *proper* time in this action. We will see that this allows many calculations to be done analytically. In the next subsections we will consider more general trial actions.

Again following Feynman, we replace the strength $C > 0$ by a parameter v via

$$v^2 = w^2 + \frac{4C}{w} . \qquad (126)$$

It is obvious that v has to be larger than w. From Eq. (110), we obtain

$$A_F(E) = \frac{v^2 + E^2}{w^2 + E^2} \quad . \qquad (127)$$

Note that as a function of the *complex* variable E, Feynman's profile function vanishes at $E = \pm iv$ which in Minkowski space determines the location of the caustics (or focal points) [40]. In addition, $A_F(E)$ has poles at $E = \pm iw$. From Eq. (120), we obtain the pseudotime

$$\mu_F^2(\sigma) = \frac{w^2}{v^2}\, \sigma + \frac{v^2 - w^2}{v^3} \left(1 - e^{-v\sigma} \right) . \qquad (128)$$

The limits (121) and (122) can be read off directly from this explicit form. Finally one obtains

$$\Omega_F = \frac{(v-w)^2}{v} \qquad (129)$$

which is the $D = 4$ generalization of the polaron result.[5]

B. An improved retardation function

The Feynman parametrization outlined above has the advantage that it is extremely simple and that many manipulations may be done analytically. It has, however, the disadvantage that for small σ, it exhibits a different behavior to the true action, which is singular at this point. We shall now indicate heuristically how one may arrive at a trial action which does exhibit this singularity behavior. To start with, we shall add a constant term to the previous action (105):

$$\int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \left[g(\tau_1 - \tau_2) + f(\tau_1 - \tau_2) \left(x(\tau_1) - x(\tau_2) \right)^2 \right] . \qquad (130)$$

This should mimic the exact action (30) as much as possible. Here, we have written the constant term (which cancels in the averaging procedure) as a double-time integral over a function $g(\tau_1 - \tau_2)$. We can determine the functions f and g approximately by requiring that on the level of the proper time integrands, the momentum averaging of (130) should be equal to the momentum averaging of the exact action. To avoid nonlinear equations, we perform the averaging with the free action. Using Eqs. (97) and (98) in the large-β limit and setting $\tilde{\lambda} = A_0 = 1$, $\mu^2(\tau_1, \tau_2) = \tau_1 - \tau_2 = \sigma$, we obtain

$$g(\sigma) + f(\sigma) \langle\langle \left(x(\tau_1) - x(\tau_2) \right)^2 \rangle\rangle_{\tilde{S}_0} \simeq -\frac{g^2}{8\pi^2} \frac{1}{\sigma} \int_0^1 du \; e\left(m\sqrt{\sigma}, -ip\sqrt{\sigma}, u \right) . \qquad (131)$$

If we approximate the u integral by taking the integrand at some $u = \bar{u}$, we obtain

$$g(\sigma) + f(\sigma) \langle\langle \left(x(\tau_1) - x(\tau_2) \right)^2 \rangle\rangle_{\tilde{S}_0} \simeq -\frac{g^2}{8\pi^2} \frac{1}{\sigma} \exp\left[-\frac{1}{2} \left(m^2 \frac{1-\bar{u}}{\bar{u}} - p^2 \bar{u} \right) \sigma \right] . \qquad (132)$$

[5] In the polaron case, the kinetic term in the variational expression for the energy is $3(v-w)^2/4v$ [11].

Furthermore, as a special case of the general averaging (55), we have

$$\langle\!\langle (x(\tau_1) - x(\tau_2))^2 \rangle\!\rangle_{\tilde{S}_0} = 4\sigma - \sigma^2 p^2 \tag{133}$$

which is well known in Brownian motion: at small times the mean square distance in a diffusion process grows linearly with the time. Expanding around $p^2 = -M^2_{\text{phys}}$ and comparing coefficients we finally obtain, for the retardation function $f(\sigma)$,

$$f_I(\sigma) \simeq \frac{g^2}{16\pi^2}\frac{\bar{u}}{\sigma^2}\exp\left[-\frac{1}{2}\left(m^2\frac{1-\bar{u}}{\bar{u}} + M^2_{\text{phys}}\bar{u}\right)\sigma\right]$$
$$= \frac{C'}{\sigma^2}\,e^{-w\sigma}\,. \tag{134}$$

The most remarkable feature of the "improved" retardation function (134) is that it is singular at small relative times and thereby simulates the singular behavior of the exact effective action. Although Eq. (134) gives explicit values for the constants C' and w, these should not be taken too seriously as they are derived from averaging with the free action. We will only use the form of the retardation function as suggested by Eq. (134) and again treat C' and w as variational parameters. The resulting profile function is

$$A_I(E) = 1 + \frac{4C'}{E}\left[\arctan\frac{E}{w} - \frac{w}{2E}\ln\left(1+\frac{E^2}{w^2}\right)\right]. \tag{135}$$

At large E, this falls off only like

$$A_I(E) \overset{E\to\infty}{\simeq} 1 + \frac{2\pi C'}{E} + \ldots, \tag{136}$$

which reflects the small-σ behavior of the retardation function. Furthermore, $A_I(E)$ now has a *branch point* at $E = \pm iw$ which will become important when we study processes like meson production and scattering in subsequent applications. Again, we can eliminate the strength parameter C' in terms of a parameter v by writing $A_I(0) = v^2/w^2$. This determines

$$C' = \frac{1}{2w}\,(v^2 - w^2)\,. \tag{137}$$

We have been unable to find analytical expressions for $\mu^2(\sigma)$ and Ω with the profile function (135). They will be calculated numerically in all the applications which follow.

C. Variational equations

The optimal choice for the retardation function is obtained if one doesn't restrict its functional form in the way we have done in the two cases above, but rather determines this form through the variational principle itself. In the polaron case, this approach was first proposed by Adamowski *et al.* [13] and Saitoh [14]. It corresponds to varying Eq. (85) with respect to λ and the profile function $A(E)$. We first recall from equation (120) that the pseudotime $\mu^2(\sigma)$ can be expressed through the profile function by

$$\mu^2(\sigma) = \frac{4}{\pi}\int_0^\infty dE\,\frac{1}{A(E)}\,\frac{\sin^2(E\sigma/2)}{E^2}\,. \tag{138}$$

We may then vary Eq. (85) with respect to λ. This gives

$$(1-\lambda)\,M^2_{\text{phys}} - \frac{\partial V}{\partial\lambda} = 0.$$

The derivative can be worked out easily [see Eqs. (82) and (62)] and we obtain the implicit equation for λ

$$\frac{1}{\lambda} = 1 + \frac{g^2}{8\pi^2}\int_0^\infty d\sigma\,\frac{\sigma^2}{\mu^4(\sigma)}\int_0^1 du\,u$$
$$\times\, e\left(m\mu(\sigma), \frac{\lambda M_{\text{phys}}\sigma}{\mu(\sigma)}, u\right). \tag{139}$$

Similarly, the variation with respect to $A(E)$,

$$\frac{\delta}{\delta A(E)}\,(\Omega + V) = 0,$$

gives

$$A(E) = 1 + \frac{g^2}{4\pi^2}\frac{1}{E^2}\int_0^\infty d\sigma\,\frac{\sin^2(E\sigma/2)}{\mu^4(\sigma)}\int_0^1 du$$
$$\times\left[1 + \frac{m^2}{2}\mu^2(\sigma)\frac{1-u}{u} - \frac{\lambda^2 M^2_{\text{phys}}\sigma^2}{2\mu^2(\sigma)}u\right]$$
$$\times e\left(m\mu(\sigma), \frac{\lambda M_{\text{phys}}\sigma}{\mu(\sigma)}, u\right), \tag{140}$$

where Eq. (138) has been used to evaluate $\delta\mu^2(\sigma)/\delta A(E)$.

Let us discuss some of the aspects of the coupled variational equations (138)–(140). We first note that we may read off the retardation function, as defined in Eq. (110), from the profile function (140); it is given by

$$f_{\text{var}}(\sigma) = \frac{g^2}{32\pi^2}\frac{1}{\mu^4(\sigma)}\int_0^1 du\left[1 + \frac{m^2}{2}\mu^2(\sigma)\frac{1-u}{u}\right.$$
$$\left. - \frac{\lambda^2 M^2_{\text{phys}}\sigma^2}{2\mu^2(\sigma)}u\right]$$
$$\times e\left(m\mu(\sigma), \frac{\lambda M_{\text{phys}}\sigma}{\mu(\sigma)}, u\right). \tag{141}$$

Obviously, it has the same $1/\sigma^2$ behavior for small relative times as the "improved" parametrization (134). Furthermore, it should be noted that *no renormalization* is needed: all integrals converge for $\sigma \to 0$. In addition, the variational equations are also well behaved in the limit $m \to 0$. From Eq. (139), we observe that

$$0 < \lambda \leq 1 \tag{142}$$

always, which allows interpretation of λ as a kind of average "velocity" [see Eq. (75)] in the proper time. From Eq. (140), we find that asymptotically

$$A_{\text{var}}(E) \overset{E\to\infty}{\simeq} 1 + \frac{g^2}{4\pi^2}\frac{1}{E^2}\int_0^\infty d\sigma\,\frac{\sin^2(E\sigma/2)}{\sigma^2} + \cdots$$
$$= 1 + \frac{g^2}{16\pi}\frac{1}{E} + \cdots \tag{143}$$

which is consistent with Eq. (136). Note that while V needs renormalization, Ω does not because the E integral in Eq. (83) is still convergent with the asymptotic behavior (143).

D. Approximate solution of the variational equations

Although we will present numerical solutions of the above variational equations in the following paper, it is very useful to first attempt to derive some approximate *analytical* results. Because the ratio of the pion mass to the nucleon mass is small ($m^2/M_{\text{phys}}^2 \simeq 0.02$), a natural approximation to make is to set the pion mass to zero. This is a meaningful thing to do because, as we have already noted, the variational equations (138)–(140) are both ultraviolet and infrared safe. For $m = 0$, the equation for λ becomes

$$\frac{1}{\lambda} = 1+\frac{g^2}{2\pi^2}\frac{1}{M_{\text{phys}}^4\lambda^4}\int_0^\infty d\sigma\frac{1}{\sigma^2}\left[1-(1+\gamma(\sigma))\,e^{-\gamma(\sigma)}\right], \tag{144}$$

and the corresponding equation for the profile function is

$$A(E) = 1 + \frac{g^2}{4\pi^2}\frac{1}{E^2}\int_0^\infty d\sigma\,\frac{\sin^2(E\sigma/2)}{\mu^4(\sigma)}\,e^{-\gamma(\sigma)}, \tag{145}$$

where

$$\gamma(\sigma) = \frac{\lambda^2 M_{\text{phys}}^2\sigma^2}{2\mu^2(\sigma)}\,. \tag{146}$$

Furthermore, as seen in Eqs. (121) and (122), the pseudotime $\mu^2(\sigma)$ is proportional to σ both in the small- and large-σ limit. Let us for the moment assume, in order to be able to do the remaining integrals in Eqs. (144) and (145), that the pseudotime is in fact always proportional to the relative proper time

$$\mu^2(\sigma) \approx r\,\sigma \tag{147}$$

with $r \le 1$. This approximation will be a good one if either the region of small or large σ dominates the integrals. One can now evaluate all the integrals. Defining the dimensionless coupling constant

$$\alpha = \frac{g^2}{4\pi}\frac{1}{M_{\text{phys}}^2}\,, \tag{148}$$

the variational equation for λ [Eq. (144)] becomes

$$\frac{1}{\lambda} \approx 1 + \frac{\alpha}{\pi r\lambda^2}\,, \tag{149}$$

while the variational equation for the profile function yields

$$A(E) \approx 1+\frac{\alpha M_{\text{phys}}^2}{2E\pi r^2}\left\{\arctan\frac{2rE}{\lambda^2M_{\text{phys}}^2} - \frac{\lambda^2M_{\text{phys}}^2}{4Er}\times\ln\left[1+\left(\frac{2rE}{\lambda^2M_{\text{phys}}^2}\right)^2\right]\right\}. \tag{150}$$

In particular,

$$A(0) \approx 1 + \frac{\alpha}{2\pi}\frac{1}{\lambda^2 r} = \frac{1}{2}\left(1+\frac{1}{\lambda}\right)\,. \tag{151}$$

This is precisely the form of the profile function (135) obtained with the "improved" retardation function in Sec. IV B, if we identify

$$C' = \frac{\alpha M_{\text{phys}}^2}{8\pi r^2} \quad\text{and}\quad w = \frac{\lambda^2 M_{\text{phys}}^2}{2r}\,. \tag{152}$$

Solving Eq. (149) for λ, one obtains

$$\lambda \approx \frac{1}{2}\left[1 \pm \sqrt{1-\frac{4\alpha}{\pi r}}\right]\,. \tag{153}$$

This equation has some rather remarkable properties. First of all, it has no real solutions for α larger than

$$\alpha_c = \frac{\pi}{4}r\,. \tag{154}$$

Below this branch point, it has two solutions, one approaching $\lambda = 1$ as the coupling α goes to zero, while the other one approaches $\lambda = 0$. The first of these limits corresponds to the perturbative limit (see Sec. III C), while $\lambda = 0$ seems unphysical [see Eq. (84) or (85)].

If one argues that mostly small-σ values matter in the respective integrals, i.e., $r \approx 1$, then

$$\alpha_c \approx \frac{\pi}{4} = 0.785. \tag{155}$$

For $\alpha > \alpha_c$, only *complex* solutions are possible. This is a sign of the instability of the model and will be studied in more detail in the following paper.

V. DISCUSSION AND SUMMARY

In this work we have introduced a variational approach to relativistic quantum field theory which is closely modeled on the very successful treatment of the polaron in condensed matter physics. The final aim is to do this for a realistic theory such as QED or a meson-nucleon theory. However, there are considerable problems in going from the nonrelativistic polaron problem to a field theory. So, in order not to be confronted with all complications at once, we have chosen to start with a toy theory (the Wick-Cutkosky model) which is not a gauge theory and where spin and isospin degrees of freedom are neglected, but where the coupling is of a similar Yukawa form as for the more physically-relevant theories mentioned above. This theory not only has the advantage of relative simplicity, but it also turned out that the action is actually extremely similar to the polaron action so that one might expect to have similar success by using the same variational treatment as was introduced by Feynman in the polaron problem.

Following this idea, we have integrated out the light mesons and represented the heavy particles' degrees of freedom by trajectories parametrized by the proper time. This step necessarily required neglect of heavy particle pair production, i.e., the quenched approximation. The resulting nonlocal effective action S_{eff} was then approximated variationally by a retarded quadratic action S_t whose parameters [the "profile function" $A(E)$ and an average "velocity" λ] have to be determined on the pole of the two-point function. Apart from technical differences, the Wick-Cutkosky model here again turned out to be very similar to the polaron problem. We have introduced two different ways of averaging over the exact action ("coordinate averaging" and "momentum averaging") which gave identical results on the pole of the two-point function. In contrast to methods which optimize perturbation theory [48, 49], ours is a truly variational approach and, as shown in the case of "coordinate averaging," even a minimum principle.

However, the model to which we applied our method clearly also has some disadvantages. One of the technical differences to the polaron problem is the need of renormalization in a relativistic field theory. In this respect, the Wick-Cutkosky model is too simplistic: only a mass renormalization is needed in the quenched approximation (i.e., the model is super renormalizable) which certainly is not enough for dealing with the (nonperturbative) renormalization of realistic theories. Of more immediate concern, however, is the fact that, unrelated to the variational approach as such, the model is unstable. This is, of course, not a feature of the more realistic problems in which one is interested in the first place. Luckily, we have been able to ignore this instability in so far as that, at least in the variational approach presented here, it only starts to manifest itself for couplings larger than some critical coupling.

Nevertheless, the instability prevents us from comparing the results of the variational calculation to a strong coupling limit of the theory. This is rather unfortunate, as for the polaron the success of the approach could be gauged by the excellent agreement of the variational treatment with *both* the strong and weak coupling limits. Here, we can only compare with the latter, a comparison with the strong coupling limit will have to wait until the method is applied to a theory where this limit exists in the first place. Actually, although a stable model would have been more welcome, the instability does allow us, through the use of this nonperturbative method, to explore the behavior of the theory around the critical coupling, something which one could not do in perturbation theory.

As was the case for the polaron, the variational calculation contains within it first-order perturbation theory, as we have seen by way of example for the self-energy of the heavy particle. Importantly, this is true for *any* value of the variational parameters so that agreement with the first-order perturbative calculation is assured. In the language of perturbation theory, variation of the parameters then allows one to effectively sum up parts of higher diagrams up to all orders. In principle, the variational approach may be improved systematically by going beyond the leading order of the cumulant expansion which is used in Feynman's variational principle. In the polaron case, this leads to results for the ground-state energy and the effective mass [50] which nearly match the exact Monte Carlo calculations [12]. In practice, however, higher order corrections become increasingly difficult to calculate and so the usefulness of the approach depends on how closely the leading orders reflect reality. In particular, the accuracy of the zeroth-order results (using the first-order variational parameters) is of interest. We will show in a subsequent paper that already the zeroth-order approximation gives a quite reasonable description of meson production and scattering processes after analytic continuation to Minkowski space.

An important ingredient of the approach advocated here is to apply the variational principle to the action expressed in terms of particle coordinates rather than fields, as has previously been done. The reason for doing this is the reduction in the number of degrees of freedom which this entails. This is important as one is restricted to generalized quadratic trial actions for practical variational calculations. Furthermore, as we have seen, the particle action makes extraction of the connected part of a Green function completely trivial. On the other hand, one might consider it to be a disadvantage that the action in the particle representation is nonlocal. Although not crucial, there is a certain loss of intuition associated with this. For example, in the formulation in terms of fields one may extend the concept of the classical potential, and the physical picture which this entails, to higher orders in the coupling through the use of the effective potential. Even at the classical level, it is immediately clear by looking at the potential in Fig. 1 that the Wick-Cutkosky model is unstable. It is rather difficult to see this in the particle representation of the action (30). Indeed, even after approximating the particle action by the trial action, one first has to solve a set of nonlinear coupled equations before any signs of the instability manifested itself. Fortunately, we could obtain very good approximative results and analytical insight for the solution of the variational equations by setting the meson mass to zero and by replacing the "pseudotime" $\mu^2(\sigma)$ by its limit when the relative proper time σ tends to zero. The success of this rather drastic approximation indicates that, to a large extent, the dynamical behavior of this relativistic system is governed by short-time processes. Although no substitute for a numerical solution, these analytical expressions prove to be rather useful guides to the general behavior of the solutions. Whether the value of the critical coupling is only an artefact of our present quadratic approximation or has some physical meaning is not fully clear. In support of the latter view, it may be argued that the critical coupling corresponds to the situation where the average heavy particle field is just large enough to overcome the barrier depicted in Fig. 2.

In conclusion, we think that the variational approach in the form advocated here looks rather promising at least for the particular model which we have examined. Not only has it provided rather simple analytical expressions which go considerably beyond perturbation theory, but it also allows for numerical investigations which will be reported in the following paper. We therefore believe that it is certainly worthwhile to apply and extend it to other more realistic cases.

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APPENDIX: REGULARIZATION

Here we perform the regularization of the averaged action $\langle S_1\rangle_{S_t}$ by subtracting the term (66) from the σ integrand. Allowing for an arbitrary subtraction point μ_0, we then have

$$\langle S_1\rangle_{S_t} = -\frac{g^2}{8\pi^2}\int_0^\beta d\sigma \int_{\sigma/2}^{\beta-\sigma/2} dT \int_0^1 du \times\left[\frac{1}{\mu^2(\sigma,T)} e\left(m\mu(\sigma,T),\ \frac{x\sigma}{\beta\mu(\sigma,T)}\ ,\ u\right) - \frac{1}{\sigma}\, e\left(\Lambda\sqrt{\sigma},\mu_0\sqrt{\sigma},u\right)\right]. \quad \text{(A1)}$$

We write the quantity in square brackets as

$$\frac{1}{\sigma}\left[e\left(m\sqrt{\sigma},\mu_0\sqrt{\sigma},u\right) - e\left(\Lambda\sqrt{\sigma},\mu_0\sqrt{\sigma},u\right)\right] + \frac{1}{\mu^2(\sigma,T)}\, e\left(m\mu(\sigma,T),\ \frac{x\sigma}{\beta\mu(\sigma,T)}\ ,\ u\right) - \frac{1}{\sigma} e\left(m\sqrt{\sigma},\mu_0\sqrt{\sigma},u\right) \quad \text{(A2)}$$

and concentrate on the term in the first line which diverges if the cut-off Λ goes to infinity. The term in the second line gives rise to the regular part (69) of the averaged action. For the first term, we can immediately perform the T integration since the integrand does not depend on T. This gives a factor $\beta-\sigma$. With the explicit form (62) of the function $e(s,t,u)$, we then have to evaluate

$$\langle S_1\rangle^{\rm div} \equiv -\frac{g^2}{8\pi^2}\int_0^1 du \int_0^\beta d\sigma\, \frac{\beta-\sigma}{\sigma} \times\left[e^{-z_{m,\mu_0}(u)\,\sigma} - e^{-z_{\Lambda,\mu_0}(u)\,\sigma}\right], \quad \text{(A3)}$$

where

$$z_{m,\mu_0}(u) = \frac{m^2}{2}\frac{1-u}{u} + \frac{\mu_0^2}{2}u\,. \quad \text{(A4)}$$

The σ integration can be done in terms of the exponential integral [41]

$$E_1(z) = \int_1^\infty dt\, \frac{1}{t}\, e^{-zt}\,. \quad \text{(A5)}$$

For $z \to 0$, this function behaves like

$$E_1(z) \longrightarrow -\gamma - \ln z - O(z), \quad \text{(A6)}$$

where $\gamma = 0.577215...$ is Euler's number and for $z\to\infty$ such as

$$E_1(z) \longrightarrow \frac{e^{-z}}{z}\left[1 + O\left(\frac{1}{z}\right)\right]. \quad \text{(A7)}$$

We easily find

$$\langle S_1\rangle^{\rm div} = -\frac{g^2}{8\pi^2}\int_0^1 du\left\{\beta\left[\ln\frac{z_{\Lambda,\mu_0}(u)}{z_{m,\mu_0}(u)} + E_1(z_{\Lambda,\mu_0}(u)\beta) - E_1(z_{m,\mu_0}(u)\beta)\right] - \frac{1}{z_{m,\mu_0}(u)} \times\left[1 - e^{-z_{m,\mu_0}(u)\beta}\right] + \frac{1}{z_{\Lambda,\mu_0}(u)}\left[1 - e^{-z_{\Lambda,\mu_0}(u)\beta}\right]\right\}. \quad \text{(A8)}$$

In the limit where the cut-off mass Λ goes to infinity, this becomes simpler because of Eq. (A7)

$$\langle S_1\rangle^{\rm div} = -\frac{g^2}{8\pi^2}\int_0^1 du\left\{\beta\left[\ln\frac{\Lambda^2}{m^2} - \ln\left(1+\frac{\mu_0^2}{m^2}\frac{u^2}{1-u}\right) - E_1(z_{m,\mu_0}(u)\beta)\right] - \frac{1}{z_{m,\mu_0}(u)}\left[1 - e^{-z_{m,\mu_0}(u)\beta}\right]\right\} \equiv -\frac{g^2}{8\pi^2}\beta\ln\frac{\Lambda^2}{m^2} + \langle\, S_1\,\rangle^{\rm fin}. \quad \text{(A9)}$$

The above expression for the finite part simplifies considerably for $\mu_0 = 0$ and/or $\beta\to\infty$. This is what we employ in the main text. Note that for $m=0$, we would need $\mu_0 \neq 0$.

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