Bias Compensation for the Bearings-Only Pseudolinear Target Track Estimator

Kutluyıl Doğançay, *Senior Member, IEEE*

***Abstract*—The bearings-only pseudolinear target track estimator is known to suffer from severe bias problems. This paper presents a bias analysis for the pseudolinear estimator and develops a method of bias compensation, resulting in a closed-form reduced-bias pseudolinear estimator. The reduced-bias estimator is then incorporated into an instrumental variable estimator to produce asymptotically unbiased target motion parameter estimates. Unlike batch iterative estimators, the proposed instrumental variable estimator has a closed-from solution and therefore avoids the convergence problems associated with iterative estimators. The performance of the proposed instrumental variable estimator is illustrated by way of simulation examples and is shown to be almost identical to that of the computationally more demanding iterative maximum likelihood estimator.**

***Index Terms*—Bearings-only tracking, bias compensation, Gauss-Newton (GN) algorithm, instrumental variables, least squares, pseudolinear estimator.**

I. Introduction

TARGET tracking is an important practical problem with many civilian and military applications. The objective of target tracking is to estimate the location, velocity and acceleration of a target from noisy sensor data collected by a moving observer or several physically dispersed stationary observers. The sensor data typically comprise bearing angle (angle of arrival), time difference of arrival or Doppler frequency measurements. In passive bearings-only target tracking, the observer "listens" for signals emitted by a target in order to acquire bearing data. The moving observer, which is also known as the ownship, can be an aircraft, a ship or an unmanned aerial vehicle (UAV).

Several estimation algorithms have been developed for target tracking over the last few decades. Early approaches made use of the state-space representation of the estimation problem and used the extended Kalman filter (EKF) [1]. The EKF estimator was later shown to exhibit divergence problems when the tracking problem is formulated in Cartesian coordinates [1]. The stability of the EKF was improved by implementing it in a new coordinate system known as modified polar coordinates [2]. Being recursive, both versions of the EKF estimator require good initialization to avoid divergence [3]. The maximum likelihood (ML) estimator for target tracking was developed in [4]. For white Gaussian bearing noise, the ML estimator reduces to a nonlinear least squares estimator. While the ML estimator enjoys certain desirable properties such as asymptotic unbiasedness and efficiency, it does not have a closed-form solution and is often implemented as an iterative numerical search algorithm. The iterative ML solution is computationally expensive because of high-complexity arithmetic operations that must be performed in each iteration. It is also vulnerable to convergence problems. In [5], a track-before-detect "empirical" maximum *a posteriori* approach was developed which avoids the gradient computation associated with the iterative ML solution, resulting in lower computational complexity. In [6] the pseudolinear estimator—a linear least squares estimator with closed-form solution—was proposed to alleviate computational complexity and convergence problems. A recursive least squares implementation of the pseudolinear estimator was developed in [7]. Despite its simplicity and low computational requirements, the pseudolinear estimator suffers from severe bias problems [8], [4].

To overcome the pseudolinear estimation bias, several approaches have been proposed in the literature. In [4] a modified instrumental variable (MIV) estimator was proposed as an iterative batch algorithm derived from the Gauss-Newton (GN) iterations for the ML estimator. In [9] a recursive instrumental variable estimator was developed by using predictions of the target motion parameters from past measurements to construct an instrumental variable (IV) matrix. These IV estimators do not have closed-form solutions and require good initialization to avoid convergence problems. The convergence properties of iterative tracking algorithms are analyzed in detail in [10].

A closed-form reduced-bias target track estimator was developed in [11] by defining the target motion parameters in a new coordinate system related to modified polar coordinates [2]. The resulting estimate was shown by simulation to be unbiased under the assumption of large range-to-baseline ratio and small bearing noise. An elegant unbiased target track estimator was developed in [12] by restricting the observer trajectory to multiple constant-velocity linear segments (legs). This solution comprises two stages whereby the measurements on each observer leg are compressed into approximate sufficient statistics and then the target motion parameters are estimated from the compressed data by employing an iterative quadratic estimator. In [13], a closed-form reduced-bias pseudolinear estimator was developed by adopting a similar approach to [12] under the assumption of multiple-leg constant-velocity observer trajectory. A common assumption made by these estimators is that the target moves with a constant velocity. This restricts the target motion to a linear trajectory. In this paper, we generalize this assumption by considering a constant-acceleration target motion model [14].

Manuscript received March 18, 2004; revised February 16, 2005. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Jan C. de Munck.

The author is with the School of Electrical and Information Engineering, University of South Australia, Adelaide SA 5095, Australia (e-mail: k.dogancay@ieee.org).

Digital Object Identifier 10.1109/TSP.2005.861088

1053-587X/$20.00 © 2006 IEEE

The paper first presents a bias analysis for the pseudolinear estimator. Based on this analysis, a new bias compensated pseudolinear estimator is developed to mitigate the estimation bias. The bias compensated algorithm is shown to outperform the pseudolinear estimator under certain conditions. A new IV estimator is presented that utilizes the bias compensated estimator to construct an IV matrix. Unlike the previously proposed iterative IV-based algorithms, the new IV estimator has a closed-form solution, thereby avoiding excessive computational requirements and convergence problems associated with iterative solutions. In simulation examples, the performance of the new IV estimator is shown to match that of the computationally more demanding ML estimator. Unlike [12] and [13], the proposed estimator does not assume a multiple-leg constant-velocity observer trajectory. The large range-to-baseline ratio assumption made in [11] is not required either. The unbiasedness of the proposed estimator depends on the availability of a sufficiently large number of bearing measurements.

The paper is organized as follows. Section II defines the target tracking problem and states the assumptions made throughout the paper. An overview of the key least squares target tracking algorithms is presented in Section III. Section IV provides a bias analysis for the pseudolinear target track estimator. The bias compensated pseudolinear estimator is developed in Section V. In Section VI the proposed IV estimator is derived. The computational requirements of the tracking algorithms are analyzed in Section VII. Comparative simulation studies are presented in Section VIII.

II. Bearings-Only Target Tracking Problem

Fig. 1 depicts the problem of two-dimensional (2-D) target tracking from bearings-only measurements. The tracking geometry shown in Fig. 1 assumes exact knowledge of bearing angles θ_k and observer locations $\boldsymbol{r}_k$. The objective of bearings-only target tracking is to estimate the target location $\boldsymbol{p}_k$ from N noisy bearing measurements taken at discrete-time instants $k \in \{0, 1, \ldots, N-1\}$. In Fig. 1, the relationship between the bearing angle, observer location and target location is given by the nonlinear equation

$$\theta_k = \tan^{-1} \frac{\Delta y_k}{\Delta x_k}, \quad k = 0, \ldots, N-1 \tag{1}$$

where $\Delta y_k = p_{y,k} - r_{y,k}$ and $\Delta x_k = p_{x,k} - r_{x,k}$ with $\boldsymbol{p}_k = [p_{x,k}, p_{y,k}]^T$ and $\boldsymbol{r}_k = [r_{x,k}, r_{y,k}]^T$ denoting the target location vector and the observer location vector at time instant k, respectively. Here the symbol ^{T} denotes the transpose operator.

Throughout the paper we make the following assumptions:

- The target moves with a constant acceleration. Constant-velocity and stationary targets are special cases with zero acceleration and zero velocity, respectively. Three commonly used target motion models are the nearly constant-velocity model, the nearly constant-acceleration model and the coordinated turn model [15]. The choice of the model depends on the bearing measurement rate. If the bearing measurement rate allows a sufficiently large number of bearing measurements to be collected during target maneuvers, then the constant-acceleration model is appropriate. For batch estimators such as the pseudolinear estimator, using the constant-velocity model would lead to large estimation errors during target maneuvers. However, if the bearing measurement rate is not large enough, it is better to use the constant-velocity model.
- The bearing and observer location measurements are taken at regular time instants $t_k = kT/N, k = 0, \ldots, N-1$, where T is the length of the bearing measurement time interval. Let $\boldsymbol{p}_0$ and $\boldsymbol{v}_0$ denote the initial target location vector and the initial velocity vector at $k = 0$, respectively, and $\boldsymbol{a}$ be the constant target acceleration vector. The target location at time $t_k, k = 0, 1, \ldots, N-1$, is then given by the kinematic equation [16]

$$\boldsymbol{p}_k = \boldsymbol{p}_0 + t_k \boldsymbol{v}_0 + \frac{t_k^2}{2}\boldsymbol{a} \tag{2a}$$

$$= \boldsymbol{M}_k \boldsymbol{\xi} \tag{2b}$$

where

$$\boldsymbol{M}_k = \begin{bmatrix} 1 & 0 & t_k & 0 & \frac{1}{2}t_k^2 & 0 \\ 0 & 1 & 0 & t_k & 0 & \frac{1}{2}t_k^2 \end{bmatrix}$$

and

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{v}_0 \\ \boldsymbol{a} \end{bmatrix}$$

is the 6×1 *target motion parameter* vector to be estimated. Given an estimate of $\boldsymbol{\xi}$, the estimated target locations can be obtained from (2) by simple substitution. For constant-velocity targets, (2) becomes

$$\boldsymbol{p}_k = \begin{bmatrix} 1 & 0 & t_k & 0 \\ 0 & 1 & 0 & t_k \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{v}_0 \end{bmatrix}. \tag{3}$$

- The bearing measurements are corrupted by independent zero-mean Gaussian noise

$$\tilde{\theta}_k = \theta_k + n_k, \quad n_k \sim \mathcal{N}\left(0, \sigma_n^2(k)\right) \tag{4}$$

where the $\tilde{\theta}_k, k = 0, \ldots, N-1$, are the bearing measurements and n_k is a Gaussian random variable with zero mean and variance $\sigma_n^2(k)$. The bearing noise variance can vary with k and is assumed to be known *a priori* for the purpose of bias compensation.
- The target is observable. This requires the observer to maneuver while collecting bearing measurements [17]–[19].

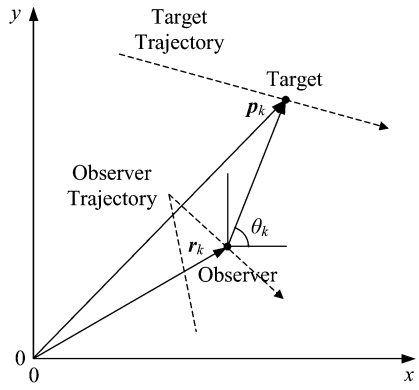


Fig. 1. Two-dimensional bearings-only target tracking geometry.

III. Overview of ML and Pseudolinear Estimation Algorithms

A. ML Estimator

The likelihood function for the bearing measurements is a multivariate Gaussian probability density function given by

$$p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\xi}) = \frac{1}{(2\pi)^{N/2}|\boldsymbol{K}|^{1/2}} \times \exp\left\{-\frac{1}{2}(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}(\boldsymbol{\xi}))^T\boldsymbol{K}^{-1}(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}(\boldsymbol{\xi}))\right\}$$

where

$$\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_0, \tilde{\theta}_2, \ldots, \tilde{\theta}_{N-1}]^T$$

is the $N \times 1$ vector of noisy bearing measurements

$$\boldsymbol{\theta}(\boldsymbol{\xi}) = \begin{bmatrix} \theta_0(\boldsymbol{\xi}) \\ \theta_1(\boldsymbol{\xi}) \\ \vdots \\ \theta_{N-1}(\boldsymbol{\xi}) \end{bmatrix} = \begin{bmatrix} \angle(\boldsymbol{M}_0\boldsymbol{\xi}-\boldsymbol{r}_0) \\ \angle(\boldsymbol{M}_1\boldsymbol{\xi}-\boldsymbol{r}_1) \\ \vdots \\ \angle(\boldsymbol{M}_{N-1}\boldsymbol{\xi}-\boldsymbol{r}_{N-1}) \end{bmatrix}$$

is the $N \times 1$ vector of bearing angles as a function of $\boldsymbol{\xi}$ with $\angle$ denoting the bearing angle, i.e., for $\boldsymbol{x} = [x_1, x_2]^T$

$$\angle\boldsymbol{x} = \tan^{-1}\frac{x_2}{x_1}$$

$\boldsymbol{K} = \mathrm{diag}(\sigma_n^2(0), \ldots, \sigma_n^2(N-1))$ is the $N \times N$ diagonal covariance matrix of the bearing measurement noise, and $|\boldsymbol{K}|$ is the determinant of $\boldsymbol{K}$.

The ML estimate of the target motion parameters $\hat{\boldsymbol{\xi}}_{\mathrm{ML}}$ is obtained from maximization of the log-likelihood function $\ln p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\xi})$ over $\boldsymbol{\xi}$, which can be equivalently written as

$$\hat{\boldsymbol{\xi}}_{\mathrm{ML}} = \underset{\boldsymbol{\xi}\in\mathbb{R}^6}{\arg\min}\, J_{\mathrm{ML}}(\boldsymbol{\xi}) \tag{5}$$

where $J_{\mathrm{ML}}(\boldsymbol{\xi})$ is the ML cost function

$$J_{\mathrm{ML}}(\boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{e}^T(\boldsymbol{\xi})\boldsymbol{K}^{-1}\boldsymbol{e}(\boldsymbol{\xi}), \quad \boldsymbol{e}(\boldsymbol{\xi}) = \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi}). \tag{6}$$

The minimization of $J_{\mathrm{ML}}(\boldsymbol{\xi})$ over $\boldsymbol{\xi}$ is a nonlinear least squares problem and does not have a closed-form solution. A numerical solution can be obtained by employing gradient or downhill simplex-based iterative search algorithms such as the steepest descent algorithm [20], the Gauss-Newton algorithm [21] or the Nelder-Mead simplex algorithm [22]. The GN algorithm for computing the ML estimate is given by

$$\hat{\boldsymbol{\xi}}(i+1) = \hat{\boldsymbol{\xi}}(i) - (\boldsymbol{J}^T(i)\boldsymbol{K}^{-1}\boldsymbol{J}(i))^{-1}\boldsymbol{J}^T(i)\boldsymbol{K}^{-1}\boldsymbol{e}(\hat{\boldsymbol{\xi}}(i)), \quad i = 0, 1, \ldots \tag{7}$$

where $\boldsymbol{J}(i)$ is the $N \times 6$ Jacobian matrix of $\boldsymbol{e}(\boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$ evaluated at $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(i)$:

$$\boldsymbol{J}(i) = \begin{bmatrix} \frac{[\sin\theta_0(\hat{\boldsymbol{\xi}}(i)), -\cos\theta_0(\hat{\boldsymbol{\xi}}(i))]\boldsymbol{M}_0}{\|\boldsymbol{M}_0\hat{\boldsymbol{\xi}}(i)-\boldsymbol{r}_0\|_2} \\ \frac{[\sin\theta_1(\hat{\boldsymbol{\xi}}(i)), -\cos\theta_1(\hat{\boldsymbol{\xi}}(i))]\boldsymbol{M}_1}{\|\boldsymbol{M}_1\hat{\boldsymbol{\xi}}(i)-\boldsymbol{r}_1\|_2} \\ \vdots \\ \frac{[\sin\theta_{N-1}(\hat{\boldsymbol{\xi}}(i)), -\cos\theta_{N-1}(\hat{\boldsymbol{\xi}}(i))]\boldsymbol{M}_{N-1}}{\|\boldsymbol{M}_{N-1}\hat{\boldsymbol{\xi}}(i)-\boldsymbol{r}_{N-1}\|_2} \end{bmatrix}.$$

Here $\|\cdot\|_2$ denotes the Euclidean norm. The GN algorithm is stopped at iteration i_f if

$$\|(\boldsymbol{J}^T(i_f)\boldsymbol{K}^{-1}\boldsymbol{J}(i_f))^{-1}\boldsymbol{J}^T(i_f)\boldsymbol{K}^{-1}\boldsymbol{e}(\hat{\boldsymbol{\xi}}(i_f))\|_2 < \epsilon \tag{8}$$

where ϵ is a threshold.

Iterative search algorithms must be initialized appropriately in order to avoid divergence problems. Despite its convergence problems and large computational complexity, the ML estimator is asymptotically unbiased and efficient (i.e., it approaches the Cramer-Rao lower bound as $N \to \infty$).

B. Pseudolinear Estimator

Rewriting (1) and (4) as

$$\frac{\sin(\tilde{\theta}_k - n_k)}{\cos(\tilde{\theta}_k - n_k)} = \frac{\Delta y_k}{\Delta x_k}$$

and lumping the noise terms together, we get [6]

$$\boldsymbol{F}\boldsymbol{\xi} = \boldsymbol{b} + \boldsymbol{\eta} \tag{9}$$

where

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{u}_0^T\boldsymbol{M}_0 \\ \boldsymbol{u}_1^T\boldsymbol{M}_1 \\ \vdots \\ \boldsymbol{u}_{N-1}^T\boldsymbol{M}_{N-1} \end{bmatrix}_{N\times 6}, \quad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{u}_0^T\boldsymbol{r}_0 \\ \boldsymbol{u}_1^T\boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{u}_{N-1}^T\boldsymbol{r}_{N-1} \end{bmatrix}_{N\times 1},$$

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{N-1} \end{bmatrix}_{N\times 1}, \quad \boldsymbol{u}_k = \begin{bmatrix} \sin\tilde{\theta}_k \\ -\cos\tilde{\theta}_k \end{bmatrix}.$$

A least squares solution to (9) is given by

$$\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} = \underset{\boldsymbol{\xi}\in\mathbb{R}^6}{\arg\min}\, \|\boldsymbol{F}\boldsymbol{\xi} - \boldsymbol{b}\|_2^2 \tag{10a}$$

$$= (\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{b} \tag{10b}$$

which is referred to as the pseudolinear estimator [4] and is also known as the orthogonal vector estimator in the context of target localization [16]. The target is observable if the Gram matrix $\boldsymbol{F}^T\boldsymbol{F}$ is nonsingular.

As we shall see in the next section, the pseudolinear estimator is biased even for $N \to \infty$. The error covariance matrix of the pseudolinear estimator is

$$\boldsymbol{C}_{\mathrm{PLE}} = E\{(\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} - \boldsymbol{\xi})(\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} - \boldsymbol{\xi})^T\} \tag{11a}$$

$$= E\{(\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{\eta}\boldsymbol{\eta}^T\boldsymbol{F}(\boldsymbol{F}^T\boldsymbol{F})^{-1}\}. \tag{11b}$$

The mean-squared error (MSE) of the pseudolinear estimator is given by the trace of $\boldsymbol{C}_{\mathrm{PLE}}$:

$$E\left\{\|\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} - \boldsymbol{\xi}\|_2^2\right\} = \mathrm{tr}\boldsymbol{C}_{\mathrm{PLE}}.$$

IV. Bias of the Pseudolinear Estimator

Substituting (9) into (10b), the pseudolinear estimate for target motion parameters can be written as

$$\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} = \boldsymbol{\xi} - (\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{\eta}. \tag{12}$$

For finite N, the bias of the pseudolinear estimator is

$$\boldsymbol{\delta} = E\{\hat{\boldsymbol{\xi}}_{\mathrm{PLE}}\} - \boldsymbol{\xi} = -E\{(\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{\eta}\}. \tag{13}$$

For large N, the pseudolinear estimation bias can be approximated by [23]

$$\boldsymbol{\delta} \approx -E\left\{\frac{\boldsymbol{F}^T\boldsymbol{F}}{N}\right\}^{-1} E\left\{\frac{\boldsymbol{F}^T\boldsymbol{\eta}}{N}\right\}. \tag{14}$$

This result also follows from Slutsky's theorem [24], [10]. We will use (14) in the bias analysis under the tacit assumption of sufficiently large N. Equation (14) reveals that the chief reason for the pseudolinear estimation bias is the correlation between the entries of the measurement matrix $\boldsymbol{F}$ and the least squares noise vector $\boldsymbol{\eta}$. In what follows, we provide a bias analysis for the pseudolinear estimator based on the approach employed in [25].

First, let us write $\boldsymbol{F}^T\boldsymbol{\eta}$ in terms of the noiseless bearing angles θ_k and the bearing noise n_k

$$\boldsymbol{F}^T\boldsymbol{\eta} = \sum_{k=0}^{N-1} \eta_k \boldsymbol{M}_k^T \boldsymbol{u}_k \tag{15a}$$

$$= \sum_{k=0}^{N-1} d_k \boldsymbol{M}_k^T \begin{bmatrix} \frac{1}{2}\sin\theta_k \sin 2n_k + \cos\theta_k \sin^2 n_k \\ -\frac{1}{2}\cos\theta_k \sin 2n_k + \sin\theta_k \sin^2 n_k \end{bmatrix} \tag{15b}$$

where $d_k = \|\boldsymbol{p}_k - \boldsymbol{r}_k\|_2$ is the target range at time instant k. Taking the expectation gives

$$E\left\{\frac{\boldsymbol{F}^T\boldsymbol{\eta}}{N}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} \mu_k^2 \boldsymbol{M}_k^T(\boldsymbol{p}_k - \boldsymbol{r}_k) \tag{16}$$

where

$$\mu_k^2 = E\{\sin^2 n_k\}. \tag{17}$$

The autocorrelation matrix $E\{\boldsymbol{F}^T\boldsymbol{F}/N\}$ can be written as

$$E\left\{\frac{\boldsymbol{F}^T\boldsymbol{F}}{N}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} \boldsymbol{M}_k^T E\{\boldsymbol{u}_k\boldsymbol{u}_k^T\}\boldsymbol{M}_k \tag{18}$$

where

$$E\left\{\boldsymbol{u}_k\boldsymbol{u}_k^T\right\} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \tag{19}$$

with

$$\begin{aligned} u_{11} &= \left(1-\mu_k^2\right)\sin^2\theta_k + \mu_k^2\cos^2\theta_k \\ u_{12} &= -\frac{1}{2}\left(1-2\mu_k^2\right)\sin 2\theta_k \\ u_{21} &= -\frac{1}{2}\left(1-2\mu_k^2\right)\sin 2\theta_k \\ u_{22} &= \mu_k^2\sin^2\theta_k + \left(1-\mu_k^2\right)\cos^2\theta_k. \end{aligned}$$

Here we have used $E\{\cos^2 n_k\} = 1 - \mu_k^2$ and $E\{\cos 2n_k\} = 1 - 2\mu_k^2$. Substituting (19) into (18), we obtain

$$E\left\{\frac{\boldsymbol{F}^T\boldsymbol{F}}{N}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} \boldsymbol{M}_k^T \left(\begin{bmatrix} \sin^2\theta_k & -\frac{1}{2}\sin 2\theta_k \\ -\frac{1}{2}\sin 2\theta_k & \cos^2\theta_k \end{bmatrix} + \mu_k^2 \begin{bmatrix} \cos 2\theta_k & \sin 2\theta_k \\ \sin 2\theta_k & -\cos 2\theta_k \end{bmatrix}\right) \boldsymbol{M}_k. \tag{20}$$

Substituting (16) and (20) into (14) gives the estimation bias for the pseudolinear estimator in terms of the tracking geometry parameters and bearing noise variance.

We see from the above analysis that the bias is dependent not only on the correlation vector $E\{\boldsymbol{F}^T\boldsymbol{\eta}/N\}$, but also on the inverse of the autocorrelation matrix $E\{\boldsymbol{F}^T\boldsymbol{F}/N\}$. To ensure a small bias, the correlation vector must be small and the autocorrelation matrix must be well-conditioned, as required by target observability. Equation (16) shows that the bias is proportional to the bearing noise variance through μ_k^2. We also observe that the correlation between the entries of $\boldsymbol{F}$ and $\boldsymbol{\eta}$ does not vanish with increasing N, which implies an asymptotically nonvanishing bias for the pseudolinear estimator as $N \to \infty$.

V. Bias Compensation for the Pseudolinear Estimator

Central to the concept of bias compensation is the estimation of the instantaneous bias from the available measurements. Using (12), the instantaneous bias of the pseudolinear estimator is given by

$$\boldsymbol{\delta}_i = -(\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{\eta}. \tag{21}$$

Because $\boldsymbol{\eta}$ is unavailable, $\boldsymbol{\delta}_i$ cannot be calculated. However, for sufficiently large N, the instantaneous bias can be approximated by

$$\hat{\boldsymbol{\delta}}_i = -(\boldsymbol{F}^T\boldsymbol{F})^{-1}E\{\boldsymbol{F}^T\boldsymbol{\eta}\} \tag{22a}$$

$$= -(\boldsymbol{F}^T\boldsymbol{F})^{-1}\sum_{k=0}^{N-1}\mu_k^2\boldsymbol{M}_k^T(\boldsymbol{p}_k - \boldsymbol{r}_k) \tag{22b}$$

where we have used (16). The approximate instantaneous bias $\hat{\boldsymbol{\delta}}_i$ cannot be calculated either because knowledge of the target locations $\boldsymbol{p}_k$ is not available. An estimate of $\hat{\boldsymbol{\delta}}_i$ can be obtained from the pseudolinear estimate $\hat{\boldsymbol{\xi}}_{\mathrm{PLE}}$ by replacing $\boldsymbol{p}_k$ with $\boldsymbol{M}_k\hat{\boldsymbol{\xi}}_{\mathrm{PLE}}$ in (22b). This results in

$$\hat{\hat{\boldsymbol{\delta}}}_i = -(\boldsymbol{F}^T\boldsymbol{F})^{-1}\sum_{k=0}^{N-1}\mu_k^2\boldsymbol{M}_k^T(\boldsymbol{M}_k\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} - \boldsymbol{r}_k). \tag{23}$$

For large N, the mean of the approximate instantaneous bias is

$$E\{\hat{\boldsymbol{\delta}}_i\} \approx -E\{\boldsymbol{F}^T\boldsymbol{F}\}^{-1}\sum_{k=0}^{N-1}\mu_k^2\boldsymbol{M}_k^T(\boldsymbol{p}_k - \boldsymbol{r}_k) \tag{24a}$$

$$\approx \boldsymbol{\delta} \tag{24b}$$

and the mean of the instantaneous bias estimate is

$$E\{\hat{\hat{\boldsymbol{\delta}}}_i\} \approx -E\{\boldsymbol{F}^T\boldsymbol{F}\}^{-1}\sum_{k=0}^{N-1}\mu_k^2\boldsymbol{M}_k^T(\boldsymbol{M}_k(\boldsymbol{\xi}+\boldsymbol{\delta}) - \boldsymbol{r}_k) \tag{25a}$$

$$\approx (\boldsymbol{I} - E\{\boldsymbol{F}^T\boldsymbol{F}\}^{-1}\boldsymbol{Q})\boldsymbol{\delta} \tag{25b}$$

where

$$\boldsymbol{Q} = \sum_{k=0}^{N-1}\mu_k^2\boldsymbol{M}_k^T\boldsymbol{M}_k. \tag{26}$$

Unlike the approximate instantaneous bias, the instantaneous bias estimate is asymptotically biased. The instantaneous bias estimate will have a small bias if

- the target tracking geometry is such that we have $E\{\boldsymbol{F}^T\boldsymbol{F}\}^{-1}\boldsymbol{Q} \approx \gamma\boldsymbol{I}$ with $0 \leq \gamma \ll 1$, or
- $\mu_k^2 \approx 0$.

The bias compensated pseudolinear estimator is obtained by subtracting the estimated bias from the pseudolinear estimator:

$$\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}} = \hat{\boldsymbol{\xi}}_{\mathrm{PLE}} - \hat{\hat{\boldsymbol{\delta}}}_i \tag{27a}$$

$$= \hat{\boldsymbol{\xi}}_{\mathrm{PLE}} + (\boldsymbol{F}^T\boldsymbol{F})^{-1}(\boldsymbol{Q}\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} - \boldsymbol{z}) \tag{27b}$$

where

$$\boldsymbol{z} = \sum_{k=0}^{N-1} \mu_k^2 \boldsymbol{M}_k^T \boldsymbol{r}_k. \tag{28}$$

The estimator $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ given in (27b) is not unbiased. However, it has a reduced bias compared with the pseudolinear estimator $\hat{\boldsymbol{\xi}}_{\mathrm{PLE}}$. The bias of $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ is

$$E\{\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}\} - \boldsymbol{\xi} \approx \boldsymbol{\delta} - E\{\hat{\hat{\boldsymbol{\delta}}}_i\} \tag{29a}$$

$$\approx E\{\boldsymbol{F}^T\boldsymbol{F}\}^{-1}\boldsymbol{Q}\boldsymbol{\delta} \tag{29b}$$

The bias reduction can be measured in terms of the bias norm $\|\boldsymbol{\delta} - E\{\hat{\hat{\boldsymbol{\delta}}}_i\}\|_2$. A reduced bias refers to having

$$\|\boldsymbol{\delta} - E\{\hat{\hat{\boldsymbol{\delta}}}_i\}\|_2 < \|\boldsymbol{\delta}\|_2. \tag{30}$$

For small bearing noise variances, we have $\boldsymbol{Q} \approx \boldsymbol{0}$ or $E\{\hat{\hat{\boldsymbol{\delta}}}_i\} \approx \boldsymbol{\delta}$, which satisfies (30). However, as the bearing noise variance increases the approximation $\boldsymbol{Q} \approx \boldsymbol{0}$ ceases to hold. In Section VIII we will show by simulation that (30) holds for a range of bearing noise variances, implying that the bias of $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ is indeed smaller than that of $\hat{\boldsymbol{\xi}}_{\mathrm{PLE}}$.

The error covariance matrix of $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ is given by

$$\boldsymbol{C}_{\mathrm{BCPLE}} = E\{(\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}} - \boldsymbol{\xi})(\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}} - \boldsymbol{\xi})^T\} \tag{31a}$$

$$= \boldsymbol{C}_{\mathrm{PLE}} - E\left\{\boldsymbol{\delta}_i\hat{\hat{\boldsymbol{\delta}}}_i^T\right\} - E\left\{\hat{\hat{\boldsymbol{\delta}}}_i\boldsymbol{\delta}_i^T\right\} + E\left\{\hat{\hat{\boldsymbol{\delta}}}_i\hat{\hat{\boldsymbol{\delta}}}_i^T\right\} \tag{31b}$$

$$\approx \boldsymbol{C}_{\mathrm{PLE}} - E\left\{\boldsymbol{\delta}_i\boldsymbol{\delta}_i^T\right\}. \tag{31c}$$

The approximation in (31c) is valid if $\hat{\hat{\boldsymbol{\delta}}}_i \approx \boldsymbol{\delta}_i$. We expect $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ to have a smaller MSE than $\hat{\boldsymbol{\xi}}_{\mathrm{PLE}}$ because of bias reduction. Given its favorable mean and MSE performance, the bias compensated pseudolinear estimator can be used in preference to the pseudolinear estimator for initializing the GN algorithm in (7).

VI. Asymptotically Unbiased Instrumental Variable Estimator

The bias compensation method described in the previous section does not lead to an unbiased estimator although it can reduce the bias significantly depending on the bearing noise variance and the tracking geometry. In this section we develop an IV estimator by utilizing the bias compensated pseudolinear estimator in (27b), producing an asymptotically unbiased estimator. We also construct a weighting matrix from the bias compensated pseudolinear estimator to reduce the estimation variance, leading to a weighted IV estimator. The pseudolinear estimator is obtained by solving the normal equations

$$\boldsymbol{F}^T\boldsymbol{F}\hat{\boldsymbol{\xi}}_{\mathrm{PLE}} = \boldsymbol{F}^T\boldsymbol{b}. \tag{32}$$

For observable targets with nonsingular $\boldsymbol{F}^T\boldsymbol{F}$, the estimation bias arises from the correlation between $\boldsymbol{F}$ and $\boldsymbol{\eta}$. This correlation can be eliminated by modifying the normal equations to

$$\boldsymbol{G}^T\boldsymbol{F}\hat{\boldsymbol{\xi}}_{\mathrm{IV}} = \boldsymbol{G}^T\boldsymbol{b} \tag{33}$$

where $\hat{\boldsymbol{\xi}}_{\mathrm{IV}}$ is the IV estimator

$$\hat{\boldsymbol{\xi}}_{\mathrm{IV}} = (\boldsymbol{G}^T\boldsymbol{F})^{-1}\boldsymbol{G}^T\boldsymbol{b}. \tag{34}$$

Here $\boldsymbol{G}$ is called the IV matrix [26]. If $\boldsymbol{G}$ is selected such that $E\{\boldsymbol{G}^T\boldsymbol{F}/N\}$ is nonsingular and $E\{\boldsymbol{G}^T\boldsymbol{\eta}/N\} = \boldsymbol{0}$, then $\hat{\boldsymbol{\xi}}_{\mathrm{IV}}$ will be asymptotically unbiased, i.e., $E\{\hat{\boldsymbol{\xi}}_{\mathrm{IV}}\} = \boldsymbol{\xi}$ as $N \to \infty$.

In general the selection of $\boldsymbol{G}$ is problem-dependent [27]. Assuming that the η_k are i.i.d., the optimal IV matrix is given by the noiseless version of the measurement matrix $\boldsymbol{F}$ [28]

$$\boldsymbol{F}_o = \begin{bmatrix} \boldsymbol{\nu}_0^T\boldsymbol{M}_0 \\ \boldsymbol{\nu}_1^T\boldsymbol{M}_1 \\ \vdots \\ \boldsymbol{\nu}_{N-1}^T\boldsymbol{M}_{N-1} \end{bmatrix}, \quad \boldsymbol{\nu}_k = \begin{bmatrix} \sin\theta_k \\ -\cos\theta_k \end{bmatrix}. \tag{35}$$

The optimal IV matrix is not available since we only have access to the noisy measurements of the bearing angles. A simple and appealing method for constructing the IV matrix $\boldsymbol{G}$ is based on bearing angle estimation from the bias compensated pseudolinear estimator. Using $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ defined in (27b), the bearing angle estimates are given by

$$\hat{\theta}_k = \angle(\hat{\boldsymbol{p}}_k - \boldsymbol{r}_k), \quad k = 0, \ldots, N-1 \tag{36}$$

where

$$\hat{\boldsymbol{p}}_k = \boldsymbol{M}_k\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}, \quad k = 0, \ldots, N-1. \tag{37}$$

Using the $\hat{\theta}_k$ obtained above, the IV matrix can be constructed as

$$\boldsymbol{G} = \begin{bmatrix} \hat{\boldsymbol{\nu}}_0^T\boldsymbol{M}_0 \\ \hat{\boldsymbol{\nu}}_1^T\boldsymbol{M}_1 \\ \vdots \\ \hat{\boldsymbol{\nu}}_{N-1}^T\boldsymbol{M}_{N-1} \end{bmatrix}, \quad \hat{\boldsymbol{\nu}}_k = \begin{bmatrix} \sin\hat{\theta}_k \\ -\cos\hat{\theta}_k \end{bmatrix}. \tag{38}$$

It is desirable for $\boldsymbol{G}$ to be close to $\boldsymbol{F}_o$ so that it correlates well with $\boldsymbol{F}$, thereby avoiding an ill-conditioned correlation matrix $E\{\boldsymbol{G}^T\boldsymbol{F}/N\}$. This requires the use of a target motion parameter estimate with small bias. In order to satisfy this requirement, we have used the bias compensated pseudolinear estimator instead of the pseudolinear estimator in the construction of $\boldsymbol{G}$ above.

The bias compensated pseudolinear estimator $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ is correlated with the noise vector $\boldsymbol{\eta}$ [see (12)]. However, the correlation between $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ and $\boldsymbol{\eta}$ vanishes as $N \to \infty$ since the covariance of $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ tends to zero as $N \to \infty$. Thus the IV

matrix $\boldsymbol{G}$, which is obtained from $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$, is also asymptotically uncorrelated with $\boldsymbol{\eta}$, i.e.,

$$E\left\{\frac{\boldsymbol{G}^T\boldsymbol{\eta}}{N}\right\}=\boldsymbol{0},\quad \text{as } N\to\infty. \tag{39}$$

For finite and sufficiently large N, the above limiting result can be approximated by $E\{\boldsymbol{G}^T\boldsymbol{\eta}/N\}\approx\boldsymbol{0}$ [29].

The correlation matrix can be written as

$$E\left\{\frac{\boldsymbol{G}^T\boldsymbol{F}}{N}\right\}=\frac{1}{N}\sum_{k=0}^{N-1}\boldsymbol{M}_k^T E\{\hat{\boldsymbol{\nu}}_k\boldsymbol{u}_k^T\}\boldsymbol{M}_k. \tag{40}$$

A comparison of $E\{\boldsymbol{G}^T\boldsymbol{F}/N\}$ and $E\{\boldsymbol{F}^T\boldsymbol{F}/N\}$ in (18) reveals that $E\{\boldsymbol{G}^T\boldsymbol{F}/N\}$ will be nonsingular if $\boldsymbol{F}^T\boldsymbol{F}$ is nonsingular, i.e., the target is observable, and $\boldsymbol{G}$ is sufficiently close to $\boldsymbol{F}_o$. If the target is unobservable, $\boldsymbol{F}$ will be rank-deficient, making $E\{\boldsymbol{G}^T\boldsymbol{F}/N\}$ singular [30]. Therefore, the observability criterion for the IV estimator is that $\boldsymbol{G}^T\boldsymbol{F}$ be nonsingular.

Because the range between the observer and the target is time-varying, the η_k would not have identical variances even for i.i.d. bearing noise. A weighting matrix that normalizes the variances of individual noise terms η_k can be introduced in order to reduce the estimation variance, as follows:

$$\boldsymbol{W}=\mathrm{diag}(\hat{d}_0^2\mu_0^2,\hat{d}_1^2\mu_1^2,\ldots,\hat{d}_{N-1}^2\mu_{N-1}^2). \tag{41}$$

Here $\hat{d}_k$ is the range estimate obtained from the bias compensated pseudolinear estimator

$$\hat{d}_k=\|\hat{\boldsymbol{p}}_k-\boldsymbol{r}_k\|_2,\quad k=0,\ldots,N-1. \tag{42}$$

The weighted IV (WIV) estimator is given by

$$\hat{\boldsymbol{\xi}}_{\mathrm{WIV}}=(\boldsymbol{G}^T\boldsymbol{W}^{-1}\boldsymbol{F})^{-1}\boldsymbol{G}^T\boldsymbol{W}^{-1}\boldsymbol{b}. \tag{43}$$

A summary of the WIV algorithm is provided in Table I. The error covariance matrix of the WIV estimator is

$$\begin{aligned}\boldsymbol{C}_{\mathrm{WIV}}&=E\{(\hat{\boldsymbol{\xi}}_{\mathrm{WIV}}-\boldsymbol{\xi})(\hat{\boldsymbol{\xi}}_{\mathrm{WIV}}-\boldsymbol{\xi})^T\} &\text{(44a)}\\ &=E\{(\boldsymbol{G}^T\boldsymbol{W}^{-1}\boldsymbol{F})^{-1}\boldsymbol{G}^T\boldsymbol{W}^{-1}\boldsymbol{\eta}\\ &\quad\times\boldsymbol{\eta}^T\boldsymbol{W}^{-1}\boldsymbol{G}(\boldsymbol{F}^T\boldsymbol{W}^{-1}\boldsymbol{G})^{-1}\}. &\text{(44b)}\end{aligned}$$

TABLE I
WIV ALGORITHM

Given

$\tilde{\theta}_k,\ \boldsymbol{r}_k,\ \mu_k^2,\quad k=0,\ldots,N-1$

Compute

- $\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}}$ (Equation (27b))
- $\hat{\boldsymbol{p}}_k=\boldsymbol{M}_k\hat{\boldsymbol{\xi}}_{\mathrm{BCPLE}},\quad k=0,\ldots,N-1$
- $\hat{\theta}_k=\angle(\hat{\boldsymbol{p}}_k-\boldsymbol{r}_k),\quad k=0,\ldots,N-1$
- $\hat{d}_k^2=\|\hat{\boldsymbol{p}}_k-\boldsymbol{r}_k\|_2^2,\quad k=0,\ldots,N-1$
- $\hat{\boldsymbol{\nu}}_k=\begin{bmatrix}\sin\hat{\theta}_k\\ -\cos\hat{\theta}_k\end{bmatrix},\quad k=0,\ldots,N-1$
- $\boldsymbol{G}=\begin{bmatrix}\hat{\boldsymbol{\nu}}_0^T\boldsymbol{M}_0\\ \vdots\\ \hat{\boldsymbol{\nu}}_{N-1}^T\boldsymbol{M}_{N-1}\end{bmatrix}$
- $\boldsymbol{W}=\mathrm{diag}(\hat{d}_0^2\mu_0^2,\ldots,\hat{d}_{N-1}^2\mu_{N-1}^2)$
- $\hat{\boldsymbol{\xi}}_{\mathrm{WIV}}=(\boldsymbol{G}^T\boldsymbol{W}^{-1}\boldsymbol{F})^{-1}\boldsymbol{G}^T\boldsymbol{W}^{-1}\boldsymbol{b}$

TABLE II
COMPARISON OF COMPUTATIONAL COMPLEXITIES

Algorithm	Muls	Divs	Mat. Invs	Roots
PLE	$48N$	–	1	–
BCPLE	$52N$	–	1	–
WIV	$110N$	N	2	–
GN	$52N$ $+64N(i_f+1)$	$N(i_f+1)$	i_f+2	$N(i_f+1)$

VII. COMPUTATIONAL COMPLEXITY

In this section, we provide a detailed comparison of the computational requirements for the pseudolinear estimator, the bias compensated pseudolinear estimator, the WIV estimator, and the GN algorithm. The computational complexity analysis presented here assumes the use of the constant-acceleration target motion model.

Given N bearing measurements and observer locations, the total computational complexity of the pseudolinear estimator is $48N+36$ multiplications and one 6×6 matrix inversion. Referring to (27b), the bias compensated pseudolinear estimator can be seen to require an additional $4N+72$ multiplications. The total computational complexity of the bias compensated pseudolinear estimator (including that of the pseudolinear estimator) is $52N+108$ multiplications and one 6×6 matrix inversion. Here we assume that $\boldsymbol{Q}$ has been precalculated since it does not depend on the angle measurements or the observer locations, thereby requiring no computational effort. If the observer locations were fixed, the same assumption would also apply to $\boldsymbol{z}$, reducing the total computational complexity by $4N$ multiplications.

The total computational complexity of the WIV estimator is $110N+144$ multiplications, N divisions, and two 6×6 matrix inversions, which includes the computational complexity of the bias compensated pseudolinear estimator. The total computational complexity of the GN algorithm per iteration is $64N+38$ multiplications, N divisions, one 6×6 matrix inversion and N square root operations. The computational complexity for obtaining an initialization for the GN algorithm must also be added to the total computation complexity of the GN algorithm.

For large N, the computational complexities of the tracking algorithms can be approximated by retaining the complexities dependent on N only. Table II summarizes the approximate computational complexities for the tracking algorithms. The GN algorithm is assumed to be initialized to the bias compensated pseudolinear estimate.

VIII. SIMULATION STUDIES

A. Constant-Acceleration Target

The simulated target tracking geometry for an accelerating target is shown in Fig. 2. The observer trajectory consists of

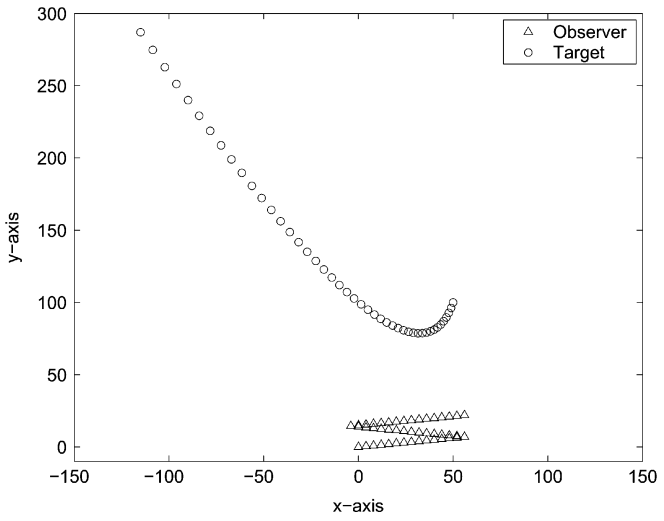


Fig. 2. Simulated tracking geometry for a constant-acceleration target.

three constant-velocity legs. This achieves the required observer maneuver to make the target observable. The target is assumed to accelerate with motion parameters $\boldsymbol{p}_0 = [50\ 100]^T, \boldsymbol{v}_0 = [-2, -8]^T$ and $\boldsymbol{a} = [-0.5, 1.5]^T$. The bearing measurements are taken at time instants $t_k = kT/N, k = 0, \ldots, N-1$, where $T = 22.5$. The observer collects $N = 45$ bearing measurements (15 measurements per leg) at marked locations shown in Fig. 2 starting from $\boldsymbol{r}_0 = [0, 0]^T$. The bearing measurements are subject to i.i.d. zero-mean Gaussian noise with a constant variance σ_n^2. The bias norm, MSE and the Cramer-Rao lower bound (CRLB) are used for performance comparison. The bias norm is defined as $\|E\{\hat{\boldsymbol{\xi}}\} - \boldsymbol{\xi}\|_2$, where $\hat{\boldsymbol{\xi}}$ is an estimate of $\boldsymbol{\xi}$, and the CRLB is given by $\sigma_n^2 \mathrm{tr}(\boldsymbol{F}_o^T \boldsymbol{W}_o^{-1} \boldsymbol{F}_o)^{-1}$ where $\boldsymbol{W}_o = \mathrm{diag}(d_0^2, \ldots, d_{N-1}^2)$ [4].

The ML estimator is implemented using the GN algorithm in (7) with the update norm threshold set to $\epsilon = 10^{-5}$. To have a fair comparison between the WIV and ML estimators, the GN algorithm is initialized to the bias compensated pseudolinear estimate. The single-iteration GN estimator, which has a computational complexity comparable to that of the WIV estimator, is also simulated for performance comparison purposes. The bias and MSE performance of the algorithms is estimated using 10 000 Monte Carlo simulation runs.

Fig. 3 shows the norm of bias estimates against the bearing noise standard deviation for $\sigma_n \in \{0.5^\circ, 0.8^\circ, 1.1^\circ, 1.4^\circ, 1.7^\circ, 2.0^\circ, 2.3^\circ\}$. The simulation results confirm the improved bias performance of the bias compensated pseudolinear estimator (BCPLE) compared with the pseudolinear estimator (PLE). The bias of the BCPLE increases with the bearing noise standard deviation as predicted in Section V. On the other hand, the best bias performance is achieved by the WIV and ML estimators. Note that the GN algorithm, which is used to compute the ML estimate, exhibits divergence at $\sigma_n = 2.3^\circ$. The average number of GN iterations for $\epsilon = 10^{-5}$ were 4.7, 5.4, 6.0, 6.5, 7.1, and 7.6 iterations for $\sigma_n = 0.5^\circ, 0.8^\circ, 1.1^\circ, 1.4^\circ, 1.7^\circ$, and 2.0°, respectively. While the bias of the WIV estimator is very small compared with that of the BCPLE and PLE, it is still larger than the bias of the ML estimator. The reason for this is the asymptotic unbiasedness of the WIV estimator, which

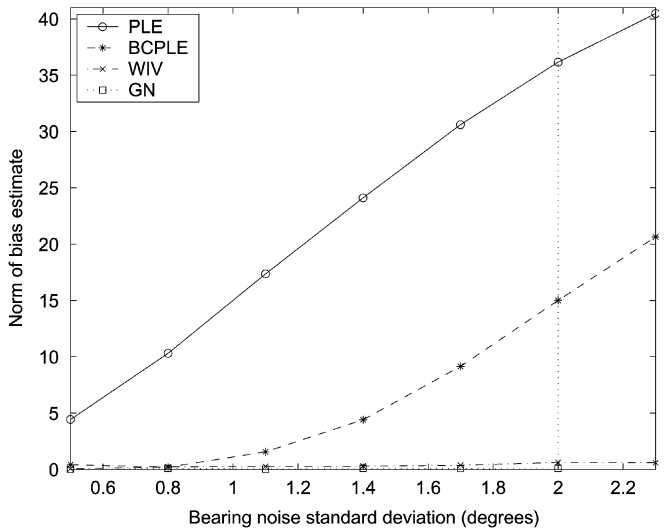


Fig. 3. Bias norm versus bearing noise standard deviation for the PLE, BCPLE, WIV, and GN. The GN algorithm diverges at $\sigma_n = 2.3^\circ$.

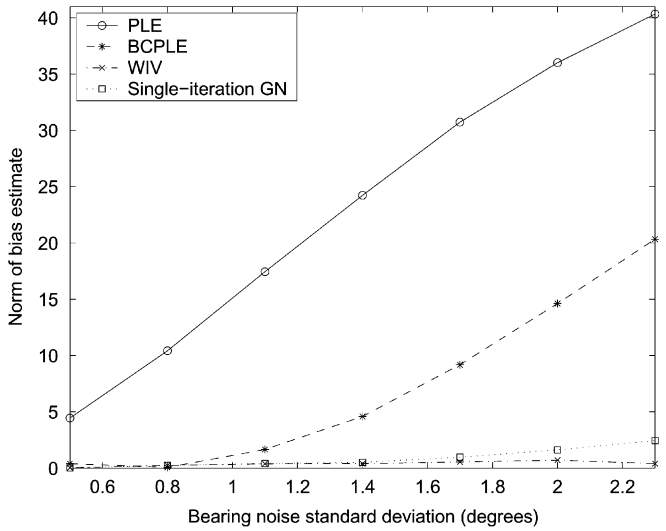


Fig. 4. Bias norm versus bearing noise standard deviation for the PLE, BCPLE, WIV, and single-iteration GN.

means that the WIV estimator will exhibit a small bias for finite N due to the correlation between the entries of $\boldsymbol{G}$ and $\boldsymbol{\eta}$. Recall that this correlation vanishes as N gets larger. The bias performance of the single-iteration GN estimator is shown in Fig. 4. Note that the bias of the single-iteration GN estimator exceeds that of the WIV estimator as the bearing standard deviation increases.

A plot of the estimated MSE versus the bearing noise standard deviation σ_n is shown in Fig. 5. As in the case of bias comparison, the BCPLE outperforms the PLE while the WIV and ML estimators attain the best MSE performance by getting very close to the CRLB. The WIV and ML estimators exhibit almost identical MSE performance for $\sigma_n \leq 1.7^\circ$ despite an almost five-fold increase in the computational complexity of the ML estimator at $\sigma_n = 1.7^\circ$. In contrast to the GN algorithm which diverges at $\sigma_n = 2.3^\circ$, the WIV algorithm appears to perform well at the same noise level.

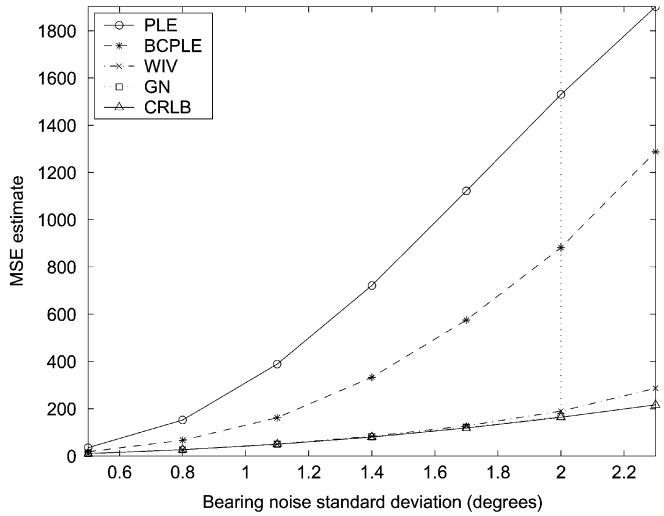


Fig. 5. MSE versus bearing noise standard deviation for the PLE, BCPLE, WIV, and GN. The GN algorithm diverges at $\sigma_n = 2.3^\circ$.

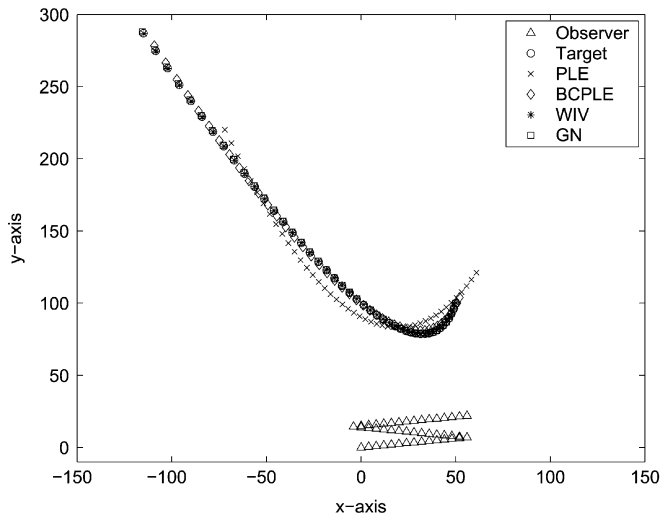


Fig. 6. Average target track estimates for the PLE, BCPLE, WIV, and GN at $\sigma_n = 1.4^\circ$.

The average target track estimates for $\sigma_n = 1.4^\circ$ are shown in Fig. 6. The bias of the PLE is visible from the large deviation of the average target track estimate from the true target track. The BCPLE yields a more accurate average target track estimate than the PLE while the most accurate target track estimates are those given by the WIV and ML estimators.

B. Constant-Velocity Target

The tracking algorithms were also applied to a constant-velocity target tracking problem which is shown in Fig. 7. The algorithms were modified according to (3) by replacing $\boldsymbol{M}_k$ with

$$\begin{bmatrix} 1 & 0 & t_k & 0 \\ 0 & 1 & 0 & t_k \end{bmatrix}$$

and $\boldsymbol{\xi}$ with $[\boldsymbol{p}_0, \boldsymbol{v}_0]^T$. The observer trajectory is the same as in Fig. 2. The target moves with a constant velocity $\boldsymbol{v}_0 = [2, -12]^T$ and its initial location is $\boldsymbol{p}_0 = [-25, 380]^T$.

Fig. 8 shows the norm of bias estimates against $\sigma_n \in \{0.4^\circ, 0.8^\circ, 1.2^\circ, 1.6^\circ, 2.0^\circ, 2.4^\circ\}$. The BCPLE outperforms

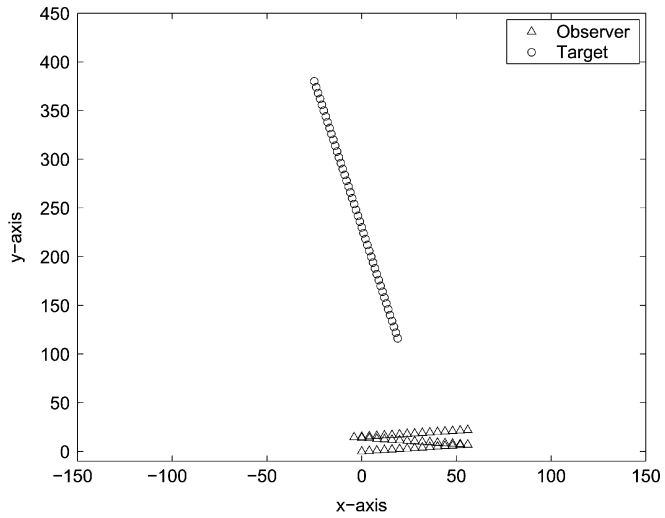


Fig. 7. Simulated tracking geometry for a constant-velocity target.

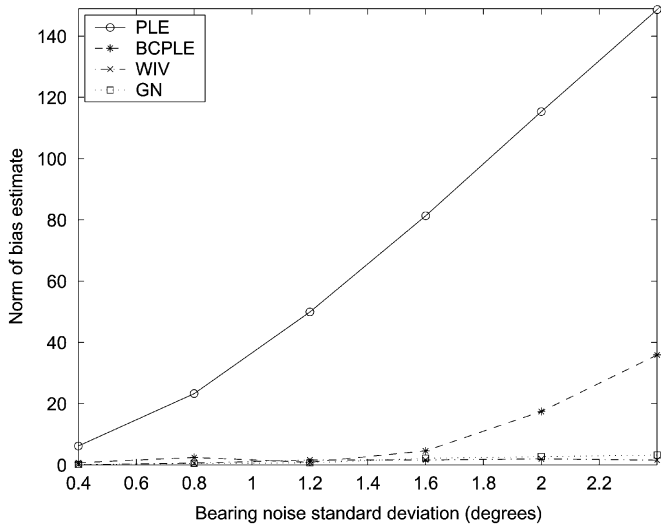


Fig. 8. Bias norm versus bearing noise standard deviation for the PLE, BCPLE, WIV, and GN.

the PLE while the smallest bias is obtained by the WIV and ML estimators. The average number of GN iterations for $\epsilon = 10^{-5}$ were 4.8, 5.8, 6.6, 7.4, 8.2, and 8.9 iterations for $\sigma_n = 0.4^\circ, 0.8^\circ, 1.2^\circ, 1.6^\circ, 2.0^\circ$ and 2.4°, respectively. The bias performance of the WIV and ML estimators is almost identical despite larger computational requirements of the GN algorithm. Fig. 9 shows the bias performance of the single-iteration GN estimator. The bias of the single-iteration GN estimator appears to exceed that of the WIV estimator by an appreciable margin for increased σ_n.

Fig. 10 shows a plot of the estimated MSE versus the bearing noise standard deviation σ_n. The MSE of the BCPLE is significantly smaller than that of the PLE while the WIV and ML estimators achieve the best MSE performance, almost equaling the CRLB.

IX. Summary and Conclusion

In this paper, we have studied four estimators for bearings-only target tracking: the pseudolinear estimator, the bias compensated pseudolinear estimator, the WIV estimator utilizing

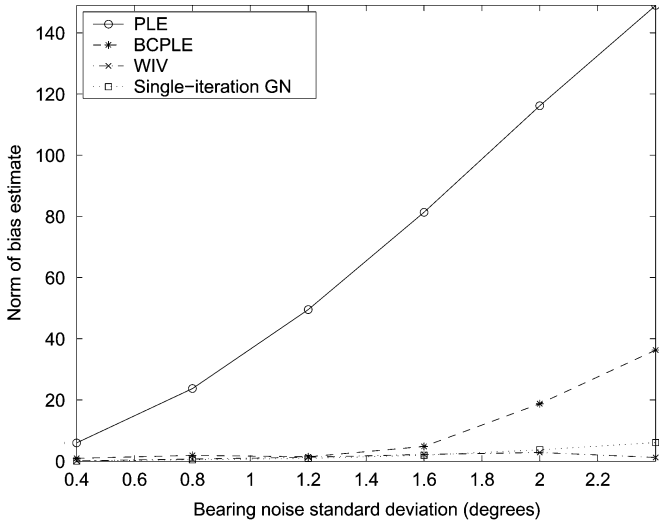


Fig. 9. Bias norm versus bearing noise standard deviation for the PLE, BCPLE, WIV, and single-iteration GN.

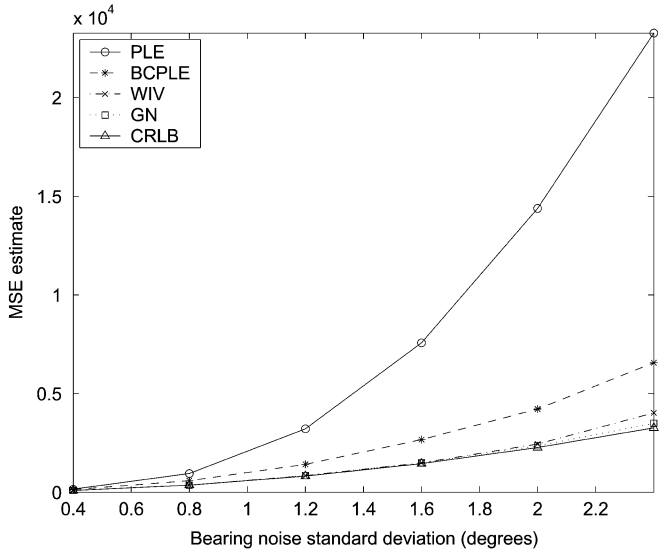


Fig. 10. MSE versus bearing noise standard deviation for the PLE, BCPLE, WIV, and GN.

the bias compensated pseudolinear estimator, and the ML estimator, listed in order of increasing computational complexity. The first three estimators are closed-form while the ML estimator is iterative with potential convergence problems. The first two estimators yield biased estimates with the bias compensated pseudolinear estimator having a smaller bias. The last two estimators are asymptotically unbiased.

The WIV estimator requires an estimate of the target motion parameters with asymptotically vanishing covariance matrix in order to construct an IV matrix. We used the bias compensated pseudolinear estimator for this purpose in view of its improved performance over the pseudolinear estimator. By way of computer simulations we showed that the WIV estimator achieves a performance very close to that of the computationally more demanding iterative ML estimator which is known to be asymptotically unbiased and efficient. Having a closed-form solution, the proposed WIV estimator avoids the convergence problems that plague the ML estimator as well as other iterative IV estimators. A major problem with iterative estimators is the nonconvex cost function topology, which necessitates a good initialization in order to avoid divergence.

In the development of the estimators we assumed a fairly general target motion model that allows tracking of constant-acceleration targets. The proposed estimators can be applied to tracking of constant-velocity targets by setting the acceleration vector equal to zero, and to localization of stationary targets by setting both the acceleration vector and the initial velocity vector equal to zero.

REFERENCES

[1] V. J. Aidala, "Kalman filter behavior in bearings-only tracking applications," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 15, pp. 29–39, Jan. 1979.

[2] V. J. Aidala and S. E. Hammel, "Utilization of modified polar coordinates for bearings-only tracking," *IEEE Trans. Autom. Control*, vol. 28, pp. 283–294, Mar. 1983.

[3] D. Lerro and Y. Bar-Shalom, "Bias compensation for improved recursive bearings-only target state estimation," in *Proc. American Control Conf.*, Seattle, WA, Jun. 1995, pp. 648–652.

[4] S. C. Nardone, A. G. Lindgren, and K. F. Gong, "Fundamental properties and performance of conventional bearings-only target motion analysis," *IEEE Trans. Autom. Control*, vol. 29, no. 9, pp. 775–787, Sep. 1984.

[5] R. L. Streit and M. J. Walsh, "A linear least squares algorithm for bearings-only target motion analysis," in *Proc. IEEE Aerospace Conf.*, vol. 4, Snowmass at Aspen, CO, Mar. 1999, pp. 443–455.

[6] A. G. Lindgren and K. F. Gong, "Position and velocity estimation via bearing observations," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 14, no. 4, pp. 564–577, Jul. 1978.

[7] S. K. Rao, "Pseudo-linear estimator for bearings-only passive target tracking," in *IEE Proc.-Radar Sonar Navig.*, vol. 148, Feb. 2001, pp. 16–22.

[8] V. J. Aidala and S. C. Nardone, "Biased estimation properties of the pseudo-linear tracking filter," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 18, pp. 432–441, Jul. 1982.

[9] Y. T. Chan and S. W. Rudnicki, "Bearings-only and Doppler-bearing tracking using instrumental variables," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 4, pp. 1076–1083, Oct. 1992.

[10] J.-P. Le Cadre and C. Jauffret, "On the convergence of iterative methods for bearings-only tracking," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 35, no. 3, pp. 801–818, Jul. 1999.

[11] A. Holtsberg and J. H. Holst, "A nearly unbiased inherently stable bearings-only tracker," *IEEE J. Ocean. Eng.*, vol. 18, no. 2, pp. 138–141, Apr. 1993.

[12] D. T. Pham, "Some quick and efficient methods for bearing-only target motion analysis," *IEEE Trans. Signal Process.*, vol. 41, no. 9, pp. 2737–2751, Sep. 1993.

[13] S. C. Nardone and M. L. Graham, "A closed-form solution to bearings-only target motion analysis," *IEEE J. Ocean. Eng.*, vol. 22, no. 1, pp. 168–178, Jan. 1997.

[14] K. Doğançay, "Least squares algorithms for constant-acceleration target tracking," in *Proc. Int. Conf. Radar 2003*, Adelaide, Australia, Sept. 2003, pp. 566–571.

[15] Y. Bar-Shalom and W. D. Blair, Eds., *Multitarget-Multisensor Tracking: Applications and Advances*. Boston, MA: Artech House, 2000, vol. III.

[16] D. Koks, "Passive geolocation for multiple receivers with no initial state estimate," Defence Sci. Technol. Org., Edinburgh, SA, Australia, Tech. Rep. DSTO RR-0222, Nov. 2001.

[17] S. C. Nardone and V. J. Aidala, "Observability criteria for bearings-only target motion analysis," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 17, pp. 162–166, Mar. 1981.

[18] J. M. Passerieux and D. Van Cappel, "Optimal observer maneuver for bearings-only tracking," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 34, no. 3, pp. 777–788, Jul. 1998.

[19] A. Farina, "Target tracking with bearings-only measurements," *Signal Processing*, vol. 78, pp. 61–78, 1999.

[20] S. Haykin, *Adaptive Filter Theory*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[21] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification*. Cambridge, MA: MIT Press, 1983.

[22] J. A. Nelder and R. Mead, "A simplex method for function minimization," *Comput. J.*, vol. 7, pp. 308–313, 1965.
[23] J. M. Mendel, *Lessons in Estimation Theory for Signal Processing, Communications, and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[24] T. S. Ferguson, *A Course in Large Sample Theory*. London, U.K.: Chapman and Hall, 1996.
[25] K. Doğançay, "On the bias of linear least squares algorithms for passive target localization," *Signal Processing*, vol. 84, no. 3, pp. 475–486, Mar. 2004.
[26] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*. London, U.K.: Griffin, 1961, vol. 2.
[27] L. Ljung, *System Identification: Theory for the User*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1999.
[28] K. Y. Wong and E. Polak, "Identification of linear discrete time systems using the instrumental variable method," *IEEE Trans. Autom. Control*, vol. 12, no. 6, pp. 707–718, Dec. 1967.
[29] K. Doğançay, "Passive emitter localization using weighted instrumental variables," *Signal Process.*, vol. 84, no. 3, pp. 487–497, Mar. 2004.
[30] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York: Cambridge Univ. Press, 1985.

Kutluyıl Doğançay (S'90–M'91–SM'01) received the B.S. degree with honors in electrical and electronic engineering from Boğaziçi University, Turkey, in 1989, the M.Sc. degree in communications and signal processing from Imperial College London, U.K., in 1992, and the Ph.D. degree in telecommunications engineering from The Australian National University in 1996.

From June 1996 to February 1998, he was a Research Fellow with the Department of Electrical and Electronic Engineering, The University of Melbourne, Melbourne, Australia. Between February 1998 and November 1999, he worked as a Lecturer in Communications Engineering with Massey University, New Zealand. In November 1999, he joined the School of Electrical and Information Engineering, University of South Australia, where he is currently a senior lecturer and program director. His research interests include statistical and adaptive signal processing, communication networks, and optimization techniques. He serves as a consultant to government and private industry in signal processing related projects.

Dr. Doğançay was the Publications Chair and a member of the Technical Program Committee in the International Conference on Radar 2003. He has been the IEEE University of South Australia Student Branch Counselor since April 2002. He received a 2005 Tall Poppy Science Award of the Australian Institute of Political Science.