

Algebraic Aspects of Lorentzian Homogeneous Spaces

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Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

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Abstract

This thesis explores Lorentzian (locally) reductive homogeneous spaces, with a focus on how algebraic conditions on the isotropy group affect the geometry and curvature of the homogeneous space. We take the approach of using Ambrose–Singer connections, and we prove that a Lorentzian (locally) reductive homogeneous space is locally isometric to a plane wave if it admits an Ambrose–Singer connection with indecomposable, non-irreducible holonomy. Along the way we prove novel results that are both algebraic and geometric in nature. This generalises several results in the literature that require certain algebraic types of torsion for the Ambrose–Singer connection.

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Chapter 1

Introduction

1.1 Homogeneous spaces and motivation

Since the beginnings of geometry, there has been ample interest in the concept of symmetry. In the framework of differential geometry, symmetry arises as a group of diffeomorphisms which leave the “geometry” invariant. When the geometry is semi-Riemannian, such symmetries arise as groups of diffeomorphisms which preserve the metric, the isometries. Of special interest, are semi-Riemannian manifolds with a “large number” of isometries. The conceptual idea of “large amount” of isometries is formalised by the notion of being (locally) homogeneous. Formally, a semi-Riemannian manifold (M, g) is (locally) homogeneous if it admits a (pseudo) group of (local) isometries that act transitively on M . Intuitively, this property implies that the local geometry of the manifold is “everywhere the same”.

Given a transitive group of isometries G and a point $o \in M$, the isotropy group H is defined as the subgroup of G that fixes o . Importantly, the isotropy group, up to conjugation, is independent of the chosen point. Since G acts transitively, M can be shown to be diffeomorphic to the quotient space G/H , where H is the isotropy of o . Additionally, the isotropy can be represented as a subgroup of $O(p, q)$, where (p, q) is the signature of the metric g , through the representation given by, $H \ni h \mapsto dh|_o \in O(p, q)$, using a fixed basis in T_oM . This representation is known as the isotropy representation of H . A homogeneous space with transitive group of isometries G with isotropy H , is termed reductive when in the Lie algebra $\mathfrak{g}$ of G , there is an $\text{Ad}(H)$ -invariant complement $\mathfrak{m}$ to the Lie algebra $\mathfrak{h}$ of H . Riemannian homogeneous spaces are always reductive.

One natural question one may want to ask is to which extent can the algebraic properties of the isotropy group H affect the geometry of (M, g) . The first condition one might consider on H is being irreducible, that is H does not leave invariant any proper subspace

of T_oM . Given this restriction, the situation is rather flexible in the Riemannian setting, we refer the reader to [5] and [35], however is more rigid in the indefinite case. Particularly, for Lorentzian homogeneous spaces this condition is very strong which is seen by the following result.

Theorem 1.1.1 (Zeghib [36]). *If a Lorentzian homogeneous space of dimension ≥ 3 admits an irreducible isotropy group, then it has constant sectional curvature.*

Given this result and following this line of thinking, then the next natural restriction to place on H is to relax the condition of being irreducible. In the indefinite context, this condition is being indecomposable (this is also known as “weakly irreducible” in the literature). We say that a subgroup $H \subset O(p, q)$ (or a subalgebra $\mathfrak{h} \subset \mathfrak{so}(p, q)$) is indecomposable if H (or $\mathfrak{h}$) has no invariant non-degenerate proper subspace of T_oM .

To our knowledge, all examples of reductive Lorentzian homogeneous spaces of dimension ≥ 4 with indecomposable, non-irreducible isotropy are a special type of Lorentzian manifolds known as plane waves. A Lorentzian plane wave is a Lorentzian manifold that,

1. admits a parallel null vector field V ,
2. the curvature tensor satisfies $R(X, Y) = 0$ for all $X, Y \in V^\perp$, and
3. $\nabla_X R = 0$ for all $X \in V^\perp$.

Homogeneous plane waves are always reductive and have been classified in [6]. Moreover, if $n + 2$ is the dimension of M , they admit indecomposable isotropy group $\mathbb{R}^n$ and holonomy group $\mathbb{R}^n$.

Given the lack of examples, the motivating conjecture of this thesis is the following:

Conjecture 1.1.2. *A reductive Lorentzian homogeneous space G/H of dimension ≥ 4 with indecomposable, non-irreducible isotropy H is a plane wave.*

Several results towards this conjecture have been proven for special cases in [3, 26, 13]. These papers take the approach of using a homogeneous structure, or equivalently, an Ambrose–Singer connection. This is a connection that preserves the metric and has parallel torsion and curvature. The existence of such a connection is equivalent to being locally reductive homogeneous, which under further global assumptions is equivalent to being reductive homogeneous (see [9]). This approach is relevant to Conjecture 1.1.2, as in some sense the holonomy of an Ambrose–Singer connection corresponds to the isotropy of an homogeneous space, we refer the reader to Section 3.2 in Chapter 3. Hence, tools from Lorentzian holonomy theory can be used to try and tackle this problem.

1.2 Results and structure of the thesis

With the above context, we look to progress towards Conjecture 1.1.2 through the machinery of Ambrose–Singer connections. The primary results towards this goal are found in Chapter 4, and Chapter 5, both made up primarily of original results. Chapter 4 contains all the needed algebraic results, and Chapter 5 the geometric results, culminating into our main result, Theorem 5.4.7. Our main result states:

Theorem (5.4.7). *Let (M, g) be a Lorentzian manifold of dimension ≥ 4 with an Ambrose–Singer connection that has indecomposable, non-irreducible holonomy. Then, the universal cover of (M, g) is a locally homogeneous plane wave.*

This makes substantial progress towards Conjecture 1.1.2, and generalises results in [26, 3, 13]. Now we describe the structure of the thesis. (The number in brackets represents the number of the theorem in the thesis.)

In Chapter 2, we introduce some of the needed background for the future chapters, however we assume the reader is familiar with basic concepts from differential geometry. Specifically, in Section 2.1 we provide the needed theory on holonomy of connections over vector bundles, with a focus on the holonomy principle and the Ambrose–Singer holonomy theorem. In Sections 2.2 and 2.3, we introduce semi-Riemannian homogeneous spaces and their associated canonical connection. Next, in Section 2.4, we introduce the local counterpart to reductive homogeneous spaces which are locally reductive homogeneous spaces. We then move onto the main part of this chapter in Section 2.5, where we expand upon the details and proof of Theorem 1.1.1 obtained in [36] which states that a Lorentzian homogeneous space with irreducible isotropy must have constant sectional curvature.

We then see what can be done through the same methods when the isotropy is indecomposable, non-irreducible and we arrived at the following result.

Theorem (2.5.11). *If $(M \cong G/H, g)$ is a Lorentzian homogeneous space of dimension ≥ 3 such that H is indecomposable, non-irreducible, then there exists a filtration of auto-parallel distributions,*

$$D \subset D^\perp \subset TM,$$

where D and $D^\perp$ are distributions of null lines and hyperplanes respectively.

This approach falls short in tackling Conjecture 1.1.2 directly since being auto-parallel is strictly weaker than being parallel, which would be the first step towards the conjecture. Hence, we decided to take the approach of using Ambrose–Singer connections which are introduced in the next chapter. Lastly in Section 2.6, we introduce the notion of Lorentzian pp-waves and plane-waves.

In Chapter 3, we introduce the key ingredient, the Ambrose–Singer connection. In Section 3.1, Ambrose–Singer connections, homogeneous structures, and their relationship to semi-Riemannian locally reductive homogeneous spaces are reviewed from [9]. Next, in Section 3.2 we present the relationship between the isotropy of a homogeneous space and the holonomy of an Ambrose–Singer connection. Importantly, using results from [22], we show that every semi-Riemannian homogeneous space admits a presentation such that the isotropy is the holonomy of an Ambrose–Singer connection. This lets us link our main result in Theorem 5.4.7 to Conjecture 1.1.2. In Section 3.3, we present the decomposition of the space of homogeneous structures, $\wedge^1 \otimes \wedge^2$, into three irreducible modules. The irreducible decomposition allows us to classify homogeneous structure tensors into eight different algebraic types, with tensors belonging to one of the three irreducible modules being called vectorial, twistorial, or totally skew. We also briefly look at the isomorphism between $\wedge^1 \otimes \wedge^2$ and $\wedge^2 \otimes \wedge^1$, which explains how the homogeneous structure completely determines the torsion of the Ambrose–Singer connection, and visa versa. We end with Section 3.4, where we present the known results in the literature relevant to Lorentzian Ambrose–Singer connections with indecomposable, non-irreducible holonomy. These results require additional assumptions on the algebraic type of the torsion.

We now move onto our main result chapters of the thesis, Chapter 4 and Chapter 5. Starting with Chapter 4, this chapter presents all the algebraic results proved en route to Theorem 5.4.7. In Section 4.1, we introduce our notation going forward for $(n + 2)$ -dimensional Minkowski space V with a basis $(e_-, e_1, \dots, e_n, e_+)$ such that e_- is a null vector. We also present indecomposable subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{so}(1, n + 1)$ which were classified into four types in [4]. All such algebras stabilise a null line which restricts them to be contained in the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, where $\mathfrak{p} = (\mathbb{R} \otimes \mathfrak{so}(n)) \ltimes \mathfrak{g}_-$ and $\mathfrak{g}_-$ is an abelian ideal in $\mathfrak{p}$. In Section 4.2, we compute the centraliser of the four types of indecomposable subalgebras, see Proposition 4.2.1. We use this result in Section 4.3 to prove one of the most important results in this thesis. However, before that, as an aside, the motivation of this chapter is for $\mathfrak{h}$ to be a holonomy algebra of an Ambrose–Singer connection in Chapter 5. Hence, the rest of this chapter is focused on algebras of type 2 and type 4. This is because these are the only two types which annihilate the null vector e_- and, as will be seen further down, this is the case for an indecomposable, non-irreducible Ambrose–Singer holonomy algebra.

Back to Section 4.3, in this section we determine the largest trivial $\mathfrak{h}$ -submodule in $\wedge^1 \otimes \wedge^2$. We determine that:

Theorem (4.3.1). *Let $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)$ with $n \geq 2$, be a indecomposable subalgebra of type 2 or type 4. If $\mathcal{S}$ is a trivial submodule of $\mathfrak{h}$ in $\wedge^1 \otimes \wedge^2$, and $S \in \mathcal{S}$, then,*

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathfrak{p},$$

such that $S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i$. Moreover, let $\mathfrak{h}_0 = \text{pr}_{\mathfrak{g}_0}(\mathfrak{h})$, then $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_0}(S(e_+)) = 0$. Furthermore, in type 2, $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$, while in type 4, $(\psi^{-1}(V_1) \oplus \mathfrak{h}'_0) \cdot \text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$.

This is one of the most important theorems in this thesis since for an Ambrose–Singer connection $\tilde{\nabla}$, its associated homogeneous structure S is parallel. Hence, the holonomy of $\tilde{\nabla}$ annihilates S and its form can be completely determined by the theorem above. We then move onto the next section, Section 4.4, and we investigate how the irreducible decomposition of $\wedge^1 \otimes \wedge^2$ interacts with the trivial submodule $\mathcal{S}$. We refer the reader to Theorem 4.4.1 for the results obtained in this section. Since the homogeneous structure completely determines the torsion of the Ambrose–Singer connection, our result Theorem 4.3.1 describes the complete form of possible torsion for Ambrose–Singer connections with indecomposable, non-irreducible holonomy. Moreover, the Ambrose–Singer curvature tensor satisfies the Bianchi symmetry for such torsion T . Hence, in Section 4.4, we prove the second most important result of this chapter, we determine the space of algebraic curvature tensors, $\mathcal{R}(V, \mathfrak{h}, T)$, taking values in $\mathfrak{h}$ with such torsion described by Theorem 4.3.1, we refer to Theorem 4.5.4 for when $\mathfrak{h}$ is of type 2, and Theorem 4.5.5 for when $\mathfrak{h}$ is of type 4. We then further determine the largest trivial $\mathfrak{h}$ -submodule since the Ambrose–Singer curvature is parallel.

Theorem (4.5.6). *Let $\mathfrak{h}$ be of type 2 or type 4 and let $R \in \mathcal{R}(V, \mathfrak{h}, T)$. If $\mathfrak{h} \cdot R = 0$, then $R(X, Y) \in \mathfrak{g}_-$ for all $X, Y \in V$.*

Lastly, in Section 4.6 we determine the possible homogeneous structures in dimension 3 and their possible algebraic curvature tensors.

In Chapter 5, we investigate Lorentzian manifolds with Ambrose–Singer connections with indecomposable holonomy through the algebraic results in the last chapter. We do this in multiple steps building up to our main result. We shall note that to prove Theorem 5.4.7, the two conditions in order to being a plane wave is that the Levi-Civita connection has holonomy $\mathbb{R}^n$ and satisfies some extra curvature conditions (see Section 2.6). Hence, our main efforts are in proving that the holonomy of the Levi-Civita connection is contained in $\mathbb{R}^n$.

Starting with Section 5.1, recall that a locally symmetric connection is a connection with parallel curvature, i.e. $\nabla R = 0$. In this section we investigate locally symmetric Lorentzian connections with indecomposable holonomy. Our result in this section is:

Theorem (5.1.4). *Let (M, g) be a simply-connected Lorentzian manifold of dimension ≥ 3 with a locally symmetric metric connection $\tilde{\nabla}$. Assume that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, $\tilde{\nabla}$ admits a $\tilde{\nabla}$ -parallel null vector field and $\mathfrak{hol}(\tilde{\nabla})$ is of types 2 or 4.*

In particular this means that if we have a locally symmetric Lorentzian connection with indecomposable holonomy, then the holonomy of this connection is contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$. We then move to looking at Lorentzian Ambrose–Singer connections with indecomposable holonomy in Section 5.3, however before that we briefly introduce the screen bundle in Section 5.2 which gives us the tools to investigate the $\mathfrak{so}(n)$ -projection of a holonomy

algebra contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$. Specifically, that the holonomy $\mathfrak{hol}(\nabla^E)$ of the connection ∇^E induced on this bundle by a connection ∇ corresponds to the $\mathfrak{so}(n)$ -projection of the holonomy of ∇ .

In Section 5.3, our first result is determining that for a Lorentzian manifold with an Ambrose–Singer connection with indecomposable holonomy, the holonomy of the Levi-Civita connection is contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$, see Theorem 5.3.5. Hence, by the results in Section 5.1, we have that both holonomy algebras of the Ambrose–Singer connection and Levi-Civita connection are contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$. We now want to compare both algebras and hopefully show that the $\mathfrak{so}(n)$ -projection of the Levi-Civita holonomy is trivial through the Ambrose–Singer holonomy. Thus, along these lines, we end this section with showing that the $\mathfrak{so}(n)$ -projections of both holonomy algebras are equal.

Theorem (5.3.11). *Let (M, g) be a simply-connected Lorentzian manifold of dimension ≥ 4 with Ambrose–Singer connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then $\mathfrak{hol}(\tilde{\nabla}^E)$ and $\mathfrak{hol}(\nabla^E)$ are equal.*

In other words, $\text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\tilde{\nabla}))$ and $\text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\nabla))$ are equal. We note that this result can be generalised to arbitrary Lorentzian connections with parallel torsion which satisfy the extra condition of having a parallel null vector field, see Remark 5.3.12.

Moving into Section 5.4, where we tie everything together and complete the last step of showing the $\mathfrak{so}(n)$ -projection of the Ambrose–Singer holonomy is trivial, then by the previous result, the Levi-Civita holonomy would be contained in $\mathbb{R}^n$. We do this with the following theorem.

Theorem (5.4.4). *Let (M, g) be a simply-connected Lorentzian manifold of dimension ≥ 4 with Ambrose–Singer connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then,*

$$\mathfrak{hol}(\tilde{\nabla}^E) = \mathfrak{hol}(\nabla^E) = 0,$$

implying that the holonomy for both $\tilde{\nabla}$ and ∇ must be abelian.

We then end the section with our main result Theorem 5.4.7, showing that the curvature must satisfy the extra constraints to be a plane-wave. We end the thesis with Section 5.5, in this section we investigate the 3-dimensional case and show that we cannot apply the same methods to drop the dimension bound in Theorem 5.4.7.

Remark 1.2.1. The main result Theorem 5.4.7 and the key results to be able to prove it contained in Chapters 4 and 5 have been rewritten and in some cases generalised and submitted as a research paper (Greenwood and Leistner [20]), and is currently in submission. The preprint is available on arXiv at arXiv:2404.17470.

1.3 Further research

Summarising our work, we prove a substantial step towards Conjecture 1.1.2. We show that a Lorentzian manifold of dimension ≥ 4 which admits an Ambrose–Singer connection with indecomposable, non-irreducible holonomy, is covered by a locally homogeneous plane-wave. Along the way, we prove novel results about certain trivial submodules, Lorentzian connections with parallel torsion, and locally symmetric Lorentzian connections.

To be able to complete the proof of Conjecture 1.1.2 through the approach of Ambrose–Singer connections, further research is needed in two possible directions.

- The first direction involves further development on the relationship between the isotropy of a homogeneous space and the holonomy of an Ambrose–Singer connection, that is, if the isotropy of any presentation can always be realised as the holonomy of an Ambrose–Singer connection without having to pass to a different presentation. If this is the case, then Theorem 5.4.7 would immediately prove Conjecture 1.1.2.

We suspect this would also involve clarifying further the relationship between canonical connections (see Section 2.3) and Ambrose–Singer connections.

- The second direction involves studying the case when the isotropy is indecomposable, non-irreducible and the Ambrose–Singer holonomy is decomposable. This is because the reason our main result does not prove Conjecture 1.1.2 is as follows: if one starts with a homogeneous space $M \cong G/H$ with indecomposable, non-irreducible isotropy H , one may have to pass down to a smaller presentation $M \cong \tilde{G}/\tilde{H}$ such that $\tilde{H} = \text{Hol}(\tilde{\nabla})$ where $\tilde{\nabla}$ is an Ambrose–Singer connection. In doing so, it is not guaranteed that $\tilde{H} \subset H$ is indecomposable, non-irreducible. Hence, the next steps would involve assuming that H is indecomposable, non-irreducible and that $\tilde{H} \subset H$ is decomposable.

Chapter 2

Semi-Riemannian Homogeneous Spaces

In this chapter we shall introduce some of the needed background for the rest of the thesis. Specifically, in Section 2.1 we introduce the concept of holonomy and some basic results which will be used in further chapters. In Section 2.2 and Section 2.4 we introduce semi-Riemannian homogeneous spaces and their local counterparts which will be a central focus of interest going forward. In Section 2.3 we introduce the canonical connection associated to any reductive homogeneous space which will be used as examples in Chapter 3. Next in Section 2.5, we move on to explain in detail the results obtained in [36] which applies to Lorentzian homogeneous spaces with irreducible isotropy. Lastly, in Section 2.6, we introduce Lorentzian pp-waves and plane waves which are a special class of Lorentzian manifolds and will be needed later in the thesis.

2.1 Preliminaries on holonomy

In this section we shall define the holonomy group and holonomy algebra for a connection ∇ on a vector bundle E over a manifold M . We shall then introduce the holonomy principle and Ambrose-Singer holonomy theorem, both of which shall be used throughout this thesis.

Let E be a vector bundle with connection ∇ over M . The curvature R^∇ of ∇ is a 2-form on M with values in the bundle $\text{End}(E) = E^* \otimes E$, i.e. $R^\nabla \in \Gamma(\wedge^2 T^*M \otimes \text{End}(E))$. It is defined as,

$$R^\nabla(X, Y)S = \nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{[X, Y]} S, \quad X, Y \in \Gamma(TM), S \in \Gamma(E).$$

With a connection ∇ , we can define the notion of parallel transport on E with respect to

∇ along a curve in M . If we let $\gamma : [0, 1] \rightarrow M$ be a smooth curve, then we can pullback the bundle E along γ to get the pullback bundle γ^*E . Moreover, we can pullback the connection ∇ to a connection ∇^γ on γ^*E .

Definition 2.1.1. A section $S \in \Gamma(\gamma^*E)$ is said to be parallel along γ if $\nabla^\gamma S = 0$.

Suppose $\gamma(0) = p \in M$, and we have $v \in E_p$, then we want some notion of parallel transporting v along γ . We do this by finding some $S \in \Gamma(\gamma^*E)$ such that $S(0) = v$ and $\nabla^\gamma S = 0$, then $S(1) \in E_{\gamma(1)}$ is the transported vector.

Proposition 2.1.2. For any $v \in E_{\gamma(0)}$, there is a unique parallel section $S \in \Gamma(\gamma^*E)$ such that $S(0) = v$.

Proof. In any coordinate neighbourhood the conditions placed on such an S define an ODE with initial condition, thus we are guaranteed existence and uniqueness. See [21] for details. $\square$

This defines an isomorphism $\mathcal{P}_\gamma^\nabla$ between the fibers $E_{\gamma(0)}$ and $E_{\gamma(1)}$ which we call the parallel transport of γ with respect to ∇ . This is defined as,

$$\mathcal{P}_\gamma^\nabla : E_{\gamma(0)} \rightarrow E_{\gamma(1)}, \quad v = S(0) \mapsto S(1),$$

where S is the unique parallel section of γ^*E such that $S(0) = v$.

This allows us to define the holonomy group $\text{Hol}_p(E, \nabla)$ of ∇ at a point $p \in M$.

Definition 2.1.3. If $p \in M$, then the holonomy group $\text{Hol}_p(E, \nabla)$ at p of ∇ is defined as,

$$\text{Hol}_p(E, \nabla) := \{\mathcal{P}_\gamma^\nabla \in \text{GL}(E_p) : \gamma(0) = \gamma(1) = p, \gamma \text{ piece-wise smooth}\}.$$

The holonomy group of any connection ∇ enjoys being a Lie group, see [21]. Hence, we shall denote its Lie algebra as $\mathfrak{hol}_p(E, \nabla)$, which we call the holonomy algebra of ∇ at $p \in M$.

Remark 2.1.4. If $p, q \in M$ and we fix some piece-wise smooth curve σ from p to q , then $\text{Hol}_q(E, \nabla) = \mathcal{P}_\sigma^\nabla \circ \text{Hol}_p(E, \nabla) \circ \mathcal{P}_{\sigma^{-1}}^\nabla$. As a consequence the holonomy groups at various points of M are in fact isomorphic. This enables us to talk of the holonomy group/algebra up to conjugation without denoting the base point. This shall be denoted $\text{Hol}(E, \nabla)$ and $\mathfrak{hol}(E, \nabla)$ respectively.

One of the most important results in the theory of holonomy is the so called holonomy principle.

Theorem 2.1.5 (Holonomy Principle [5]). *Let M be a connected smooth manifold and E a vector bundle over M . Moreover, suppose that ∇ is a connection on E . Then, there is a bijective correspondence between,*

- *sections of E which are invariant under parallel transport of ∇ ,*
- *$\text{Hol}_p(\nabla, E)$ -invariant vectors in E_p for $p \in M$,*
- *∇ -parallel sections of E .*

A consequence of the holonomy principle, which shall be used extensively throughout Chapter 4, is that if $A \in \Gamma(E)$ is parallel, i.e $\nabla A = 0$, then its value at $p \in M$ is invariant under $\text{Hol}_p(\nabla, E)$, which implies $\mathfrak{hol}_p(E, \nabla) \cdot A = 0$.

Moreover, we have the following theorem concerning the holonomy algebra proved by Ambrose and Singer [1], which shall be used throughout this thesis.

Theorem 2.1.6 (Ambrose-Singer Holonomy Theorem [1]). *The holonomy algebra of ∇ on E is given by,*

$$\mathfrak{hol}_p(E, \nabla) = \text{span} \left\{ \mathcal{P}_\gamma^\nabla (R_{\gamma(1)}^\nabla(X, Y)) : X, Y \in T_{\gamma(1)}M \right\},$$

where $\gamma : [0, 1] \rightarrow M$ is any curve such that $\gamma(0) = p$.

Remark 2.1.7. We note above that $\mathcal{P}_\gamma^\nabla$ is mapping a endomorphism of $T_{\gamma(1)}M$ to one of $T_{\gamma(0)}M$, this is explicitly given by,

$$\mathcal{P}_\gamma^\nabla (R_{\gamma(1)}^\nabla(X, Y)) = \mathcal{P}_{\gamma^{-1}}^\nabla \circ R_{\gamma(1)}^\nabla(X, Y) \circ \mathcal{P}_\gamma^\nabla.$$

If we have an additional connection $\tilde{\nabla}$ on the tangent bundle of M , we can extend ∇ to $\bigotimes^r T^*M \otimes E$, by,

$$(\nabla_X A)(X_1, \dots, X_r) = \nabla_X(A(X_1, \dots, X_r)) - A(\tilde{\nabla}_X X_1, \dots, X_r) - \dots - A(X_1, \dots, \tilde{\nabla}_X X_r).$$

Where we are abusing notation by denoting the induced connection as ∇ . If $E = TM$, then no auxiliary connection is needed.

Proposition 2.1.8. *Let ∇ and $\tilde{\nabla}$ be connections on E and TM respectively. Moreover, denote D as the induced connection on $\wedge^2 T^*M \otimes E$. If $DR^\nabla = 0$, then $\mathfrak{hol}_p(E, \nabla) = \text{span}\{R_p^\nabla(X, Y) : X, Y \in T_pM\}$.*

Proof. By Theorem 2.1.6,

$$\mathfrak{hol}_p(E, \nabla) = \text{span} \left\{ \mathcal{P}_\gamma^\nabla (R_{\gamma(1)}^\nabla(X, Y)) : X, Y \in T_{\gamma(1)}M, \gamma(0) = p \right\},$$

where $\gamma : [0, 1] \rightarrow M$ is any curve such that $\gamma(0) = p$. Before we continue, we shall first state how $\mathcal{P}_\gamma^D$ maps elements of $(\wedge^2 \otimes E)|_{\gamma(1)}$ to $(\wedge^2 \otimes E)|_{\gamma(0)}$. Let $Q \in \Gamma(\wedge^2 \otimes E)$, then,

$$\mathcal{P}_\gamma^D(Q_{\gamma(1)})(X, Y)S = \mathcal{P}_{\gamma^{-1}}^\nabla \circ Q_{\gamma(1)}(\mathcal{P}_\gamma^{\tilde{\nabla}}(X), \mathcal{P}_\gamma^{\tilde{\nabla}}(Y)) \circ \mathcal{P}_\gamma^\nabla(S),$$

where $X, Y \in T_{\gamma(0)}M$ and $S \in E_{\gamma(0)}$. Hence, for the curvature R^∇ , if we consider the term $\mathcal{P}_\gamma^\nabla(R_{\gamma(1)}^\nabla(X, Y))$ for $X, Y \in T_{\gamma(1)}M$, then,

$$\begin{aligned} \mathcal{P}_\gamma^\nabla(R_{\gamma(1)}^\nabla(X, Y)) &= \mathcal{P}_{\gamma^{-1}}^\nabla \circ R_{\gamma(1)}^\nabla(X, Y) \circ \mathcal{P}_\gamma^\nabla, \\ &= \mathcal{P}_\gamma^D(R_{\gamma(1)}^\nabla)(\mathcal{P}_{\gamma^{-1}}^{\tilde{\nabla}}(X), \mathcal{P}_{\gamma^{-1}}^{\tilde{\nabla}}(Y)). \end{aligned}$$

Now, since $DR^\nabla = 0$, by Theorem 2.1.5, R^∇ is invariant under parallel transport of D , hence, $\mathcal{P}_\gamma^D(R_{\gamma(1)}^\nabla) = R_{\gamma(0)}^\nabla$. Thus, $\mathcal{P}_\gamma^\nabla(R_{\gamma(1)}^\nabla(X, Y)) = R_{\gamma(0)}^\nabla(\mathcal{P}_{\gamma^{-1}}^{\tilde{\nabla}}(X), \mathcal{P}_{\gamma^{-1}}^{\tilde{\nabla}}(Y))$, and since parallel transport is an isomorphism,

$$\mathfrak{hol}_p(E, \nabla) = \text{span}\{R_p^\nabla(X, Y) : X, Y \in T_pM\}. \quad \square$$

We now introduce the notion of irreducibility and indecomposability for subgroups of $\text{GL}(n, \mathbb{R})$ and $\text{O}(p, q)$. These algebraic assumptions shall be a focal point later in this chapter, and shall reappear in Chapter 4 and Chapter 5.

Definition 2.1.9. If V is a vector space, let $H \subset \text{GL}(V)$ (or algebra $\mathfrak{h} \subset \mathfrak{gl}(V)$). We say that H (or $\mathfrak{h}$) is irreducible if there is no non-trivial proper subspace of V which is invariant under H (or $\mathfrak{h}$).

Definition 2.1.10. If V is a semi-Euclidean vector space, let $H \subset \text{O}(V)$ (or algebra $\mathfrak{h} \subset \mathfrak{so}(V)$). We say that H (or $\mathfrak{h}$) is indecomposable if there is no non-trivial proper non-degenerate subspace of V which is invariant under H (or $\mathfrak{h}$).

We can apply the irreducible assumption to the holonomy of any connection ∇ on a vector bundle E of rank n since $\text{Hol}_p(E, \nabla) \subset \text{GL}(E_p)$, and $\mathfrak{hol}_p(E, \nabla) \subset \mathfrak{gl}(E_p)$. Moreover, if the vector bundle admits a parallel metric g , then by the holonomy principle $\text{Hol}_p(E, \nabla) \subset \text{O}(E_p)$ and $\mathfrak{hol}_p(E, \nabla) \subset \mathfrak{so}(E_p)$. Hence, in such a situation the indecomposability assumption is relevant.

Remark 2.1.11. Note that for Riemannian metrics, every subspace is non-degenerate, hence irreducibility and indecomposability are equivalent.

We shall use the term “indecomposable, non-irreducible” to exclude the irreducible case when assuming indecomposability.

2.2 Semi-Riemannian homogenous spaces

In this section we introduce semi-Riemannian homogeneous spaces. This includes reductive homogeneous spaces and the isotropy representation which shall be a key focus throughout this thesis.

We start with the following theorem.

Theorem 2.2.1 ([21]). *Let G be a Lie group and H a closed subgroup of G . Then there is a unique smooth structure on G/H such that the action of G on G/H is smooth. That is, the mapping $G \times G/H \rightarrow G/H$, $(g_1, g_2H) \mapsto g_1g_2H$ is smooth. In particular $\pi : G \rightarrow G/H$ is a submersion and $\dim(G/H) = \dim(G) - \dim(H)$.*

This leads to the definition of a homogeneous space.

Definition 2.2.2. A homogeneous space is a quotient G/H of a Lie group G by a closed subgroup H , such that G/H is equipped with the unique smooth structure making $\pi : G \rightarrow G/H$ a submersion.

An equivalent definition of a homogeneous space is the following.

Definition 2.2.3. A manifold M is homogeneous if there is a Lie group G acting smoothly and transitively on M .

We see that these definitions are equivalent. Take G/H as a homogeneous space given the first definition. That is by a quotient of a Lie group G and a closed subgroup H . Then, G acts on G/H by left translations, and this action must be transitive. This is because the left translation by $g_2g_1^{-1}$ takes g_1H to g_2H , and this can be done for any point.

Conversely, let M be a homogeneous space given by the second definition. That is M a manifold on which G acts transitively. We define the isotropy group H_p as follows.

Definition 2.2.4. If M is a homogeneous space, the isotropy group at a point $p \in M$ is the subgroup,

$$H_p := \{g \in G : g \cdot p = p\}.$$

The isotropy group H_p is a closed subgroup of G and the map,

$$G/H_p \rightarrow M, \quad gH \mapsto g \cdot p,$$

is a diffeomorphism between G/H_p and M . See [21] for details.

Remark 2.2.5. Since the action of G is transitive, all isotropy groups are conjugate,

$$H_{g \cdot p} = gH_pg^{-1}.$$

Hence, we will often abuse notation and omit the point.

In the context of semi-Riemannian geometry, which shall be the focus for the rest of the thesis.

Definition 2.2.6. A semi-Riemannian manifold (M, g) is called homogeneous if there is a Lie group G of isometries acting transitively on M .

Throughout this thesis the main focus will be on a distinguished subclass of homogeneous spaces. The so-called *reductive* homogeneous spaces.

Definition 2.2.7. A homogeneous space G/H is called reductive if the Lie algebra $\mathfrak{g}$ of G admits a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, such that $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\text{Ad}(H)$ -invariant subspace. That is, $\text{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$.

Lemma 2.2.8 ([21]). *For a reductive homogeneous space, the condition $\text{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$ implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.*

Remark 2.2.9. If H is connected then the converse is also true. This is a consequence of the fact that if $[X, Y] \in \mathfrak{m}$, then $\text{ad}(X)(\mathfrak{m}) \subset \mathfrak{m}$ and thus it can be shown that $\text{Ad}(h)(Y) \in \mathfrak{m}$ for all h in the subgroup of H generated by $\exp(\mathfrak{h})$. If H is connected, then this subgroup is exactly H .

For reductive homogeneous spaces, we can identify the subspace $\mathfrak{m}$ with $T_o M$, where $o = H \in G/H$. This is because the projection $\pi : G \rightarrow G/H$ gives rise to an isomorphism,

$$d\pi_e|_{\mathfrak{m}} : \mathfrak{m} \rightarrow T_o M.$$

More explicitly, consider the differential of π , $d\pi_e : \mathfrak{g} \rightarrow T_o(G/H)$. Let $X \in \mathfrak{g}$ and $\exp(tX)$ its one parameter subgroup. Then,

$$d\pi_e(X) = \left. \frac{d}{dt} \right|_{t=0} (\pi \circ \exp(tX)) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX)H).$$

Note, $d\pi_e(X) = 0$ if and only if $X \in \mathfrak{h}$. Hence, $\ker(d\pi_e) = \mathfrak{h}$, and since $d\pi$ is surjective, we obtain the isomorphism, $\mathfrak{g}/\mathfrak{h} \cong T_o(M)$. But for reductive homogeneous spaces this is exactly $\mathfrak{m}$.

Remark 2.2.10. For reductive homogeneous spaces, given any $X, Y \in \mathfrak{g}$, then $[X, Y]_{\mathfrak{m}}$ and $[X, Y]_{\mathfrak{h}}$ shall denote the projections of $[X, Y] \in \mathfrak{g}$ onto $\mathfrak{m}$ and $\mathfrak{h}$ respectively.

In the semi-Riemannian setting, not every homogeneous space is reductive. However, remarkably in the Riemannian signature, every homogeneous space is reductive.

Theorem 2.2.11 ([9]). *Every Riemannian homogeneous space (M, g) is reductive.*

Proof. See [9] for proof. However, the key idea is to use that the metric is positive definite to define a symmetric bilinear form on $\mathfrak{g}$ which is $\text{Ad}(H)$ -invariant and definite on $\mathfrak{h}$. Then, we define $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to this bilinear form. $\square$

We will now present some examples of homogeneous spaces.

Example 2.2.12 (Lie groups). Every connected Lie group G is a homogeneous space since G acts naturally on itself, giving the coset space $G = G/\{e\}$.

Another way to see them as homogeneous spaces, which shall be used in later chapters as examples, is to consider $K = G \times G$ acting on G by,

$$K \times G \ni ((g_1, g_2), g_3) \mapsto g_1 g_3 g_2^{-1} \in G.$$

Then the isotropy subgroup H of K at $e \in G$ is the diagonal $\{(g, g) : g \in G\}$. Let $\mathfrak{g}$ be the Lie algebra of G so that $\mathfrak{k} = \mathfrak{g} + \mathfrak{g}$ and $\mathfrak{h} = \{(X, X) : X \in \mathfrak{g}\}$. The homogeneous space $G = K/H$ is reductive and there are three decompositions of $\mathfrak{k}$:

$$\begin{aligned} \mathfrak{k} &= \mathfrak{h} + \mathfrak{g}_+, & \mathfrak{g}_+ &= \{(0, X) : X \in \mathfrak{g}\}, \\ \mathfrak{k} &= \mathfrak{h} + \mathfrak{g}_-, & \mathfrak{g}_- &= \{(X, 0) : X \in \mathfrak{g}\}, \\ \mathfrak{k} &= \mathfrak{h} + \mathfrak{g}_0, & \mathfrak{g}_0 &= \{(X, -X) : X \in \mathfrak{g}\}. \end{aligned}$$

Example 2.2.13 (Spheres). The sphere $\mathbb{S}^n$ is a reductive homogeneous space. This is because $\text{Iso}^0(\mathbb{S}^n) = \text{SO}(n+1)$ acts transitively on $\mathbb{S}^n$. To see that its reductive, $\mathfrak{g} = \mathfrak{so}(n+1)$ and $\mathfrak{h} = \mathfrak{so}(n)$. Since the Killing form on $\mathfrak{so}(n)$ is definite, we can define $\mathfrak{m}$ as $\mathfrak{h}^\perp$ with respect the Killing form on $\mathfrak{so}(n)$. Given the Killing form on $\mathfrak{so}(n)$ is $B(X, Y) = \text{tr}(XY)$ for $X, Y \in \mathfrak{so}(n)$, it can be verified that,

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & c_1 & \dots & c_n \\ -c_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -c_n & 0 & \dots & 0 \end{pmatrix} : c_1, \dots, c_n \in \mathbb{R} \right\},$$

and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition.

Example 2.2.14 (Spaces of constant sectional curvature). Some more examples of homogeneous spaces are the other spaces of constant sectional curvature,

$$\mathbb{R}^n = \frac{\text{SO}(n) \ltimes \mathbb{R}^n}{\text{SO}(n)}, \quad \mathbb{H}^n = \frac{\text{SO}_0(1, n)}{\text{SO}(n)},$$

and the Lorentzian constant curvature counterparts,

$$\mathbb{R}^{1, n-1} = \frac{\text{SO}_0(1, n-1) \ltimes \mathbb{R}^{1, n-1}}{\text{SO}_0(1, n-1)}, \quad \text{AdS}^n = \frac{\text{SO}(2, n-1)}{\text{SO}_0(1, n-1)}, \quad \text{dS}^n = \frac{\text{SO}_0(1, n)}{\text{SO}_0(1, n-1)}.$$

Example 2.2.15 (Projective spaces). The various projective spaces are homogeneous,

$$\mathbb{RP}_n = \frac{\mathrm{SO}(n+1)}{\mathrm{S}(\mathbb{Z}_2 \times \mathrm{O}(n))}, \quad \mathbb{CP}_n = \frac{\mathrm{SU}(n+1)}{\mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))}, \quad \mathbb{HP}_n = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\mathrm{Sp}(1)}.$$

We will now introduce the isotropy representation. Given a semi-Riemannian homogeneous space, $(M = G/H, g)$, then the isotropy H acts in a natural way on the tangent space. Let H be the isotropy at $o = eH \in G/H$, then for every $h \in H$, $h \cdot o = o$. Hence, for any $h \in H$, we can take the differential at o ,

$$\mathrm{d}h|_o : T_oM \rightarrow T_oM,$$

producing an element of $\mathrm{O}(T_oM)$. This results in H acting on T_oM and we get the corresponding representation,

$$\rho : H \rightarrow \mathrm{O}(T_oM), \quad h \mapsto \mathrm{d}h|_o.$$

Definition 2.2.16. The representation $\rho : H \rightarrow \mathrm{O}(T_oM)$ defined above is called the isotropy representation.

We can always assume G acts effectively on M by replacing G with G/N , where N is the normal subgroup which acts trivially. This implies that the isotropy representation is always injective. This can be seen as all elements apart from the identity act not trivially, and elements of G are isometries which are uniquely determined at a point and its differential. Hence, if $g, h \in H$ and $\mathrm{d}g|_o = \mathrm{d}h|_o$, then $g = h$. Thus, we can identify H in $\mathrm{O}(T_oM)$, and we can look at the isotropy being irreducible, and indecomposable in the sense explained in Section 2.1.

Definition 2.2.17. We say the isotropy is irreducible or the homogeneous space is isotropy irreducible if the isotropy group/algebra is irreducible with respect to the isotropy representation i.e. $\rho(H) \subset \mathrm{GL}(T_oM)$ is irreducible.

Definition 2.2.18. For a semi-Riemannian homogeneous space, we say the isotropy is indecomposable or the homogeneous space is isotropy indecomposable if and only if the isotropy group/algebra is indecomposable with respect to the isotropy representation i.e. $\rho(H) \subset \mathrm{O}(T_oM)$ is indecomposable.

Example 2.2.19. Examples of homogeneous spaces with irreducible isotropy are the Lorentzian symmetric spaces of non-zero constant curvature,

$$\mathrm{AdS}^{n+1} = \frac{\mathrm{SO}(2, n)}{\mathrm{SO}_0(1, n)}, \quad \mathrm{dS}^{n+1} = \frac{\mathrm{SO}_0(1, n+1)}{\mathrm{SO}_0(1, n)}.$$

Both have irreducible isotropy $\mathrm{SO}(1, n)$.

Examples of Lorentzian homogeneous spaces with indecomposable, non-irreducible isotropy are indecomposable homogenous pp-waves/plane waves which will be introduced in Section 2.6. They have indecomposable, non-irreducible isotropy algebra $\mathfrak{h} = \mathbb{R}^n$.

2.3 Canonical connections

Let $M = G/H$ be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Any G -invariant connection on G/H is uniquely determined by its value at the origin $o := eH$ by applying left translations. Moreover, since we can identify $\mathfrak{m} = T_o M$, then by Nomizu [27], an invariant metric connection on M corresponds to $\text{Ad}(H)$ -equivariant map,

$$\Lambda : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m}),$$

called the *Nomizu map*. Explicitly, the Nomizu map and the associated connection $\tilde{\nabla}$ are related by,

$$(\tilde{\nabla}_X Y)_o = \Lambda(X)Y_o, \quad X, Y \in \mathfrak{m},$$

where we have identified $X, Y \in \mathfrak{m}$ with their associated vector fields on M . Moreover, this can be extended to act on any invariant tensor.

Proposition 2.3.1 ([21]). *Let Λ be a Nomizu map and $\tilde{\nabla}$ its associated connection. The torsion and curvature corresponding to $\tilde{\nabla}$ expressed at the origin o are given as follows:*

$$\begin{aligned} \tilde{T}(X, Y)_o &= \Lambda(X)Y - \Lambda(Y)X - [X, Y]_{\mathfrak{m}}, \\ \tilde{R}(X, Y)_o &= [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]_{\mathfrak{m}}) - \text{ad}([X, Y]_{\mathfrak{h}}), \end{aligned}$$

for all $X, Y \in \mathfrak{m}$.

We can take this notion and apply it to the Levi-Civita connection, as it will be an invariant metric connection on G/H and hence has an associated Nomizu map.

Proposition 2.3.2 ([21]). *Let $(M = G/H, g)$ be a semi-Riemannian reductive homogeneous space. Then the Nomizu map $\Lambda^g : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$ corresponding to the Levi-Civita connection is given by,*

$$\Lambda^g(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y),$$

for all $X, Y \in \mathfrak{m}$, where $U(X, Y)$ is a symmetric tensor defined by,

$$2g(U(X, Y), Z) = g(X, [Z, Y]_{\mathfrak{m}}) + g([Z, X]_{\mathfrak{m}}, Y).$$

The main interest of this section is a invariant connection called the *canonical connection*. It is a connection whose horizontal distribution is generated by left translations of $\mathfrak{m}$, see [21]. Such a connection is unique after fixing a reductive decomposition.

Definition 2.3.3. The canonical connection ∇^c associated to a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the connection whose Nomizu map $\Lambda^c : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$ is the zero map.

Proposition 2.3.4 ([21]). *If a tensor on M is invariant by G , then it is parallel with respect to the canonical connection.*

As a result, using Proposition 2.3.1, the torsion and curvature of the canonical connection are given by

$$T^c(X, Y) = -[X, Y]_{\mathfrak{m}}, \quad \text{and} \quad R^c(X, Y) = -[X, Y]_{\mathfrak{h}}.$$

Moreover, $\nabla^c T = \nabla^c R = 0$ by Proposition 2.3.4.

For an example, we can look at the canonical connections associated with the reductive decompositions of G viewed as a homogeneous space given in second description of Example 2.2.12. Their defining feature was how their $\mathfrak{m}$ term was defined, these were labeled $\mathfrak{g}_+$, $\mathfrak{g}_-$, or $\mathfrak{g}_0$. Thus, the canonical connections associated to these decompositions are called the $(+)$ -connection, $(-)$ -connection, and the (0) -connection respectively, and were first introduced by Cartan and Schouten [31].

Proposition 2.3.5 ([21]). *Let G be a connected Lie group, and let $X, Y, Z \in \mathfrak{g}$ with reductive decompositions as in Example 2.2.12. Then,*

- *for the $(+)$ -connection: $T(X, Y) = [X, Y]$, and $R = 0$,*
- *for the $(-)$ -connection: $T(X, Y) = -[X, Y]$, and $R = 0$, and*
- *for the (0) -connection: $T = 0$, and $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$.*

2.4 Semi-Riemannian locally homogeneous spaces

In this section we introduce semi-Riemannian locally homogeneous spaces as presented in [9]. These shall be of importance in Chapter 2 when we define Ambrose-Singer connections and their relationship with locally reductive homogeneous spaces.

We shall first introduce the notion of a semi-Riemannian locally homogeneous space.

Definition 2.4.1. A semi-Riemannian manifold (M, g) is locally homogeneous if for every two points $p, q \in M$, there are neighborhoods U and V of p and q , respectively, and an isometry $f : U \rightarrow V$ taking p to q .

Remark 2.4.2. We call such isometries local isometries. That is, $f : M \rightarrow M$ is a local isometry if for each $p \in M$, there is a neighborhood U of p such that $f|_U$ is an isometry onto an open subset of M .

There are multiple notions of being locally homogeneous in the literature, we refer to [11] for a comprehensive list. For example, the conditions of Definition 2.4.1 are the case if $p \in M$ admits a neighbourhood U of p and Killing vector fields on U that span $T_p M$. Here we recall that a Killing vector field is a vector field $X \in \Gamma(TM)$ such that $\mathcal{L}_X g = 0$.

We now want to make sense of what it means for a locally homogeneous space to be reductive. To do this we shall follow [9] in their approach. To define some sort of reductivity condition, we would like some reductive decomposition of an isometry group acting transitively on M . However, now that our isometries are only local isometries, they do not necessarily form a group, let alone a Lie group. Thus, before we continue we need the following definitions.

Definition 2.4.3. A (smooth) pseudo-group on a smooth manifold M is a set $\mathcal{G}$ of diffeomorphisms $\psi : U \rightarrow V$, defined on open sets $U, V \subset M$, such that,

- $\text{Id}_U \in \mathcal{G}$,
- if $\psi \in \mathcal{G}$, then $\psi^{-1} \in \mathcal{G}$,
- if $\psi : U \rightarrow V$ belongs to $\mathcal{G}$ and $W \subset U$ is open, then $\psi|_W \in \mathcal{G}$,
- if $\psi : U \rightarrow V$, with $U = \bigcup_i U_i$ a union of open sets, then $\psi \in \mathcal{G}$ if $\psi|_{U_i} \in \mathcal{G}$,
- if $\psi : U \rightarrow V$ and $\phi : U' \rightarrow V'$ and $V \cap U'$ is non-empty, then $\phi \circ \psi : \psi^{-1}(V \cap U') \rightarrow \phi(V \cap U') \in \mathcal{G}$.

Definition 2.4.4. A transitive Lie algebra is a pair $(\mathfrak{g}, \mathfrak{g}_0)$, where $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}_0$ is a proper subalgebra such that the only ideal of $\mathfrak{g}$ in $\mathfrak{g}_0$ is $\{0\}$.

Let (M, g) be a locally homogeneous semi-Riemannian manifold. Consider the set of all local isometries $\mathcal{I}$. This is not a group, however, it is a pseudo-group, more specifically it is a Lie pseudo-group, we refer to [32, 9] for details. This means we can associate a Lie algebra $\mathfrak{i}$ to $\mathcal{I}$, furthermore, we can construct an $\mathfrak{i}^0$, making $(\mathfrak{i}, \mathfrak{i}^0)$ a transitive Lie algebra. We shall skip over its construction and the general theory of Lie pseudo-groups, however, we will describe a way to view $(\mathfrak{i}, \mathfrak{i}^0)$.

First introduced by Nomizu in [28], we start by defining a Killing generator.

Definition 2.4.5. A Killing generator at $p \in M$ is an element $(X, A) \in T_p M \oplus \mathfrak{so}(T_p M)$ which are solutions to the following equation,

$$\nabla_X \nabla^i R_p + A \cdot \nabla^i R_p = 0, \quad i \geq 0,$$

where ∇ is the Levi-Civita connection and R its curvature. Moreover, $\nabla^i R_p$ denotes the i -th iterated covariant derivative of R at the point $p \in M$.

Using the notation from [14], the space of Killing generators at a point $p \in M$ shall be denoted $K_p^{R,\infty}$. We can equip it with the Lie bracket,

$$[(X, A), (Y, B)] = (AY - BX, [A, B] - R_p(X, Y)),$$

which turns $K_p^{R,\infty}$ into a real Lie algebra. We define,

$$H_p^{R,\infty} := \{(X, A) \in K_p^{R,\infty} : X = 0\}.$$

Lemma 2.4.6 ([32, 9]). *The pair $(K_p^{R,\infty}, H_p^{R,\infty})$ is a transitive Lie algebra isomorphic to $(\mathfrak{i}, \mathfrak{i}^0)$.*

Hence, this is one way view the transitive Lie algebra $(\mathfrak{i}, \mathfrak{i}^0)$ associated to $\mathcal{I}$. Now, since M is locally homogeneous, consider the Lie pseudo-group of local isometries $\mathcal{G} \subset \mathcal{I}$ acting transitively on (M, g) . We can attach a Lie algebra $\mathfrak{g}$ to $\mathcal{G}$, this is the set of germs of local Killing vector fields with 1-parameter group contained in $\mathcal{G}$. We can also consider the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ where it is formed by all $[\xi] \in \mathfrak{g}$ which vanish at $p \in M$. It turns out that $\mathfrak{g} \subset \mathfrak{i}$ and $\mathfrak{h} \subset \mathfrak{i}^0$, moreover $(\mathfrak{g}, \mathfrak{h})$ is a transitive Lie algebra, we refer to [32, 9] for details.

Definition 2.4.7. Let $\mathcal{G}$ be a Lie pseudo-group of isometries acting transitively on (M, g) . The isotropy pseudo-group at a point $p \in M$ is,

$$\mathcal{H}_p = \{f \in \mathcal{G} : f(p) = p\}.$$

The linear isotropy group of $\mathcal{G}$ at $p \in M$ is,

$$H_p = \{F : T_p M \rightarrow T_p M : F = df|_p, f \in \mathcal{H}_p\}.$$

The isotropy pseudo-group is not always a Lie pseudo-group, hence we work with the linear isotropy group which is a Lie subgroup of $O(T_p M)$.

Lemma 2.4.8 ([9]). *The Lie algebra of H_p is isomorphic to $\mathfrak{h}$.*

We can now define what a semi-Riemannian locally reductive homogeneous space is.

Definition 2.4.9. Let (M, g) be a semi-Riemannian locally homogeneous manifold, and let $\mathcal{G}$ be the Lie pseudo-group of isometries acting transitively on (M, g) . We say that the triple $(M, g, \mathcal{G})$ is reductive if the transitive Lie algebra $(\mathfrak{g}, \mathfrak{h})$ associated with $\mathcal{G}$ can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{m}$ is $\text{Ad}(H_p)$ -invariant.

We also note that this definition does not depend on the point $p \in M$ which is in the background of this definition.

Proposition 2.4.10 ([9]). *If $(M, g, \mathcal{G})$ is reductive at a point $p \in M$, then it is reductive at every point.*

Remark 2.4.11. We remark that this definition of locally reductive coincides with the definition provided in Definition 2.2.7 if (M, g) is a global homogeneous space. We refer to [9] for details.

2.5 Zeghib's results

In [36], Zeghib classifies Lorentzian homogeneous spaces with irreducible isotropy. Such spaces can be classified via algebraic methods in deducing that a subgroup of $O(1, n-1)$ that acts irreducibly on $\mathbb{R}^{1, n-1}$ contains the connected component of $O(1, n-1)$, see [7, 12]. Zeghib however provides a direct geometric proof. In this section we shall explore the details of the proof in [36] and apply these results to the setting when the isotropy is indecomposable, non-irreducible.

2.5.1 Construction of totally geodesic hypersurfaces

Let $(M \cong G/H, g)$ be a Lorentzian homogeneous space. Fix $o \in M$ such that H is the isotropy of o . That is, $H = \{g \in G : g(o) = o\}$ and $\mathfrak{h}$ the Lie algebra of H . Now define the subspace,

$$L := \text{span}\{v \in T_o M : v \text{ null and } \exp_o(v^\perp) \text{ is a totally geodesic hypersurface}\}, \quad (2.1)$$

where null means $g(v, v) = 0$, and $v^\perp := \{w \in T_o M : g(v, w) = 0\}$.

Proposition 2.5.1. *The subspace L is invariant under the isotropy representation.*

For the proof we need the following Lemma.

Lemma 2.5.2. *If $\exp : TM \rightarrow M$ is the Riemannian exponential map and $\phi : M \rightarrow M$ is an isometry then $\phi \circ \exp_p = \exp_{\phi(p)} \circ d\phi|_p$.*

Proof. Let $p \in M$ and $v \in T_p M$ in the domain of $\exp_p$. Since ϕ is an isometry it sends geodesics of M onto geodesics of M . As a result, $\gamma : t \mapsto \phi(\exp_p(tv))$ is a geodesic of M and satisfies $\gamma(0) = \phi(p)$ and $\gamma'(0) = d\phi|_p(v)$. However, note that the curve $t \mapsto \exp_{\phi(p)}(t d\phi|_p(v))$ is another geodesic which also satisfies the same conditions. Hence, they are equal. $\square$

We now prove Proposition 2.5.1.

Proof. If $\phi \in H$ and $v \in L$, then $d\phi|_o(v)$ is null as the pushforward of ϕ preserves the metric, and by Lemma 2.5.2, $\exp_o((d\phi|_o(v))^\perp) = \exp_o(d\phi_o(v^\perp)) = \phi \circ \exp_o(v^\perp)$. Thus, as ϕ is an isometry $\exp_o((d\phi|_o(v))^\perp)$ is totally geodesic. Hence, $d\phi|_o(L) \subseteq L$. $\square$

Now we impose that G/H is isotropy indecomposable. In doing so, $\rho(H) \subset O(1, n+1)$ is not precompact as seen from the following lemma.

Lemma 2.5.3. *If $H \subset O(p, q)$ is indecomposable, then H is not precompact.*

Proof. For a contradiction, assume that H is precompact. Then this means the closure of H is compact and contained in $O(p, q)$. Take the connected component $SO_0(p, q)$, then the maximal compact subgroup is $SO(p) \times SO(q)$ and thus up to conjugation $H \subset SO(p) \times SO(q)$. But this would imply that H stabilises a non-degenerate subspace, this is impossible due to H being indecomposable. $\square$

This brings us to the following theorem proved by Zeghib [36].

Theorem 2.5.4 (Zeghib [36]). *If G/H is a Lorentzian homogeneous space such that $\rho(H)$ is non-precompact. Then, G/H contains a totally geodesic null hypersurface through a point $o \in M$.*

Proof. Without loss of generality, let H be the isotropy of $o \in M$. Consider $f \in H$ and its graph $\text{Graph}(f) = \{(p, f(p)) : p \in M\} \subset M \times M$. Then, $\text{Graph}(f)$ is a d -dimensional submanifold of $M \times M$ where $d = \dim(M)$. Moreover, $\text{Graph}(f)$ is null when equipped with the metric $g \oplus (-g)$, and is totally geodesic as isometries preserve geodesics.

We now consider a sequence $f_n \in H$ such that $d f_n|_o \in \rho(H)$ is divergent, that is, it does not have a converging sub-sequence. Such a sequence exists since $\rho(H)$ is non-precompact. Now, consider the sequence of graphs $\text{Graph}(f_n)$. We shall work locally and let O be a convex neighbourhood of $o \in M$, then define E_n to be the connected component of (o, o) in a convex neighbourhood of $(O \times O) \cap \text{Graph}(f_n)$. Furthermore, define $V_n = \text{Graph}(d f_n|_o) \subset T_o M \times T_o M$ under the isotropy representation, then $E_n = \exp_{(o,o)}(V_n \cap U_n)$, where U_n is an open neighbourhood of 0. Then $V_n \in \mathbf{Gr}_d(T_o M \times T_o M)$, where $\mathbf{Gr}_d(T_o M \times T_o M)$ is the Grassmannian of d -dimensional subspaces of $T_o M \times T_o M$. Moreover, for any vector space F , $\mathbf{Gr}_d(F)$ is compact with respect to the metric $d(A, B) = \|\text{Pr}_A - \text{Pr}_B\|_{op}$, $A, B \in \mathbf{Gr}_d(F)$. Hence, without loss of generality, V_n will convergence to some $V \subset T_o M \times T_o M$, which is null due to continuity. This lets us construct a limiting null submanifold E of E_n defined by $E = \exp_{(o,o)}(V \cap U)$, where U is an open neighbourhood of 0. It should be noted that V cannot be a graph, as if it were the graph of some $A \in O(T_o M)$, then $d f_n|_o \rightarrow A$ which is a contradiction.

Now we prove that E is totally geodesic. If $X \in T_p E$ for $p \in E$, take any curve $\gamma : I \rightarrow E$, where $\gamma(0) = p$. Let $P_\gamma : T_p E \rightarrow T_{\gamma(1)} E$ be the parallel transport along γ . We need to show that $P_\gamma(T_p E) \subseteq T_{\gamma(1)} E$. First, note that there exists a curve $c : I \rightarrow V$ such that $\gamma(t) = \exp \circ c(t)$. Thus, define $c_n = \text{Pr}_{V_n}^\perp \circ c : I \rightarrow V_n$ then $c_n \rightarrow c$ and $c'_n \rightarrow c'$ by definition of the metric on the Grassmannian. Let $\gamma_n : I \rightarrow E_n$ where $\gamma_n(t) = \exp \circ c_n(t)$, then clearly $\gamma_n \rightarrow \gamma$ and $\gamma'_n \rightarrow \gamma'$ as $\gamma'_n(t) = d\exp|_{c_n(t)} \circ c'_n(t)$ and $\gamma'(t) = d\exp|_{c(t)} \circ c'(t)$.

As a result, fix coordinates $(x^1, \dots, x^n)$, $\partial_i = \frac{\partial}{\partial x^i}$ then $X = X^i \partial_i|_p$ and if,

$$X(t) = P_{\gamma|t}(X) = \sum_i P^i(t) \partial_i|_{\gamma(t)},$$

then due to parallel transport we have the collection of ODEs,

$$(P^k)' = -F(t, P), \quad P^k(0) = X^k,$$

where $F^k(t, P) = \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)P^j(t)$. Now, for each $\gamma_n(t) : I \rightarrow E_n$ we have $X_n \in T_{\gamma_n(0)}E_n$ such that $X_n \rightarrow X$. Hence, if we take $P_{\gamma_n|t}(X_n)$ we have the ODEs,

$$(P_{(n)}^k)' = -F_{(n)}^k(t, P_{(n)}), \quad P_{(n)}^k(0) = X_{(n)}^k,$$

where $F_{(n)}^k(t, P_{(n)}) = \Gamma_{ij}^k(\gamma_n(t))(\gamma_n^i)'(t)P_{(n)}^j(t)$. Now, due to [33], we can impose,

$$\begin{aligned} |P^k(1) - P_{(n)}^k(1)| &\leq |P^k(0) - P_{(n)}^k(0)| e^{L_{(n)}^k} + \frac{M_{(n)}^k}{L_{(n)}^k} (e^{L_{(n)}^k} - 1) \\ &= |X^k - X_{(n)}^k| e^{L_{(n)}^k} + \frac{M_{(n)}^k}{L_{(n)}^k} (e^{L_{(n)}^k} - 1) \end{aligned}$$

where $M_{(n)}^k, L_{(n)}^k$ are constants. Specifically,

$$\begin{aligned} M_{(n)}^k &= \sup_{(t,h)} |F_{(n)}^k(t, h) - F^k(t, h)| \\ &= \sup_{(t,h)} |\Gamma_{ij}^k(\gamma_n(t))(\gamma_n^i)'(t)h^j(t) - \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)h^j(t)| \\ &\leq \sup_{(t,h)} |\Gamma_{ij}^k(\gamma_n(t))(\gamma_n^i)'(t) - \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)| |h(t)|^j \end{aligned}$$

But, $\gamma_n \rightarrow \gamma$ and $\gamma_n' \rightarrow \gamma'$. Thus, $M_{(n)}^k \rightarrow 0$ and hence $P_{(n)}^k(1) \rightarrow P^k(1)$. This implies that $X_n(1) \rightarrow X(1)$ and as each E_n is totally geodesic we must have $X(1) \in T_{\gamma(1)}E$ as required.

Now, define $V^2 = V \cap (\{0\} \times T_o M)$. This is non-trivial as otherwise the function,

$$F : T_o M \rightarrow T_o M, \quad F(X) = Y \text{ when } (X, Y) \in V,$$

is well-defined and $V = \text{Graph}(F)$, a contradiction since V is not a graph. The dimension of V^2 is one since V is null and M is Lorentzian. If Pr_1 is the projection onto the first factor, then $\text{Ker}(\text{Pr}_1(V)) = V^2$. Hence, $V^1 = \text{Pr}_1(V) \subset T_o M$ is a null hypersurface.

We now claim that $E^1 = \text{Pr}_1(E)$ is a totally geodesic null hypersurface. It is totally geodesic since E is totally geodesic, and is a null hypersurface since $E^1 = \text{Pr}_1(E) = \text{Pr}_1(\exp_{(o,o)}(V \cap U)) = \exp_o(V^1 \cap U^1)$. $\square$

As a result, if G/H is isotropy indecomposable, then by Lemma 2.5.3, $\rho(H)$ is non-precompact. This immediately leads to the corollary.

Corollary 2.5.5. *If G/H is a isotropy indecomposable Lorentzian homogeneous space then,*

$$L = \text{span}\{v \in T_oM : v \text{ null and } \exp_o(v^\perp) \text{ is a totally geodesic hypersurface}\}$$

is non-trivial.

Proof. By Theorem 2.5.4 there exists a totally geodesic null hypersurface at $o \in M$. Thus, if we consider the orthogonal line to its tangent space, this is spanned by a non-zero element of L . $\square$

2.5.2 Consequences in the irreducible case

In the case that G/H is isotropy irreducible, then L cannot be trivial by Corollary 2.5.5, hence $L = T_oM$. This leads to the following result proved by Zeghib [36].

Theorem 2.5.6 ([36]). *If G/H is a isotropy irreducible Lorentzian homogeneous space of dimension ≥ 3 . Then G/H has constant sectional curvature.*

Before we give the proof, we need the following Lemma.

Lemma 2.5.7 (Weak Schur's lemma, [21] p. 202). *If a semi-Riemannian manifold of dimension ≥ 3 has constant sectional curvature at each of its points, then this constant is the same everywhere on the manifold. Hence, the manifold has constant sectional curvature.*

We will now give the proof of Theorem 2.5.6.

Proof. The homogeneous space is isotropy irreducible, since L is isotropy invariant and non-trivial, $L = T_oM$. As a result, we can choose n null vectors η^i such that $T_oH^i = (\eta^i)^\perp$ for some null totally geodesic hypersurface H^i . For $u \in T_oM$, let A_u denote the curvature operator $A_u : T_oM \ni v \rightarrow R(u, v)u \in T_oM$. Then, for any $u \in T_oH^i$, A_u preserves T_oH^i . This is seen by the equation relating the curvature of submanifolds (see [29]),

$$\langle R(X, Y)W, Z \rangle = \langle \bar{R}(X, Y)W, Z \rangle + \langle \Pi(X, W), \Pi(Y, Z) \rangle - \langle \Pi(X, Z), \Pi(Y, W) \rangle,$$

where $\bar{R}$ is the induced curvature on H^i , and for a totally geodesic submanifold, $\Pi = 0$. Moreover, for $v \in T_oM$ then $g(A_u(\eta^i), v) = g(A_u(v), \eta^i)$ by pairwise symmetry of the curvature tensor. As a result this equals zero if $u, v \in T_oH^i$, since $A_u(v) \in T_oH^i$ and η^i is null. Hence $A_u(\eta^i) \in (T_oH^i)^\perp = \mathbb{R}\eta^i$, so $A_u(\eta^i)$ is collinear to η^i .

Now, fix i and let $E = \cap_{j \neq i} T_oH^j$, this can't be the trivial intersection as otherwise this would imply that

$$T_oM = (\cap_{j \neq i} T_oH^j)^\perp \subseteq \text{span}_{j \neq i}(\eta^j)$$

thus T_oM has dimension less than n which is absurd.

As a result, let e_i be a unit directional vector of E , and consider A_{e_i} . Since, $e_i \in T_oH^j$ for $j \neq i$, by collinearity there is $\lambda_{i,j}$ such that $A_{e_i}\eta^j = \lambda_{i,j}\eta^j$ (for $i \neq j$). However, note that as A_{e_i} is symmetric, $\lambda_{i,j}g(\eta^j, \eta^k) = g(A_{e_i}\eta^j, \eta^k) = g(A_{e_i}\eta^k, \eta^j) = \lambda_{i,k}g(\eta^j, \eta^k)$, which implies that $\lambda_{i,j} = \lambda_{i,k}$ (for $j \neq k$, $g(\eta^j, \eta^k) \neq 0$ because both η^j and η^k are null). Thus, write $\lambda_i = \lambda_{i,j}$ and hence the sectional curvature of any non-degenerate plane which contains e_i equals λ_i . From this, we conclude that $\lambda_1 = \dots = \lambda_n$ (consider 2-planes generated by two vectors e_i and e_j). As a result, all the 2-planes in T_oM have the same sectional curvature. We use Lemma 2.5.7 to complete the proof. $\square$

As a result, we get the immediate corollaries

Corollary 2.5.8. *If (M, g) is a connected Lorentzian homogenous space of dimension ≥ 3 with irreducible isotropy group. Then it also has irreducible holonomy group.*

Proof. Let (M, g) be isotropy irreducible, then from Theorem 2.5.6 M must have constant sectional curvature. That is

$$R(X, Y)Z = c(g(Z, X)Y - g(Z, Y)X), \quad c \in \mathbb{R},$$

which also implies that the curvature is parallel, i.e. $\nabla R = 0$. Then by the Theorem 2.1.6, the holonomy algebra is given by

$$\begin{aligned} \mathfrak{hol}_p(\nabla) &= \text{span}\{P_{\gamma(t)}^{-1} \circ R(P_{\gamma(t)}X, P_{\gamma(t)}Y) \circ P_{\gamma(t)} : \gamma(0) = p, X, Y \in T_pM\}, \\ &= \text{span}\{R(X, Y) : X, Y \in T_pM\} \quad (\text{ using Theorem 2.1.5 as } \nabla R = 0). \end{aligned}$$

However, note that $R(X, Y)$ spans $\mathfrak{so}(1, n+1)$, which is seen by the following identification of $\wedge^2 T^*M$ and $\mathfrak{so}(TM)$,

$$(X \wedge Y)Z = g(X, Z)Y - g(Y, Z)X.$$

Hence we will generate all skew-symmetric endomorphisms by choosing basis vectors. As a result, the holonomy is irreducible. $\square$

2.5.3 Consequences in the indecomposable case

Now, we can consider Lorentzian homogeneous spaces which are isotropy indecomposable, but not irreducible. However, we first need some properties from Lorentzian geometry.

Lemma 2.5.9. *Let V be a Lorentzian inner product space. If two null vectors are orthogonal, then they are linearly dependent.*

Proof. Let $p, q \in V$ be null vectors and assume they are orthogonal. Then, take a unit timelike vector $v \in V$ and decompose V such that $V = v^\perp \oplus \mathbb{R}v$. Hence, we have that $p = p_0 + \alpha v$, $q = q_0 + \beta v$ where $\alpha, \beta \in \mathbb{R}$. Since v is timelike, then $v^\perp$ is a spacelike hyperplane. Hence, p_0 and q_0 are spacelike. This gives us the relation

$$\begin{aligned}\langle p_0, p_0 \rangle &= \langle p - \alpha v, p - \alpha v \rangle, \\ &= \langle p, p \rangle - 2\alpha \langle p, v \rangle + \alpha^2 \langle v, v \rangle, \\ &= \alpha^2.\end{aligned}$$

Similarly, we get that $\langle q_0, q_0 \rangle = \beta^2$ and $\langle p_0, q_0 \rangle = \alpha\beta$. Then,

$$\begin{aligned}\langle \beta p_0 - \alpha q_0, \beta p_0 - \alpha q_0 \rangle &= \beta^2 \langle p_0, p_0 \rangle - 2\alpha\beta \langle p_0, q_0 \rangle + \alpha^2 \langle q_0, q_0 \rangle, \\ &= \beta^2 \alpha^2 - 2\alpha^2 \beta^2 + \alpha^2 \beta^2 = 0.\end{aligned}$$

But since $\beta p_0 - \alpha q_0$ is spacelike by $\beta p_0 - \alpha q_0 \in v^\perp$, this implies that $\beta p_0 - \alpha q_0 = 0$. Thus, as $\alpha, \beta \neq 0$ since p, q are null we get that

$$\beta p - \alpha q = \beta p_0 + \alpha\beta v - \alpha q_0 - \alpha\beta v = \beta p_0 - \alpha q_0 = 0$$

Thus, $\{p, q\}$ are linearly dependent. □

Lemma 2.5.10. *The span of two null lines is a non-degenerate subspace.*

Proof. Let $\ell_i = \mathbb{R}e_i$ be two null lines. Consider $S = \text{span}\{e_1, e_2\}$. Then, by Lemma 2.5.9 $\langle e_1, e_2 \rangle = 1$ by rescaling. For a contradiction, assume that S is degenerate. Then, there exists $x \in S$ with $x \neq 0$ such that $\langle x, y \rangle = 0$ for all $y \in S$. If this was the case,

$$0 = \langle x, y \rangle = \langle \alpha v_1 + \beta v_2, y \rangle = \alpha \langle v_1, y \rangle + \beta \langle v_2, y \rangle.$$

If $y = v_1$, then $\beta = 0$, and likewise if $y = v_2$, $\alpha = 0$. Thus, S must be non-degenerate. □

Now, we have everything to prove the following result.

Theorem 2.5.11. *If $(M \cong G/H, g)$ is a Lorentzian homogeneous space of dimension ≥ 3 such that H is indecomposable not irreducible, then there exists a filtration of auto-parallel distributions,*

$$D \subset D^\perp \subset TM,$$

where D and $D^\perp$ are distributions of lines and hyperplanes respectively.

Proof. Without loss of generality, fix a point $o \in M$ and let H be the isotropy of o . Then H is non-precompact from Lemma 2.5.3. As a result, applying Theorem 2.5.4 there exists a totally geodesic null hypersurface at o . Thus, define

$$L := \text{span}\{v \in T_o M : v \text{ null and } \exp_o(v^\perp) \text{ is a totally geodesic hypersurface}\}.$$

By Proposition 2.5.1 L is invariant under the isotropy representation and by Corollary 2.5.5 we know that L is nontrivial. Since H is indecomposable, this implies that $L \cap L^\perp \neq \{0\}$, so $L \cap L^\perp$ is a null line. However, as H is indecomposable, by Lemma 2.5.10, L can only be spanned by one vector. Thus $L \cap L^\perp = \mathbb{R}v$ where v null and $\exp_o(v^\perp)$ is totally geodesic.

Using the fact that M is homogeneous, G acts transitively, and thus for any $p \in M$ there exists $\phi \in G$ such that $\phi(o) = p$. Hence, define

$$D|_p := d\phi|_o(\mathbb{R}v) \subset T_p M.$$

This is well defined as if we have another $\psi \in G$ to ϕ such that $\psi(o) = p$, then $\phi^{-1}\psi \in H$ and thus

$$\begin{aligned} d\psi|_o(\mathbb{R}v) &= d(\phi\phi^{-1}\psi)|_o(\mathbb{R}v) \\ &= d\phi|_o \circ d(\phi^{-1}\psi)|_o(L) \\ &\subseteq d\phi|_o(\mathbb{R}v). \end{aligned}$$

Conversely, $d\phi|_o(\mathbb{R}v) \subseteq d\psi|_o(\mathbb{R}v)$ which implies $d\phi|_o(\mathbb{R}v) = d\psi|_o(\mathbb{R}v)$. As a result, we get a distribution D of null lines together with a distribution of hyperplanes $D^\perp$ such that the leafs $\exp(v^\perp)$ are totally geodesic null hypersurfaces. Hence, we get a filtration of distributions

$$D \subset D^\perp \subset TM,$$

which are auto-parallel as the Levi-Civita connection is torsion free. $\square$

Remark 2.5.12. If $D^\perp$ is auto-parallel, then $[\Gamma(D^\perp), \Gamma(D^\perp)] \subseteq \Gamma(D^\perp)$. Which implies that $D^\perp$ is integrable.

Being auto-parallel is a strictly weaker condition for a distribution than being parallel, which would be the first step towards proving that $(M \cong G/H, g)$ is a plane wave. This is because it would be the first step in trying to show that M admits a parallel null vector field. We do not know how to proceed from here in showing that D is parallel and progress towards Conjecture 1.1.2. Hence, we will follow the approach of Ambrose–Singer connections and homogeneous structures which are introduced in the next chapter.

2.6 Homogeneous plane waves

In this section we introduce Lorentzian homogeneous pp-waves and plane waves as they will appear in future chapters. *Plane waves* are a special case of a distinguished class of Lorentzian manifolds, namely the so-called *pp-waves*. One defining characteristic is that a general pp-wave has abelian holonomy algebra.

Definition 2.6.1. A pp-wave is a Lorentzian manifold (M, g) that admits a null parallel vector field V , i.e.,

$$g(V, V) = 0, \quad \nabla V = 0,$$

such that the curvature tensor is non-zero and satisfies,

$$R(X, Y) = 0, \quad \text{for all } X, Y \in V^\perp.$$

This is equivalent to $R(X, Y)Z \in \mathbb{R}V$ for all $Z \in \Gamma(V^\perp)$ and $X, Y \in \Gamma(TM)$. In particular, this implies that pp-waves have abelian holonomy $\mathbb{R}^n$. Locally, an $(n+2)$ -dimensional pp-wave has coordinates $(x_-, x_1, \dots, x_n, x_+)$ such that the metric takes the form,

$$g = 2 dx_- dx_+ + 2Q(x_1, \dots, x_n, x_+) dx_+^2 + \sum_{i=1}^n dx_i^2. \quad (2.2)$$

Here Q denotes a smooth function which does not depend on the coordinate x_- . In these coordinates the parallel null vector field is given by $V = \partial_-$, where

$$\partial_- := \frac{\partial}{\partial x_-}, \quad \partial_i := \frac{\partial}{\partial x_i}, \quad \partial_+ := \frac{\partial}{\partial x_+}.$$

In such coordinates, the Levi-Civita connection, is given by,

$$\nabla \partial_- = 0, \quad \nabla \partial_i = (\partial_i Q) dx_+ \otimes \partial_-, \quad \nabla \partial_+ = dQ \otimes \partial_- - dx_+ \otimes \text{grad}(Q). \quad (2.3)$$

Where $\text{grad}(Q) = \delta^{ij} \partial_i(Q) \partial_j$ denotes the gradient of H with respect to the flat metric $\sum_{i=1}^n dx_i^2$.

The curvature tensor of a pp-wave and its first covariant derivative are given by,

$$R = 4 \sum_{i,j=1}^n Q_{ij} (dx_i \wedge dx_+) \cdot (dx_j \wedge dx_+), \quad \nabla R = 4 \sum_{i,j=1}^n dQ_{ij} \otimes (dx_i \wedge dx_+) \cdot (dx_j \wedge dx_+),$$

where $Q_{ij} := \partial_i \partial_j Q$.

The special case that we are interested in are the so called plane waves. These are pp-waves for which the curvature tensor is also parallel in the direction of $V^\perp$.

Definition 2.6.2. A plane wave is a pp-wave for which the curvature satisfies,

$$\nabla_X R = 0, \text{ for all } X \in V^\perp.$$

This means that for a pp-wave to be a plane wave, in local coordinates the function Q must satisfy $\partial_i \partial_j \partial_k Q = 0$ for all $i, j, k = 1, \dots, n$. Therefore, forcing Q to be a quadratic function on the x_i 's. This means,

$$Q(\mathbf{x}, x_+) = \frac{1}{2} \mathbf{x}^\top S(x_+) \mathbf{x},$$

where $\mathbf{x}$ denotes the vector $(x_1, \dots, x_n)^\top$, and $S(x_+)$ is a symmetric $n \times n$ -matrix which depends on x_+ .

We now restrict to the homogeneous case. Locally homogeneous pp-waves were studied by Globke and Leistner [19]. They showed that if the rank of the curvature endomorphism is greater than 1, they must be homogeneous plane waves. Moreover, they showed that locally homogeneous plane waves are reductive.

Homogeneous plane waves were classified by Blau and O'Loughlin [6] into two families. They found that homogeneous plane waves are parametrised by a constant symmetric matrix S_0 and a constant matrix $F \in \mathfrak{so}(n)$.

With homogenous plane waves being in the first family if,

$$S(x_+) = e^{x_+ F} S_0 e^{-x_+ F}, \quad (2.4)$$

and the second,

$$S(x_+) = \frac{1}{(x_+)^2} e^{F \log(x_+)} S_0 e^{-F \log(x_+)}. \quad (2.5)$$

The first family generalise the so called *Cahen-Wallach spaces* with constant matrix S to which they reduce to when $[S, F] = 0$, whereas the second family generalise the so called *singular plane waves* when $F = 0$. Thus, we will say (M, g) is regular if $S(x_+)$ is as (2.4), and singular if $S(x_+)$ is as (2.5).

Moreover, Blau and O'Loughlin [6] showed that a transitive isometry algebra $\mathfrak{g}$ of a homogeneous plane wave is spanned by $\{X, A^i, B^j, Z\}$ for $i, j = 1, \dots, n$, with the relations,

$$\begin{aligned} [A^i, B^j] &= -\delta_{ij} Z, \\ [X, A^i] &= F_{is} A^s + (a_0 x_0 + b_0) (S_0)_{is} B^s, \\ [X, B^i] &= (a_0 x_0 - b_0) A^i + (a_0 \delta_{is} + F_{is}) B^s, \\ [X, Z] &= a_0 Z, \end{aligned}$$

for constants a_0, b_0, x_0 . The regular plane waves have an isometry algebra with $b_0 = x_0 = 1$ and $a_0 = 0$, while the singular plane waves have an algebra with $a_0 = x_0 = 1$ and $b_0 = 0$.

It can be seen that $\{A^i, B^j, Z\}$ generate the $2n + 1$ dimensional Heisenberg algebra $\mathfrak{he}(n)$, resulting in $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{he}(n)$ and as a vector space $\mathfrak{g} = \mathfrak{he}(n) \oplus \mathbb{R}$. The isotropy algebra is given by $\text{span}\{B^j : j = 1, \dots, n\}$ and is isomorphic to $\mathbb{R}^n$, hence indecomposable.

Chapter 3

Ambrose–Singer Connections and Homogeneous Structures

In this chapter, we introduce the key machinery of this thesis, Ambrose–Singer connections and their associated homogeneous structure and clarify the relationship between the holonomy of an Ambrose–Singer connection and the isotropy of a homogeneous space. We begin this chapter in Section 3.1 by introducing Ambrose–Singer connections and their associated homogeneous structure. Much of this section follows Calvaruso’s and Castrillon Lopez’s *Pseudo-Riemannian Homogeneous Structures* [9]. In Section 3.2, we present the relation between holonomy of Ambrose–Singer connections and isotropy groups of a homogeneous space. Specifically, that there always exists a presentation as G/H such that the isotropy is the holonomy of an Ambrose–Singer connection. Then, in Section 3.3, we present the eight different algebraic types of homogeneous structures (and torsion) based on the irreducible decomposition of $\wedge^1 \otimes \wedge^2$. We end this chapter in Section 3.4, presenting the known results in the literature towards Conjecture 1.1.2 which all require the homogeneous structure (torsion) to have a specific algebraic type.

3.1 Ambrose–Singer connections

When considering reductive homogeneous semi-Riemannian manifolds, an alternative interpretation stems from a theorem of Ambrose and Singer. This was first proved in the Riemannian case [2], and later generalised to metrics of arbitrary signature [16]. It states that (M, g) is a reductive homogeneous semi-Riemannian manifold if and only if there exists a metric affine connection with parallel torsion and parallel curvature.

Theorem 3.1.1 (Ambrose–Singer). *Let (M, g) be a simply-connected and complete semi-Riemannian manifold. The following properties are equivalent:*

1. (M, g) is a reductive homogeneous semi-Riemannian manifold.
2. (M, g) admits a affine connection $\tilde{\nabla}$ satisfying

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0,$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$.

For a proof of Theorem 3.1.1 we refer to [9].

Remark 3.1.2. If global conditions are dropped, i.e. simply-connected or completeness, homogeneous is weakened to locally homogeneous. See [9] for more details.

A connection $\tilde{\nabla}$ as in (2) of Theorem 3.1.1 is called a Ambrose–Singer connection (AS-connection). A major class of AS-connections were introduced in Section 2.3, the canonical connection associated to a reductive decomposition. The difference between the Levi-Civita connection ∇ and the connection $\tilde{\nabla}$ is a $(2, 1)$ -tensor field $S : TM \rightarrow \text{End}(TM)$, defined by

$$S(X)Y := \nabla_X Y - \tilde{\nabla}_X Y,$$

and since both connections are metric connections,

$$g(S(X)Y, Z) + g(Y, S(X)Y) = 0,$$

hence $S : TM \rightarrow \mathfrak{so}(TM)$.

Definition 3.1.3. Let $\tilde{\nabla}$ be an AS-connection on (M, g) . The tensor field $S = \nabla - \tilde{\nabla}$ is called a homogeneous structure of M .

The curvature and torsion of a AS-connection can be expressed in terms of its homogeneous structure and the curvature of the Levi-Civita connection.

Lemma 3.1.4. *The torsion and curvature of $\tilde{\nabla}$ can be expressed as*

$$\begin{aligned} \tilde{T}(X, Y) &= S(Y)X - S(X)Y, \\ \tilde{R}(X, Y) &= R(X, Y) + [S(X), S(Y)] + S(\tilde{T}(X, Y)). \end{aligned}$$

Proof. By the definition of the torsion,

$$\begin{aligned} \tilde{T}(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y], \\ &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - (\nabla_X Y - \nabla_Y X), \\ &= S(Y)X - S(X)Y. \end{aligned}$$

Likewise, by the definition of the curvature,

$$\begin{aligned}
\tilde{R}(X, Y) &= [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X, Y]}, \\
&= [\nabla_X - S(X), \nabla_Y - S(Y)] - \nabla_{[X, Y]} + S([X, Y]), \\
&= R(X, Y) - [\nabla_X, S(Y)] - [S(X), \nabla_Y] + [S(X), S(Y)] + S([X, Y]), \\
&= R(X, Y) + [S(X), S(Y)] + S(\tilde{T}(X, Y)).
\end{aligned}$$

□

Proposition 3.1.5. *The equations in Theorem 3.1.1 are equivalent to*

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0,$$

where R is the curvature with respect to the Levi-Civita connection.

Proof. The equivalence is a direct consequence of Lemma 3.1.4. □

Remark 3.1.6. Since the tensor field S is invariant under the action of the Lie group G acting transitively on (M, g) (also because its parallel), it is completely determined by its value at the origin $o \in M$.

It must be stated that homogeneous structures are not unique and a homogeneous space can admit more than one homogeneous structure. Different homogeneous structures on (M, g) may give rise to different presentations of M as a coset space. Before this can be explained, we need the following definitions and theorems.

Definition 3.1.7. Two homogeneous structures S_1 and S_2 on a semi-Riemannian manifold (M, g) are isomorphic if there exists an isometry $\phi : M \rightarrow M$ such that

$$\phi^*S_1 = S_2.$$

Now, let $\mathfrak{g}_1 = \mathfrak{m}_1 \oplus \mathfrak{h}_1$ and $\mathfrak{g}_2 = \mathfrak{m}_2 \oplus \mathfrak{h}_2$ be two reductive decompositions of M and their associated homogeneous structures S_1 and S_2 such that $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ are the Ambrose–Singer connections associated to each S . Then, we have the following result.

Theorem 3.1.8 ([9]). *If S_1 and S_2 are isomorphic on M , then there exists a Lie algebra isomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $\phi(\mathfrak{m}_1) = \mathfrak{m}_2$ and $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$. Moreover, the restriction $\phi|_{\mathfrak{m}_1}$ is an isometry.*

The converse theorem is the following.

Theorem 3.1.9 ([9]). *Let (M_1, g_1) and (M_2, g_2) be simply-connected and complete semi-Riemannian manifolds with homogeneous structures S_1 and S_2 respectively. If there exists a Lie algebra isomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $\phi(\mathfrak{m}_1) = \mathfrak{m}_2$, $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$ and $\phi|_{\mathfrak{m}_1}$ is an isometry, then S_1 and S_2 are isomorphic.*

We can now expand upon the situation when a homogeneous space admits more than one homogeneous structure. In such a situation, two cases can occur as described in [9, 34]:

- There exist two homogeneous structures S_1 and S_2 which are not isomorphic but which give rise to the same Lie algebra $\mathfrak{g}$ but with different decompositions,

$$\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{h}_1 = \mathfrak{m}_2 \oplus \mathfrak{h}_2.$$

This means that there does not exist a Lie algebra automorphism ϕ of $\mathfrak{g}$ such that $\phi(\mathfrak{m}_1) = \mathfrak{m}_2$, $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$, and $\phi|_{\mathfrak{m}_1}$ is an isometry.

- There exist two homogeneous structures S_1 and S_2 with non-isomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. This means that we have different representations of the homogeneous space as a coset space G/H .

We now present some examples of the situations described above occurring.

Example 3.1.10. An example of the first situation occurring is on the Heisenberg group G . The Heisenberg group G is diffeomorphic to $\mathbb{R}^3$ and,

$$g = dx^2 + dz^2 + (dy - xdz)^2, \quad (3.1)$$

is a left invariant metric on G . Moreover, let $\theta_1 = dx$, $\theta_2 = dy$, and $\theta_3 = dy - xdz$. Based on the following theorem H admits an infinite number of homogeneous structures.

Theorem 3.1.11 ([34]). *Let (G, g) denote the Heisenberg group with the left invariant metric (3.1). Then all the homogeneous structures S on (H, g) are given by,*

$$S_\mu = 2\mu\theta_3 \otimes (\theta_1 \wedge \theta_2) + \theta_1 \otimes (\theta_2 \wedge \theta_3) + \theta_2 \otimes (\theta_3 \wedge \theta_1),$$

where $\mu \in \mathbb{R}$.

Moreover, all S_μ , $\mu \in \mathbb{R}$ are non-isomorphic. When $\mu = -1/2$, the corresponding group of isometries is the group G itself acting by left translations. For any other $\mu \neq -1/2$, the corresponding group is the connected component of the identity of $G \ltimes \mathrm{SO}(2)$ but giving rise to different decompositions. For more details, see [34].

Example 3.1.12. An example of the second situation occurring can be seen on the 6-dimensional sphere $\mathbb{S}^6$. It has the two presentations,

$$\mathbb{S}^6 = \frac{\mathrm{SO}(7)}{\mathrm{SO}(6)} = \frac{\mathrm{G}_2}{\mathrm{SU}(2)},$$

which correspond to two different homogeneous structures with non-isomorphic Lie algebras $\mathfrak{so}(7)$ and $\mathfrak{g}_2$.

3.2 Holonomy of Ambrose-Singer connections

In this section, the link between isotropy algebras $\mathfrak{h}$ of reductive decompositions and holonomy algebras $\mathfrak{hol}(\tilde{\nabla})$ of AS-connections shall be discussed. Recall that given a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then $[\cdot, \cdot]_{\mathfrak{h}}$ and $[\cdot, \cdot]_{\mathfrak{m}}$ shall denote the projections onto $\mathfrak{h}$ and $\mathfrak{m}$ respectively.

Proposition 3.2.1 ([22]). *Let G/H be a connected reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the subspace $\mathfrak{g}' = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} + \mathfrak{m}$ is an ideal of $\mathfrak{g}$ and the corresponding connected normal subgroup $G' \subset G$ acts transitively on G/H by left translations.*

Proof. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition, the following holds $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Thus,

$$\begin{aligned} [\mathfrak{g}', \mathfrak{g}] &= [[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} + \mathfrak{m}, \mathfrak{m} + \mathfrak{h}] \\ &= [[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{m}] + [[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{h}] + [\mathfrak{m}, \mathfrak{m}] + [\mathfrak{m}, \mathfrak{h}] \\ &\subseteq [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} + \mathfrak{m} + [[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{h}]. \end{aligned}$$

Thus, we just need to show that $[[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{h}]$ is contained in $\mathfrak{m}$ or $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$. From the Jacobi identity it can be seen that $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{h}] \subseteq [\mathfrak{m}, \mathfrak{m}]$, and thus taking projections,

$$[[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}, \mathfrak{h}] + [[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{h}] \subseteq [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} + [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}.$$

But note that $[[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{h}]$ must be in $\mathfrak{h}$ and hence $[[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{h}] \subseteq [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$ and we are done. Thus, $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$ and the corresponding connected Lie subgroup $G' \subset G$ is normal.

For transitivity of the Lie group G' with lie algebra $\mathfrak{g}'$, consider for every $gH \in G/H$ the projection $\pi_{gH} : G \rightarrow G/H \cong M$ defined by $s \mapsto sgH$. Then, $\pi_{gH}|_{G'} : G' \rightarrow G/H$ has constant rank and $d_e \pi_{gH}(\mathfrak{g}') = d_e \pi_{gH}(\mathfrak{m}) = T_{gH}(G/H)$. As a result $\pi_{gH}|_{G'}$ is a submersion and $\pi_{gH}(G') \subset G/H$ is open. Thus all the orbits are open by definition of π . Now, fix $o \in G/H$ where $o = eH$ and denote $S = \{gH \in G/H : gH \notin \pi_o(G')\}$, then consider

$$G/H = \pi_o(G') \cup \bigcup_{gH \in S} \pi_{gH}(G').$$

But then we have written the connected set G/H as a disjoint union of open sets, and since the first set is nonempty, this must mean that the second one is empty, and hence $\pi_{eH}(G') = G/H$. This implies that G' acts transitively on G/H . $\square$

Proposition 3.2.2. *Let H' be the isotropy group of G' at $o \in M$. Then the Lie algebra $\mathfrak{h}'$ of H' is isomorphic to the holonomy algebra $\mathfrak{hol}(\tilde{\nabla})$ of the canonical connection $\tilde{\nabla}$ of the reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$.*

Proof. The Lie algebra of H' is isomorphic to $\mathfrak{g}' \cap \mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \cap \mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$. Thus, $\mathfrak{h}' = \text{span}\{[X, Y]_{\mathfrak{h}} : X, Y \in \mathfrak{m}\}$ which is equal to $\text{span}\{\tilde{R}_o(X, Y) : X, Y \in \mathfrak{m}\}$, which, since $\tilde{\nabla}\tilde{R} = 0$, by the Ambrose–Singer holonomy theorem is equal to $\mathfrak{hol}(\tilde{\nabla})$. $\square$

As a result, we have that $M = G'/H'$ where $\mathfrak{g}' = \mathfrak{hol}(\tilde{\nabla}) \oplus \mathfrak{m}$ is the reductive decomposition. This leads us to the following definition for any reductive homogeneous space M with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Definition 3.2.3. The presentation $M = G/H$, is called the transvection presentation of M when $\mathfrak{g} = \mathfrak{hol}(\tilde{\nabla}) \oplus \mathfrak{m}$ is the reductive decomposition and $\tilde{\nabla}$ is a AS-connection on M .

Thus, if we start with a reductive homogeneous space, we can always “reduce” the presentation of M to the transvection presentation, where the isotropy is equal to the holonomy of an AS-connection on M .

Conversely, given any homogeneous structure S on a homogeneous pseudo-Riemannian manifold (M, g) , if we fix a point $o \in M$, let $\mathfrak{m} = T_o M$, and consider the holonomy algebra $\mathfrak{hol}(\tilde{\nabla})$ of $\tilde{\nabla} = \nabla - S$, which is spanned by all endomorphisms $\tilde{R}_o(X, Y)$, where $X, Y \in \mathfrak{m}$, according to Theorem 2.1.6. Then, we can define a Lie algebra $\mathfrak{g} = \mathfrak{hol}(\tilde{\nabla}) \oplus \mathfrak{m}$ with Lie bracket defined by

$$\begin{aligned} [A, B] &= AB - BA, \quad A, B \in \mathfrak{hol}(\tilde{\nabla}), \\ [A, X] &= A \cdot X, \quad A \in \mathfrak{hol}(\tilde{\nabla}), X \in \mathfrak{m}, \\ [X, Y] &= \tilde{R}_o(X, Y) + \tilde{T}_o(X, Y), \quad X, Y \in \mathfrak{m}. \end{aligned}$$

Proposition 3.2.4 ([22]). *The brackets above form a Lie algebra, moreover the splitting is reductive.*

Proof. To show it does indeed define a Lie algebra on $\mathfrak{g}$, the only non-trivial part is to show the Jacobi identity. This follows however from the first and second Bianchi identities together with the fact that both $\tilde{R}$ and S are parallel and $\tilde{T}(X, Y) = S(Y)X - S(X)Y$.

Reductivity follows from the definition of the brackets. $\square$

We recover a transvection algebra and one has the unique simply connected Lie group G with Lie algebra $\mathfrak{g}$ and the unique connected subgroup H with Lie algebra $\mathfrak{hol}(\tilde{\nabla})$. The data $(\mathfrak{m}, \tilde{R}_o, \tilde{T}_o)$ defining such a splitting is called an infinitesimal model of the locally homogeneous space (M, g) .

Definition 3.2.5. An infinitesimal model is said to be regular if H is closed in G .

In the case that the infinitesimal model is regular, we can take the homogeneous space G/H which is sometimes called the homogeneous model of S .

Proposition 3.2.6 ([9]). *The manifold (M, g) is locally isometric to $(G/H, g)$.*

Remark 3.2.7. We remark that during the process of constructing the homogeneous space from a homogeneous structure S , one can loose information about the original group of isometries. For example, take Euclidean space $\mathbb{R}^n$, then the full group of isometries is $\mathrm{SO}(n) \ltimes \mathbb{R}^n$ and as $\mathbb{R}^n$ is a symmetric space with such isometry group, one gets the homogeneous structure $S = 0$. However, if one starts with $S = 0$ on $\mathbb{R}^n$, then one can only construct the group of translations $\mathbb{R}^n$ acting on $\mathbb{R}^n$. Hence, one has lost information about the rotations $\mathrm{SO}(n)$.

3.3 Algebraic types of homogeneous structures

The space of tensor fields which are homogeneous structures are contained in sections of the vector bundle $\wedge^1(T^*M) \otimes \mathfrak{so}(TM, g)$, using the metric, this is isomorphic to the sub-bundle,

$$\wedge^1(T^*M) \otimes \wedge^2(T^*M) \subset \otimes^3(T^*M),$$

where we are thinking of S as a $(0, 3)$ -tensor field,

$$S(X, Y, Z) = g(S(X, Y), Z).$$

Remark 3.3.1. We shall often abuse notation and drop the bundle/vector space when referring to spaces such as $\wedge^2(T^*M)$. For example, the sub-bundle above might be denoted $\wedge^1 \otimes \wedge^2$.

In dimension ≥ 3 , the bundle $\wedge^1 \otimes \wedge^2$ breaks up into the sum of three sub-bundles,

$$\wedge^1 \otimes \wedge^2 = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3,$$

each one corresponding to an irreducible representation of the orthogonal group. In terms of Young tableaux [15], this decomposition is given by

$$\wedge^1 \otimes \wedge^2 = \square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_0 \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad (3.2)$$

where the subscript 0 denotes being trace free with respect to g . More explicitly, the bundles can be described as follows:

1. $\mathcal{T}_1$ corresponds to elements S of $\wedge^1 \otimes \wedge^2$ that are pure trace, that is,

$$S(X, Y, Z) = g(X, Y)\omega(Z) - g(X, Z)\omega(Y)$$

for some one-form ω .

2. $\mathcal{T}_2$ corresponds to elements S of $\wedge^1 \otimes \wedge^2$ that are trace free, that is,

$$S(X, Y, Z) + S(Z, X, Y) + S(Y, Z, X) = 0,$$

and are in the kernel of the map $C : \bigotimes^3 \square \rightarrow \square$ defined by:

$$C(S)(Z) := \sum_{a,b} g(e_a, e_b) S(e_a, e_b, Z),$$

where $\{e_a\}$ is a pseudo-orthonormal frame.

3. $\mathcal{T}_3$ corresponds to elements S of $\wedge^1 \otimes \wedge^2$ which are totally skew-symmetric. That is:

$$S(X, Y, Z) + S(Y, X, Z) = 0.$$

Throughout this thesis, we will often use the Young tableaux to denote where S belongs, and it should be clear from context if S is a section/element of the respective bundle/vector space. However, we shall also call S ,

- totally skew if and only if $S \in \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$,
- twistorial if and only if $S \in \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}_0$,
- vectorial if and only if $S \in \square$.

We can write down the explicit expressions for each of the projections of S onto each factor. Specifically, using abstract index notation [30], where the Young tableaux denotes the projection onto the corresponding module:

$$S_{abc} = \square(S_{abc}) \oplus \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}_0(S_{abc}) \oplus \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}(S_{abc}),$$

where,

$$\square(S_{abc}) = -\frac{2}{n-1} S_{e[b}^e g_{c]a},$$

$$\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}_0(S_{abc}) = S_{abc} - \square(S_{abc}) - \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}(S_{abc}),$$

$$\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}(S_{abc}) = \frac{1}{3}(S_{abc} + S_{bca} + S_{cab}).$$

Given a reductive locally homogeneous space (M, g) , the Ambrose-Singer theorem (Theorem 3.1.1) guarantees the existence of a tensor field $S \in \wedge^1 \otimes \wedge^2$. We can distinguish eight types of homogeneous structures from the eight invariant subspaces of $\wedge^1 \otimes \wedge^2$,

$$\{0\}, \quad \square, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_0, \quad \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad \square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_0, \quad \square \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_0 \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad \square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_0 \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}.$$

We also saw in Lemma 3.1.4, that the torsion $T \in \wedge^2 \otimes \wedge^1$ of an AS-connection can be given as $T_{abc} = -2S_{[ab]c}$. In fact, if $A \in \wedge^1 \otimes \wedge^2$, and $B \in \wedge^2 \otimes \wedge^1$, the map,

$$\wedge^1 \otimes \wedge^2 \xrightarrow{\phi} \wedge^2 \otimes \wedge^1,$$

where $\phi(A)_{abc} = A_{[ab]c}$ is an isomorphism between $\wedge^1 \otimes \wedge^2$ and $\wedge^2 \otimes \wedge^1$. Its inverse is given by $\phi^{-1}(B)_{abc} = B_{abc} + 2B_{c(ab)}$. Thus, given this isomorphism, the torsion completely determines the homogeneous structure and visa versa.

Clearly both $\wedge^1 \otimes \wedge^2$ and $\wedge^2 \otimes \wedge^1$ can be split into irreducibles as given in (3.2), and ϕ descends to maps between these which must respect this splitting. It is clear that the isomorphism induced from ϕ between the 3-forms must be the identity map. However, the isomorphism induced from ϕ between $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_0$ can be viewed as the adjoint map respect to g .

Lemma 3.3.2. *Let $\tilde{\phi}$ be the isomorphism induced from ϕ ,*

$$\wedge^1 \otimes \wedge^2 \supset \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \xrightarrow{\tilde{\phi}} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \subset \wedge^2 \otimes \wedge^1.$$

Then $\tilde{\phi}$ can be identified with the adjoint map respect to g . That is if $A \in \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \subset \wedge^1 \otimes \wedge^2$, then

$$\tilde{g}(A(X), \omega) = g(\tilde{\phi}(A)(\omega), X),$$

where $\tilde{g}$ is the inner product induced on $\wedge^2$ from g .

Proof. Let $A \in \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \subset \wedge^1 \otimes \wedge^2$ and denote $B = \phi(A)$. Then, we want to show,

$$(A^a_{bc} X_a) \omega^{bc} = (B^{ab}_c \omega_{ab}) X^c,$$

where $\omega \in \wedge^2$. We shall start with the RHS. We know $B_{abc} = A_{[ab]c}$, and $B^{ab}_c = g^{sa} g^{tb} B_{stc}$. Hence,

$$\begin{aligned} (B^{ab}_c \omega_{ab}) X^c &= g^{sa} g^{tb} A_{[st]c} \omega_{ab} X^c, \\ &= \frac{1}{2} (A_{stc} - A_{tsc}) g^{sa} g^{tb} \omega_{ab} X^c, \\ &= \frac{1}{2} (A_{stc} - A_{tsc}) \omega^{st} X^c. \end{aligned}$$

Now, A also enjoys $A_{abc} + A_{cab} + A_{bca} = 0$, and since $A \in \wedge^1 \otimes \wedge^2$, is skew in the last two indices. Meaning,

$$\frac{1}{2}(A_{stc} - A_{tsc})\omega^{st}X^c = -\frac{1}{2}A_{cst}\omega^{st}X^c.$$

Thus, $(B^{ab}{}_c\omega_{ab})X^c = -\frac{1}{2}A_{cst}\omega^{st}X^c$, and contracting with respect to c we get,

$$\begin{aligned} (B^{ab}{}_c\omega_{ab})X^c &= -\frac{1}{2}A_{cst}\omega^{st}X^c, \\ &= -\frac{1}{2}g_{ci}A^i{}_{st}\omega^{st}X^c, \\ &= -\frac{1}{2}(A^i{}_{st}X_i)\omega^{st}. \end{aligned}$$

Which by relabeling indices gives $(B^{ab}{}_c\omega_{ab})X^c = -\frac{1}{2}(A^a{}_{bc}X_a)\omega^{bc}$ as needed. Implying that $\tilde{\phi}$ is the adjoint map with respect to g . $\square$

Proposition 3.3.3. *The isomorphism between $\wedge^1 \otimes \wedge^2$ and $\wedge^2 \otimes \wedge^1$ is given by the identity map on $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, and the adjoint on $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$.*

3.4 Results for Lorentzian homogeneous structures

Let $(M, g, \tilde{\nabla})$ be a semi-Riemannian homogeneous space with AS-connection $\tilde{\nabla}$. In this section we shall compile the known results in the Lorentzian signature about the geometry of M assuming $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. These results assume extra constraints on S (or $\tilde{T}$) through the different irreducible sub-modules of $\wedge^1 \otimes \wedge^2$.

In the case when S is vectorial, then S takes the form,

$$S(X)Y = g(X, Y)\xi - g(Y, \xi)X,$$

for some vector field $\xi \in \Gamma(TM)$. Thus, we say that S is degenerate if ξ is null, and S non-degenerate otherwise. The condition that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible forces S to be degenerate. This can be seen by first noting that ξ is $\tilde{\nabla}$ -parallel since S is $\tilde{\nabla}$ -parallel. Then, if $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible ξ must be null. The case when S is degenerate was investigated by Montesinos Amilibia [3].

Theorem 3.4.1 (Montesinos Amilibia [3]). *Let (M, g) be a connected Lorentzian homogeneous space of dimension $n + 2$ admitting a degenerate vectorial homogeneous structure. Then (M, g) is locally isometric to a singular plane wave.*

Remark 3.4.2. The theorem proved in [3] actually finds the explicit form of the metric g , and can be applied to semi-Riemannian manifolds of any signature. When we restrict to the Lorentzian signature, we get the metric of a singular plane wave.

Now, consider the case when S is totally skew. These were studied by Ernst and Galaev [13] in the broader context of Lorentzian connections with totally skew parallel torsion. When applied to AS-connections, they classified all naturally reductive Lorentzian homogeneous spaces of dimension ≥ 4 .

Theorem 3.4.3 (Ernst and Galaev [13]). *Let (M, g) be a simply-connected indecomposable Lorentzian homogeneous space of dimension ≥ 4 admitting a totally skew homogeneous structure. Moreover, assume $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, (M, g) must be a regular homogeneous plane wave.*

We move on to the case when S has a non-zero vectorial part and a non-zero totally skew part. Then, as in the case when S is vectorial, we can distinguish the vectorial component as either degenerate or non-degenerate. If $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, then as in the pure vectorial case, the $\mathcal{T}_1$ component of S must be degenerate. Such geometries were investigated by Meessen in [26].

Theorem 3.4.4 (Meessen [26]). *A connected Lorentzian homogeneous space that admits a $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure such that both components are non-zero, and the $\mathcal{T}_1$ component is degenerate, is a singular homogeneous plane wave.*

For the other irreducible modules of $\wedge^1 \otimes \wedge^2$, we are unaware of any results in the literature involving S having a twistorial component. Hence, we make the following conjecture based on the above results.

Conjecture 3.4.5. *Let (M, g) be a simply-connected Lorentzian homogeneous space of dimension ≥ 4 with an AS-connection $\tilde{\nabla}$. If $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible then $\mathfrak{hol}(\nabla) \subseteq \mathbb{R}^n$. Moreover, (M, g) is a homogeneous plane wave.*

The reminder of the thesis will be dedicated to proving this conjecture.

Chapter 4

Algebraic Results

In this chapter, we consider indecomposable, non-irreducible subalgebras of $\mathfrak{so}(1, n+1)$. Specifically, in Section 4.1, we introduce indecomposable, non-irreducible subalgebras of $\mathfrak{so}(1, n+1)$, our notation going forward in the thesis, and the classification by Bérard-Bergery and Ikemakhen [4] of these algebras into four types. Section 4.2 will be dedicated to computing their centralisers. Next, Section 4.3 and Section 4.4 will investigate tensors in $\wedge^1 \otimes \wedge^2$ annihilated by algebras of type 2 and type 4, and how the irreducible decomposition of $\wedge^1 \otimes \wedge^2$ classifies these tensors. In Section 4.5 we look at the spaces of algebraic curvature tensors with a specific form of torsion for algebras of type 2 and type 4. Lastly, in Section 4.6, we apply the same analysis as previous sections but in dimension 3.

4.1 Indecomposable subalgebras of $\mathfrak{so}(1, n+1)$

In this section we introduce indecomposable, non-irreducible subalgebras $\mathfrak{h}$ of $\mathfrak{so}(1, n+1)$ that will be heavily used in the rest of the thesis.

Definition 4.1.1. A subalgebra $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ is called indecomposable if it does not preserve any non-degenerate proper subspace of $\mathbb{R}^{1, n+1}$.

Lemma 4.1.2. *If $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ is indecomposable, non-irreducible, then $\mathfrak{h}$ preserves a null line in $\mathbb{R}^{1, n+1}$.*

Proof. Since $\mathfrak{h}$ is indecomposable, non-irreducible, then it preserves a degenerate proper subspace $U \subset \mathbb{R}^{1, n+1}$. Hence, $L := U \cap U^\perp$ is a null line and must also be preserved by $\mathfrak{h}$. $\square$

Thus, we find that any indecomposable, non-irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ must be contained in the stabiliser of a null line.

Henceforth, let (V, g) be Minkowski space of dimension $n + 2 \geq 3$ where g is a inner product on $\mathbb{R}^{n+2}$ of signature $(1, n + 1)$. Fix a basis $(e_-, e_1, \dots, e_n, e_+)$ of V such that,

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.1)$$

Definition 4.1.3. A basis $(e_-, e_1, \dots, e_n, e_+)$ of V such that the inner product takes the form of (4.1) will be called a Witt basis.

This gives us a grading of V as,

$$V = V_- \oplus V_0 \oplus V_+, \quad (4.2)$$

where $V_- := \mathbb{R}e_-$, $V_0 := \text{span}(e_1, \dots, e_n) \cong \mathbb{R}^n$, and $V_+ := \mathbb{R}e_+$. Moreover, we define a basis of V^* as,

$$e^- := g(e_+, \cdot), \quad e^i := g(e_i, \cdot), \quad e^+ := g(e_-, \cdot),$$

which is dual to $(e_-, e_1, \dots, e_n, e_+)$. We also set,

$$V^- := \mathbb{R}e^- = V_-^*, \quad V^0 := \text{span}(e^i : i = 1, \dots, n) = V_0^*, \quad V^+ := \mathbb{R}e^+ = V_+^*.$$

With respect to a Witt basis, the Lie algebra $\mathfrak{so}(1, n + 1)$ is conjugated to the matrix Lie algebra,

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & z^\top & 0 \\ w & A & -z \\ 0 & -w^\top & -a \end{pmatrix} : a \in \mathbb{R}, z, w \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.$$

If we now define,

$$\begin{aligned} \mathfrak{g}_+ &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ w & 0 & 0 \\ 0 & -w^\top & 0 \end{pmatrix} : w \in \mathbb{R}^n \right\}, \\ \mathfrak{g}_0 &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, A \in \mathfrak{so}(n) \right\}, \\ \mathfrak{g}_- &= \left\{ \begin{pmatrix} 0 & z^\top & 0 \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{pmatrix} : z \in \mathbb{R}^n \right\}. \end{aligned}$$

We get the decomposition,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-,$$

with $\mathfrak{g}_+ \cong \mathbb{R}^n$, $\mathfrak{g}_- \cong \mathbb{R}^n$, and $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathfrak{so}(n)$ corresponding to (a, A) . It can be checked that this turns $\mathfrak{g}$ into a graded Lie algebra. Moreover, it interacts with the grading of V given in (4.2) as follows,

$$\mathfrak{g}_i(V_j) \subset V_{i+j}, \quad i, j \in \{-, 0, +\}, \quad \mathfrak{g}_i(V_j) = \{0\} \quad i + j \notin \{-, 0, +\}. \quad (4.3)$$

Remark 4.1.4. It should be remarked that the convention in the literature concerning the grading of $\mathfrak{g}$ is often with the labels of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ swapped. That is, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, see [10]. We adopt this non-standard convention in this thesis for (4.3) to remain true.

Any element of $\mathfrak{g}$ can be identified with a triple. For example,

$$\begin{pmatrix} a & z^\top & 0 \\ w & A & -z \\ 0 & -w^\top & -a \end{pmatrix} \in \mathfrak{g},$$

can be identified with the triple $(w, (a, A), z) \in \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$. The commutator relations are given as follows:

$$\begin{aligned} [(w, (a, A), z), (u, (b, B), v)] = \\ ((A - aI)u - (B - bI)w, (z^\top u - v^\top w, [A, B]_{\mathfrak{so}(n)}), (A + aI)v - (B + bI)z). \end{aligned}$$

The stabiliser of the null line $\mathbb{R}e_-$ can be identified with the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{g}_-$, or equivalently,

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & z^\top & 0 \\ 0 & A & -z \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, z \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.$$

Any element of $\mathfrak{p}$ can be identified with $(0, (a, A), z)$ and the commutator relations are given as follows:

$$[(0, (a, A), z), (0, (b, B), v)] = (0, (0, [A, B]_{\mathfrak{so}(n)}), (A + aI)v - (B + bI)z).$$

Moreover, since $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{g}_-$, $\mathbb{R}$, $\mathfrak{so}(n)$, and $\mathfrak{g}_-$ define subalgebras of $\mathfrak{p}$. Specifically, $\mathbb{R}$ is an abelian subalgebra of $\mathfrak{p}$, commuting with $\mathfrak{so}(n)$, and $\mathfrak{g}_-$ is an abelian ideal in $\mathfrak{p}$. Moreover, $\mathfrak{so}(n)$ is the semisimple part and $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{so}(n)$ the reductive part of $\mathfrak{p}$. Since $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{g}_-$, we get the further decomposition,

$$\mathfrak{p} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathfrak{g}_-.$$

We will assign to a subalgebra $\mathfrak{h} \subset \mathfrak{p}$ the projections $\mathfrak{h}_0 := \text{pr}_{\mathfrak{g}_0}(\mathfrak{h})$ and $\mathfrak{h}_- := \text{pr}_{\mathfrak{g}_-}(\mathfrak{h})$. We can also consider the projections of $\mathfrak{h}$ onto $\mathbb{R}$ and $\mathfrak{so}(n)$ inside $\mathfrak{g}_0$. It should be noted that if $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$, then $\text{pr}_{\mathfrak{g}_0}(\mathfrak{h}) = \text{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$. The following properties hold.

Lemma 4.1.5 ([17]). *If $\mathfrak{h}$ is an indecomposable subalgebra of $\mathfrak{p}$, then*

- $\mathfrak{h}_- = \mathfrak{g}_-$,

- $\mathfrak{h}$ has a trivial sub-representation in V if and only if $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$,
- $\mathfrak{h}$ is abelian if and only if $\mathfrak{h} = \mathfrak{g}_-$,
- $\text{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$ is a compact Lie algebra and hence reductive.

As a result, any indecomposable, non-irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ must be conjugated to an indecomposable subalgebra contained in $\mathfrak{p}$ such that the null line is exactly $\mathbb{R}e_-$. Specifically, such algebras have been classified by Bérard-Bergery and Ikemakhen [4].

Theorem 4.1.6 (Bérard-Bergery and Ikemakhen [4]). *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{p} \subset \mathfrak{so}(1, n+1)$, acting indecomposably on $\mathbb{R}^{n+2}$, and let $\mathfrak{h}_{\mathfrak{so}} := \text{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$. Then $\mathfrak{h}$ belongs to one of the following types.*

1. If $\mathfrak{h}$ contains $\mathfrak{g}_-$, then there are three types:

Type 1 : $\mathfrak{h}$ contains $\mathbb{R}$. Then $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{h}_{\mathfrak{so}}) \ltimes \mathfrak{g}_-$.

Type 2 : $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$, i.e. $\mathfrak{h} = \mathfrak{h}_{\mathfrak{so}} \ltimes \mathfrak{g}_-$.

Type 3 : Neither type 1 nor type 2. In this case there exists an epimorphism $\phi : \mathfrak{h}_{\mathfrak{so}} \rightarrow \mathbb{R}$ such that $\psi|_{[\mathfrak{h}_{\mathfrak{so}}, \mathfrak{h}_{\mathfrak{so}}]} = 0$, in which case

$$\mathfrak{h} = \mathfrak{l} \ltimes \mathfrak{g}_-$$

where $\mathfrak{l} := \text{Graph}(\phi) = \{(\phi(T), T) : T \in \mathfrak{h}_{\mathfrak{so}}\} \subset \mathbb{R} \oplus \mathfrak{h}_{\mathfrak{so}}$, or written in matrix form

$$\mathfrak{h} = \left\{ \begin{pmatrix} \phi(B) & v^\top & 0 \\ 0 & B & -v \\ 0 & 0 & -\phi(B) \end{pmatrix} : B \in \mathfrak{h}_{\mathfrak{so}}, v \in \mathbb{R}^n \right\}.$$

2. In the case where $\mathfrak{h}$ does not contain $\mathfrak{g}_-$, we have:

Type 4 : There exists a non-trivial decomposition $\mathfrak{g}_- = V_1 \oplus V_2$, and an epimorphism $\psi : \mathfrak{h}_{\mathfrak{so}} \rightarrow V_1$ such that $\psi|_{[\mathfrak{h}_{\mathfrak{so}}, \mathfrak{h}_{\mathfrak{so}}]} = 0$. In which case $\mathfrak{h}_{\mathfrak{so}} \subset \mathfrak{so}(V_2)$ and,

$$\mathfrak{h} = \mathfrak{l} \ltimes V_2$$

where $\mathfrak{l} := \{(\psi(T), T) : T \in \mathfrak{h}_{\mathfrak{so}}\} = \text{Graph}(\psi) \subset V_1 \oplus \mathfrak{h}_{\mathfrak{so}}$, or written in matrix form

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(B)^\top & v^\top & 0 \\ 0 & 0 & 0 & -\psi(B) \\ 0 & 0 & B & -v \\ 0 & 0 & 0 & 0 \end{pmatrix} : B \in \mathfrak{h}_{\mathfrak{so}}, v \in V_2 \right\}.$$

4.2 The centraliser of indecomposable subalgebras

In this section we compute the centraliser of indecomposable, non-irreducible subalgebras $\mathfrak{h} \subset \mathfrak{g}$ as this will be needed in subsequent sections. We shall denote the centraliser of $\mathfrak{h} \subset \mathfrak{g}$ as $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$.

Proposition 4.2.1. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable, non-irreducible subalgebra and $\dim(V) \geq 3$. If $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$, then the centraliser of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$, satisfies $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}_-$, otherwise $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = 0$.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable, non-irreducible subalgebra. To determine the centraliser of $\mathfrak{h}$ in $\mathfrak{g}$, we need to determine all $X \in \mathfrak{g}$ such that $[\mathfrak{h}, X] = 0$.

Let $X = (w, (a, A), v) \in \mathfrak{g}$ be an arbitrary element of $\mathfrak{g}$, and let $(0, (b, B), z) \in \mathfrak{h}$. If $[(w, (a, A), v), (0, (b, B), z)] = 0$, then this implies

$$((B - bI)w, (z^{\top}w, [A, B]|_{\mathfrak{so}(n)}), (B + bI)v - (A + aI)z) = 0.$$

Resulting in the equations,

$$\begin{aligned} (B - bI)w &= 0, \\ z^{\top}w &= 0, \\ [A, B]|_{\mathfrak{so}(n)} &= 0, \\ (B + bI)v - (A + aI)z &= 0, \end{aligned}$$

for all $(b, B) \in \mathfrak{h}_0$ and $z \in \mathfrak{g}_-$. Since for indecomposable, non-irreducible subalgebras $\mathfrak{h}_- = \mathfrak{g}_-$, the second equation must imply that $w = 0$.

Now, suppose that $\text{pr}_{\mathbb{R}}(\mathfrak{h}) \neq 0$. Then, using the equation $(B + bI)v - (A + aI)z = 0$, taking the inner product with respect to $z \neq 0$,

$$\langle Bv, z \rangle = \langle az - bv, z \rangle.$$

Since $\text{pr}_{\mathbb{R}}(\mathfrak{h}) \neq 0$ and $\mathfrak{g}_- \subset \mathfrak{h}$ when $\text{pr}_{\mathbb{R}}(\mathfrak{h}) \neq 0$, we can let $z = v$ and $b \neq 0$ such that $a - b \neq 0$. The resulting equation becomes $(a - b)\langle v, v \rangle = 0$ implying $v = 0$. This turns the last equation into the eigenvalue problem $Az = -az$, however $A \in \mathfrak{so}(n)$ meaning all eigenvalues are imaginary. Thus, $a = 0$ which forces $A = 0$. As a result, $X = 0$ implying $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = 0$.

Now, suppose that $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$. Since $w = 0$, from the above equations,

$$\begin{aligned} [A, B]|_{\mathfrak{so}(n)} &= 0, \\ Bv - (A + aI)z &= 0, \end{aligned}$$

for all $(0, B) \in \mathfrak{h}_0$ and $z \in \mathfrak{h}_-$. The last equation implies $a = 0$ by considering when $z = v$ resulting in the eigenvalue problem. All we have to now show is $A = 0$. To do so we will consider each type of indecomposable, non-irreducible subalgebras in which $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$ separately.

Case 1 ($\mathfrak{h}$ type 2) :

The last equation has become $Bv - Az = 0$. We can let $B = 0$ in type 2 with no condition on z . Hence, by $\mathfrak{h}_- = \mathfrak{g}_-$, $A = 0$.

Case 2 ($\mathfrak{h}$ type 4) :

In this case,

$$z = \begin{pmatrix} \psi(B) \\ x \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Thus, let

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ and } A = \begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_3 \end{pmatrix}.$$

The equation $Bv - Az = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_3 \end{pmatrix} \begin{pmatrix} \psi(B) \\ x \end{pmatrix} = 0.$$

Resulting in the two equations

$$\begin{aligned} A_1\psi(B) + A_2x &= 0, \\ Bv_2 + A_2^t\psi(B) - A_3x &= 0. \end{aligned}$$

We can pick $B = 0$ so that $\psi(B) = 0$, resulting in $A_2x = 0$ and $A_3x = 0$. But this must imply that $A_2 = A_3 = 0$ by freedom in x . Then, when $B \neq 0$, the first equation becomes $A_1\psi(B) = 0$. As ψ is surjective, $A_1 = 0$. Hence, we have just shown $A = 0$.

In both cases $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}_-$. □

4.3 Trivial $\mathfrak{h}$ -modules of $\wedge^1 \otimes \wedge^2$

In this section, suppose that $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ is indecomposable, non-irreducible such that $\text{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$, i.e type 2 or type 4. Then, we are interested in determining the $\mathfrak{h}$ -trivial submodule $\mathcal{S} \subset \wedge^1 \otimes \wedge^2$, that is,

$$\mathcal{S} := \{S \in \wedge^1 \otimes \wedge^2 : \mathfrak{h} \cdot S = 0\}.$$

Theorem 4.3.1. *Let $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ with $n \geq 2$ be a indecomposable, non-irreducible subalgebra of type 2 or type 4. Suppose that $S \in \mathcal{S}$, then,*

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathfrak{p},$$

such that,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i.$$

Moreover, let $\mathfrak{h}_0 = \text{pr}_{\mathfrak{g}_0}(\mathfrak{h})$, then $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_0}(S(e_+)) = 0$. Furthermore, in type 2, $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$, while in type 4, $(\psi^{-1}(V_1) \oplus \mathfrak{h}'_0) \cdot \text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$.

Proof. Consider a Witt basis $(e_-, e_1, \dots, e_n, e_+)$ of Minkowski space V and the corresponding dual basis $e^+, e^1, \dots, e^n, e^-$ where,

$$e^+ = g(e_-, \cdot), \quad e^i = g(e_i, \cdot), \quad e^- = g(e_+, \cdot).$$

Let $S \in \wedge^1 V \otimes \wedge^2 V$, that is

$$S = S_{AB}{}^C e^A \otimes e^B \otimes e_C. \quad (4.4)$$

We shall identify $\wedge^2 V$ with $\mathfrak{g}$ through the metric. Hence, set

$$S(e_-) = \begin{pmatrix} a_- & \omega_-^\top & 0 \\ \eta_- & Q_- & -\omega_- \\ 0 & -\eta_-^\top & -a_- \end{pmatrix}, S(e_i) = \begin{pmatrix} a_i & \omega_i^\top & 0 \\ \eta_i & Q_i & -\omega_i \\ 0 & -\eta_i^\top & -a_i \end{pmatrix}, S(e_+) = \begin{pmatrix} a_+ & \omega_+^\top & 0 \\ \eta_+ & Q_+ & -\omega_+ \\ 0 & -\eta_+^\top & -a_+ \end{pmatrix}, \quad (4.5)$$

where $\omega_A = S_{A+}{}^j e_j \in \mathbb{R}^n$, $\eta_A = S_{A-}{}^j e_j \in \mathbb{R}^n$, $a_A = S_{A-}{}^- \in \mathbb{R}$ and $Q_A = S_{Ai}{}^j \in \mathfrak{so}(n)$.

Since S is annihilated by $\mathfrak{h}$,

$$[H, SX] - S(HX) = 0, \quad \forall H \in \mathfrak{h}. \quad (4.6)$$

Suppose $\mathfrak{h}$ is of type 2 or type 4, we shall consider

$$H = \begin{pmatrix} 0 & z^\top & 0 \\ 0 & B & -z \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h},$$

where $z \in \mathbb{R}^n$ and $B \in \mathfrak{h}_0$. In (4.6), let $X = e_-$, then $H \cdot e_- = 0$. Therefore, equation (4.6) becomes,

$$[H, S(e_-)] = 0.$$

This implies $S(e_-)$ is in the centraliser of $\mathfrak{h}$ and from Proposition 4.2.1 we know $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_-$. Thus,

$$S(e_-) = \begin{pmatrix} 0 & \omega_-^\top & 0 \\ 0 & 0 & -\omega_- \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.7)$$

Now, to continue, we shall see what equation (4.6) implies when $X = e_i$. Let $X = e_i$, then $He_i = g(z, e_i)e_- + Be_i$, hence (4.6) takes the form,

$$[H, S(e_i)] - g(z, e_i)S(e_-) + \sum_{j=1}^n B_{ij}S(e_j) = 0. \quad (4.8)$$

From (4.5) and (4.7), equation (4.8) is equivalent to the matrix equation,

$$\left[\begin{pmatrix} 0 & z^\top & 0 \\ 0 & B & -z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha_i & \omega_i^\top & 0 \\ \eta_i & Q_i & -\omega_i \\ 0 & -\eta_i^\top & -\alpha_i \end{pmatrix} \right] - g(z, e_i) \begin{pmatrix} 0 & \omega_-^\top & 0 \\ 0 & 0 & -\omega_- \\ 0 & 0 & 0 \end{pmatrix} + \sum_{j=1}^n B_{ij} \begin{pmatrix} \alpha_j & \omega_j^\top & 0 \\ \eta_j & Q_j & -\omega_j \\ 0 & -\eta_j^\top & -\alpha_j \end{pmatrix} = 0.$$

Giving rise to the equations,

$$z^\top \eta_i + \sum_{j=1}^n B_{ij} \alpha_j = 0, \quad (4.9)$$

$$Q_i z + \alpha_i z + g(e_i, z) \omega_- - B \omega_i - \sum_{j=1}^n B_{ij} \omega_j = 0. \quad (4.10)$$

We shall now consider each type of $\mathfrak{h}$ separately. That is, when $\mathfrak{h}$ is of type 2 or type 4.

Case 1 ($\mathfrak{h}$ of type 2) :

Since $\mathfrak{h}$ is of type 2, both equations (4.9) and (4.10) must hold for all $B \in \mathfrak{h}_0$ and all $z \in \mathbb{R}^n$. We shall first see what both equations imply when $B = 0$. Thus, we have that,

$$\begin{aligned} z^\top \eta_i &= 0, \\ Q_i z + \alpha_i z + g(e_i, z) \omega_- &= 0. \end{aligned}$$

Since both must hold for all $z \in \mathbb{R}^n$, the first equation implies that $\eta_i = 0$. For the second equation, we aim to show that $\alpha_i = 0$, $Q_i = 0$, and $\omega_- = 0$. Hence, since $n > 1$, take $z = e_j$ such that $j \neq i$, then,

$$Q_i e_j + \alpha_i e_j = 0,$$

implying $\alpha_i = 0$, since $Q_i \in \mathfrak{so}(n)$. Now, aiming to show that $\omega_- = 0$, take $z = \omega_-$, then together with $\alpha_i = 0$,

$$Q_i \omega_- + g(e_i, \omega_-) \omega_- = 0,$$

which again means $g(e_i, \omega_-) = 0$ since $Q_i \in \mathfrak{so}(n)$, however since this is true for each i , $\omega_- = 0$. As a result, equation (4.10) when $B = 0$ has been reduced to,

$$Q_i z = 0,$$

resulting in $Q_i = 0$. We currently have,

$$S(e_-) = 0, \quad S(e_i) = \begin{pmatrix} 0 & \omega_i^\top & 0 \\ 0 & 0 & -\omega_i \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.11)$$

Now, we look at (4.6) when $X = e_+$. It follows that $He_+ = -z$, and (4.6) becomes,

$$[H, S(e_+)] + S(z) = 0.$$

This is equivalent to the matrix equation,

$$\left[\begin{pmatrix} 0 & z^\top & 0 \\ 0 & B & -z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha_+ & \omega_+^\top & 0 \\ \eta_+ & Q_+ & -\omega_+ \\ 0 & -\eta_+^\top & -\alpha_+ \end{pmatrix} \right] + \sum_{i=1}^n g(z, e_i) \begin{pmatrix} 0 & \omega_i^\top & 0 \\ 0 & 0 & -\omega_i \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

which grants the equations,

$$z^\top \eta_+ = 0, \quad (4.12)$$

$$[B, Q_+] = 0, \quad (4.13)$$

$$Q_+ z + \alpha_+ z - B\omega_+ - \sum_{i=1}^n g(e_i, z)\omega_i = 0. \quad (4.14)$$

Equation (4.12) implies $\eta_+ = 0$, and equation (4.13) is equivalent to $\mathfrak{h}_0 \cdot Q_+ = 0$. Left with the third equation, we aim to use it to deduce the relationship between ω_i and the $S(e_+)$ terms.

Setting $B = 0$ in equation (4.14) results in,

$$Q_+ z + \alpha_+ z - \sum_{i=1}^n g(e_i, z)\omega_i = 0.$$

Let $z = e_i$, then

$$\omega_i = (Q_+ + \alpha_+ I)e_i.$$

Now, consider when $z = 0$ in equation (4.14), leaving us with $B\omega_+ = 0$ implying $\mathfrak{h}_0 \cdot \omega_+ = 0$.

As a result,

$$S(e_-) = 0, S(e_i) = \begin{pmatrix} 0 & \omega_i^\top & 0 \\ 0 & 0 & -\omega_i \\ 0 & 0 & 0 \end{pmatrix}, S(e_+) = \begin{pmatrix} \alpha_+ & \omega_+^\top & 0 \\ 0 & Q_+ & -\omega_+ \\ 0 & 0 & -\alpha_+ \end{pmatrix},$$

where $\omega_i = (Q_+ + \alpha_+ I)e_i$, $\mathfrak{h}_0 \cdot Q_+ = 0$, and $\mathfrak{h}_0 \cdot \omega_+ = 0$. Hence,

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathfrak{p},$$

and the relation $\omega_i = (Q_+ + \alpha_+)e_i$ is equivalent to,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i.$$

Moreover, $\mathfrak{h}_0 \cdot Q_+ = 0$ and $\mathfrak{h}_0 \cdot \omega_+ = 0$ imply $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_0}(S(e_+)) = 0$ and $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$ respectively.

Case 2 ($\mathfrak{h}$ of type 4) :

Since, $\mathfrak{h}$ is of type 4, there exists a splitting of $\mathfrak{g}_- = V^1 \oplus V^2$ such that $\mathfrak{h}_0 \subset \mathfrak{so}(V^2)$. Moreover, there exists a surjective linear map $\psi : \mathfrak{h}_0 \rightarrow V^1$ such that $\psi|_{\mathfrak{h}'_0} = 0$. Thus, the H we have been using previously is of the form $z = (\psi(B), x)^\top$ and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, B \in \mathfrak{h}_0.$$

Before we start, our convention in this case is that $\dim(V^1) = l$ and $\dim(V^2) = m$. Moreover, the indices i, j, k will run over the dimension of $\mathfrak{g}_- \cong \mathbb{R}^n$, the indices r, s, t will run the dimension of V^1 , and the indices a, b, c will run over the dimension of V^2 . Explicitly, $i, j, k \in \{1, \dots, n\}$, $r, s, t \in \{1, \dots, l\}$, and $a, b, c \in \{l+1, \dots, l+m\}$. Furthermore, $(e_1, \dots, e_l)$ and $(e_{l+1}, \dots, e_{l+m})$ will be a basis for V^1 and V^2 respectively. Lastly, we note that $m \geq 2$ for the surjection ψ to exist.

Now, just like in the previous case, we consider the equations (4.9) and (4.10). We first aim to show that $\alpha_i = 0$ for all i . Thus, taking (4.10) when $B = 0$,

$$Q_r z + \alpha_r z = 0,$$

and,

$$Q_a z + \alpha_a z + g(e_a, z)\omega_- = 0,$$

where $z \in V^2$.

The first equation implies that $\alpha_r = 0$ for all r since $Q_r \in \mathfrak{so}(n)$, which further implies $Q_r|_{V^2} = 0$. For the second equation, since $m \geq 2$, we can take $z = e_b$ such that $b \neq a$ to get,

$$Q_a e_b + \alpha_a e_b = 0,$$

resulting in $\alpha_a = 0$ for all a by the same reasoning as above. Hence, $\alpha_i = 0$ for all $i \in \{1, \dots, n\}$, which together with equation (4.9) implies $\eta_i = 0$ for all $i \in \{1, \dots, n\}$. Thus, (4.10) is now,

$$Q_i z + g(e_i, z)\omega_- - B\omega_i - \sum_{j=1}^n B_{ij}\omega_j = 0. \quad (4.15)$$

We next aim to show that $\omega_- = 0$. Hence, decompose ω_- as $\omega_- = (\omega_-^1, \omega_-^2)^\top$, and take $B = 0$ and $z = (0, \omega_-^2)^\top$ in equation (4.15), giving the equation,

$$Q_a \omega_-^2 + g(e_a, \omega_-^2)\omega_- = 0,$$

implying $\omega_-^2 = 0$. To show that $\omega_-^1 = 0$, look at (4.15) when $i = r$ and $z = (\psi(B), 0) \in V^1$, then $B_{rj} = 0$ and $B\omega_r = (0, B\omega_r^2)^\top$, hence

$$Q_r \begin{pmatrix} \psi(B) \\ 0 \end{pmatrix} + g \left(e_r, \begin{pmatrix} \psi(B) \\ 0 \end{pmatrix} \right) \begin{pmatrix} \omega_-^1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ B\omega_r^2 \end{pmatrix} = 0,$$

and by surjectivity in ψ , there exists a $B \in \mathfrak{h}_0$ such that $\psi(B) = \omega_-^1$ which in the equation above implies $\omega_-^1 = 0$. Hence, $\omega_- = 0$ and this means that $Q_i|_{V^2} = 0$ for all $i \in \{1, \dots, n\}$, which follows from (4.15) when $B = 0$. However, this means that Q_i is of the form,

$$Q_i = \begin{pmatrix} Q_i^1 & 0 \\ 0 & 0 \end{pmatrix}, Q_i^1 \in \mathfrak{so}(l),$$

thus we want to show that $Q_i^1 = 0$. Equation (4.15) is now of the form,

$$Q_i z - B\omega_i - \sum_{j=1}^n B_{ij}\omega_j = 0.$$

Take $z = (\psi(B), 0)^\top$,

$$\begin{pmatrix} Q_i^1 \psi(B) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \omega_i^1 \\ \omega_i^2 \end{pmatrix} - \sum_{j=1}^n \begin{pmatrix} 0 & 0 \\ 0 & B_{ij} \end{pmatrix} \begin{pmatrix} \omega_j^1 \\ \omega_j^2 \end{pmatrix} = 0,$$

implying $Q_i^1 \psi(B) = 0$ which means $Q_i^1 = 0$ since ψ is surjective. Hence, we are left with the equation,

$$B\omega_i + \sum_{j=1}^n B_{ij}\omega_j = 0, \tag{4.16}$$

and we currently have,

$$S(e_-) = 0, \quad S(e_i) = \begin{pmatrix} 0 & \omega_i^\top & 0 \\ 0 & 0 & -\omega_i \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.17}$$

Lastly, turning our attention to when $X = e_+$, then exactly as when $\mathfrak{h}$ is of type 2, we get the equations,

$$z^\top \eta_+ = 0, \tag{4.18}$$

$$[B, Q_+] = 0, \tag{4.19}$$

$$Q_+ z + \alpha_+ z - B\omega_+ - \sum_{i=1}^n g(e_i, z)\omega_i = 0. \tag{4.20}$$

Equation (4.18) implies $\eta_+ = 0$, and equation (4.19) is equivalent to $\mathfrak{h}_0 \cdot Q_+ = 0$. Left with the third equation, just like when $\mathfrak{h}$ is of type 2, we aim to use it to deduce the relationship between ω_i and the $S(e_+)$ terms.

Hence, take equation (4.20) when $B = 0$ and $z = (0, x)^\top$,

$$Q_+ \begin{pmatrix} 0 \\ x \end{pmatrix} + \alpha_+ \begin{pmatrix} 0 \\ x \end{pmatrix} - \sum_b g(e_b, x) \omega_b = 0.$$

Taking $x = e_a$ implies

$$\omega_a = (Q_+ + \alpha_+ I) e_a.$$

Now, to determine what ω_r looks like, take equation (4.20) when $z = (\psi(B), 0)^\top$,

$$Q_+ \begin{pmatrix} \psi(B) \\ 0 \end{pmatrix} + \alpha_+ \begin{pmatrix} \psi(B) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \omega_+^1 \\ \omega_+^2 \end{pmatrix} - \sum_s g(e_s, \psi(B)) \omega_s = 0.$$

Taking $\psi(B) = e_r$ implies

$$\omega_r = (Q_+ + \alpha_+ I) e_r - B_r \omega_+, \quad \psi(B_r) = e_r.$$

We would like to show that $B_r \omega_+ = 0$ for type 4 to be consistent with type 2. We can use equation (4.16), which states,

$$B \omega_i + \sum_{j=1}^n B_{ij} \omega_j = 0.$$

Hence, applying this for $i = r \in \{1, \dots, l\}$, in which case $B_{rj} = 0$ leaving us with $B \omega_r = 0$. Moreover, $\omega_r = (Q_+ + \alpha_+ I) e_r - B_r \omega_+$. Thus,

$$\begin{aligned} 0 &= B \omega_r \\ &= B \cdot Q_+ e_r + \alpha_+ B e_r - B \cdot B_r \omega_+. \end{aligned}$$

This implies,

$$B \cdot B_r \omega_+ = 0, \quad \forall B \in \mathfrak{h}_0, \tag{4.21}$$

since $\mathfrak{h}_0 \cdot Q_+ = 0$ and $B e_r = 0$. However, picking $B = B_r$ in equation (4.21) implies

$$B_r^2 \omega_+ = 0,$$

which must imply that $B_r \omega_+ = 0$ since $B_r \in \mathfrak{so}(n)$. Thus, we have that $\omega_i = (Q_+ + \alpha_+ I) e_i$ for all $i \in \{1, \dots, n\}$. Moreover, we also get from above that $(\psi^{-1}(V^1) \oplus \mathfrak{h}'_0) \cdot \omega_+ = 0$.

As a result,

$$S(e_-) = 0, S(e_i) = \begin{pmatrix} 0 & \omega_i^\top & 0 \\ 0 & 0 & -\omega_i \\ 0 & 0 & 0 \end{pmatrix}, S(e_+) = \begin{pmatrix} \alpha_+ & \omega_+^\top & 0 \\ 0 & Q_+ & -\omega_+ \\ 0 & 0 & -\alpha_+ \end{pmatrix},$$

where $\omega_i = (Q_+ + \alpha_+ I)e_i$, $\mathfrak{h}_0 \cdot Q_+ = 0$, and $(\psi^{-1}(V^1) \oplus \mathfrak{h}'_0) \cdot \omega_+ = 0$. Hence,

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathfrak{p},$$

and the relation $\omega_i = (Q_+ + \alpha_+ I)e_i$ is equivalent to,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i.$$

Moreover, $\mathfrak{h}_0 \cdot Q_+ = 0$ implies $\mathfrak{h}_0 \cdot \text{pr}_{\mathfrak{g}_0}(S(e_+)) = 0$, and $(\psi^{-1}(V^1) \oplus \mathfrak{h}'_0) \cdot \omega_+ = 0$ implies $(\psi^{-1}(V^1) \oplus \mathfrak{h}'_0) \cdot \text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$. $\square$

Remark 4.3.2. It should be highlighted, that a $S \in \wedge^1 \otimes \wedge^2$ such that $\mathfrak{h} \cdot S = 0$, is completely determined by $S(e_+) \in \mathfrak{p}$. This is a result of the condition,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i,$$

which completely controls $S(e_i)$. This can be seen clearly by writing S out explicitly. Based on the proof above, S can be written explicitly as,

$$S(e_-) = 0, S(e_i) = \begin{pmatrix} 0 & e_i^\top(\alpha I - A) & 0 \\ 0 & 0 & -(\alpha I + A)e_i \\ 0 & 0 & 0 \end{pmatrix}, S(e_+) = \begin{pmatrix} \alpha & \eta^\top & 0 \\ 0 & A & -\eta \\ 0 & 0 & -\alpha \end{pmatrix},$$

where $\alpha \in \mathbb{R}$, $\eta \in \mathbb{R}^n$, and $A \in \mathfrak{so}(n)$.

4.4 Algebraic types of trivial $\mathfrak{h}$ -modules

In Chapter 3, we introduced the irreducible decomposition of $\wedge^1 \otimes \wedge^2$ and how it allows the classification of homogeneous structures depending on which collection of irreducible modules it belongs to.

In this section, we are interested in how each irreducible module of $\wedge^1 \otimes \wedge^2$ intersects the vector space $\mathcal{S}$ defined in the last section. That is, determining the vector subspaces:

$$\mathcal{S}_1 := \{S \in \mathcal{S} : S \in \square\}, \quad \mathcal{S}_2 := \left\{S \in \mathcal{S} : S \in \begin{array}{|c|c|} \hline \square & \\ \hline \square & 0 \\ \hline \end{array}\right\}, \quad \mathcal{S}_3 := \left\{S \in \mathcal{S} : S \in \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}\right\}.$$

Theorem 4.4.1. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable, non-irreducible subalgebra of type 2 or type 4. Suppose $S \in \mathcal{S}$, then,*

- $S \in \mathcal{S}_3$ if and only if $S(e_+) \in \mathfrak{so}(n)$,
- $S \in \mathcal{S}_2$ if and only if $S(e_+) \in \mathfrak{g}_-$,
- $S \in \mathcal{S}_1$ if and only if $S(e_+) \in \mathbb{R}$.

We highlight that the theorem above enjoys a nice correspondence between the decomposition of an $S \in \Lambda^1 \otimes \Lambda^2$ into its irreducible modules,

$$\Lambda^1 \otimes \Lambda^2 = \square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array},$$

and $S(e_+) \in \mathfrak{p}$ in terms of the decomposition of the parabolic subalgebra,

$$\mathfrak{p} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathfrak{g}_-.$$

Written explicitly, for an $S \in \mathcal{S}$,

$$\mathfrak{p} \ni S(e_+) = \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix}}_{\text{pr}_{\mathcal{S}_1}(S)(e_+)} \oplus \underbrace{\begin{pmatrix} 0 & \eta^\top & 0 \\ 0 & 0 & -\eta \\ 0 & 0 & 0 \end{pmatrix}}_{\text{pr}_{\mathcal{S}_2}(S)(e_+)} \oplus \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{pr}_{\mathcal{S}_3}(S)(e_+)}.$$

Before, we can prove Theorem 4.4.1, we need to prove the following two Lemmas that will be used in the proof.

Lemma 4.4.2. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable, non-irreducible subalgebra of type 2 or type 4. Suppose $S \in \mathcal{S}$, then $S \in \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$ if and only if $S(e_+) \in \mathfrak{so}(n)$.*

Proof. Let $S \in \Lambda^1 \otimes \Lambda^2$ such that $\mathfrak{h} \cdot S = 0$, where $\mathfrak{h} \subset \mathfrak{g}$ is indecomposable, non-irreducible of type 2 or type 4. Then, Theorem 4.3.1 states,

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathfrak{p},$$

such that,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i. \quad (4.22)$$

Suppose, $S \in \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$. Then, $S(e_+)e_- = 0$ since $S(e_-) = 0$, thus implying $\text{pr}_{\mathbb{R}}(S(e_+)) = 0$. Moreover, $S(e_i)e_+ = -S(e_+)e_i$, but using (4.22), and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_-$, this is equivalent to

$$\text{pr}_{\mathfrak{g}_-}(S(e_+))e_i = 0,$$

which implies $\text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$. Hence, since $\text{pr}_{\mathbb{R}}(S(e_+)) = 0$ and $\text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$, then $S(e_+) \in \mathfrak{so}(n)$.

Conversely, let S be as stated in the Lemma. We aim to show that S is totally skew. Writing out S explicitly, it would be of the form,

$$S(e_-) = 0, \quad S(e_i) = \begin{pmatrix} 0 & (Ae_i)^\top & 0 \\ 0 & 0 & -Ae_i \\ 0 & 0 & 0 \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.23)$$

where $A \in \mathfrak{so}(n)$. From (4.23), it is clear that $S(X)X = 0$ since $A \in \mathfrak{so}(n)$. Moreover, $S(X)e_- = -S(e_-)X = 0$ since $S(e_i) \in \mathfrak{g}_-$ and $\text{pr}_{\mathbb{R}}(S(e_+)) = 0$. Thus, we just need to check $S(e_i)e_j = -S(e_j)e_i$ and $S(e_i)e_+ = -S(e_+)e_i$. The second follows from $S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i$ since $\text{pr}_{\mathfrak{g}_0}(S(e_+)) = S(e_+)$, while the first follows from checking that $S(e_i)e_j = A(e_i, e_j)$ and $S(e_j)e_i = A(e_j, e_i) = -A(e_i, e_j)$. $\square$

Remark 4.4.3. Lemma 4.4.2 recovers the second part of Lemma 1 in (Ernst and Galaev [13]). They show that if $S \in \wedge^3$ such that it is annihilated by an indecomposable non-irreducible subalgebra $\mathfrak{h}$ of type 2 or type 4, then $S = e^+ \wedge A$ such that $A \in \wedge^2 \mathbb{R}^n$ where $\text{pr}_{\mathfrak{so}(n)}(\mathfrak{h}) \cdot A = 0$.

Lemma 4.4.4. Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable, non-irreducible subalgebra of type 2 or type 4. Suppose $S \in \mathcal{S}$, then $S \in \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$ if and only if $S(e_+) \in \mathbb{R} \ltimes \mathfrak{g}_-$.

Proof. Let $S \in \wedge^1 \otimes \wedge^2$ such that $\mathfrak{h} \cdot S = 0$. From Theorem 4.3.1,

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathfrak{p}, \quad (4.24)$$

such that,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i. \quad (4.25)$$

Since $S \in \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$ if and only if $\begin{smallmatrix} \square \\ \square \end{smallmatrix}(S) = 0$, Lemma 4.4.2 implies $S \in \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$ if and only if

$$\text{pr}_{\mathfrak{so}(n)}\left(\begin{smallmatrix} \square \\ \square \end{smallmatrix}(S)(e_+)\right) = 0. \quad (4.26)$$

This means that,

$$\begin{smallmatrix} \square \\ \square \end{smallmatrix}(S)(e_+, e_i, e_j) = 0. \quad (4.27)$$

However,

$$\begin{smallmatrix} \square \\ \square \end{smallmatrix}(S)(e_+)(X, Y) = \langle S(e_+)X, Y \rangle + \langle S(Y)e_+, X \rangle - \langle S(X)e_+, Y \rangle.$$

Thus, equation (4.27) is equivalent to

$$\langle S(e_+)e_i, e_j \rangle + \langle S(e_j)e_+, e_i \rangle - \langle S(e_i)e_+, e_j \rangle = 0,$$

which applying the condition (4.25), is

$$\begin{aligned} 0 &= \langle S(e_+)e_i, e_j \rangle - \langle \text{pr}_{\mathfrak{g}_0}(S(e_+))e_j, e_i \rangle + \langle \text{pr}_{\mathfrak{g}_0}(S(e_+))e_i, e_j \rangle \\ &= 3 \text{pr}_{\mathfrak{so}(n)}(S(e_+))(e_i, e_j). \end{aligned}$$

Hence, $\text{pr}_{\mathfrak{so}(n)}(S(e_+)) = 0$, implying $S(e_+) \in \mathbb{R} \ltimes \mathfrak{g}_-$.

Conversely, let S be as stated in the Lemma. From Theorem 4.3.1, it is clear that $S \in \wedge^1 \otimes \wedge^2$ such that $\mathfrak{h} \cdot S = 0$. However, since $S(e_+) \in \mathbb{R} \ltimes \mathfrak{g}_-$, then $S(e_+) \notin \mathfrak{so}(n)$.

This implies $S \notin \begin{smallmatrix} \square \\ \square \end{smallmatrix}$ from lemma 4.4.2, so we conclude that $S \in \begin{smallmatrix} \square & \square \\ & \end{smallmatrix}$. $\square$

Now Theorem 4.4.1 can be proved.

Proof. We shall prove each case in order, as we use some cases to prove the latter cases.

Case 1: Totally Skew. This is proved in Lemma 4.4.2.

Case 2: Twistorial. Let $S \in \begin{smallmatrix} \square & \square \\ & \end{smallmatrix}$ such that $\mathfrak{h} \cdot S = 0$ where $\mathfrak{h} \subset \mathfrak{g}$ is indecomposable, non-irreducible of type 2 or type 4. Then, Lemma 4.4.4 states,

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathbb{R} \ltimes \mathfrak{g}_-, \quad (4.28)$$

such that,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i. \quad (4.29)$$

Suppose $S \in \begin{smallmatrix} \square & \square \\ & \end{smallmatrix}_0$, thus S is trace free, implying,

$$S(e_-, e_+, X) + S(e_+, e_-, X) + \sum_{i=1}^n S(e_i, e_i, X) = 0, \quad \forall X \in V. \quad (4.30)$$

Let $X = e_+$, then $S(e_-, e_+, e_+) = \text{pr}_{\mathbb{R}}(S(e_+))$, and using the condition (4.29), $S(e_i, e_i, e_+) = \text{pr}_{\mathbb{R}}(S(e_+))$. Hence, (4.30) would imply,

$$\text{pr}_{\mathbb{R}}(S(e_+))(1 + n) = 0,$$

and thus $\text{pr}_{\mathbb{R}}(S(e_+)) = 0$. As a result, $S(e_+) \in \mathfrak{g}_-$.

Conversely, let S be as stated in the Theorem. That is, $S(e_+) \in \mathfrak{g}_-$. Then $S \in \begin{smallmatrix} \square & \square \\ & \end{smallmatrix}$ from Lemma 4.4.4, so we just need to show that S is trace-free. Based on (4.30), the trace condition reduces to,

$$S(e_+, e_-, X) = 0, \quad \forall X \in V.$$

However, $S(e_+)e_- = 0$ since $S(e_+) \in \mathfrak{g}_-$. Thus, we conclude that S is trace-free.

Case 3: Vectorial. Let $S \in \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ such that $\mathfrak{h} \cdot S = 0$ where $\mathfrak{h} \subset \mathfrak{g}$ is indecomposable, non-irreducible of type 2 or type 4. Then, Lemma 4.4.4 states,

$$S(e_-) = 0, \quad S(e_i) \in \mathfrak{g}_-, \quad S(e_+) \in \mathbb{R} \ltimes \mathfrak{g}_-, \quad (4.31)$$

such that,

$$S(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S(e_+))e_i. \quad (4.32)$$

Suppose $S \in \square$, then there exists a $U \in V$ such that,

$$S(X)Y = g(X, Y)U - g(Y, U)X.$$

However,

$$0 = S(e_-)Y = g(e_-, Y)U - g(Y, U)e_-,$$

implies U must be a multiple of e_- . If U is a multiple of e_- , then,

$$S(e_+)e_i = g(e_+, e_i)U - g(e_i, U)e_- = 0.$$

Which implies $\text{pr}_{\mathfrak{g}_-}(S(e_+)) = 0$. Hence, $S(e_+) \in \mathbb{R}$.

Conversely, let S be as stated in the proposition. Then, $S \in \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ by Lemma 4.4.4, however since $S(e_+) \notin \mathfrak{g}_+$, Case 2 implies $S \notin \begin{smallmatrix} \square & \square \\ \square & 0 \end{smallmatrix}$. Thus, $S \in \square$. $\square$

For completeness,

Corollary 4.4.5. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable, non-irreducible subalgebra of type 2 or type 4. Suppose $S \in \mathcal{S}$, then,*

- $S \in \mathcal{S}_1 \oplus \mathcal{S}_3$ if and only if $S(e_+) \in \mathbb{R} \oplus \mathfrak{so}(n)$,
- $S \in \mathcal{S}_2 \oplus \mathcal{S}_3$ if and only if $S(e_+) \in \mathfrak{so}(n) \ltimes \mathfrak{g}_-$,
- $S \in \mathcal{S}_1 \oplus \mathcal{S}_2$ if and only if $S(e_+) \in \mathbb{R} \ltimes \mathfrak{g}_-$.

Proof. This follows immediately from Theorem 4.4.1. $\square$

4.5 Algebraic curvature tensors

In this section we shall describe the spaces of curvature tensors for indecomposable subalgebras $\mathfrak{h} \subset \mathfrak{g}$ of type 2 and type 4 that have torsion of the form,

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^0 \wedge V^+) \otimes V_0. \quad (4.33)$$

These results will be very similar to the results obtained by Galaev [18], who considers curvature tensors without torsion.

The spaces of algebraic curvature tensors with torsion T for algebra $\mathfrak{h}$ are given by,

$$\mathcal{R}(V, \mathfrak{h}, T) = \left\{ R \in \wedge^2 V^* \otimes \mathfrak{h} : \mathfrak{S}_{XYZ} R(X, Y)Z = -\mathfrak{S}_{XYZ} T(T(X, Y), Z), \quad \forall X, Y, Z \in V \right\}.$$

If $T = 0$ we shall denote the space as $\mathcal{R}(V, \mathfrak{h})$. Moreover, we shall call the identity,

$$\mathfrak{S}_{XYZ} R(X, Y)Z = -\mathfrak{S}_{XYZ} T(T(X, Y), Z),$$

Bianchi symmetry, see [30]. We shall also define the space of weak curvature tensors $\mathcal{P}(V, \mathfrak{h})$ as,

$$\mathcal{P}(V, \mathfrak{h}) := \{ P \in V^* \otimes \mathfrak{h} : \langle P(X)Y, Z \rangle + \langle P(Z)X, Y \rangle + \langle P(Y)Z, X \rangle = 0 \}.$$

We begin by motivating why we have chosen our torsion to adhere to (4.33). In the previous sections we deduced that if $\mathfrak{h} \cdot S = 0$ for an $S \in \wedge^1 \otimes \wedge^2$, then,

$$S \in (V^0 \otimes \mathfrak{g}_-) \oplus (V^+ \otimes \mathfrak{p}). \quad (4.34)$$

In such a situation, the torsion can be given by the skew symmetrisation of S . Hence, if S is of the form (4.34), then we can take the skew-symmetrisation, resulting in,

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^0 \wedge V^+) \otimes V_0.$$

Hence, T must be restricted to the subspace given in (4.33).

We start by investigating Bianchi symmetry assuming T is of the form given in (4.33). Since the torsion is skew, only distinct combinations need to be investigated. For example,

$$\mathfrak{S}T(T(e_+, X), e_+) = 0, \quad X \in V_0.$$

Lemma 4.5.1. *Let,*

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^0 \wedge V^+) \otimes V_0.$$

Then,

$$\mathfrak{S}T(T(X, Y), Z) = 0, \quad \mathfrak{S}T(T(X, e_+), e_-) = 0, \quad \mathfrak{S}T(T(X, Y), e_+) \in V_-,$$

where $X, Y, Z \in V_0$.

Proof. Based on the form of T , $T(X, Y) \in V_-$, and $T(e_-, X) = 0$ for all $X, Y \in V_0$. Hence,

$$\mathfrak{S}T(T(X, Y), Z) = 0, \quad X, Y, Z \in V_0.$$

For the second equality, since $T(e_-, X) = 0$ for all $X \in V_0$, we are left with,

$$T(T(X, e_+), e_-) + T(T(e_+, e_-), X).$$

Then, $T(X, e_+) \in V_- \oplus V_0$, and $T(e_+, e_-) \in V_-$. Which implies,

$$T(T(X, e_+), e_-) + T(T(e_+, e_-), X) = 0.$$

Now, for $\mathfrak{S}T(T(X, Y), e_+)$, we know $T(X, Y) \in V_-$, $T(e_+, X) \in V_- \oplus V_0$. This implies,

$$T(T(X, Y), e_+) \in V_-, \quad T(T(e_+, X), Y) \in V_-.$$

Hence, $\mathfrak{S}T(T(X, Y), e_+) \in V_-$. □

Based on Lemma 4.5.1 above, we can define $B_T \in V^0 \wedge V^0$ as,

$$\mathfrak{S}T(T(X, Y), e_+) = B_T(X, Y)e_-, \quad X, Y \in V_0. \quad (4.35)$$

Then, based on Lemma 4.5.1, Bianchi symmetry tells us that,

$$R(X, Y)e_+ + R(e_+, X)Y + R(Y, e_+)X = B_T(X, Y)e_-, \quad X, Y \in V_0,$$

and all other cyclic permutations of the curvature are zero. We shall now see to which extent pairwise symmetry holds for a $R \in \mathcal{R}(V, \mathfrak{h}, T)$.

Lemma 4.5.2. *Let $R \in \mathcal{R}(V, \mathfrak{h}, T)$ such that the torsion satisfies,*

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^0 \wedge V^+) \otimes V_0.$$

Moreover, let $B_T \in \mathfrak{so}(V_0)$ be as defined in (4.35). Then, for $X, Y, W, Z \in V_0 \oplus V_+$,

$$\begin{aligned} R(X, Y, W, Z) &= R(W, Z, X, Y), \\ R(X, Y, e_+, W) &= R(e_+, W, X, Y), \\ R(X, e_+, e_-, e_+) &= R(e_-, e_+, X, e_+), \end{aligned}$$

and otherwise,

$$R(X, e_+, Y, e_+) = R(Y, e_+, X, e_+) + B_T(Y, X), \quad X, Y \in V_0. \quad (4.36)$$

Proof. We only need to check pairwise symmetry for $R(X, Y)e_+$ where $X, Y \in V_0$. All others follow from having the cyclic permutations of the curvature equal to zero like in the torsion free case. We know that,

$$R(X, Y)e_+ + R(e_+, X)Y + R(Y, e_+)X = B_T(X, Y)e_-,$$

where $B_T(X, Y) = g(\mathfrak{S}T(T(X, Y), e_+), e_+)$. Thus,

$$R(X, Y, e_+, Z) + R(e_+, X, Y, Z) + R(Y, e_+, X, Z) = B_T(X, Y)g(e_-, Z), \quad (4.37)$$

where $X, Y \in V_0$, and $Z \in V$. The identity given by (4.36) is obtained when $Z = e_+$ in equation (4.37). Otherwise, take $Z \in V_-^\perp$, then,

$$R(X, Y, e_+, Z) + R(e_+, X, Y, Z) + R(Y, e_+, X, Z) = 0,$$

and using that all other cyclic sums of the curvature are zero,

$$\begin{aligned} 0 &= R(X, Y, e_+, Z) + R(e_+, X, Y, Z) + R(Y, e_+, X, Z) \\ &\quad - (R(X, Y, Z, e_+) + R(Z, X, Y, e_+) + R(Y, Z, X, e_+)) \\ &\quad - (R(Z, X, e_+, Y) + R(e_+, Z, X, Y) + R(X, e_+, Z, Y)) \\ &\quad - (R(Y, Z, e_+, X) + R(e_+, Y, Z, X) + R(Z, e_+, Y, X)), \\ &= 2R(X, Y, e_+, Z) - 2R(e_+, Z, X, Y). \end{aligned}$$

Hence, $R(X, Y, e_+, Z) = R(e_+, Z, X, Y)$ for $X, Y \in V_0$ and $Z \in V_-^\perp$. $\square$

This results in the following Lemma.

Lemma 4.5.3. *Let $\mathfrak{h}$ be a indecomposable, non-irreducible subalgebra of type 2, or type 4. Moreover, let $R \in \mathcal{R}(V, \mathfrak{h}, T)$ such that the torsion satisfies,*

$$T \in ((V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_-) \oplus ((V^0 \wedge V^+) \otimes V_0).$$

Then, $R(e_-, \cdot) = 0$.

Proof. If $R \in \mathcal{R}(V, \mathfrak{h}, T)$, then $R(X, Y) \in \mathfrak{h}$ for $X, Y \in V$. Hence, $R(X, Y)e_- = 0$, and we can employ Lemma 4.5.2 to deduce that $R(e_-, \cdot) = 0$. $\square$

We can now classify $\mathcal{R}(V, \mathfrak{h}, T)$ for $\mathfrak{h}$ of type 2 and type 4. We shall start in the setting when $\mathfrak{h}$ is type 2 then proceed to $\mathfrak{h}$ being of type 4.

Theorem 4.5.4. *Let $\mathfrak{h}$ be a indecomposable, non-irreducible subalgebra of type 2, and let $\mathfrak{h}_0 = \text{pr}_{\mathfrak{g}_0}(\mathfrak{h})$. Moreover, let T be such that,*

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^+ \wedge V^0) \otimes V_0.$$

Then there exists a $\mathfrak{h}$ -module isomorphism,

$$\mathcal{R}(V, \mathfrak{h}, T) \cong \mathcal{R}(V_0, \mathfrak{h}_0) \oplus \mathcal{P}(V_0, \mathfrak{h}_0) \oplus \mathcal{L}(T),$$

where $B_T \in V^0 \wedge V^0$ is as defined in (4.35), and,

$$\mathcal{L}(T) := \{L \in V^0 \otimes V^0 : L(X, Y) - L(Y, X) = B_T(Y, X)\}.$$

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a indecomposable subalgebra of type 2, and let $R \in \mathcal{R}(V, \mathfrak{h}, T)$ for T such that,

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^+ \wedge V^0) \otimes V_0.$$

We know that if $R \in \mathcal{R}(V, \mathfrak{h}, T)$, then $R(e_-, \cdot) = 0$ from Lemma 4.5.3. So, define,

$$R_0(X, Y) := \text{pr}_{\mathfrak{g}_0}(R(X, Y)), \quad P(X) := \text{pr}_{\mathfrak{g}_0}(R(e_+, X)), \quad L(X, Y) := R(e_+, X, e_+, Y),$$

for $X, Y \in V_0$. We claim that,

$$\mathcal{R}(V, \mathfrak{h}, T) \rightarrow \mathcal{R}(V_0, \mathfrak{h}_0) \oplus \mathcal{P}(V_0, \mathfrak{h}_0) \oplus \mathcal{L}(T), \quad R \mapsto (R_0, P, L),$$

is well defined and an injection. We first show that $R_0 \in \mathcal{R}(V_0, \mathfrak{h}_0)$, $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$, and $L \in \mathcal{L}(T)$. As a consequence of Lemma 4.5.1, $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$ for all $X, Y, Z \in V_0$. This implies,

$$\begin{aligned} R_0(X, Y)Z + R_0(Z, X)Y + R_0(Y, Z)X \\ + \text{pr}_{\mathfrak{g}_-}(R(X, Y))Z + \text{pr}_{\mathfrak{g}_-}(R(Z, X))Y + \text{pr}_{\mathfrak{g}_-}(R(Y, Z))X = 0. \end{aligned} \tag{4.38}$$

Since the first line is in V_0 and the second in V_- , we conclude that $\mathfrak{S}R_0(X, Y)Z = 0$. Meaning $R_0 \in \mathcal{R}(V_0, \mathfrak{h}_0)$ since $R_0 \in \wedge^2 V^0 \otimes \mathfrak{h}_0$ based on its definition. Moving onto P . By definition $P \in V^0 \otimes \mathfrak{h}_0$, thus only $\langle P(X)Y, Z \rangle + \langle P(Z)X, Y \rangle + \langle P(Y)Z, X \rangle = 0$ for all $X, Y, Z \in V_0$, needs to be confirmed. Based on its definition,

$$\begin{aligned} \mathfrak{S}_{XYZ} \langle P(X)Y, Z \rangle &= \mathfrak{S}_{XYZ} \langle \text{pr}_{\mathfrak{g}_0}(R(e_+, X))Y, Z \rangle, \\ &= \mathfrak{S}_{XYZ} \langle R(e_+, X)Y, Z \rangle, \quad \text{since } \text{pr}_{\mathfrak{g}_-}(R(e_+, V_0))V_0 \in V_-, \\ &= 0, \quad \text{as in the proof of Lemma 4.5.2.} \end{aligned}$$

Hence, $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$. Finally, L being in $\mathcal{L}(T)$ follows directly from (4.36) in Lemma 4.5.2.

We now aim to show that this map is an injection. Let $R_0 = P = L = 0$, we need to show $R = 0$. We shall start by showing that $R(X, Y) = 0$ for all $X, Y \in V_0$. Since $R_0 = 0$ then $\text{pr}_{\mathfrak{g}_0}(R(X, Y)) = 0$ for all $X, Y \in V_0$. Meaning, if we can show $\text{pr}_{\mathfrak{g}_-}(R(X, Y)) = 0$ then we are done. Hence, let $Z \in V_0$, then,

$$\begin{aligned} \langle R(X, Y)e_+, Z \rangle &= \langle R(e_+, Z)X, Y \rangle \\ &= \langle \text{pr}_{\mathfrak{g}_-}(R(e_+, Z))X + \text{pr}_{\mathfrak{g}_0}(R(e_+, Z))X, Y \rangle \\ &= 0. \end{aligned}$$

Where we have used that $P = 0$ and that $\text{pr}_{\mathfrak{g}_-}(R(e_+, Z))X \in V_-$. This shows that $\text{pr}_{\mathfrak{g}_-}(R(X, Y)) = 0$, meaning $R(X, Y) = 0$ for all $X, Y \in V_0$. We now aim to show $R(e_+, X) = 0$ for all $X \in V_0$, as this together with what we have just shown would imply $R = 0$. By $P = 0$, we know $\text{pr}_{\mathfrak{g}_0}(R(e_+, X)) = 0$, so like above we just need to investigate its $\mathfrak{g}_-$ projection. However, if $X, Y \in V_0$, then,

$$\langle R(e_+, X)e_+, Y \rangle = L(X, Y) = 0,$$

which immediately implies that $\text{pr}_{\mathfrak{g}_-}(R(e_+, X)) = 0$ for all $X \in V_0$. We can now conclude that $R = 0$, and thus the map is injective.

To show that it is also a surjection. Let $R_0 \in \mathcal{R}(V_0, \mathfrak{h}_0)$, $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$, and $L \in \mathcal{L}(T)$. Define,

$$\begin{aligned} R(X, Y) &= (0, (0, R_0(X, Y)), P(X)Y - P(Y)X), \\ R(e_+, X) &= (0, (0, P(X)), -L(X)), \\ R(e_-, \cdot) &= 0. \end{aligned}$$

We need to show that $R \in \mathcal{R}(V, \mathfrak{h}, T)$. It is clear that $R \in \wedge^2 V^* \otimes \mathfrak{h}$, thus we just need to show that R satisfies Bianchi symmetry. Again, we just need to check distinct combinations. Let $X, Y, Z \in V_0$,

$$\begin{aligned} \mathfrak{S}R(X, Y)Z &= R_0(X, Y)Z + R_0(Z, X)Y + R_0(Y, Z)X \\ &\quad + g(P(X)Y, Z)e_- - g(P(Y)X, Z)e_- + g(P(Z)X, Y)e_- \\ &\quad - g(P(X)Z, Y)e_- + g(P(Y)Z, X)e_- - g(P(Z)Y, X)e_-, \\ &= 0, \end{aligned}$$

by $R_0 \in \mathcal{R}(V_0, \mathfrak{h}_0)$ and $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$. Now, to check the X, Y, e_+ Bianchi symmetry,

$$\begin{aligned} \mathfrak{S}R(X, Y)e_+ &= P(Y)X - P(X)Y + P(X)Y - L(X, Y)e_- - P(Y)X + L(Y, X)e_-, \\ &= L(Y, X)e_- - L(X, Y)e_- = B(X, Y)e_-. \end{aligned}$$

Thus, $R \in \mathcal{R}(V, \mathfrak{h}, T)$, implying we have a surjection and,

$$\mathcal{R}(V, \mathfrak{h}, T) \cong \mathcal{R}(V_0, \mathfrak{h}_0) \oplus \mathcal{P}(V_0, \mathfrak{h}_0) \oplus \mathcal{L}(T)$$

□

Theorem 4.5.5. *Let $\mathfrak{h}$ be a indecomposable, non-irreducible subalgebra of type 4, and let $\mathfrak{h}_0 = \text{pr}_{\mathfrak{g}_0}(\mathfrak{h})$. Moreover, let T be such that,*

$$T \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^+ \wedge V^0) \otimes V_0.$$

Then,

$$\mathcal{R}(V, \mathfrak{h}, T) = \{(R, P, L) \in \mathcal{R}(V_0, \ker(\psi)) \oplus \mathcal{P}(V_2, \mathfrak{h}_0) \oplus \mathcal{L}(T) : \text{pr}_{V_1} \circ L = \psi \circ P\},$$

where,

$$\mathcal{L}(T) = \{L \in V^0 \otimes V^0 : L(X, Y) - L(Y, X) = B_T(Y, X)\}.$$

Proof. Proceeding in the same vein as [18]. Note that if $\mathfrak{h}$ is of type 4 and $\mathfrak{h}_0 = \text{pr}_{\mathfrak{g}_0}(\mathfrak{h})$, then $\mathfrak{h} \subset \mathfrak{h}_0 \ltimes \mathfrak{g}_-$. Thus, if $R \in \mathcal{R}(V, \mathfrak{h}, T)$, then $R \in \mathcal{R}(V, \mathfrak{h}_0 \ltimes \mathfrak{g}_-, T)$. However, $\mathfrak{h}_0 \ltimes \mathfrak{g}_-$ is a subalgebra of type 2. The space of such algebraic curvature tensors was classified in Theorem 4.5.4. It states,

$$\mathcal{R}(V, \mathfrak{h}, T) \cong \mathcal{R}(V_0, \mathfrak{h}_0) \oplus \mathcal{P}(V_0, \mathfrak{h}_0) \oplus \mathcal{L}(T),$$

where $\mathcal{L}(T) = \{L \in V^0 \otimes V^0 : L(X, Y) - L(Y, X) = B_T(Y, X)\}$, where B is defined from T in (4.35). Since, $R \in \mathcal{R}(V, \mathfrak{h}_0 \ltimes \mathfrak{g}_-, T)$, then there exists an $R \in \mathcal{R}(V_0, \mathfrak{h}_0)$, $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$, and $L \in \mathcal{L}(T)$ which defines R . We will next proceed to show the further conditions on such components if $\mathfrak{h}$ is an algebra of type 4.

Since $\mathfrak{h}$ is a subalgebra of type 4, there exists a splitting $\mathfrak{g}_- = V_1 \oplus V_2$ such that $\mathfrak{h}_0 \subset \mathfrak{so}(V_2)$. Meaning,

$$\mathfrak{h}_0 \cdot V_1 = 0, \quad \mathfrak{h}_0 \cdot V_2 \subset V_2. \quad (4.39)$$

Let $X \in V_1$, and $Y, Z \in V_2$, then since $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$,

$$g(P(X)Y, Z) + g(P(Z)X, Y) + g(P(Y)Z, X) = 0.$$

However, by (4.39), $P(Z)X = 0$ since $P(Z) \in \mathfrak{h}_0$ and $X \in V_1$, and $g(P(Y)Z, X) = 0$ since $P(Y)Z \in V_2$ and $X \in V_1$. Hence, $g(P(X)Y, Z) = 0$ for all $X \in V_1$, and $Y, Z \in V_2$, implying $P(V_1) = 0$. Thus, $P \in \mathcal{P}(V_2, \mathfrak{h}_0)$.

Moreover, let $X, Y \in V_0$, then $R(X, Y) = R_0(X, Y) + e^+ \wedge (P(Y)X - P(X)Y) \in \mathfrak{h}$. This implies that $R_0(X, Y) \in \ker(\psi)$, since $e^+ \wedge (P(Y)X - P(X)Y) \in V_2$ and as such there exists no V_1 component. Hence, $R_0 \in \mathcal{R}(V_0, \ker(\psi))$.

We now consider $R(e_+, X)$ where $X \in V_0$. Then, $R(e_+, X) = P(X) + e^+ \wedge L(X) \in \mathfrak{h}$, where $P(X) \in \mathfrak{h}_0$, and $L(X) \in V^0$. Hence, we can denote $L_1(X) = \text{pr}_{V_1} \circ L(X)$, and $L_2(X) = \text{pr}_{V_2} \circ L(X)$. Then, by virtue of $\mathfrak{h}$ being an algebra of type 4, $L_1(X) = \psi \circ P(X)$, equivalently $\text{pr}_{V_1} \circ L = \psi \circ P$.

Conversely, given an $R_0 \in \mathcal{R}(V_0, \ker(\psi))$, $P \in \mathcal{P}(V_0, \mathfrak{h}_0)$, and $L \in \mathcal{L}(T)$ such that $\text{pr}_{V_1} \circ T = \psi \circ P$. Then we aim to show that $R \in \mathcal{R}(V, \mathfrak{h}, T)$. From Theorem 4.5.4 it follows that these determine a $R \in \mathcal{R}(V, \mathfrak{h}_0 \ltimes \mathfrak{g}_-, T)$. Moreover, by construction it follows that $R(X, Y) \in \mathfrak{h}$ for all $X, Y \in V$. Hence, $R \in \mathcal{R}(V, \mathfrak{h}, T)$. $\square$

Based on the results above, if $\mathfrak{h}$ is of type 2 or type 4, each algebraic curvature tensor $\mathcal{R}(V, \mathfrak{h}, T)$ is uniquely determined by the elements R_0, P , and L satisfying certain conditions. We now make one further assumption on R , that is,

$$\mathfrak{h} \cdot R = 0.$$

The following Lemma tells us how this impacts an $R \in \mathcal{R}(V, \mathfrak{h}, T)$.

Theorem 4.5.6. *Let $\mathfrak{h}$ be of type 2 or type 4 and let $R \in \mathcal{R}(V, \mathfrak{h}, T)$. If $\mathfrak{h} \cdot R = 0$, then $R_0 = 0$ and $P = 0$.*

Proof. Since R is annihilated by $\mathfrak{h}$,

$$[H, R(X, Y)] - R(HX, Y) - R(X, HY) = 0, \quad \forall H \in \mathfrak{h}, \quad \forall X, Y \in V. \quad (4.40)$$

We shall now consider each type of $\mathfrak{h}$ separately.

Case 1 ($\mathfrak{h}$ of type 2) :

In type 2 there exists no coupling between $\mathfrak{g}_-$ and $\mathfrak{h}_0$. Hence, consider when $X, Y \in V_0$ and $H \in \mathfrak{g}_- \subset \mathfrak{h}$. Then $HX \in V_-$ implying $R(HX, Y) = 0$ by Theorem 4.5.4, and similarly for $R(X, HY)$. Hence, $[H, R(X, Y)] = 0$, which using Theorem 4.5.4 implies,

$$[H, (0, (0, R_0(X, Y)), P(X)Y - P(Y)X)] = 0,$$

for all $X, Y \in V_0$ and $H \in \mathfrak{g}_-$. Identifying H with an element of V_0 , this results in,

$$R_0(X, Y)Z = 0, \quad \forall X, Y \in V_0, \quad \forall Z \in V_0,$$

from which we conclude $R_0 = 0$. Now, consider equation (4.40) for when $X \in V_0, Y = e_+$ and $H \in \mathfrak{g}_- \subset \mathfrak{h}$. Then, by the same reasoning above $R(HX, e_+) = 0$, giving,

$$[H, R(X, e_+)] - R(X, He_+) = 0, \quad \forall X \in V_0, \quad \forall H \in \mathfrak{g}_+.$$

Using Theorem 4.5.4, this is equivalent to the equation,

$$[(0, (0, 0), Z), (0, (0, P(X)), -L(X))] + (0, (0, R_0(X, Z)), P(X)Z - P(Z)X) = 0.$$

Expanding this out and using that $R_0 = 0$, the resulting equation is,

$$(0, (0, 0), -P(X)Z) + (0, (0, 0), P(X)Z - P(Z)X) = 0.$$

Hence, $P(Z)X = 0$ for any $Z, X \in V_0$. It follows that $P = 0$.

Case 2 ($\mathfrak{h}$ of type 4) :

In type 4 recall that there exists a splitting of $\mathfrak{g}_-$ as $\mathfrak{g}_- = V_1 \oplus V_2$, such that there exists $\psi : \mathfrak{h}_0 \rightarrow V_1$ where $\mathfrak{h}_- = \psi(\mathfrak{h}_0) \oplus V_2$. Moreover, ψ vanishes on $[\mathfrak{h}_0, \mathfrak{h}_0]$, is surjective, and $\mathfrak{h}_0 \subset \mathfrak{so}(V_2)$. The space of curvature tensors $\mathcal{R}(V, \mathfrak{h}, T)$ is given by Theorem 4.5.5, which states,

$$\mathcal{R}(V, \mathfrak{h}, T) = \{(R, P, L) \in \mathcal{R}(V_0, \ker(\psi)) \oplus \mathcal{P}(V_2, \mathfrak{h}_0) \oplus \mathcal{L}(T) : \text{pr}_{V_1} \circ L = \psi \circ P\}.$$

Using the same argument in type 2 to show $R_0 = 0$, using equation (4.40), let $X, Y \in V_0$ and $H = (0, (0, 0), (0, Z)) \in \mathfrak{h}_-$ where $Z \in V_2$. Then, by the same arguments in type 2, $R(HX, Y) = R(X, HY) = 0$ meaning $[H, R(X, Y)] = 0$. From Theorem 4.5.5, that is,

$$\begin{aligned} 0 &= [(0, (0, 0), (0, Z)), (0, (0, R_0(X, Y)), P(Y)X - P(X)Y)], \\ &= (0, (0, 0), (0, R_0(X, Y)Z)). \end{aligned}$$

Hence, $R_0(X, Y)Z = 0$ for all $X, Y \in V_0$ and $Z \in V_2$. As $R_0(X, Y) \in \mathfrak{h}_0 \subset \mathfrak{so}(V_2)$ this implies $R_0 = 0$. Now, look at equation (4.40) when $X \in V_2, Y = e_+$, and $H = (0, (0, 0), (0, Z)) \in \mathfrak{h}_-$. Then, equation (4.40) becomes,

$$[H, R(X, e_+)] + R(X, Z) = 0.$$

Employing Theorem 4.5.5, this is equivalent to,

$$\begin{aligned} &[(0, (0, 0), (0, Z)), (0, (0, P(X)), (-\psi(P(X)), -L_2(X)))] \\ &+ (0, (0, 0), (0, P(X)Z - P(Z)X)) = 0. \end{aligned}$$

Which results in the equation,

$$P(Z)X = 0, \quad \forall Z, X \in V_2.$$

Since $P(V_1) = 0$ in type 4, we conclude that $P = 0$. □

This means if $R \in \mathcal{R}(V, \mathfrak{h}, T)$ such that $\mathfrak{h} \cdot R = 0$. Then, R is completely determined by an $L \in V^0 \otimes V^0$ such that,

$$L(X, Y) - L(Y, X) = B_T(Y, X), \quad X, Y \in V_0,$$

where $B_T(X, Y)e_- = \mathfrak{S}T(T(X, Y), e_+)$ for $X, Y \in V_0$. Specifically, as follows:

$$R(e_-, e_+) = R(e_-, X) = R(X, Y) = 0, \quad R(e_+, X) = e^+ \wedge L(X).$$

4.6 The case of dimension 3

In this section we shall look at what happens to an $S \in \mathcal{S}$ and an $R \in \mathcal{R}(V, \mathfrak{h}, T)$ such that $\mathfrak{h} \cdot R = 0$ when $\dim(V) = 3$. We first show that Theorem 4.3.1 is false in dimension 3.

Let $\dim(V) = 3$, and consider a Witt basis (e_-, e_1, e_+) of V . The only indecomposable, non-irreducible subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of type 2 and type 4 is $\mathfrak{h} = \mathfrak{g}_- \subset \mathfrak{so}(1, 2)$.

Theorem 4.6.1. *Let $\dim(V) = 3$ and $\mathfrak{h} = \mathfrak{g}_-$. Moreover, let $S \in \mathcal{S} \subset \wedge^1 \otimes \wedge^2$ as detailed in Section 4.3. Then,*

$$S(e_-) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad S(e_1) = \begin{pmatrix} -a & b & 0 \\ 0 & 0 & -b \\ 0 & 0 & a \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} b & c & 0 \\ a & 0 & -c \\ 0 & -a & -b \end{pmatrix},$$

for $a, b, c \in \mathbb{R}$. In particular, $\mathcal{S} \subset \wedge^1 \otimes \wedge^2$ is of dimension 3.

Proof. Let,

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathfrak{h} = \mathfrak{g}_-$, that is,

$$\mathfrak{h} = \mathbb{R} \cdot X$$

Define $S \in \wedge^1 \otimes \wedge^2$ by,

$$S(e_-) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad S(e_1) = \begin{pmatrix} -a & b & 0 \\ 0 & 0 & -b \\ 0 & 0 & a \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} b & c & 0 \\ a & 0 & -c \\ 0 & -a & -b \end{pmatrix},$$

for $a, b, c \in \mathbb{R}$. We will show that $\mathfrak{h} \cdot S = 0$, equivalently,

$$[X, S(Y)] - S(XY) = 0, \quad Y \in V.$$

We first note,

$$[X, S(e_-)] = 0, \quad [X, S(e_1)] = aX, \quad [X, S(e_+)] = -S(e_1).$$

Hence,

$$\begin{aligned} [X, S(e_-)] - S(Xe_-) &= 0, \\ [X, S(e_1)] - S(Xe_1) &= aX - S(e_-) = 0, \end{aligned}$$

and

$$[X, S(e_+)] - S(Xe_+) = -S(e_1) + S(e_1) = 0.$$

Thus, $\mathfrak{h} \cdot S = 0$.

Conversely, let S be a general element of $\wedge^1 \otimes \wedge^2$. Specifically,

$$S(e_-) = \begin{pmatrix} o & p & 0 \\ q & 0 & -p \\ 0 & -q & -o \end{pmatrix}, \quad S(e_1) = \begin{pmatrix} k & i & 0 \\ j & 0 & -i \\ 0 & -j & -k \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} d & e & 0 \\ f & 0 & -e \\ 0 & -f & -d \end{pmatrix},$$

for all entries in $\mathbb{R}$. Then if,

$$0 = [X, S(e_-)] - S(Xe_-) = [X, S(e_-)],$$

then this forces $q = o = 0$. Hence,

$$S(e_-) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & -p \\ 0 & 0 & 0 \end{pmatrix}, \quad S(e_1) = \begin{pmatrix} k & i & 0 \\ j & 0 & -i \\ 0 & -j & -k \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} d & e & 0 \\ f & 0 & -e \\ 0 & -f & -d \end{pmatrix}.$$

Next, if,

$$0 = [X, S(e_1)] - S(Xe_1) = [X, S(e_1)] - S(e_-),$$

then $j = 0$, and $k + p = 0$. Thus, $k = -p$, and,

$$S(e_-) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & -p \\ 0 & 0 & 0 \end{pmatrix}, \quad S(e_1) = \begin{pmatrix} -p & i & 0 \\ 0 & 0 & -i \\ 0 & 0 & p \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} d & e & 0 \\ f & 0 & -e \\ 0 & -f & -d \end{pmatrix}.$$

Lastly, if,

$$0 = [X, S(e_+)] - S(Xe_+) = [X, S(e_+)] + S(e_1),$$

then $f - p = 0$, and $d - i = 0$, meaning $f = p$ and $d = i$. So, S must be of the form,

$$S(e_-) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & -p \\ 0 & 0 & 0 \end{pmatrix}, \quad S(e_1) = \begin{pmatrix} -p & i & 0 \\ 0 & 0 & -i \\ 0 & 0 & p \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} i & e & 0 \\ p & 0 & -e \\ 0 & -p & -i \end{pmatrix}.$$

Relabeling $p = a$, $i = b$, and $e = c$, we are done. $\square$

Theorem 4.6.1 tells us that if $S \in \wedge^1 \otimes \wedge^2$ such that $\mathfrak{h} \cdot S = 0$, then there can exist S such that,

$$S(e_-) \neq 0, \quad S(e_1) \notin \mathfrak{g}_-, \quad S(e_+) \notin \mathfrak{p}.$$

We see that this shows that Theorem 4.3.1 is not true in dimension 3.

Moving onto the possible algebraic curvature tensors, we shall see for our purposes in Chapter 4 that we only need to consider when $\mathfrak{h} \cdot R = 0$ in dimension 3. We shall first need the following Lemma which is true for any dimension $\dim(V) = n + 2$.

Lemma 4.6.2. *Let $\mathfrak{h} = \mathfrak{g}_-$ and $R \in \wedge^2 V^* \otimes \mathfrak{h}$ such that $\mathfrak{h} \cdot R = 0$. Then $R(e_-, \cdot) = 0$.*

Proof. Let $\mathfrak{h} = \mathfrak{g}_-$ and $R \in \wedge^2 V^* \otimes \mathfrak{g}_-$. If $\mathfrak{g}_- \cdot R = 0$, then since $\mathfrak{g}_-$ is abelian, this implies,

$$\begin{aligned} 0 &= (H \cdot R)(e_-, e_+) = [H, R(e_-, e_+)] - R(He_-, e_+) - R(e_-, He_+) \\ &= -R(e_-, He_+). \end{aligned}$$

Meaning $R(e_-, Z) = 0$ for all $Z \in V_0$. Similarly,

$$\begin{aligned} 0 &= (H \cdot R)(e_i, e_+) = [H, R(e_i, e_+)] - R(He_i, e_+) - R(e_i, He_+) \\ &= -R(e_i, e_+), \end{aligned}$$

if we let $H = (0, (0, 0), e_i)$. Hence, $R(e_-, e_+) = 0$, resulting in $R(e_-, Z) = 0$ for all $Z \in V$. $\square$

Theorem 4.6.3. *If $\dim(V) = 3$, $\mathfrak{h} = \mathfrak{g}_- = \mathbb{R} \cdot X$, and $R \in \wedge^2 V^* \otimes \mathfrak{h}$ such that $\mathfrak{h} \cdot R = 0$. Then,*

$$R(e_-, e_+) = R(e_-, e_1) = 0, \quad R(e_1, e_+) = X.$$

Proof. By Lemma 4.6.2, $R(e_-, \cdot) = 0$. Hence, since $\dim(V) = 3$, $R \in V^0 \wedge V^+ \otimes \mathfrak{h}$ which is of dimension 1. Hence, $R(e_1, e_+)$ is equal to (a multiple of) X . $\square$

Moreover, as a consequence we have that such an R always satisfies the Bianchi symmetry,

$$R(e_-, e_1)e_+ + R(e_+, e_-)e_1 + R(e_1, e_+)e_- = 0.$$

Chapter 5

Indecomposable Non-irreducible Homogeneous Structures

In this chapter, starting with Section 5.1, we first look at locally symmetric Lorentzian connections with indecomposable, non-irreducible holonomy. In Section 5.2, a brief interlude is then taken to introduce the *screen bundle* associated to a recurrent null vector field along with some key properties. In Section 5.3, we then apply these results to manifolds admitting Ambrose-Singer connections (locally reductive homogeneous spaces), together with our algebraic results in Chapter 4 which can be applied to homogeneous structures. We then move into Section 5.4, where we show that all such spaces must be homogeneous plane waves. We end this chapter with Section 5.5, investigating the 3-dimensional case and showing that our dimension bound for our main theorem in the previous section cannot be dropped using our approach.

5.1 Locally symmetric Lorentzian connections

Consider a triple $(M, g, \tilde{\nabla})$, where g is a Lorentzian metric on a smooth manifold M , and $\tilde{\nabla} : TM \otimes TM \rightarrow TM$ is an affine connection on M such that $\tilde{\nabla}g = 0$. Then, we define a $\tilde{\nabla}$ -recurrent vector field.

Definition 5.1.1. A vector field $\xi \in \Gamma(TM)$ is $\tilde{\nabla}$ -recurrent if and only if $\tilde{\nabla}\xi = \Theta \otimes \xi$ for some one-form $\Theta \in \Gamma(T^*M)$.

We are interested when such a recurrent vector field can be rescaled to a $\tilde{\nabla}$ -parallel vector field.

Lemma 5.1.2. *If $\xi \in \Gamma(TM)$ is a recurrent vector field, then $R(X, Y)\xi = d\Theta(X, Y)\xi$ for $X, Y \in \Gamma(TM)$.*

Proof. Let $\xi \in \Gamma(TM)$ be a recurrent, thus $\tilde{\nabla}\xi = \Theta \otimes \xi$ for some one-form $\Theta \in \Gamma(T^*M)$. By the definition of the curvature,

$$\begin{aligned}
 R(X, Y)\xi &= \tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_Y \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X, Y]}\xi \\
 &= \tilde{\nabla}_X(\Theta(Y)\xi) - \tilde{\nabla}_Y(\Theta(X)\xi) - \Theta([X, Y])\xi \\
 &= X(\Theta(Y))\xi + \Theta(Y)\Theta(X)\xi - Y(\Theta(X))\xi - \Theta(X)\Theta(Y)\xi - \Theta([X, Y])\xi \\
 &= (X(\Theta(Y)) - Y(\Theta(X)) - \Theta([X, Y]))\xi \\
 &= d\Theta(X, Y)\xi.
 \end{aligned}
 \quad \square$$

Lemma 5.1.3. *If M is a simply-connected Lorentzian manifold with a $\tilde{\nabla}$ -recurrent null vector field ξ , i.e $\tilde{\nabla}\xi = \Theta \otimes \xi$. Then ξ can be rescaled to a parallel null vector field X if and only if Θ is closed.*

Proof. By the assumption that M is simply-connected this means that Θ is exact, thus $\Theta = dh$ for some $h \in C^\infty(M)$. As a result, consider $X = e^{-h}\xi$, then $g(X, X) = 0$ and

$$\begin{aligned}
 \tilde{\nabla}_Y X &= \tilde{\nabla}_Y e^{-h}\xi \\
 &= d e^{-h}(Y)\xi + e^{-h}\tilde{\nabla}_Y \xi \\
 &= -e^{-h}dh(Y)\xi + e^{-h}\Theta(Y)\xi = 0
 \end{aligned}$$

by exactness of Θ . Conversely, if ξ can be rescaled to a parallel null vector field $\tilde{\xi}$, where $\tilde{\xi} = f\xi$ for some function $f \in C^\infty(M)$. Then, using Lemma 5.1.2,

$$0 = R(X, Y)\xi = d\Theta(X, Y)\xi$$

for all $X, Y \in \Gamma(TM)$. $\square$

We shall now assume that $\tilde{\nabla}$ is locally symmetric, i.e $\tilde{\nabla}\tilde{R} = 0$ and that the holonomy $\text{hol}(\tilde{\nabla})$ of $\tilde{\nabla}$ is indecomposable, non-irreducible. We shall prove the following result.

Theorem 5.1.4. *Let (M, g) be a simply-connected Lorentzian manifold of dimension ≥ 3 with a locally symmetric metric connection $\tilde{\nabla}$. Moreover, assume that $\text{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, there exists a $\tilde{\nabla}$ -parallel null vector field, and $\text{hol}(\tilde{\nabla})$ is of type 2 or 4.*

We shall first need the following Lemma.

Lemma 5.1.5. *If $\tilde{\nabla}R = 0$ and ξ is a recurrent null vector field, i.e $\tilde{\nabla}\xi = \Theta \otimes \xi$, then $d\Theta \in \Gamma(\wedge^2(T^*M))$ is $\tilde{\nabla}$ -parallel.*

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then,

$$\begin{aligned} (\tilde{\nabla}_X R)(Y, Z)\xi &= \tilde{\nabla}_X(R(Y, Z)\xi) - R(\tilde{\nabla}_X Y, Z)\xi - R(Y, \tilde{\nabla}_X Z) - R(Y, Z)\tilde{\nabla}_X \xi \\ &= \tilde{\nabla}_X(d\Theta(Y, Z)\xi) - d\Theta(\tilde{\nabla}_X Y, Z)\xi - d\Theta(Y, \tilde{\nabla}_X Z)\xi - d\Theta(Y, Z)\Theta(X)\xi. \end{aligned}$$

Note that $\tilde{\nabla}_X(d\Theta(Y, Z)\xi) = X(d\Theta(Y, Z))\xi + d\Theta(Y, Z)\Theta(X)\xi$. Hence, taking into account that $d\Theta(Y, Z)\Theta(X)\xi$ terms cancel, the above expression becomes

$$(\tilde{\nabla}_X R)(Y, Z)\xi = X(d\Theta(Y, Z))\xi - d\Theta(\tilde{\nabla}_X Y, Z)\xi - d\Theta(Y, \tilde{\nabla}_X Z)\xi.$$

However, using the fact that $\tilde{\nabla}R = 0$, this implies $X(d\Theta(Y, Z)) = d\Theta(\tilde{\nabla}_X Y, Z) + d\Theta(Y, \tilde{\nabla}_X Z)$. But this is exactly the condition $\tilde{\nabla}d\Theta = 0$, i.e $d\Theta$ is parallel. $\square$

As a result, we have that $d\Theta$ is a $\tilde{\nabla}$ -parallel two form. Thus, before continuing, we will briefly look at general $\tilde{\nabla}$ -parallel two forms where the holonomy is indecomposable, non-irreducible. From henceforth, let $\omega \in \Gamma(\wedge^2(T^*M))$ be a arbitrary $\tilde{\nabla}$ -parallel two form. Using the fact that ω is parallel, then,

$$\mathfrak{hol} \cdot \omega = 0. \quad (5.1)$$

If we define $A \in \mathfrak{so}(TM, g)$ such that for any $X, Y \in \Gamma(TM)$, $\omega(X, Y) = g(A \cdot X, Y)$, then (5.1) is equivalent to the condition,

$$[\mathfrak{hol}, A] = 0. \quad (5.2)$$

However, fixing a Witt basis, $\mathfrak{so}(TM, g) \cong \mathfrak{g}$ and since $\mathfrak{hol}$ is indecomposable non-irreducible, $\mathfrak{hol} \subset \mathfrak{p}$. Hence, from equation (5.2), $A \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{hol})$, and Proposition 4.2.1 in Chapter 4 is applicable. Which implies,

- $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{hol}) = 0$ if $\text{pr}_{\mathbb{R}}(\mathfrak{hol}) \neq 0$,
- $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{hol}) = \mathfrak{g}_-$ if $\text{pr}_{\mathbb{R}}(\mathfrak{hol}) = 0$.

We now prove Theorem 5.1.4.

Proof. Since the holonomy algebra of $\tilde{\nabla}$ is indecomposable, non-irreducible, it admits an invariant null line. Thus, there exists a $\tilde{\nabla}$ -parallel null line bundle. Since M is simply-connected this implies that there exists a global $\tilde{\nabla}$ -recurrent null vector field ξ , i.e $\tilde{\nabla}\xi = \Theta \otimes \xi$. By assumption $\tilde{\nabla}R = 0$, hence by Lemma 5.1.5, the 2-form $d\Theta$ is $\tilde{\nabla}$ -parallel. This means that at a point $p \in M$, $\mathfrak{hol}(\tilde{\nabla}) \cdot d\Theta = 0$, thus as explained above, the corresponding skew endomorphism A must be in the centraliser in $\mathfrak{so}(T_p M)$ of $\mathfrak{hol}(\tilde{\nabla})$. If $\mathfrak{hol}(\tilde{\nabla})$ is of types 1 or 3, the centraliser is trivial, and hence $d\Theta = 0$ since $d\Theta$ is parallel. Thus, we can invoke Lemma 5.1.3 and ξ can be rescaled to a parallel vector field. If $\mathfrak{hol}(\tilde{\nabla})$ is of types 2 or 4, then it annihilates null vector and hence M admits a parallel null vector field. $\square$

Which leads into the following corollary,

Corollary 5.1.6. *Every simply-connected indecomposable, non-irreducible Lorentzian locally symmetric space admits a parallel null vector field.*

Remark 5.1.7. This is known from the classification of locally symmetric spaces in [8], however above we have a direct proof.

5.2 The screen bundle

In the section we introduce the *screen bundle* associated to a recurrent null vector field. This will be important in the following sections of this chapter and provide a scheme for us to investigate the holonomy of manifolds admitting a Ambrose-Singer connection, i.e locally reductive homogeneous spaces.

Let $(M, g, \tilde{\nabla})$ be a Lorentzian manifold with a metric connection $\tilde{\nabla}$, suppose M admits a recurrent null vector field ξ . Then the holonomy at $p \in M$ admits a 1-dimensional null invariant subspace $\mathbb{R}\xi_p$. The orthogonal complement of this space $\xi_p^\perp$ is a $n+1$ -dimensional subspace which is holonomy invariant and contains $\mathbb{R}\xi_p$. Thus, we can construct two $\tilde{\nabla}$ -parallel distributions, a 1-dimensional, totally null distribution $\mathbb{R}\xi$ with $\xi \in \Gamma(\mathbb{R}\xi)$, and its $n+1$ -dimensional orthogonal complement,

$$\mathbb{R}\xi \subset \xi^\perp := \{X \in TM : g(X, \xi) = 0\}.$$

We can now consider the quotient space $\xi_p^\perp / \mathbb{R}\xi_p$ at every point $p \in M$ to define the vector bundle over M ,

$$E := \xi^\perp / \mathbb{R}\xi,$$

which is called the *screen bundle*. The metric g defines a metric g^E on E , by,

$$g^E([X], [Y]) := g(X, Y), \quad X, Y \in \Gamma(\xi^\perp).$$

However, we will often make no distinction between g^E and g . Moreover, since $\mathbb{R}\xi$ is $\tilde{\nabla}$ -parallel, the connection $\tilde{\nabla}$ induces a connection $\tilde{\nabla}^E$ on E , defined as,

$$\tilde{\nabla}_X^E[Y] := [\tilde{\nabla}_X Y], \quad X \in \Gamma(TM), Y \in \Gamma(\xi^\perp).$$

This connection is a metric connection with respect to g^E since $\tilde{\nabla}$ is a metric connection with respect to g . We can also define the curvature $\tilde{R}^E$ of $\tilde{\nabla}^E$ in the standard way,

$$\tilde{R}^E(X, Y)[Z] = \tilde{\nabla}_X^E \tilde{\nabla}_Y^E[Z] - \tilde{\nabla}_Y^E \tilde{\nabla}_X^E[Z] - \tilde{\nabla}_{[X, Y]}^E[Z], \quad X, Y \in \Gamma(TM), Z \in \Gamma(\xi^\perp).$$

Lemma 5.2.1. *The curvature $\tilde{R}^E$ of the induced connection $\tilde{\nabla}^E$ is equal to,*

$$\tilde{R}^E(X, Y)[Z] = [\tilde{R}(X, Y)Z], \quad X, Y \in \Gamma(TM), Z \in \Gamma(\xi^\perp),$$

where $\tilde{R}$ is the curvature of $\tilde{\nabla}$.

Proof. The definition of $\tilde{R}^E$ is given by,

$$\tilde{R}^E(X, Y)[Z] = \tilde{\nabla}_X^E \tilde{\nabla}_Y^E[Z] - \tilde{\nabla}_Y^E \tilde{\nabla}_X^E[Z] - \tilde{\nabla}_{[X, Y]}^E[Z], \quad X, Y \in \Gamma(TM), Z \in \Gamma(\xi^\perp).$$

Now, expanding out the definition of $\tilde{\nabla}^E$,

$$\begin{aligned} \tilde{R}^E(X, Y)[Z] &= \tilde{\nabla}_X^E[\tilde{\nabla}_Y Z] - \tilde{\nabla}_Y^E[\tilde{\nabla}_X Z] - [\tilde{\nabla}_{[X, Y]} Z], \\ &= [\tilde{\nabla}_X \tilde{\nabla}_Y Z] - [\tilde{\nabla}_{a_Y} \tilde{\nabla}_X Z] - [\tilde{\nabla}_{[X, Y]} Z] = [\tilde{R}(X, Y)Z]. \end{aligned}$$

□

The main result about screen bundles, which will be relevant to us later in this section, is the following.

Theorem 5.2.2. *Let E be the screen bundle defined above with induced connection $\tilde{\nabla}^E$. Then,*

$$\mathfrak{hol}(\tilde{\nabla}^E) = \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\tilde{\nabla})).$$

Proof. This result was proved for the Levi-Civita connection in [23, 24], and generalised to connections with torsion in [20]. □

5.3 Lorentzian homogeneous structures

We now consider an Ambrose-Singer connection $\tilde{\nabla}$ on a simply-connected Lorentzian manifold, where its associated homogeneous structure $S \in \mathcal{S}$ has no restrictions on its algebraic type. Moreover, we assume that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, by Lemma 3.1.4, Ambrose-Singer connections satisfy $\tilde{\nabla}\tilde{R} = 0$, and hence we can employ Theorem 5.1.4. As a result, there exists a null vector field ξ such that

$$\tilde{\nabla}\xi = 0, \tag{5.3}$$

and $\mathfrak{hol}(\tilde{\nabla})$ must be either of type 2 or type 4. We now wish to see, given the condition $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, what a general homogeneous structure S looks like.

Theorem 5.3.1. *Let (M, g) be a simply connected Lorentzian manifold of dimension $n + 2 \geq 4$ with Ambrose-Singer connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, then the associated homogeneous structure $S \in \wedge^1 \otimes \mathfrak{so}(TM)$ satisfies,*

1. $S \in \wedge^1 \otimes \mathfrak{stab}(\mathbb{R}\xi)$,
2. $S(\xi) = 0$,
3. $S(X)Y \in \mathbb{R}\xi$ for all $X \in \xi^\perp$, $Y \in TM$.

Proof. We have already established that there exists a $\tilde{\nabla}$ -parallel null vector field ξ . Moreover, $\mathfrak{hol}(\tilde{\nabla})$ is of type 2 or type 4. Thus, we can impose Theorem 4.3.1 at each point, identifying $T_p M \cong V$ such that $e_- = \xi_p$. $\square$

As in the proof above, we will often work at a point $p \in M$ and consider a Witt basis $(e_-, e_1, \dots, e_n, e_+)$ of $T_p M$ along side the corresponding dual basis $(e^+, e^1, \dots, e^n, e^-)$ as in Chapter 3. In such situations, our homogeneous structure at a point will look exactly as in Theorem 4.3.1, that is,

$$S_p(e_-) = 0, \quad S_p(e_i) \in \mathfrak{g}_-, \quad S_p(e_+) \in \mathfrak{p},$$

such that,

$$S_p(e_i)e_+ = -\text{pr}_{\mathfrak{g}_0}(S_p(e_+))e_i.$$

From henceforth we shall let,

$$A = \text{pr}_{\mathfrak{so}}(S(e_+)), \quad \alpha = \text{pr}_{\mathbb{R}}(S(e_+)), \quad \eta = \text{pr}_{\mathfrak{g}_-}(S(e_+)).$$

Moreover, if we denote $\mathfrak{hol}_0(\tilde{\nabla}) := \text{pr}_{\mathfrak{g}_0}(\mathfrak{hol}(\tilde{\nabla}))$, then $\mathfrak{hol}_0(\tilde{\nabla}) \cdot A = 0$. Furthermore, in type 2, $\mathfrak{hol}_0(\tilde{\nabla}) \cdot \eta = 0$, while in type 4, $(\psi^{-1}(V^1) \oplus (\mathfrak{hol}_0(\tilde{\nabla}))') \cdot \eta = 0$.

Now, that the form of a general homogeneous structure is known when $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable non-irreducible, we can investigate the implications on the torsion $\tilde{T}$ and curvature $\tilde{R}$ of $\tilde{\nabla}$.

Proposition 5.3.2. *Let S be a Lorentzian homogeneous structure and $(e_-, e_1, \dots, e_n, e_+)$ a Witt basis of $T_p M$ in which S is of the form of Theorem 4.3.1. Then the torsion satisfies,*

$$\begin{aligned} \tilde{T}_p(e_-, X) &= 0, \\ \tilde{T}_p(e_-, e_+) &= \alpha e_-, \\ \tilde{T}_p(X, Y) &= 2A(Y, X)e_-, \\ \tilde{T}_p(e_+, X) &= -(\alpha I + 2A)X - \eta(X)e_-, \end{aligned}$$

where $X, Y \in V_0$.

Proof. Pick a point $p \in M$, then from Lemma 3.1.4, the torsion of $\tilde{\nabla}$ is given by,

$$\tilde{T}_p(X, Y) = S_p(Y)X - S_p(X)Y, \quad X, Y \in T_pM.$$

Hence, using Theorem 4.3.1 the torsion is computed for each e_-, e_+ and $X, Y \in V_0$. $\square$

Proposition 5.3.3. *Let (M, g) be a simply-connected Lorentzian manifold of dimension $n + 2 \geq 4$ with AS-connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, if $(e_-, e_1, \dots, e_n, e_+)$ is a Witt basis of T_pM , the following holds,*

$$\begin{aligned} \tilde{R}_p(U, W) &= R_p(U, W), \quad U, V \in V_-^\perp, \\ \tilde{R}_p(e_+, e_-) &= R_p(e_+, e_-), \end{aligned}$$

and otherwise,

$$\tilde{R}_p(e_+, X) = R_p(e_+, X) - (0, (0, 0), A^2(X) + \alpha A(X)), \quad X \in V_0.$$

Proof. From Lemma 3.1.4, the curvature of $\tilde{\nabla}$ is related to the curvature of ∇ by,

$$\tilde{R}(X, Y) = R(X, Y) + [S(X), S(Y)] + S(\tilde{T}(X, Y)).$$

Hence, pick a point $p \in M$ and using Theorem 5.3.1 for S_p and Proposition 5.3.2 for $\tilde{T}_p$, we compute for each basis element. We will only show that,

$$\tilde{R}_p(e_+, X) = R_p(e_+, X) - (0, (0, 0), A^2(X) + \alpha A(X)),$$

as the others follow clearly.

$$\begin{aligned} \tilde{R}(e_+, X) &= R(e_+, X) + [S(e_+), S(X)] + S(\tilde{T}(e_+, X)) \\ &= R(e_+, X) + [S(e_+), S(X)] - \alpha S(X) - 2S(A(X)). \end{aligned}$$

From Theorem 5.3.1, $S(X) = (0, (0, 0), \alpha X + A(X))$, $S(A(X)) = (0, (0, 0), \alpha A(X) + A^2(X))$, and $S(e_+) = (0, (\alpha, A), \eta)$. Thus,

$$\tilde{R}(e_+, X) = R(e_+, X) + [S(e_+), S(X)] - (0, (0, 0), 3\alpha AX + \alpha^2 X + 2A^2X).$$

We now compute $[S(e_+), S(X)]$,

$$\begin{aligned} [S(e_+), S(X)] &= [(0, (\alpha, A), \eta), (0, (0, 0), \alpha X + AX)], \\ &= (0, (0, 0), (A + \alpha I)(\alpha X + AX)), \\ &= (0, (0, 0), 2\alpha AX + A^2X + \alpha^2 X). \end{aligned}$$

Hence,

$$\tilde{R}(e_+, X) = R(e_+, X) - (0, (0, 0), A^2(X) + \alpha A(X)).$$

$\square$

Remark 5.3.4. We highlight that based on the result above, if $A = 0$ then $\tilde{R} = R$, which implies $\mathfrak{hol}(\tilde{\nabla}) \subset \mathfrak{hol}(\nabla)$. This happens exactly when S has no totally skew component.

We now wish to see if the $\tilde{\nabla}$ -parallel vector field ξ is also ∇ -parallel or if it can be rescaled to one. To do this we will investigate $S(\cdot)\xi$, since $\nabla = \tilde{\nabla} + S$, meaning $\nabla_X \xi = S(X)\xi$.

Theorem 5.3.5. *Let (M, g) be a simply-connected Lorentzian manifold of dimension $n + 2 \geq 4$ with Ambrose-Singer connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, then the $\tilde{\nabla}$ -parallel null vector field ξ can be rescaled to a ∇ -parallel null vector field.*

Proof. Since $S = \nabla - \tilde{\nabla}$, then $\nabla_X \xi = S_X \xi$. Thus, pick a point $p \in M$ and let $e_- = \xi_p$. Starting with e_- , we can always extend this to form a Witt basis of $T_p M$, hence according to Theorem 5.3.1,

$$S_p(\cdot, e_-, \cdot) = \alpha e^+ \otimes e^+. \quad (5.4)$$

This implies that $\nabla_X \xi = \alpha g(\xi, X)\xi$, where α is now a smooth function. Moreover, since S and ξ are $\tilde{\nabla}$ -parallel, this implies that $S(\cdot, \xi, \cdot)$ is $\tilde{\nabla}$ -parallel. Hence, (5.4) implies that α is constant.

Consequently, ξ is recurrent for the Levi-Civita connection with $\Theta = \alpha \xi^b$ and by Lemma 5.1.3 can be rescaled to a parallel vector field if Θ is closed. In this case Θ is indeed closed as,

$$d\Theta = d\alpha \wedge \xi^b + \alpha d\xi^b = 0,$$

where $d\alpha = 0$, since α is constant, and $d\xi^b = 0$ since,

$$\begin{aligned} (d\xi^b)(X, Y) &= g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi), \\ &= \alpha (g(\xi, Y)g(\xi, X) - g(\xi, Y)g(\xi, X)), \quad \text{since } \xi \text{ is recurrent and } \Theta = \xi^b, \\ &= 0. \end{aligned}$$

□

Hence, we have a $\tilde{\nabla}$ -parallel null vector field ξ and a ∇ -parallel null vector field $\hat{\xi} = f\xi$ for some $f \in C^\infty(M)$. This implies that both holonomy algebras of the connections $\tilde{\nabla}$ and ∇ are contained in $\mathfrak{so}(n) \ltimes \mathfrak{g}_-$. We wish to further compare both algebras. One way to do this is to look at their $\mathfrak{so}(n)$ -projection. We shall do this through the respective screen holonomies.

Since $\hat{\xi}$ is a multiple of ξ by some smooth function, ξ and $\hat{\xi}$ generate the same totally null distribution, $\xi^\perp = \hat{\xi}^\perp$. Thus, since the metric is parallel with respect to both connections,

the distribution $\xi^\perp$ is parallel with respect to both connections. Hence, as described in Section 5.2, we can consider the screen bundle,

$$E := \xi^\perp / \mathbb{R} \xi,$$

with induced connections $\tilde{\nabla}^E$ and ∇^E , from $\tilde{\nabla}$ and ∇ respectively. Moreover, we can define the curvature $\tilde{R}^E$ and R^E respectively as defined in Section 5.2. Also, E carries a parallel metric as defined in Section 5.2, so the vector bundle $\mathfrak{so}(E)$ can be defined. In regards to holonomy, from Theorem 5.2.2, we know that,

$$\mathfrak{hol}(\tilde{\nabla}^E) = \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\tilde{\nabla})), \quad \text{and} \quad \mathfrak{hol}(\nabla^E) = \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\nabla)).$$

Proposition 5.3.6. *If $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, then,*

$$\tilde{R}^E(X, Y) = R^E(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

where $\tilde{R}^E$ and R^E are the curvatures of the induced connections $\tilde{\nabla}^E$ and ∇^E respectively.

Proof. By Lemma 5.2.1,

$$\tilde{R}^E(X, Y)[Z] = [\tilde{R}(X, Y)Z], \quad R^E(X, Y)[Z] = [R(X, Y)Z], \quad (5.5)$$

for $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(\xi^\perp)$. We can now apply Proposition 5.3.3 at each point $p \in M$, which states $\tilde{R}_p = R_p$ apart from,

$$\tilde{R}_p(e_+, X) = R_p(e_+, X) - (0, (0, 0), A^2(X) + \alpha A(X)).$$

However, $(0, (0, 0), A^2(X) + \alpha A(X))[Z] \in \mathbb{R}e_-$ when $Z \in \text{span}(e_1, \dots, e_n)$. Hence, using equations (5.5), we conclude that $\tilde{R}^E = R^E$. $\square$

Proposition 5.3.7. *If $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, then the associated homogeneous structure $S \in \Gamma(\wedge^1 \otimes \mathfrak{so}(TM))$ induces a well defined section $S^E \in \Gamma(\wedge^1 \otimes \mathfrak{so}(E))$, defined as,*

$$S^E(X)[Y] := [S(X)Y], \quad X \in \Gamma(TM), Y \in \Gamma(\xi^\perp),$$

such that,

$$\nabla^E = \tilde{\nabla}^E + S^E.$$

Proof. Theorem 5.3.1 states that $S \in \Gamma(\wedge^1 \otimes \mathfrak{stab}(\mathbb{R} \xi))$. Hence,

$$S(X)Y \in \xi^\perp, \quad \forall X \in \Gamma(TM), Y \in \Gamma(\xi^\perp).$$

Thus, $[S(X)Y] \in E$ for all X and when $Y \in \Gamma(\xi^\perp)$. To show it does not depend on the representative in the equivalence class $[Y]$, take a $\tilde{Y} \sim Y$, then $\tilde{Y} = f\xi + Y$ for some $f \in C^\infty(M)$. Therefore,

$$\begin{aligned} S^E(X)[\tilde{Y}] &= [S(X)\tilde{Y}] \\ &= [S(X)(f\xi + Y)] \\ &= f[S(X)\xi] + S^E(X)[Y]. \end{aligned}$$

However, since $S(X) \in \mathfrak{stab}(\mathbb{R}\xi)$, $S(X)\xi \in \mathbb{R}\xi$ for all X , meaning $S^E(X)[\tilde{Y}] = S^E(X)[Y]$. Hence, $S^E \in \Gamma(\wedge^1 \otimes \mathfrak{so}(E))$ is well defined. Moreover, as $\nabla = \tilde{\nabla} + S$, then based on definitions, it is immediate that $\nabla^E = \tilde{\nabla}^E + S^E$. $\square$

Lemma 5.3.8. *There exists a section A of $\mathfrak{so}(E)$ such that $S^E = \xi^\flat \otimes A$.*

Proof. Theorem 5.3.1 dictates that,

$$S(X)Y \in \mathbb{R}\xi, \quad \forall X, Y \in \xi^\perp.$$

This implies $S^E(\xi^\perp) = 0$, which means $S^E \in \Gamma(\wedge^1 \otimes \mathfrak{so}(E))$ must be of the form,

$$S^E = \xi^\flat \otimes A, \quad A \in \Gamma(\mathfrak{so}(E)).$$

$\square$

Based on the above, if we evaluate S^E at a point $p \in M$ with respect to a Witt basis, then $A_p \in \mathfrak{so}(E|_p) \cong \mathfrak{so}(n)$ and is equal to,

$$A_p = \text{pr}_{\mathfrak{so}(n)}(S_p(e_+)).$$

Note that since $S^E(\xi^\perp) = 0$, then $S_p^E(e_+)$ does not depend on the choice of e_+ . Take for example $\tilde{e}_+$ not equal to e_+ such that $(e_-, \tilde{e}_1, \dots, \tilde{e}_n, \tilde{e}_+)$ is a Witt basis. Then $\tilde{e}_+ = e_+ + c^i e_i + c^- e_-$ for constants c_i and c^- subject to restraints such that $g(\tilde{e}_+, \tilde{e}_+) = 0$, and

$$S_p^E(\tilde{e}_+) = S_p^E(e_+) + c^i S_p^E(e_i) + c^- S_p^E(e_-) = S_p^E(e_+).$$

In contrast, it may not be possible to extend $S_p(e_+)$ into a global section of $\mathfrak{so}(TM)$ as there may not be a global vector field that is transversal to $\xi^\perp$.

We want to compare the screen holonomy of both connections. However, we will first need the following lemma.

Lemma 5.3.9. *If A is as in Lemma 5.3.8, then $\tilde{\nabla}^E A = 0$ and $\nabla^E A = 0$.*

Proof. Consider the connection $D := \tilde{\nabla} \otimes \tilde{\nabla}^E$ on the vector bundle $\wedge^1 \otimes \mathfrak{so}(E)$. Then, to show $\tilde{\nabla}^E A = 0$, it is sufficient to show $DS^E = 0$. This is because, by Lemma 5.3.8,

$$DS^E = \tilde{\nabla} \xi^\flat \otimes A + \xi^\flat \otimes \tilde{\nabla}^E A = \xi^\flat \otimes \tilde{\nabla}^E A,$$

where $\xi^\flat$ is $\tilde{\nabla}$ -parallel since $\tilde{\nabla} \xi = 0$. However, $DS^E = 0$ follows from $\tilde{\nabla} S = 0$, as we see below,

$$\begin{aligned} (D_X S^E)(Y)[Z] &= \tilde{\nabla}_X^E(S^E(Y)[Z]) - S^E(\tilde{\nabla}_X Y)[Z] - S^E(Y)\tilde{\nabla}_X^E[Z], \\ &= \tilde{\nabla}_X^E([S(X)Z]) - [S(\tilde{\nabla}_X Y)Z] - S^E(Y)[\tilde{\nabla}_X Z], \\ &= [\tilde{\nabla}_X S(X)Z] - [S(\tilde{\nabla}_X Y)Z] - [S(Y)\tilde{\nabla}_X Z], \\ &= [(\tilde{\nabla}_X S)(Y)Z] = 0, \quad (\text{from } \tilde{\nabla} S = 0). \end{aligned}$$

Thus, $\tilde{\nabla}^E A = 0$.

Now to show that $\nabla^E A = 0$, it is sufficient to show,

$$S^E(X) \cdot A = 0.$$

This is because,

$$\nabla^E = \tilde{\nabla}^E + S^E,$$

and from above $\tilde{\nabla}^E A = 0$. Explicitly, $S^E(X) \cdot A$ is equal to,

$$S^E(X) \cdot A = [S^E(X), A] = g(X, \xi)[A, A] = 0,$$

since $S^E = \xi^\flat \otimes A$. Hence, $\nabla^E A = 0$. □

Corollary 5.3.10. *Let $A \in \Gamma(\mathfrak{so}(E))$ be as defined by S^E as in Lemma 5.3.8. Then,*

$$\mathfrak{hol}(\tilde{\nabla}^E) \cdot A = 0, \quad \text{and} \quad \mathfrak{hol}(\nabla^E) \cdot A = 0.$$

Proof. This follows directly from Lemma 5.3.9, since A is parallel with respect to both $\tilde{\nabla}^E$ and ∇^E . □

Theorem 5.3.11. *Let (M, g) be a simply-connected Lorentzian manifold of dimension $n + 2 \geq 4$ with AS-connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then the holonomy algebras $\mathfrak{hol}(\tilde{\nabla}^E)$ and $\mathfrak{hol}(\nabla^E)$ are equal.*

Proof. Since,

$$\tilde{\nabla}_X^E = \nabla_X^E - S_X^E,$$

from Lemma 5.3.8,

$$\tilde{\nabla}_X^E = \nabla_X^E - g(\xi, X)A, \quad \forall X \in \Gamma(TM). \quad (5.6)$$

Following the same argument as Ernst and Galaev [13], define the bundle $\mathfrak{z}(A) \subset \mathfrak{so}(E)$ such that,

$$\mathfrak{z}(A) = \{B \in \mathfrak{so}(E) : [B, A] = 0\}.$$

We first know both connections $\tilde{\nabla}^E$ and ∇^E preserve this bundle as,

$$[\nabla_X^E B, A] = \nabla_X^E([B, A]) + [\nabla_X^E A, B] = 0,$$

implying $\nabla_X^E B \in \mathfrak{z}(A)$ for all $B \in \mathfrak{z}(A)$, and also for $\tilde{\nabla}_X^E$, where we have used that A is parallel with respect to both connections, see Lemma 5.3.9. Moreover, the connections are equal when restricted to $\mathfrak{z}(A)$ by equation (5.6).

Furthermore, Proposition 5.3.6 guarantees the curvatures of $\tilde{\nabla}^E$ and ∇^E are equal, and they take their values in $\mathfrak{z}(A)$ by,

$$R^E(X, Y) = \tilde{R}^E(X, Y) \in \mathfrak{hol}(\tilde{\nabla}^E) \subset \mathfrak{z}(A).$$

Thus, we have the curvatures are equal, the parallel transport is equal on $\mathfrak{z}(A)$, the connections preserve $\mathfrak{z}(A)$, and the curvatures take their values in $\mathfrak{z}(A)$. Thus, by the Ambrose-Singer holonomy theorem, the holonomy algebras are given by,

$$\begin{aligned} \mathfrak{hol}_p(\tilde{\nabla}^E) &= \text{span}\{\tilde{\mathcal{P}}_\gamma(\tilde{R}_q^E(X, Y)) : \gamma(0) = q, \gamma(1) = p, X, Y \in T_q M\}, \\ \mathfrak{hol}_p(\nabla^E) &= \text{span}\{\mathcal{P}_\gamma(R_q^E(X, Y)) : \gamma(0) = q, \gamma(1) = p, X, Y \in T_q M\}, \end{aligned}$$

and we conclude that both algebras must be equal. $\square$

We highlight the above Theorem generalises Proposition 2 from [13] when applied to manifolds admitting AS-connections (locally reductive homogeneous spaces). Their result states the screen holonomies are equal for Lorentzian metric connections with indecomposable, non-irreducible holonomy such that the torsion is parallel and totally skew. Hence, when applied to AS-connections, it only holds for homogeneous structures that are totally skew, whereas our result needs no restriction on the homogeneous structure.

Remark 5.3.12. We also note that the proofs leading to this result can be altered to be true in the more general setting of Lorentzian metric connections with indecomposable, non-irreducible holonomy such that the torsion is parallel. However, we need the extra assumption that there exists a parallel null vector field with respect to this connection. In the case of AS-connections, this is a consequence of the curvature being parallel, and in [13] this is a consequence of the torsion being totally skew.

Corollary 5.3.13 ([20]). *Let (M, g) be a Lorentzian manifold of dimension ≥ 4 with a Lorentzian connection $\tilde{\nabla}$ with parallel torsion. Assume that $\tilde{\nabla}$ admits a parallel null vector field and has indecomposable, non-irreducible holonomy. Then, the curvatures and the holonomy algebras of $\tilde{\nabla}^E$ and ∇^E are equal.*

Proof. See [20] for full details. $\square$

Corollary 5.3.14. *If the holonomy of an Ambrose-Singer connection is indecomposable, non-irreducible, then its $\mathfrak{so}(n)$ -projection is equal to a Riemannian holonomy algebra.*

Proof. This follows from the fact that the screen holonomy $\mathfrak{hol}(\nabla^E)$ is equal to a Riemannian holonomy algebra. See Theorem 1.1 of [25]. $\square$

Corollary 5.3.15. *Let (M, g) be a simply-connected reductive Lorentzian homogeneous space of dimension $n + 2 \geq 4$. If the isotropy of the transvection presentation is indecomposable, non-irreducible, then the $\mathfrak{so}(n)$ -projection of the isotropy algebra is a Riemannian holonomy algebra.*

Proof. The isotropy of a transvection presentation is equal to the holonomy of an AS-connection as established in Chapter 3. Hence, using the corollary above we are done. $\square$

5.4 Proof of main result

Let $S \in \wedge^1 \otimes \wedge^2$, be a homogeneous structure such that the associated AS-connection has indecomposable holonomy. Then the holonomy is of type 2 or type 4. Moreover, as described in Chapter 3, the torsion $\tilde{T} \in \wedge^2 \otimes \wedge^1$ is defined as,

$$\tilde{T}(X, Y) = S(Y)X - S(X)Y.$$

Since S is $\tilde{\nabla}$ -parallel, then $\tilde{T}$ is also $\tilde{\nabla}$ -parallel. Hence, the space of curvature tensors for $\tilde{\nabla}$ at a point with torsion $\tilde{T}$ is defined by,

$$\mathcal{R}(T_p M, \mathfrak{hol}(\tilde{\nabla}), \tilde{T}) = \left\{ \tilde{R} \in \wedge^2 V^* \otimes \mathfrak{hol}(\tilde{\nabla}) : \mathfrak{S}_{XYZ} \tilde{R}(X, Y)Z = - \mathfrak{S}_{XYZ} \tilde{T}(\tilde{T}(X, Y), Z), \quad \forall X, Y, Z \in T_p M \right\},$$

We remark that usually Bianchi's symmetry as seen above is seen with a change of sign in front of the torsion. In our case this is a consequence of our convention when we define $\tilde{T}$.

Identifying $T_p M$ with V , the torsion $\tilde{T}$ was given explicitly in Proposition 5.3.2, and it can be seen to satisfy,

$$\tilde{T}_p \in (V \wedge V^+ \oplus V^0 \wedge V^0) \otimes V_- \oplus (V^+ \wedge V^0) \otimes V_0. \quad (5.7)$$

Hence, we are in the exact situation as Section 4.5, where such spaces $\mathcal{R}(V, \mathfrak{h}, T)$ were described. We will first explicitly find $B_{\tilde{T}} \in V^0 \wedge V^0$ for $\tilde{T}_p$. For a T in (5.7), it was defined by,

$$\mathfrak{S}T(T(X, Y), e_+) = B_T(X, Y)e_-, \quad X, Y \in V_0.$$

Lemma 5.4.1. *Let $\tilde{T}$ be the torsion for a AS-connection with indecomposable non-irreducible holonomy. Moreover, let S be the associated homogeneous structure. Then, with respect to a Witt basis $(e_-, e_1, \dots, e_n, e_+)$, $B_{\tilde{T}} \in \wedge^2 V^0$ is given by,*

$$B_{\tilde{T}}(X, Y) = 2\alpha A(X, Y), \quad X, Y \in V_0$$

where $(\alpha, A) = \text{pr}_{\mathfrak{g}_0}(S(e_+))$.

Proof. We refer to Proposition 5.3.2 to get the explicit expressions for the torsion $\tilde{T}_p$. Then,

$$\begin{aligned} \mathfrak{S}_{e_i e_j e_+} \tilde{T}_p(\tilde{T}_p(e_i, e_j), e_+) &= 2A(e_j, e_i) \tilde{T}_p(e_-, e_+) - 2\alpha \tilde{T}_p(e_i, e_j) - 2\tilde{T}_p(A(e_i), e_j) \\ &\quad + 2\tilde{T}_p(A(e_j), e_i) \\ &= 2\alpha A(e_j, e_i) e_- - 4\alpha A(e_j, e_i) e_- \\ &= -2\alpha A(e_j, e_i) e_- = 2\alpha A(e_i, e_j) e_-. \end{aligned}$$

□

We can now use the results in Section 4.5, and apply them to the curvature tensors of an AS-connection. However, we first make the following observation.

Corollary 5.4.2. *Let (M, g) be a Lorentzian manifold with AS connection $\tilde{\nabla}$ and associated homogeneous structure S . Assume the holonomy $\mathfrak{hol}(\tilde{\nabla})$ of $\tilde{\nabla}$ is indecomposable, non-irreducible. If $\text{pr}_{\mathcal{S}_3}(S) = 0$ or $\text{pr}_{\mathcal{S}_1}(S) = 0$, then,*

$$\mathcal{R}(V, \mathfrak{hol}(\tilde{\nabla}), \tilde{T}) = \mathcal{R}(V, \mathfrak{hol}(\tilde{\nabla})).$$

Proof. This is a consequence of Theorem 4.4.1. If $\text{pr}_{\mathcal{S}_3}(S) = 0$ then $A = 0$, and if $\text{pr}_{\mathcal{S}_1}(S) = 0$ then $\alpha = 0$. Thus, if either of these were true, $B_{\tilde{T}} = 0$, and the Bianchi symmetry would be torsion free. □

We now employ the results from Section 4.5 for a AS-connection with indecomposable non-irreducible holonomy, such that,

$$B_{\tilde{T}}(X, Y) = 2\alpha A(X, Y), \quad X, Y \in V_0, \quad (\alpha, A) = \text{pr}_{\mathfrak{g}_0}(S(e_+)).$$

If $\mathfrak{hol}(\tilde{\nabla})$ is of type 2, then in this context Theorem 4.5.4 states,

$$\mathcal{R}(V, \mathfrak{hol}(\tilde{\nabla}), \tilde{T}) \cong \mathcal{R}(V_0, \mathfrak{hol}(\tilde{\nabla})_0) \oplus \mathcal{P}(V_0, \mathfrak{hol}(\tilde{\nabla})_0) \oplus \mathcal{L}(\tilde{T}),$$

where,

$$\mathcal{L}(\tilde{T}) := \{L \in V^0 \otimes V^0 : L(X, Y) - L(Y, X) = B_{\tilde{T}}(Y, X)\}.$$

Moreover, if $\mathfrak{hol}(\tilde{\nabla})$ is of type 4, then using Theorem 4.5.5,

$$\mathcal{R}(V, \mathfrak{hol}(\tilde{\nabla}), \tilde{T}) = \{(R, P, L) \in \mathcal{R}(V_0, \ker(\psi)) \oplus \mathcal{P}(V_2, \mathfrak{hol}(\tilde{\nabla})_0) \oplus \mathcal{L}(\tilde{T}) : \text{pr}_{V_1} \circ L = \psi \circ P\}.$$

Additionally, the curvature of the AS-connection is parallel. Hence, we have the extra condition $\mathfrak{hol}(\tilde{\nabla}) \cdot \tilde{R} = 0$. This means from Theorem 4.5.6, that the $\mathcal{R}$ and $\mathcal{P}$ components of the curvature vanish. Meaning $\tilde{R}$ is completely determined by its $\mathcal{L}(\tilde{T})$ component. Specifically:

$$\begin{aligned} \tilde{R}_p(e_-, e_+) &= \tilde{R}_p(e_-, X) = \tilde{R}_p(X, Y) = 0 \\ \tilde{R}_p(X, e_+) &= L(X) \wedge e^+, \end{aligned}$$

where $L \in \mathcal{L}(\tilde{T})$, and $X, Y \in V_0$. We can now get the form of the Levi-Civita curvature.

Lemma 5.4.3. *Let (M, g) be a simply connected Lorentzian manifold of dimension $n+2 \geq 4$ with AS-connection $\tilde{\nabla}$. Assume the holonomy $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, with respect to a Witt basis at a point $p \in M$, the ∇ -curvature satisfies,*

$$\begin{aligned} R_p(X, Y) &= R_p(X, e_-) = R_p(e_-, e_+) = 0, \\ R_p(X, e_+) &\in \mathfrak{g}_-. \end{aligned}$$

Proof. Using Proposition 5.3.3, the curvatures $\tilde{R}$ and R are related by,

$$\begin{aligned} R_p(X, Y) &= \tilde{R}_p(X, Y), \quad \forall X, Y \in V_-^\perp, \\ R_p(e_-, e_+) &= \tilde{R}_p(e_-, e_+), \\ R_p(e_+, X) &= \tilde{R}_p(e_+, X) + Q(X), \quad \forall X \in V_0, \end{aligned}$$

where $Q(X) \in \mathfrak{g}_-$, given explicitly by $(0, (0, 0), A^2(X) + \alpha A(X))$. Moreover, from the results above, for $X, Y \in V_0$,

$$\tilde{R}_p(X, Y) = \tilde{R}_p(X, e_-) = \tilde{R}_p(e_-, e_+) = 0, \quad \tilde{R}_p(X, e_+) \in \mathfrak{g}_-.$$

Hence, the result follows. □

Theorem 5.4.4. *Let (M, g) be a simply connected Lorentzian manifold of dimension $n+2 \geq 4$ with Ambrose-Singer connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then,*

$$\mathfrak{hol}(\tilde{\nabla}^E) = \mathfrak{hol}(\nabla^E) = 0,$$

implying that the holonomy for both $\tilde{\nabla}$ and ∇ must be abelian, and (M, g) is a pp-wave.

Proof. Pick a point $p \in M$ and identify $T_p M$ with V . At such a point we can find a Witt basis $(e_-, e_1, \dots, e_n, e_+)$ and identify the screen bundle at a point E_p with $V_0 = \text{span}(e_1, \dots, e_n)$. We are assuming that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible, hence must be of type 2 or type 4. Thus we are in the setting of Theorem 4.5.4 and Theorem 4.5.5. Moreover, the AS-connection satisfies $\tilde{\nabla}\tilde{R} = 0$, implying $\mathfrak{hol}(\tilde{\nabla}) \cdot \tilde{R} = 0$. Hence, we can apply Theorem 4.5.6 to the space of curvature tensors $\mathcal{R}(V, \mathfrak{hol}(\tilde{\nabla}), \tilde{T})$. It follows that every curvature tensor $\tilde{R}_p$ is uniquely determined by a $L \in V^0 \otimes V^0$ such that,

$$\tilde{R}_p(X, Y) = \tilde{R}_p(X, e_-) = \tilde{R}_p(e_-, e_+) = 0, \quad \tilde{R}_p(X, e_+) = L(X) \wedge e^+. \quad (5.8)$$

However, by Lemma 5.2.1, $\tilde{R}_p^E(X, Y)[Z] = [\tilde{R}_p(X, Y)Z]$, meaning equation (5.8) implies $\tilde{R}_p^E = 0$. This exactly means that $\mathfrak{hol}(\tilde{\nabla}^E) = 0$. It then follows from Theorem 5.3.11,

$$\mathfrak{hol}(\nabla^E) = \mathfrak{hol}(\tilde{\nabla}^E) = 0,$$

and since,

$$\mathfrak{hol}(\tilde{\nabla}^E) = \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\tilde{\nabla})), \quad \mathfrak{hol}(\nabla^E) = \text{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\nabla)),$$

we conclude that $\mathfrak{hol}(\tilde{\nabla}) = \mathfrak{g}_-$ and $\mathfrak{hol}(\nabla) \subset \mathfrak{g}_-$, hence both are abelian. Hence, (M, g) is a pp-wave, see Section 2.6. $\square$

We can apply this to the context of simply-connected Lorentzian reductive homogeneous spaces (M, g) which always admit an AS-connection. It tells us that for any simply-connected Lorentzian reductive homogeneous space, such that it has a AS-connection with indecomposable, non-irreducible holonomy, then the Levi-Civita holonomy must be $\mathbb{R}^n$ and be a homogeneous pp-wave. In full generality, since every Lorentzian manifold admitting a AS-connection is a locally reductive homogeneous space, we get the following corollary.

Corollary 5.4.5. *Let (M, g) be a Lorentzian manifold of dimension $n + 2 \geq 4$ with AS-connection $\tilde{\nabla}$. Suppose $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then, the universal cover of (M, g) is a locally homogeneous pp-wave.*

We lastly prove that it in fact must be a plane wave.

Theorem 5.4.6. *Let (M, g) be a simply connected Lorentzian manifold of dimension $n + 2 \geq 4$ with AS-connection $\tilde{\nabla}$. Assume the holonomy $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then (M, g) is a plane wave.*

Proof. Based on Theorem 5.4.4 we know that (M, g) is a pp-wave. Thus, for (M, g) to be a plane wave, we need to show the condition,

$$\nabla_X R = 0, \quad \forall X \in \Gamma(\xi^\perp).$$

Since $\tilde{\nabla}R = 0$, this reduces to showing, $S(X) \cdot R = 0$ for all $X \in \Gamma(\xi^\perp)$. Pick a point $p \in M$ and consider a Witt basis $(e_-, e_1, \dots, e_n, e_+)$ such that $e_- = \xi_p$. Then, $\xi_p^\perp \cong V_- \oplus V_0$, and now we make use of the fact that $S_p(e_-) = 0$, and that the only non-zero curvature is $R_p(Y, e_+)$ for $Y \in V_0$. Hence, we only need to show that,

$$(S_p(X) \cdot R_p)(e_+, Y) = 0, \quad \forall X, Y \in V_0.$$

Let $X, Y \in V_0$, then,

$$(S_p(X) \cdot R_p)(e_+, Y) = [S_p(X), R_p(e_+, Y)] - R_p(S_p(X)e_+, Y) - R_p(e_+, S_p(X)Y).$$

We highlight that the commutator is zero as both $S_p(X)$ and $R_p(e_+, Y)$ are elements of $\mathfrak{g}_-$, see Lemma 5.4.3, which is abelian. Both curvature terms also vanish since, $S_p(X)e_+ \in V_0$ and $S_p(X)(Y) \in V_-$, hence using Lemma 5.4.3 they vanish. As a result, we conclude that (M, g) must be a plane wave. $\square$

This leaves us with the following results.

Theorem 5.4.7. *Let (M, g) be a Lorentzian manifold of dimension $n + 2 \geq 4$ with AS-connection $\tilde{\nabla}$. Suppose $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible. Then the universal cover of (M, g) is a locally homogeneous plane wave.*

Proof. We apply Theorem 5.4.6 to Corollary 5.4.5. $\square$

Corollary 5.4.8. *Let $(G/H, g)$ be a Lorentzian reductive homogeneous space of dimension $n + 2 \geq 4$. Assume the isotropy of the transvection presentation is indecomposable, non-irreducible, then the universal cover of (M, g) must be a homogeneous plane wave.*

5.5 The case of dimension 3

In this section we shall investigate when (M, g) is of dimension 3 and has an AS-connection with indecomposable, non-irreducible holonomy. We shall see that we cannot lower the dimension of results obtained in the previous section.

Let (M, g) be a 3-dimensional Lorentzian manifold with AS-connection $\tilde{\nabla}$. We shall assume that $\tilde{\nabla}$ is an Ambrose–Singer connection that is defined by S as in Theorem 4.6.1. That is, for $a, b, c \in \mathbb{R}$,

$$S_p(e_-) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad S_p(e_1) = \begin{pmatrix} -a & b & 0 \\ 0 & 0 & -b \\ 0 & 0 & a \end{pmatrix}, \quad S_p(e_+) = \begin{pmatrix} b & c & 0 \\ a & 0 & -c \\ 0 & -a & -b \end{pmatrix},$$

where (e_-, e_1, e_+) is a Witt basis for $T_p M$. Since S is parallel, it is sufficient to specify S at a point $p \in M$.

Lemma 5.5.1. *The torsion $\tilde{T}_p$ corresponding the such a homogeneous structure is given by,*

$$\tilde{T}_p(e_-, e_1) = -2ae_-, \quad \tilde{T}_p(e_-, e_+) = be_- + 2ae_1, \quad \tilde{T}_p(e_1, e_+) = be_1 + ce_- - 2ae_+.$$

Proof. The torsion is equal to $\tilde{T}_p(X, Y) = S(Y)X - S(X)Y$. □

Remark 5.5.2. It should be noted that such torsion is totally skew when $b = c = 0$.

Moreover, since $\mathfrak{g}_-$ is the holonomy algebra of $\tilde{\nabla}$ at p , by Theorem 4.6.3,

$$\tilde{R}_p(e_1, e_+) = (0, (0, 0), e_1).$$

We also saw that such an $\tilde{R}$ must satisfy the Bianchi symmetry,

$$\tilde{R}_p(e_-, e_1)e_+ + \tilde{R}_p(e_+, e_-)e_1 + \tilde{R}_p(e_1, e_+)e_- = 0.$$

However this implies that,

$$\tilde{T}_p(\tilde{T}_p(e_-, e_1), e_+) + \tilde{T}_p(\tilde{T}_p(e_+, e_-), e_1) + \tilde{T}_p(\tilde{T}_p(e_1, e_+), e_-) = 0.$$

Using Lemma 5.5.1, this implies $4abe_- = 0$. Hence, S_p has to lie in the subset of P that is defined by the union of the hyperplanes $P_1 = \{a = 0\}$ and $P_2 = \{b = 0\}$. If $S \in P_1$, then S_p satisfies the conclusions of Theorem 4.3.1 which are needed for the proof of Theorem 5.4.6. Thus, we shall focus on the case when $S_p \in P_2$. In such a case,

$$S_p(e_-) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad S_p(e_1) = \begin{pmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad S_p(e_+) = \begin{pmatrix} 0 & c & 0 \\ a & 0 & -c \\ 0 & -a & 0 \end{pmatrix}. \quad (5.9)$$

Meaning,

$$S_p(\cdot, e_-, \cdot) = a(e^+ \otimes e^1 - e^1 \otimes e^+),$$

which shows that $\mathbb{R}e_-$ is no longer invariant under the holonomy of the Levi-Civita connection and we cant rescale to a Levi-Civita parallel vector field like we did in Theorem 5.3.5. We now now compute the Levi-Civita curvature with S , $\tilde{T}$, and $\tilde{R}$.

Proposition 5.5.3. *Let (M, g) be a 3-dimensional Lorentzian manifold with AS-connection $\tilde{\nabla}$. Suppose that $\mathfrak{hol}(\tilde{\nabla})$ is indecomposable, non-irreducible and S is given by (5.9). Then, with respect to a Witt basis at $p \in M$,*

$$\begin{aligned} R_p(e_-, e_1) &= (0, (0, 0), a^2), \\ R_p(e_-, e_+) &= (0, (a^2, 0), 0), \\ R_p(e_1, e_+) &= (a^2, (0, 0), 1 + 2ac). \end{aligned}$$

Proof. The relationship between the Levi-Civita curvature and S , $\tilde{T}$, and $\tilde{R}$ is given in Lemma 3.1.4. It states,

$$R(X, Y) = \tilde{R}(X, Y) - [S(X), S(Y)] - S(\tilde{T}(X, Y)).$$

Hence, let $X = e_-$, $Y = e_1$,

$$\begin{aligned} R_p(e_-, e_1) &= \tilde{R}_p(e_-, e_1) - [S_p(e_-), S_p(e_1)] - S_p(\tilde{T}_p(e_-, e_1)) \\ &= -[S_p(e_-), S_p(e_1)] + 2aS_p(e_-) \\ &= (0, (0, 0), a^2). \end{aligned}$$

Now, let $X = e_-$, $Y = e_+$,

$$\begin{aligned} R_p(e_-, e_+) &= \tilde{R}_p(e_-, e_+) - [S_p(e_-), S_p(e_+)] - S_p(\tilde{T}_p(e_-, e_+)) \\ &= -[S_p(e_-), S_p(e_+)] - 2aS_p(e_1) \\ &= (0, (a^2, 0), 0). \end{aligned}$$

Finally, let $X = e_1$, $Y = e_+$,

$$\begin{aligned} R_p(e_1, e_+) &= \tilde{R}_p(e_1, e_+) - [S_p(e_1), S_p(e_+)] - S_p(\tilde{T}(e_1, e_+)) \\ &= (0, (0, 0), e_1) - [S_p(e_1), S_p(e_+)] - cS_p(e_-) + 2aS_p(e_+) \\ &= (a^2, (0, 0), 1 + 2ac). \end{aligned}$$

□

Henceforth, if we assume $a \neq 0$, then we get the following corollary.

Corollary 5.5.4. *The Levi-Civita holonomy is equal to $\mathfrak{so}(1, 2)$.*

Proof. If $a \neq 0$, then from Proposition 5.5.3, we know the Levi-Civita holonomy algebra is of dimension 3. Hence, in dimension 3, $\mathfrak{hol}(\nabla) = \mathfrak{so}(1, 2)$. □

Hence, in such a case (M, g) cannot be a plane wave and this shows we cannot reduce the dimension in the assumptions for Theorem 5.4.6. In fact, the dimension bound is sharp since the existence of a homogeneous space in dimension 3 with isotropy $\mathbb{R} \subset \mathfrak{so}(1, 2)$ and of a homogeneous structure such that $\mathfrak{hol}(\tilde{\nabla}) = \mathbb{R} \subset \mathfrak{so}(1, 2)$ that is not a pp-wave was shown in [20].

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