# On the Classification of Enriques Surfaces

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# Signed Statement

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# Abstract

In 1958, André Weil formulated a number of famous conjectures concerning the structure of K3 surfaces: all such surfaces are deformation equivalent; they all admit Kähler structures; every point of the period domain arises as the period point of a K3 surface; and the biholomorphism class of a K3 surface is determined by its period point. By appealing to the fact that every Enriques surface is double-covered by a K3 surface, one can in a natural way extend the considerations of the period map of K3 surfaces to the case of Enriques surfaces. In this case, one obtains analogous results for Enriques surfaces, and it is with their proof that the present work is concerned. By making use primarily of the classification results of K3 surfaces, as well as their moduli space of Einstein metrics, we provide short, independent proofs of the results concerning Enriques surfaces which taken together fully classify this important class of compact complex surface of Kodaira dimension 0. We conclude by extending the above considerations to the case of quotients of Enriques surfaces by free anti-holomorphic involutions where we prove that these resulting compact Einstein four-manifolds are diffeomorphic, and in fact of the same deformation type.

# Chapter 1

# Introduction

## 1.1 Context and Background

The main purpose of this thesis is to provide new proofs of the long-established classification results of Enriques surfaces which together form the analogue of the conjectures of Weil concerning K3 surfaces. Before elaborating on what these are, it is certainly instructive to make some brief remarks about the more general class of objects of primary interest of this thesis, namely compact complex surfaces. Investigations into such surfaces, particularly those defining smooth subvarieties of complex projective space, were made as far back as the nineteenth century. At some point, many algebraic geometers, particularly those of the Italian school, such as Federigo Enriques (1871 - 1946) and Guido Castelnuovo (1865 - 1952), turned their attention to classifying the class of objects we would now call *algebraic surfaces*. The key to obtaining said classification was the use of *invariants*, that is, quantities which can be used to differentiate the classes of object under study (here, sufficient to characterise them up to *birational equivalence*). In modern terms, the *Kodaira dimension* – an invariant depending *a priori* on the complex analytic structure of the surface and which takes values in the set $\{-\infty, 0, 1, 2\}$ – is the key quantity that was ultimately used in coarsely dividing the classes of algebraic surfaces.

Nonetheless, it must be remarked that the contributions of these great classical geometers were made at a time where neither algebraic nor complex geometry had been developed on any rigorous foundation. From the early to the mid-twentieth century, much foundational work in modern mathematics was taking place, including but not limited to developments in algebraic, differential and complex geometry, as well as modern topology and sheaf theory, and it was on this solid foundation that the task of classifying compact complex surfaces was undertaken anew. The chief figure, who arguably made the majority of contributions to this end single-handedly, was Japanese mathematician Kunihiko Kodaira (1915-1997). In his work, he modernised and made precise the work done decades before by the Italian school, but extended it in an essential way by including non-algebraic compact complex surfaces. The result is what we now rightly call *the Enriques-Kodaira classification of (minimal) compact complex surfaces*, and marks an important milestone in gaining a deeper understanding of higher-dimensional algebraic varieties. Kodaira's contributions to the area of complex (algebraic) geometry were far reaching, and his insight and mastery of a variety of mathematical techniques paved the way for many exciting developments in modern geometry, notably in the theory of deformations of compact complex manifolds.

By now, a large amount of work has been conducted on virtually all classes of the Enriques-Kodaira classification (for which there are ten), and a complete or otherwise sufficiently good understanding of many of them has been obtained. It could generally be said that, in much the same way that the space of compact Riemann surfaces with fixed genus $g$ becomes increasingly complex as $g$ increases, the same is true for the complexity of the space of complex surfaces as their Kodaira dimension increases; compact complex surfaces of maximal Kodaira dimension (i.e. those of general type) are difficult to get an explicit grasp of, while those of minimal Kodaira dimension (e.g. rational surfaces, ruled surfaces, etc.) are easier to describe and classify. K3 surfaces (named after 'Kummer, Kähler and Kodaira' by André Weil) represent an important subclass of surface of Kodaira dimension 0 with a slew of interesting properties and whose classification theory is neither completely intractable nor entirely trivial. As a result, K3 surfaces have played a central role in many investigations into the structure of complex surfaces, but more generally four-manifolds.

Explicit investigations into the classification theory of K3 surfaces can be said to begin with André Weil (who also credits Aldo Andreotti) in his report [53] from 1958. Note that in this context, by *classification* we do not mean the obtainment of a complete enumerable list of (the isomorphism classes of) K3 surfaces, which is certainly unthinkable, but rather, an explicit description of a space whose points correspond to (isomorphism classes of) K3 surfaces – one usually speaks of a *moduli space* in this context, i.e. a geometric object parameterising the class of objects of interest. In his short but thought-provoking text, Weil observes that any K3 surface naturally defines a point (known as its *period point*) in an open subset of a 20-dimensional projective hyperplane (what we now call the *period domain*). In his report, he raises a number of central questions pertaining to this process of assigning period periods to K3 surfaces, specifically: whether any two K3 surfaces defining the same period point must be biholomorphic, and whether any arbitrary point in the period domain corresponds to the period point of some K3 surface. He also speculated that all K3 surfaces are Kähler and form one connected family of surfaces (that is, that they can all be deformed into one another, in the sense of Kodaira and Spencer). As it turns out, all of these statements are true, and have been proved (and reproved) by a large number of authors. Chapter 4 of this thesis is dedicated solely to outlining the classification theory of K3 surfaces in detail.

Closely related to K3 surfaces are Enriques surfaces (named, of course, after their creator, Federigo Enriques). The close connection between them consists precisely in the fact that every Enriques surface is double-covered by a K3 surface, and conversely that any free holomorphic $\mathbb{Z}_2$-quotient of a K3 surface produces an Enriques surface. Thus, by passing to the double-cover, one can transport the classification theory of K3 surfaces to the case of Enriques surfaces, and in fact, one can formulate in a one-to-one fashion a direct analogue of the conjectures of Weil mentioned above. Specifically, any Enriques surface defines a point in a subset of the period domain for K3 surfaces, and this point determines its biholomorphism class. Moreover, any point in this subset arises as the period point of an Enriques surfaces, and it remarkably also turns out that all Enriques surfaces are deformation equivalent. This latter fact was long known to many mathematicians [44, p. 372], but investigations into the period domain for Enriques surfaces were first conducted by Japanese mathematician Eiji Horikawa in 1978 (see [23] and [24]). In the main chapter of this thesis, Chapter 5, we supply independent proofs of these results by relying for the most part on the results concerning K3 surfaces presented in Chapter 4.

## 1.2 Structure of the Work

The structure of this thesis is fairly linear, and begins in Chapter 2 with an overview of some of the most important concepts from differential and complex geometry that we explicitly make use of in later chapters. In Section 2.1, we provide a rapid overview of some of the standard characteristic classes associated to real and complex vector bundles over smooth manifolds. The following section – building on the previous – provides a more or less self-contained introduction into the intersection form of compact oriented $4k$-manifolds, with obvious particular emphasis on the case $k = 1$. We then make the shift from smooth to complex manifolds in Section 2.3 and provide an elementary overview of the splitting of complex differential forms as well as the integrability of almost complex structures. Section 2.4 deals with holomorphic vector bundles, and the main result of interest introduced here is the Riemann-Roch-Hirzebruch theorem, arguably one of the most important results in the area of complex geometry. Section 2.5 gives a concise overview of the basics of harmonic forms and concludes with the famous Serre Duality theorem – another indispensable tool for any complex geometer. An extremely superficial, but nonetheless important summary of Einstein, Kähler, Kähler-Einstein and hyperkähler metrics is given in Section 2.6, the highlight being Yau's theorem, which guarantees the existence of Ricci-flat Kähler-Einstein metrics for a certain class of compact Kähler manifolds. Even though the theory of divisors on a complex manifold will only play a tangential role in Chapters 4 and 5, i.e. the key chapters of this thesis, the classification theory of compact complex surfaces relies heavily on this, and so Section 2.7 acquaints the reader with the basics of divisors. The background chapter concludes in Section 2.8 with a brief discussion of two important invariants of compact connected complex manifolds: the algebraic and Kodaira dimension.

Chapter 3 restricts the focus of the reader to the theory of compact complex surfaces. It begins with a brief account of the intersection of *curves* on surfaces; no results of particular significance are introduced, but some important refinements of the notions of Sections 2.2 and 2.4 are presented. Section 3.2 provides a near self-contained explication of the notion of *blowing up* on a complex surface and its most important properties. The section concludes with a useful criterion for blowing surfaces down as well as the concept of minimality. Given that the classification of complex surfaces is that of birational classification, an extremely short description of bimeromorphic maps is provided in Section 3.3. The main result of this section is the fact that, in some sense, the only bimeromorphic map one is concerned with in the case of complex surfaces is precisely the blow up. Having set the stage in the previous sections, Chapter 3 concludes with the famous Enriques-Kodaira classification of minimal compact complex surfaces; in Section 3.4, we first motivate the classification by considering the case of compact Riemann surfaces. Thereafter, we introduce for the first time the two central objects of study in this work: K3 surfaces and Enriques surfaces, and provide a short proof of the fact that K3 surfaces equipped with free holomorphic $\mathbb{Z}_2$-actions are in one-to-one correspondence with Enriques surfaces (Theorem 3.4.1) – a fact which pervades the entirety of the investigations of Chapter 5. From there, the section continues by familiarising the reader with the remaining classes of the Enriques-Kodaira classification by providing brief descriptions of them.

In Chapter 4, we restrict our attention even further to the class of K3 surfaces. Given the indispensability of the notion of a *lattice* in modern mathematics, and, of course, in this thesis, the chapter begins by introducing some basic notions from the theory of lattices. Immediately after, we show that the intersection form of K3 surfaces has a fixed structure. Before diving straight into the classification theory of K3 surfaces, we make a brief detour in Section 4.3 and discuss the concept of a *Kähler cone* and determine its structure for the case of a K3 surface. The main exposition on the theory of K3 surfaces then begins in Section 4.4, where we introduce the key spaces and maps which classify (up to isomorphism) the complex and Kähler structures of K3 surfaces – the aforementioned *period maps* and *period domains*. We conclude our summary of results concerning K3 surfaces by investigating the moduli space of Einstein metrics on K3 surfaces in Section 4.5. The main object we consider in this context is the *Teichmüller space* – a smooth finite-dimensional quotient of the space of unit-volume Einstein metrics on a K3 surface. The chapter closes off by considering a natural period map associated to this space.

Chapter 5 is the main chapter of this work and concerns itself solely with proving the classification results concerning Enriques surfaces. In the first section, we show how the period domains and maps previously constructed for K3 surfaces can be extended to the case of Enriques surfaces, and conclude with a precise statement of their properties (Theorem 5.1.2 and Theorem 5.1.3). One important feature which makes the classification theory of Enriques surfaces via K3 surfaces possible is the fact that free holomorphic (in fact, smooth) involutions on a K3 surface induce conjugate actions on its intersection form. Section 5.2 is devoted to proving this result, and begins by providing an overview of the most important results pertaining to the study of unimodular lattices equipped with involutions. The classification results of Subsection 5.2.1 supply us with a purely algebraic proof of this conjugacy fact, which is finally explicated in Subsection 5.2.3. In between, the work of Edmonds [14] allows us to conclude in Subsection 5.2.2 that smooth involutions of K3 surfaces inducing actions on the K3 lattice of a specific form must have no fixed points. Having established these central results, Section 5.3 wastes no time in getting around to proving the surjectivity of the period maps associated to Enriques surfaces. With the help of a technical lemma related to the lifting of involutions preserving points in the Teichmüller space of K3 surfaces (Lemma 5.3.1), we obtain a quick proof of the surjectivity of the refined period map for Enriques surfaces, and the surjectivity of the ordinary period map quickly follows from this as a corollary. Section 5.4 outlines simple proofs of the fact that the period domains and maps of Enriques surfaces determine their structure up to isomorphism, and it concludes with a cursory exposition on a Teichmüller-type space for the space of Einstein metrics on an Enriques surface. Our investigations into Enriques surfaces conclude with Section 5.5, where we finally prove the fact that all Enriques surfaces are deformation equivalent using the results of previous sections.

The final chapter of this thesis, Chapter 6, extends the investigations of Chapter 5 to the case of quotients of Enriques surfaces by free anti-holomorphic involutions, often referred to as *Enriques-Einstein-Hitchin manifolds*. We show that such four-manifolds are all diffeomorphic (Theorem 6.2.2), and moreover, that all Ricci-flat Einstein structures on them are deformation equivalent (Theorem 6.2.3) – exactly as in the case of Enriques and K3 surfaces, and despite the fact that these four-manifolds do not admit complex or even almost complex structures.

A short list of the most important notation that is consistently used throughout the entirety of this work can be found in Appendix A.

## 1.3 Scope and Directions for Further Research

As one may have gathered, a lot of work has been done in this area of geometry, and as we have mentioned, the actual results we will be elaborating on have been known for quite some time. It would take us too far astray to discuss all the past and recent developments, as well as the refinements of the results that we will be studying. Besides, a great deal of informative texts in the literature exist which provide extensive accounts of the objects of interest of this work; the standard reference text [44] and its first edition [43], which we will often refer to in the sequel, provide comprehensive outlines of a lot of what is known about compact complex surfaces, with an entire chapter (Chapter VIII) being dedicated solely to the theory of K3 and Enriques surfaces and detailed notes on their development. Otherwise, [26] provides an extremely thorough look at the theory of K3 surfaces, and [9] gives an in-depth overview of Enriques surfaces. In particular, given that all Enriques surfaces are algebraic (Proposition 5.1.3), and many K3 surfaces are algebraic (19 of the 20 dimensions of the moduli space of K3 surfaces consist of algebraic K3 surfaces), the algebraic aspects of their theory, which are naturally also vast, were not able to be touched upon in this work.

As such, the contents of this thesis, including those related to the study of Enriques surfaces, are not entirely novel and certainly far from exhaustive in nature, but nonetheless elucidate quite a simple and down-to-earth approach to this well-studied area of complex geometry. The main noteworthy feature of the approach taken in arriving at the analogous classification results for Enriques surfaces is that no explicit constructions of Enriques surfaces are employed, particularly in proving the deformation equivalence of Enriques surfaces. Given the scope of the present work, this is certainly sufficient, but it naturally must be said that if one is interested in studying, for example, families of Enriques surfaces, then the methods presented here will not immediately be applicable. As a result, it could be interesting to investigate the exact points of departure of the methods outlined in Chapters 4 and 5 to the more standard constructions of the moduli of K3 and Enriques surfaces one finds in the literature, and to even ask if something new could be learned about K3 or Enriques surfaces in light of the explications contained therein.

Other possible avenues for research pertain to the actual proofs of the results elaborated in the main chapter of this work, Chapter 5. As mentioned, the fact that free smooth involutions on a K3 surface induce conjugate actions on its intersection form is a result which plays a crucial role in setting up the classification theory of Enriques surfaces. An interesting problem could be to show this fact merely using the investigations of Chapter 4, that is, results on the period map of K3 surfaces, as opposed to a purely algebraic argument, as was done here. Similarly, although the use of the Teichmüller space for K3 surfaces gives us quite straightforward proofs of the classification results of interest, particularly the deformation equivalence of Enriques surfaces, it could also be interesting to wonder whether the results could be proved without relying on the metric-related properties of K3 surfaces and obtained rather through more direct complex-geometric arguments. A similar remark can be made about the work done in Chapter 6 on free anti-holomorphic $\mathbb{Z}_2$-quotients of Enriques surfaces.

One will also observe that in Sections 5.4 and 6.2, we have provided quite cursory descriptions of Teichmüller-like spaces for the case of Enriques surfaces and Enriques-Einstein-Hitchin manifolds. Another possibility for further research would be to determine what kind of structure these spaces have, beyond that of a topological space, and moreover to investigate the proper analogue of the Teichmüller space for K3 surfaces in these other cases, and determine if these spaces are smooth or not (to the author's knowledge, these rather difficult problems remain open).

# Chapter 2

# General Background

In this heterogeneous chapter, we cover some of the most fundamental concepts from differential and complex geometry that we will need moving forward. So that we do not have to start from ground zero, we assume the reader is familiar with the more standard notions from differential geometry and algebraic topology that would typically be introduced in a first course, e.g. smooth vector bundles, differential forms, de Rham and singular cohomology, homotopy theory, and so on. We also assume a basic understanding of sheaves and their cohomology. Given the scope of many of these areas, it naturally must be noted that none of the sections in this chapter will be exhaustive.

For the basics of complex geometry, we follow Huybrechts' *Complex Geometry: An Introduction* [25], Griffiths and Harris' *Principles of Algebraic Geometry* [17] and Voisin's *Hodge Theory and Complex Algebraic Geometry* [52]. For information on Einstein metrics and Kähler-Einstein metrics, Besse's text *Einstein Manifolds* [3] provides an extensive account. The section on intersection theory will be largely self-contained. However, note that we will implicitly be making use of standard transversality results which can be found in most introductory texts on differential topology (for example, see [18]).

## 2.1 Characteristic Classes

Vector bundles are a fundamental structure in the mathematical landscape, and their characteristic classes play a crucial role in modern differential geometry and topology. For our purposes, characteristic classes will enable us to calculate a variety of important quantities which will be used to obtain geometric or algebraic information about the objects of focus of this work. In this short section, we introduce the more standard characteristic classes associated to smooth vector bundles and the relationships between them.

The first set of characteristic classes we introduce are those associated to real vector bundles. These are the so-called *Stiefel-Whitney classes*. They define cohomology classes with $\mathbb{Z}_2$-coefficients:

**Proposition 2.1.1.** *Given a real vector bundle $E \to X$ of rank $k$ over a smooth manifold, there exist classes $w_i(E) \in H^i(X, \mathbb{Z}_2)$, $i \in \mathbb{N}$ uniquely defined by the following four properties:*

*(i)* $w_0(E) = 1$ *and* $w_i(E) = 0$ *for* $i > k$;

*(ii) For any smooth map* $f : Y \to X$, *we have* $f^* w_i(E) = w_i(f^* E)$;

*(iii) If the total Stiefel-Whitney class is denoted* $w(E) = 1 + w_1(E) + \ldots + w_k(E)$, *then for any other bundle* $F \to X$, *we have* $w(E \oplus F) = w(E) \smile w(F)$;

*(iv) The first Stiefel-Whitney class of the tautological line bundle over* $\mathbb{RP}^1$ *is non-trivial.*

Any standard text covers their construction (see, for example, [37]), and is usually done by making use of the classifying space of the group $\mathbb{Z}_2$. In particular, the total Stiefel-Whitney class of the tangent bundle $TX$ of a smooth manifold $X$ is denoted $w(X)$.

Stiefel-Whitney classes are useful in obtaining important geometric information about a space. For example, an important class of real vector bundles are those with an orientation. A standard result in the theory tells us:

**Proposition 2.1.2.** *A real vector bundle $E$ is orientable if and only if $w_1(E) = 0$. In particular, a smooth manifold $X$ is orientable if and only if $w_1(X) = 0$.*

We also make the following definition:

**Definition 2.1.1.** A smooth, oriented manifold $X$ is called *spin* if $w_2(X) = 0$.

**Remark 2.1.1.** This definition coincides with the more familiar notion of a spin structure in terms of lifts of the frame bundle of a smooth manifold. However, we opted for this definition as we will not need to work with spin structures for the purposes of this thesis.

The main characteristic class associated to an oriented real vector bundle $E \to X$ of rank $k$ is known as the *Euler class*. It defines a $\mathbb{Z}$-valued cohomology class $e(E) \in H^k(X, \mathbb{Z})$ which enjoys many of the standard properties that characteristic classes possess. When we come to investigate the intersection form of manifolds in the next section, we will see that the Euler class has a very concrete geometric description (Theorem 2.2.3). For now, let us note that the Euler class is related to the Stiefel-Whitney classes in the following way (see [37, p. 99]):

**Proposition 2.1.3.** *For a real oriented vector bundle $E \to X$ of rank $k$, we have*

$$w_k(E) = e(E) \bmod 2.$$

Moving on, we can consider vector bundles with even more structure, namely complex vector bundles. Naturally, these too have their own associated characteristic classes, known as the *Chern classes*. These define $\mathbb{Z}$-valued cohomology classes and are determined effectively by analogous properties as listed in Proposition 2.1.1:

**Proposition 2.1.4.** *Given a complex vector bundle $E \to X$ of rank $k$ over a smooth manifold, there exist classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for $i \in \mathbb{N}$ uniquely defined by the following properties:*

*(i)* $c_0(E) = 1$ *and* $c_i(E) = 0$ *for* $i > k$*;*

*(ii) For any smooth map $f : Y \to X$, we have $f^*c_i(E) = c_i(f^*E)$;*

*(iii) If the total Chern class is denoted $c(E) = 1 + c_1(E) + \ldots + c_k(E)$, then for any other complex bundle $F \to X$, we have $c(E \oplus F) = c(E) \smile c(F)$;*

*(iv) If $L$ is the hyperplane bundle over $\mathbb{P}^1$, then $c_1(L)[\mathbb{P}^1] = 1$, where $[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$ denotes the fundamental class of $\mathbb{P}^1$.*

One can construct the Chern classes in a similar way to the Stiefel-Whitney classes (i.e. by considering classifying spaces). Another standard way of constructing them is via Chern-Weil theory, which defines representatives for the Chern classes in terms of the curvature of an arbitrary connection for the vector bundle. An easy result in the theory says that the first Chern class determines the isomorphism class of a complex line bundle (analogously, the first Stiefel-Whitney class determines a real line bundle):

**Proposition 2.1.5.** *Let $X$ be a smooth manifold and denote the space of isomorphism classes of complex line bundles over $X$ by $\mathcal{L}_{\mathbb{C}}(X)$. Then, $\mathcal{L}_{\mathbb{C}}(X)$ forms an abelian group under the tensor product operator, and the first Chern class map*

$$c_1 : \mathcal{L}_{\mathbb{C}}(X) \to H^2(X, \mathbb{Z}),\ L \mapsto c_1(L)$$

*is an isomorphism of abelian groups.*

Given that every complex vector bundle defines a canonically oriented real vector bundle, one would hope that the Chern classes are related to both the Euler class and Stiefel-Whitney classes. In fact, yet another way to construct the Chern classes is by making use of the Euler class and defining them inductively (as in [37]). We have:

**Theorem 2.1.1** (Theorem 20.10.6, [5])**.** *If $E$ is a complex vector bundle of rank $k$, and $E_{\mathbb{R}}$ denotes the underlying oriented real bundle of rank $2k$, then*

$$c_k(E) = e(E_{\mathbb{R}}).$$

**Proposition 2.1.6.** *If $E$ is a complex vector bundle, and $E_{\mathbb{R}}$ denotes the underlying real bundle, then we have*

$$w(E) = c(E) \bmod 2.$$

*In particular, all odd Stiefel-Whitney classes are zero and we have*

$$w_{2k}(E_{\mathbb{R}}) = c_k(E) \bmod 2.$$

Having introduced the Chern classes, we need to also define a few other characteristic classes in terms of them. They will be particularly relevant when we come to look at holomorphic vector bundles in Section 2.4. By invoking the splitting principle for complex vector bundles, we can define them as follows:

**Definition 2.1.2.** Given a complex vector bundle $E \to X$ over a smooth manifold of rank $k$, suppose $x_1, \ldots, x_k$ are its Chern roots, i.e. so that its Chern classes can be represented as symmetric polynomials in $x_1, \ldots, x_k$:

$$c_j(E) = \sum_{1 \leq i_1 < \ldots < i_j \leq k} x_{i_1} \ldots x_{i_j}.$$

Then the total *Todd class* of $E$ is given by expanding

$$\mathrm{td}(E) = \left(\frac{x_1}{1 - e^{-x_1}}\right)\left(\frac{x_2}{1 - e^{-x_2}}\right) \cdots \left(\frac{x_k}{1 - e^{-x_k}}\right),$$

and the total *Chern character* of $E$ is given by expanding

$$\mathrm{ch}(E) = e^{x_1} + e^{x_2} + \ldots + e^{x_k}.$$

Using the Taylor expansions

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{and} \quad \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \ldots$$

and expanding out the terms above, we can express the Chern character and Todd class completely in terms of the Chern classes of $E$. The first few terms of their expansions are shown below:

$$\mathrm{td}(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1^2(E) + c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} + \dots$$
$$\mathrm{ch}(E) = k + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} + \frac{c_1^3(E) - 3c_1(E)c_2(E) + c_3(E)}{6} + \dots$$

If we define $\mathrm{td}_k(E)$ to be the element of the expansion of $\mathrm{td}(E)$ of degree $2k$, and similarly for $\mathrm{ch}_k(X, \mathbb{Z})$, e.g. $\mathrm{td}_2(E) = (c_1^2(E) + c_2(E))/12$ and $\mathrm{ch}_2(E) = (c_1^2(E) - 2c_2(E))/2$, then the total Todd class of $E$ can be written as

$$\mathrm{td}(E) = \mathrm{td}_0(E) + \mathrm{td}_1(E) + \mathrm{td}_2(E) + \dots + \mathrm{td}_{\mathrm{rk}(E)}(E),$$

and similarly for $\mathrm{ch}(E)$.

Using the Chern class, we introduce one final characteristic class associated to real vector bundles – the *Pontryagin class*:

**Definition 2.1.3.** Given a real vector bundle $E$ over a smooth manifold $X$, define its $k^{\text{th}}$ *Pontryagin class* by setting

$$p_k(E) := (-1)^k c_{2k}(E_{\mathbb{C}}) \in H^{4k}(X, \mathbb{Z}),$$

where $E_{\mathbb{C}}$ denotes the complexification of $E$, i.e. $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$. If we suppose the total Pontryagin class is given by

$$p(E) = \prod_{i=1}^{k} (1 + x_i),$$

where $k$ is the rank of $E$, then the total *L-genus of* $E$ is given by expanding the product

$$L(E) = \left( \frac{\sqrt{x_1}}{\tanh \sqrt{x_1}} \right) \dots \left( \frac{\sqrt{x_k}}{\tanh \sqrt{x_k}} \right).$$

The first few terms of the L-genus are then given by

$$L(E) = L_0(E) + L_1(E) + L_2(E) + \dots = 1 + \frac{1}{3} p_1(E) + \frac{1}{45}(7p_2(E) - p_1^2(E)) + \dots$$

As usual, we denote the $k^{\text{th}}$ Pontryagin class of the tangent bundle of a smooth manifold $X$ by $p_k(X)$, and the term of degree $4k$ in the L-genus of the tangent bundle is denoted $L_k(X)$.

We conclude by noting that the Pontryagin classes of a complex vector bundle can be completely expressed in terms of the Chern classes of the bundle:

**Proposition 2.1.7.** *If $E$ is a complex vector bundle, then its Pontryagin classes are given by*

$$p_k(E) = \sum_{i=0}^{2k} (-1)^{k+i} c_i(E) \smile c_{2k-i}(E).$$

*In particular, the first Pontryagin class is given by*

$$p_1(E) = c_1^2(E) - 2c_2(E).$$

*Proof.* This follows by the fact the complexification of $E_{\mathbb{R}}$ is isomorphic to the bundle $E \otimes \overline{E}$, and $c_k(\overline{E}) = (-1)^k c_k(E)$. □

## 2.2 Poincaré Duality and Intersection Theory

As we will see, intersection theory is absolutely crucial in getting a grasp on the structure of compact complex surfaces, and will play a central role throughout all the work we do in Chapters 4 and 5. Given the importance of this area, we will go through in some detail some of the basic definitions and results which are needed in understanding the intersection form of a smooth four-manifold. We will only explicitly focus on the results that will make their appearance later.

We begin our discussion in the category of topological manifolds. After having introduced some standard terminology, we move to the smooth setting. To this end, recall the famous Poincaré duality theorem:

**Theorem 2.2.1** (Poincaré Duality). *If $X$ is a closed oriented topological manifold of dimension $n$ with fundamental class $[X] \in H_n(X, \mathbb{Z})$, then the map*

$$\mathscr{P}_X : H^k(X, \mathbb{Z}) \to H_{n-k}(X, \mathbb{Z}),\ \alpha \mapsto \alpha \frown [X]$$

*is an isomorphism for every non-negative integer $k$.*

We also introduce some notation:

**Definition 2.2.1.** For any abelian group $G$ and topological space $X$, we will denote the canonical pairing of a cohomology class $\alpha \in H^k(X, G)$ and homology class $c \in H_k(X, G)$ either by $\alpha(c) \in G$ or $\langle \alpha, c \rangle \in G$. If $X$ is a closed oriented topological manifold of dimension $n$, we also define the following intersection product on cohomology:

$$H^k(X, \mathbb{Z}) \times H^{n-k}(X, \mathbb{Z}) \to \mathbb{Z},\ (\alpha, \beta) \mapsto \alpha \cdot \beta := \langle \alpha \smile \beta, [X] \rangle.$$

**Remark 2.2.1.** Via Poincaré duality, we can define a pairing $H_k(X,\mathbb{Z})\times H_{n-k}(X,\mathbb{Z}) \to \mathbb{Z}$. Observe that the intersection pairing vanishes on torsion elements.

By the definition of the cup product, we have

$$\alpha \cdot \beta = (-1)^{k(n-k)}(\beta \cdot \alpha)$$

for classes $\alpha \in H^k(X,\mathbb{Z})$ and $\beta \in H^{n-k}(X,\mathbb{Z})$. With this simple observation in mind, we make the following important definition:

**Definition 2.2.2.** Let $X$ be a compact oriented topological $4k$-manifold. Then the $\mathbb{Z}$-valued symmetric bilinear form on $H^{2k}(X,\mathbb{Z})$ defined by the intersection product

$$H^{2k}(X,\mathbb{Z}) \times H^{2k}(X,\mathbb{Z}) \to \mathbb{Z}, (\alpha, \beta) \mapsto \alpha \cdot \beta$$

is called the *intersection form* of $X$.

In particular, if $T_k \subseteq H^k(X,\mathbb{Z})$ denotes the torsion subgroup, then $H^{2k}(X,\mathbb{Z})/T_{2k}$ is a free $\mathbb{Z}$-module of rank $b_{2k}(X)$. Equipping it with its intersection form, the free abelian group $H^{2k}(X,\mathbb{Z})/T_{2k}$ carries the structure of a *lattice*. We will look at these objects in more detail in the chapters on K3 and Enriques surfaces (cf. Section 4.1 and Section 5.2.1). Given that all real symmetric bilinear forms are diagonalisable, we can state one of the most important invariants of compact oriented $4k$-manifolds:

**Definition 2.2.3.** Let $X$ be a compact oriented topological $4k$-manifold. Extending its intersection form by $\mathbb{R}$-linearity, we define its *signature* by the formula

$$\sigma(X) := b^+(X) - b^-(X),$$

where $b^{\pm}(X)$ denotes the number of positive/negative eigenvalues in some (hence any) diagonalisation of its intersection form. In particular, we have $b_{2k}(X) = b^+(X) + b^-(X)$.

The *parity* of the intersection form on a manifold can often (as we will see in the sequel) give us useful information. The following definition will appear in greater generality later:

**Definition 2.2.4.** The intersection form of a compact, oriented $4k$-manifold $X$ is said to be *even* if for every $x \in H^{2k}(X,\mathbb{Z})$, we have $x^2 := x \cdot x \in 2\mathbb{Z}$. Otherwise, it is said to be *odd*.

Characteristics of the intersection form (or more generally, of lattices) are central to determining the parity of a lattice:

**Definition 2.2.5.** If $X$ is a compact, oriented $4k$-manifold, an element $c \in H^{2k}(X, \mathbb{Z})$ is said to be a *characteristic* of the intersection form for $X$ if for every $x \in H^{2k}(X, \mathbb{Z})$, we have

$$x^2 \equiv c \cdot x \pmod 2.$$

In particular, the intersection form is even if $c$ is zero modulo torsion.

Before moving on to investigating the smooth case, let us introduce some final definitions arising from Poincaré duality:

**Definition 2.2.6.** If $f : X \to Y$ is a continuous map between compact oriented topological manifolds of the same dimension, then we can define the pushforward on cohomology by setting:

$$f_! := \mathscr{P}_Y^{-1} \circ f_* \circ \mathscr{P}_X : H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z}).$$

Similarly, we can define a pullback on homology by setting:

$$f^! := \mathscr{P}_X \circ f^* \circ \mathscr{P}_Y^{-1} : H_k(Y, \mathbb{Z}) \to H_k(X, \mathbb{Z}).$$

We have the useful result, which follows immediately by the naturality of the cap product on cohomology:

**Proposition 2.2.1.** *Suppose $f : X \to Y$ is a continuous map between compact oriented manifolds of degree $d$. Then for every non-negative integer $k$:*

$$f_! \circ f^* = d \cdot \mathrm{id}_{H^k(X,\mathbb{Z})} \quad \textit{and} \quad f_* \circ f^! = d \cdot \mathrm{id}_{H_k(X,\mathbb{Z})}.$$

Let us now restrict our focus to the case of smooth manifolds. By making use of the smooth structure, the intersection form defined above in purely topological terms will be able to be understood in much more geometric terms:

**Definition 2.2.7.** Two smooth submanifolds $A$ and $B$ on a manifold $X$ are said to intersect *transversely* at a point $x \in A \cap B$ if $T_xA + T_xB = T_xX$. They are said to be *transverse* if they intersect transversely everywhere.

It is never guaranteed that the intersection of two embedded submanifolds is an embedded submanifold, however, the following standard result from differential topology tells us this is always the case for transverse submanifolds:

**Proposition 2.2.2.** *If $A$ and $B$ are transverse submanifolds of a smooth manifold $X$, then $A \cap B$ is an embedded submanifold and*

$$\mathrm{codim}(A \cap B) = \mathrm{codim}(A) + \mathrm{codim}(B).$$

In particular, if $A$ and $B$ have complementary dimension, then $\dim(A \cap B) = 0$, and hence $A \cap B$ is a finite set of a points when $X$ is compact. It is a standard fact (for example, see [18]) that two submanifolds of complementary dimension generically (i.e. can be perturbed so that they) intersect transversely. With this in mind, we can then talk about intersection numbers:

**Definition 2.2.8.** Suppose $A$ and $B$ are oriented submanifolds of complementary dimension on a smooth compact oriented manifold $X$ of dimension $n$ which intersect transversely at $x \in A \cap B$ such that $a_1, \ldots, a_k$ is a basis for $T_xA$ and $b_1, \ldots, b_{n-k}$ is a basis for $T_xB$ which are compatible with the given orientations of $A$ and $B$, respectively. The *sign* of the transverse intersection at $x \in A \cap B$ is defined to be

$$\iota_x(A \cdot B) := \begin{cases} 1 \text{ if } a_1, \ldots, a_k, b_1, \ldots, b_{n-k} \text{ is compatible with the given orientation of } T_xX, \\ -1 \text{ otherwise.} \end{cases}$$

If $A$ and $B$ intersect transversely everywhere, then we can define their *topological intersection number* by

$$A \cdot B := \sum_{x \in A \cap B} \iota_x(A \cdot B) \in \mathbb{Z},$$

where the sum is finite by the compactness of $X$.

Note that a compact, oriented submanifold $A \subseteq X$ of dimension $k$ defines a fundamental class $[A] \in H_k(A, \mathbb{Z})$, which we always identify with its image under the embedding $i : A \hookrightarrow X$, i.e. $[A] \in H_k(X, \mathbb{Z})$. However, keep in mind that it is not true in general that an arbitrary homology class can be represented as the fundamental class of an embedded, oriented submanifold (for more details, one can consult the foundational work of René Thom in [49]). We are now in a position to be able to state the following non-trivial result which allows one to compute intersections in a rather concrete way:

**Theorem 2.2.2.** *Let $X$ be a smooth compact oriented manifold of dimension $n$, and suppose that $a \in H_k(X, \mathbb{Z})$ and $b \in H_{n-k}(X, \mathbb{Z})$ can be represented by smooth oriented submanifolds $A$ and $B$, respectively. If $\alpha := \mathscr{P}_X^{-1}(a)$ and $\beta := \mathscr{P}_X^{-1}(b)$, then after possibly perturbing $A$ or $B$, we can assume that they are transverse, and we obtain*

$$A \cdot B = \alpha \cdot \beta = \langle \alpha \smile \beta, [X] \rangle.$$

As we alluded in the previous section, the Euler class can be given a very geometric description based on the terminology we have introduced. If we are given a real vector bundle $E \to X$ of rank $k$ over a compact manifold of dimension $n$, and a section $\sigma : X \to E$ that is transversal to the zero section, which we identify with $X$, then its zero set $Z := \sigma^{-1}(0) = \sigma(X) \cap X \subseteq X$ is an embedded submanifold of dimension $n - k$ by Proposition 2.2.2. The following is a standard result in differential topology:

**Theorem 2.2.3.** *If $E \to X$ is a real oriented vector bundle of rank $k$ over a smooth compact manifold of dimension $n$ and $\sigma : X \to E$ is a section transversal to the zero section of $E$, then the fundamental class of $Z := \sigma^{-1}(0)$ is Poincaré dual to the Euler class $e(E) \in H^k(X, \mathbb{Z})$, i.e. $[Z] = \mathscr{P}_X(e(E)) \in H_{n-k}(X, \mathbb{Z})$.*

From this, we can derive some important corollaries. The first fact it supplies is a simple proof of the representability of degree-two homology classes on four-manifolds:

**Corollary 2.2.1.** *Suppose $X$ is a smooth compact oriented four-manifold and let $x \in H_2(X, \mathbb{Z})$. Then there exists an embedded, oriented submanifold $S \subseteq X$ with $[S] = x$.*

*Proof.* Let $x \in H_2(X, \mathbb{Z})$. Then $\mathscr{P}_X^{-1}(x) \in H^2(X, \mathbb{Z})$. Since $c_1 : \mathcal{L}_{\mathbb{C}} \to H^2(X, \mathbb{Z})$ is an isomorphism (Proposition 2.1.5), there exists a complex line bundle $L \to X$ with $c_1(L) = \mathscr{P}_X^{-1}(x)$. Generically, there exists a section $\sigma : X \to L$ which is transverse to the zero section. Given that $e(L_{\mathbb{R}}) = c_1(L)$ (Theorem 2.1.1), the fundamental class of $S := \sigma^{-1}(0) \subseteq X$ satisfies $[S] = \mathscr{P}_X(c_1(L)) = x$ by Theorem 2.2.3. □

**Corollary 2.2.2.** *Let $X$ be a smooth compact oriented manifold of even dimension. If $Y$ is an oriented submanifold of $X$ such that $\dim Y = \frac{1}{2}\dim X$, then*

$$e(N)[Y] = Y \cdot Y,$$

*where $N$ is the normal bundle of $Y$ in $X$.*

*Proof.* Using standard transversality arguments, choose a smooth section $\sigma : Y \to N$ which is transverse to the zero section $\zeta : Y \to N$. Given that intersection is Poincaré dual to cupping (Theorem 2.2.2), combining with Theorem 2.2.3, we see that

$$\zeta(Y) \cdot \sigma(Y) = \langle \mathscr{P}_Y^{-1}[Z], [Y] \rangle = \langle e(N), [Y] \rangle,$$

where $Z = \sigma(Y) \cap \zeta(Y)$ is the zero set of $\sigma$. It remains to show that $Y \cdot Y = \zeta(Y) \cdot \sigma(Y)$. If we choose a Riemannian metric for $X$, then we can consider its exponential map $\exp : TX \to X$. Moreover, the metric yields $N = (TY)^{\perp} \subseteq TX\big|_Y$, and so we can consider the restriction of exp to $N$. We then have that it maps a neighbourhood $U$ of the zero section in $N$ diffeomorphically onto a neighbourhood of $Y$ in $X$. By scaling $\sigma$ by an appropriately small scalar, we can assume that $\sigma(Y) \subseteq U$, and then $\exp(\sigma(Y))$ is a submanifold transverse and diffeomorphic to $Y$, and so $Y \cdot Y = Y \cdot \exp(\sigma(Y)) = \zeta(Y) \cdot \sigma(Y)$. □

**Corollary 2.2.3.** *If $X$ is a smooth compact oriented manifold of dimension $n$, then we have*

$$e(X)[X] = \chi(X),$$

*where $e(X)$ is the Euler class of the tangent bundle of $X$, and $\chi(X)$ is the topological Euler characteristic of $X$, i.e. the alternating sum of its Betti numbers.*

*Proof.* If $f : X \to X$ is a smooth map, then its graph $\Gamma(f) \subseteq X \times X$ is a compact, oriented submanifold. If $\Delta \subseteq X \times X$ denotes the diagonal and $\Lambda_f$ denotes the ordinary topological Lefschetz trace of $f$, which is given as

$$\Lambda_f = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(f^* : H^k(X, \mathbb{R}) \to H^k(X, \mathbb{R})),$$

then writing expressions for the Poincaré duals of the classes of $\Gamma(f)$ and $\Delta$ yields:

$$\Lambda_f = \Delta \cdot \Gamma(f).$$

Note that $f$ can always be homotoped so that $\Gamma(f)$ is transverse to $\Delta$ (see, for example, [18, Chapter 3]). Noting that the graph of the identity on $X$ is exactly $\Delta$, we have

$$\chi(X) = \Lambda_{\mathrm{id}_X} = \Delta \cdot \Delta.$$

However, by Corollary 2.2.2, we obtain

$$e(N)[\Delta] = \Delta \cdot \Delta = \chi(X),$$

where $N$ is the normal bundle of $\Delta$ in $X \times X$. Given that $\Delta$ is canonically diffeomorphic to $X$, we just have to show that $N \cong TX$ (as oriented bundles). This follows at once by considering the standard exact sequence

$$0 \longrightarrow TX \xrightarrow{\alpha} (TX \oplus TX)|_\Delta \xrightarrow{\beta} TX \longrightarrow 0,$$

where $\alpha$ maps $v \in T_xX$ to $(v, v) \in T_xX \oplus T_xX$, and $\beta$ maps $(v, w) \in T_xX \oplus T_xX$ to $v - w \in T_xX$. □

With these tools, we can give a short proof of a much more general result known as *Wu's theorem* (which relates the Stiefel-Whitney classes of a smooth manifold to its *Wu classes*) in the case of four-manifolds. The statement reads:

**Theorem 2.2.4.** *If $X$ is a smooth compact oriented four-manifold, then for any $x \in H^2(X, \mathbb{Z})$, we have*

$$x^2 \equiv w_2(X) \cdot x \pmod 2.$$

*Proof.* Suppose $x \in H^2(X, \mathbb{Z})$ is arbitrary. Corollary 2.2.1 implies there exists an oriented submanifold $S \subseteq X$ with $[S] = \mathscr{P}_X(x)$. Given the normal bundle sequence

$$0 \longrightarrow TS \longrightarrow TX\big|_S \longrightarrow N \longrightarrow 0,$$

where $N$ is the normal bundle of $S$ in $X$, we have $TX\big|_S \cong TS \oplus N$. By the universal coefficient theorem, we have $H^2(X, \mathbb{Z}_2) \cong \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}_2)$, and so $w_2(X)$ is determined by its value on compact, oriented surfaces $[S]$ (again applying Corollary 2.2.1). In particular, $w_2(X) \cdot x \bmod 2 = w_2(X)[S] \in \mathbb{Z}_2$. By Proposition 2.1.1 (iii) and using the fact that $S$ and $N$ are oriented (i.e. $w_1(S) = w_1(N) = 0$), we get:

$$w_2\left(TX\big|_S\right) = w_2(S) + w_2(N).$$

Now, given that $H_2(S, \mathbb{Z}) \cong \mathbb{Z}$ is generated by $[S]$, the class $w_2(S) \in H^2(S, \mathbb{Z}_2) \cong \mathrm{Hom}(H_2(S, \mathbb{Z}), \mathbb{Z}_2)$ is determined by its value on $[S]$. Applying Proposition 2.1.3, and then Corollary 2.2.3, we see that

$$w_2(S)[S] = e(S)[S] \bmod 2 = \chi(S) \bmod 2 = 0,$$

which follows by the fact that $\chi(S) = 2(1 - g)$, where $g$ is the genus of $S$. So $w_2(S) = 0$, i.e. all compact oriented surfaces are spin. Then:

$$w_2\left(TX\big|_S\right) = w_2(N).$$

Since $w_2\left(TX\big|_S\right)[S] = w_2(X)[S]$, we have

$$w_2(X) \cdot x \bmod 2 = w_2(N)[S] = e(N)[S] \bmod 2 = x^2 \bmod 2,$$

again using Proposition 2.1.3 and Corollary 2.2.2. □

This immediately implies:

**Corollary 2.2.4.** *If $X$ is a smooth compact oriented four-manifold, then an integral lift of $w_2(X)$ is a characteristic element for the intersection form.*

An important result in the theory of four-manifolds is that integral lifts of $w_2(X)$ (or, equivalently, spin$^c$-structures on smooth oriented four-manifolds) always exist:

**Theorem 2.2.5** (Lemma 3.1.2, [39]). *Let $X$ be a smooth oriented four-manifold. Then there exists a class $c \in H^2(X, \mathbb{Z})$ such that $w_2(X) = c \bmod 2$.*

The following is another important consequence of Theorem 2.2.4, which follows at once from it and the universal coefficient theorem:

**Corollary 2.2.5.** *Let $X$ be a smooth oriented compact four-manifold with no 2-torsion in $H_1(X, \mathbb{Z})$. Then, $X$ is spin if and only if it has even intersection form.*

We conclude this section by introducing two important results regarding the signature of a smooth manifold. The first is known as the Hirzebruch signature theorem, which states that the $L$-genus that we introduced in Section 2.1 can be used to compute the signature:

**Theorem 2.2.6** (Hirzebruch Signature Theorem)**.** *If $X$ is a smooth compact oriented $4k$-manifold, then*

$$\sigma(X) = \langle L_k(X), [X] \rangle.$$

*In particular, if $X$ is a four-manifold, then*

$$\sigma(X) = \frac{1}{3}\langle p_1(X), [X] \rangle.$$

Given the fact that the Pontryagin classes are natural with respect to pulling back, we immediately obtain the useful corollary:

**Corollary 2.2.6.** *If $\pi : X \to Y$ is a smooth finite covering of smooth compact oriented $4k$-manifolds of degree $n$, then:*

$$\sigma(X) = n \cdot \sigma(Y).$$

**Remark 2.2.2.** Seeing as the signature of a closed oriented $4k$-manifold is purely a topological consideration, one would hope that the signature is multiplicative under continuous coverings of closed oriented topological $4k$-manifolds. It turns out that this is in fact true, but not at all obvious to show. In [46], the author proves this (see Theorem 8), and arrives at it by defining the $L$-genus for the tangent bundle (in the continuous setting).

The second is an important result on the signature of a spin four-manifold:

**Theorem 2.2.7** (Rokhlin)**.** *If $X$ is a smooth compact oriented spin four-manifold, then $\sigma(X) \equiv 0 \pmod{16}$.*

Let us now leave the smooth setting and investigate the complex setting.

## 2.3 Complex Manifolds and Differential Forms

In this section, we will go through some of the foundational definitions pertaining to (almost) complex manifolds. We will not touch on many of the more technical aspects of the theory, such as the obstructions to admitting an almost complex structure or the proof of the Newlander-Nirenberg theorem. The material contained in these pages is certainly important, though very much standard, and so we will only very briefly cover the elements of the theory.

Before considering complex manifolds and complex differential forms, we fix some notation (the reader may refer to Appendix A as necessary throughout). As we saw, the tangent bundle of a smooth manifold $X$ is denoted $TX$. Its cotangent bundle will always be denoted $T^*X$. The space of sections of the bundle $\bigwedge^k T^*X$ is denoted by $\Omega^k_{\mathbb{R}}(X)$ for $k = 1, \ldots, \dim X$. As is customary, we make the convention $\Omega^0_{\mathbb{R}}(X) := C^\infty(X)$, where $C^\infty(X)$ is the $\mathbb{R}$-algebra of smooth $\mathbb{R}$-valued functions on $X$.

**Definition 2.3.1.** A complex manifold is a second countable, Hausdorff topological space equipped with a maximal holomorphic atlas.

Given that any holomorphic map is smooth, any complex manifold has a smooth structure underlying it. Moreover, all complex manifolds have a natural orientation consistent with the complex structure. If $X$ is a complex manifold of dimension $n$, then $\mathcal{T}_X$ will denote its *holomorphic* tangent bundle (cf. Definition 2.4.1). By $TX$, we will still mean the tangent bundle of the underlying smooth manifold of dimension $2n$. The bundles $\mathcal{T}_X$ and $TX$ are naturally isomorphic as oriented real vector bundles.

There is an alternative point of view of a complex manifold which will be central to our analysis in later chapters. We start by considering the intermediary notion of an *almost complex* manifold:

**Definition 2.3.2.** Given a smooth manifold $X$, an *almost complex structure* for $X$ is a smooth bundle morphism $I : TX \to TX$ such that $I^2 = -\mathrm{id}_{TX}$. The pair $(X, I)$ will sometimes be referred to as an *almost complex manifold.* Note that every almost complex manifold is canonically oriented.

Given that an almost complex structure on a smooth manifold gives its tangent bundle the structure of a complex vector bundle, it makes sense to make the following definition:

**Definition 2.3.3.** Given an almost complex manifold $(X, I)$, set:

$$c_k(X) := c_k(TX), \quad \mathrm{td}_k(X) := \mathrm{td}_k(TX), \quad \mathrm{ch}_k(X) := \mathrm{ch}_k(TX).$$

By Theorem 2.1.1 and Corollary 2.2.3, we immediately obtain:

**Corollary 2.3.1.** *If $X$ is a compact, almost complex manifold of dimension $n$, then*

$$c_n(X)[X] = \chi(X).$$

Moreover, given the relationship between the Pontryagin classes and Chern classes of a complex vector bundle (Proposition 2.1.7), we state the following particular case of the Hirzebruch signature theorem:

**Theorem 2.3.1.** *The signature of a compact, almost complex four-manifold $(X, I)$ is given by*

$$\sigma(X) = \frac{1}{3}\langle c_1^2(X) - 2c_2(X), [X]\rangle.$$

If $X$ is a complex manifold, then as we noted, we have an isomorphism of real oriented vector bundles $\mathcal{T}_X \cong TX$. Under this isomorphism, multiplication by $i$ on $\mathcal{T}_X$ defines a natural almost complex structure for $X$. Thus, we have:

**Proposition 2.3.1.** *Every complex manifold admits a natural almost complex structure.*

Now, the natural complex extension of an almost complex structure $I$ on a smooth manifold $X$ to the complexified tangent bundle $T_{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C}$ induces a decomposition given by

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

where $T^{1,0}X$ and $T^{0,1}X$ are the $\pm i-$eigensubbundles coming from $I$, respectively. Conversely, providing such a splitting of $T_{\mathbb{C}}X$ into conjugate subbundles defines an almost complex structure whose $\pm i-$eigensubbundles correspond to these conjugate subbundles. The complexified cotangent bundle $T^*_{\mathbb{C}}X$ decomposes in an analogous way.

**Definition 2.3.4.** For an almost complex manifold $(X, I)$, define the complex vector bundle

$$\bigwedge^{p,q} T^*X := \bigwedge^{p} (T^*X)^{1,0} \otimes_{\mathbb{C}} \bigwedge^{q} (T^*X)^{0,1},$$

and denote its global sections by $\Omega^{p,q}(X)$. An element $\alpha \in \Omega^{p,q}(X)$ is referred to as a *differential form of bidegree* $(p, q)$, or simply a $(p, q)$*-form.* Moreover, for an arbitrary smooth manifold $X$, let $\Omega^k_{\mathbb{C}}(X)$ be the space of global sections of the complex vector bundle $\bigwedge^k T^*_{\mathbb{C}}X$. The space of smooth $\mathbb{C}$-valued functions on $X$ is then $\Omega^0_{\mathbb{C}}(X)$.

From standard linear algebra, we get the bundle decomposition

$$\bigwedge^{k} T^*_{\mathbb{C}}X = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*X,$$

which gives us a corresponding decomposition on sections

$$\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

As is standard, we identify a form $\alpha \in \Omega^{p,q}(X)$ with its image under the canonical map $\Omega^{p,q}(X) \hookrightarrow \Omega^{p+q}_{\mathbb{C}}(X)$. Furthermore, note that conjugation gives us an $\mathbb{R}$-linear isomorphism $\Omega^{p,q}(X) \cong \Omega^{q,p}(X)$.

With this in mind, we can come to define two important operators in general complex geometry:

**Definition 2.3.5.** For an almost complex manifold $(X, I)$, let $d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)$ be the complex extension of the ordinary exterior derivative. If $\pi^{p,q} : \Omega^k_{\mathbb{C}}(X) \to \Omega^{p,q}(X)$ denotes the canonical projection coming from the decomposition of $\Omega^k_{\mathbb{C}}(X)$, define

$$\partial := \pi^{p+1,q} \circ d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \ \ \overline{\partial} := \pi^{p,q+1} \circ d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X).$$

The operator $\overline{\partial}$ is usually referred to as the *Dolbeault* operator.

The question of whether an almost complex structure comes from a complex structure is very natural to ask, and such structures have a special name:

**Definition 2.3.6.** An almost complex structure $I$ on a smooth manifold $X$ is called *integrable* if it is induced by a complex structure (i.e. holomorphic atlas) for $X$, that is, if $X$ can be equipped with a complex structure and a self-diffeomorphism $f : X \to X$ which pulls the natural almost complex structure $J$ associated to the complex structure for $X$ back to $I$, i.e. $f^*(J) := df^{-1} \circ J \circ df = I$.

The following theorem of Newlander and Nirenberg is central in the theory and provides necessary and sufficient conditions for the integrability of an almost complex structure:

**Theorem 2.3.2** (Newlander-Nirenberg)**.** *An almost complex structure on a smooth manifold $X$ is integrable if and only if the following equivalent conditions are satisfied*

*(i)* $d = \partial + \overline{\partial}$ *on all forms;*

*(ii) For* $\alpha \in \Omega^{1,0}(X)$*,* $d\alpha$ *has no* $(0,2)-$*component;*

*(iii)* $\overline{\partial}^2 = 0$ *on all forms;*

*(iv) The space* $T^{1,0}X$ *(or equivalently* $T^{0,1}X$*) is preserved by the Lie bracket on vector fields.*

The forward direction is elementary, however, the reverse implication is much more difficult to show and requires some non-trivial analysis for the proof. Voisin [52] treats a proof in the case of a real analytic manifold.

Given this dual view of complex manifolds, let us reformulate what it means for a map between complex manifolds to be holomorphic in terms of their almost complex structures:

**Proposition 2.3.2.** *A map $f : X \to Y$ between complex manifolds is holomorphic if and only if $f : X \to Y$ is smooth and $df \circ I = J \circ df$, where $df : TX \to TY$ is the tangent map and $I$ and $J$ are the canonical complex structures associated to $X$ and $Y$, respectively.*

A useful consequence of this fact is that we can show a complex manifold and a smooth manifold with an integrable almost complex structure represent exactly the same geometric object, which we will make heavy use of in our investigations in Chapters 4 and 5.

**Corollary 2.3.2.** *Any integrable almost complex structure on a smooth manifold $X$ is induced up to biholomorphism by at most one complex structure.*

*Proof.* Take $f = \mathrm{id}_X$ in Proposition 2.3.2. □

Seeing as the induced Dolbeault operator of an integrable almost complex structure satisfies $\overline{\partial}^2 = 0$, we obtain a natural chain complex, and can thus speak of *Dolbeault* cohomology:

**Definition 2.3.7.** For a complex manifold $X$, define the $(p, q)$-*Dolbeault cohomology group* by

$$H^{p,q}(X) := \frac{\ker(\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X))}{\mathrm{im}(\overline{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X))}.$$

In direct analogy to de Rham's theorem, one can quite easily prove *Dolbeault's theorem* using the fact that the Dolbeault operator is locally exact and its kernel consists precisely of holomorphic forms:

**Theorem 2.3.3** (Dolbeault's Theorem)**.** *For a complex manifold $X$ of dimension $n$, we have the following isomorphism of cohomology groups for every $0 \le p, q \le n$:*

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X),$$

*where $\Omega^p_X$ denotes the sheaf of germs of holomorphic $p$-forms on $X$.*

To conclude this section, we introduce the concept of a *deformation* of a complex manifold. Deformation theory is a central, but rather difficult and technical subject in complex geometry. For our purposes, we will only need to be familiar with the very basics of it. For a more in-depth look, one can consult, e.g. [32]. Most literature on the subject usually begins with the definition:

**Definition 2.3.8.** By a *complex analytic family (of compact complex manifolds)* we mean a proper surjective holomorphic submersion $f : X \to B$ between connected complex manifolds, which we assume has connected fibres. Then, each fibre $X_b := f^{-1}(b)$ is a compact complex manifold of dimension $\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} B$. Note that $B$ is often taken to be the unit disc in $\mathbb{C}$.

Using a classic result from differential geometry by Ehresmann, we see that the fibres of a complex analytic family are diffeomorphic:

**Theorem 2.3.4** (Ehresmann)**.** *Let $f : X \to Y$ be a smooth proper surjective submersion between smooth manifolds, then $f$ defines a locally trivial fibration. In particular, all fibres are diffeomorphic.*

Thus, a complex analytic family captures the variation of the complex structures on some fixed compact smooth manifold, with this variation being parameterised by some base space $B$. Deformation theory then makes precise the notion of 'deforming' the complex structure of one complex manifold to another:

**Definition 2.3.9.** A *deformation* of a connected compact complex manifold $Y$ consists of a complex analytic family $f : X \to B$, a basepoint $b_0$ and a biholomorphism between $Y$ and the fibre $X_{b_0}$. In other words, all the fibres of a complex analytic family are deformations of each other. If two connected compact complex manifolds $Y_1$ and $Y_2$ arise as fibres of the same complex analytic family $f : X \to B$, then we say they are *deformation equivalent*.

Note that deformation equivalence as stated above is *not* a transitive relation. As a result, one usually takes its transitive closure.

As we have seen in this section, a complex manifold and a smooth manifold with an integrable almost complex structure represent exactly the same geometric object (Proposition 2.3.2). The theory of deformations is certainly not made any easier by opting for this point of view, however, for our purposes we will need to have a definition at hand for deformation equivalence from this viewpoint:

**Definition 2.3.10.** Two complex manifolds $(X, I)$ and $(X', I')$ are said to be *deformation equivalent* if there is a diffeomorphism $f : X \to X'$ such that $f^*(I') = df^{-1} \circ I' \circ df$ and $I$ can be joined by a continuous path of integrable almost complex structures on $X$.

**Remark 2.3.1.** Note that Definition 2.3.10 and Definition 2.3.9 do not exactly coincide. However, importantly, it is true that if two complex structures $I_1$ and $I_2$ on a compact manifold $M$ can be connected by a continuous path through complex structures, then the complex manifolds $X_1 := (M, I_1)$ and $X_2 := (M, I_2)$ are deformation equivalent (with respect to the transitive closure of this relation), that is, there exists a finite sequence of compact complex manifolds $Y_1 := X_1, Y_2, \dots, Y_{n-1}, Y_n := X_2$ such that $Y_i$ is deformation equivalent to $Y_{i+1}$ in the sense of Definition 2.3.9 for $i = 1, \dots, n-1$. This is a result of Kuranishi's theory on deformations (see [8, Corollary 6]). For our purposes, this is all that is required, given that Definition 2.3.9 is the standard notion of deformation equivalence that people usually take. Note that the space of (integrable) almost complex structures is some kind of infinite-dimensional space (both are collections of specific kinds of sections of the endomorphisms of the tangent bundle of a smooth manifold). As a result, there are many appropriate topologies one could consider for this space. We bypass this for the moment, and make some additional remarks on this at a later stage (see Section 4.5).

## 2.4 Holomorphic Vector Bundles

Having set up some of the formalism related to complex manifolds, we can now turn to studying holomorphic vector bundles on them. The key result that we mention here is the Riemann-Roch-Hirzebruch theorem.

In the previous section on complex manifolds and forms, we mentioned the holomorphic tangent bundle $\mathcal{T}_X$ of a complex manifold $X$ without formally giving the definition of a holomorphic vector bundle. We do that now for the sake of completeness:

**Definition 2.4.1.** A *holomorphic* vector bundle is a complex vector bundle whose base space and total space are complex manifolds such that the projection is holomorphic and each point in the base has a neighbourhood which has a holomorphic trivialisation. A *holomorphic line bundle* (or simply *line bundle*) over a complex manifold is a holomorphic vector bundle of rank 1.

The key thing to note about holomorphic vector bundles is that all standard constructions on real or complex vector bundles carry over, e.g. given holomorphic vector bundles $E$ and $F$ over the same complex manifold, we can form the direct sum bundle $E \oplus F$, tensor product bundle $E \otimes F$, and so on. In particular, we make the following definition:

**Definition 2.4.2.** Let $X$ be a complex manifold of dimension $n$. Its *canonical bundle* is the holomorphic line bundle

$$K_X := \bigwedge^n \mathcal{T}_X^* \cong \det(\mathcal{T}_X^*).$$

We can extend some of the constructions in the previous section to the case of differential forms with coefficients in a holomorphic vector bundle. Specifically, let $E \to X$ be a holomorphic vector bundle over a complex manifold, and define $\Omega^{p,q}(E)$ to be the space of global sections of the complex bundle $\bigwedge^{p,q} T^*X \otimes E$.

**Definition 2.4.3.** If $E \to X$ is a holomorphic vector bundle of rank $k$, then its Dolbeault operator

$$\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$$

is defined locally over a trivialising subset $U \subseteq X$ by

$$\overline{\partial}_E \left( \sum_{i=1}^{k} \alpha_i \otimes e_i \right) = \sum_{i=1}^{k} \overline{\partial}\alpha_i \otimes e_i,$$

where $e_1, \ldots, e_k$ form a local holomorphic frame over $U$ and $\alpha_i \in \Omega^{p,q}(U)$ for $i = 1, \ldots, k$.

This gives a well-defined global differential operator precisely because the transition maps of a holomorphic vector bundle are holomorphic. We clearly have $\overline{\partial}_E^2 = 0$, and so we can define the Dolbeault cohomology of $E$:

**Definition 2.4.4.** For a holomorphic vector bundle $E \to X$ over a complex manifold, define *the* $(p,q)$*-Dolbeault cohomology group of* $E$ by

$$H^{p,q}(X, E) := \frac{\ker(\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E))}{\operatorname{im}(\overline{\partial}_E : \Omega^{p,q-1}(E) \to \Omega^{p,q}(E))}.$$

Moreover, the operator $\overline{\partial}_E$ is locally exact and its kernel consists precisely of the holomorphic sections of $E \otimes \bigwedge^p \mathcal{T}_X^*$, and hence we obtain a natural extension of Dolbeault's theorem (Theorem 2.3.3):

**Theorem 2.4.1.** *Given a holomorphic vector bundle $E \to X$ over a complex manifold of dimension $n$, we have the following isomorphisms for $0 \leq p, q \leq n$:*

$$H^q(X, E \otimes \Omega_X^p) \cong H^{p,q}(X, E).$$

In the above statement, and as usual, we identified a holomorphic vector bundle with its sheaf of holomorphic sections.
In the next section on Hodge theory, we will see that for a holomorphic vector bundle $E \to X$ over a compact complex manifold, each cohomology group $H^q(X, E)$ is a finite-dimensional complex vector space which is zero for $q > \dim_{\mathbb{C}} X$ (see Theorem 2.5.1). We then define:

**Definition 2.4.5.** For a holomorphic vector bundle $E \to X$ over a compact complex manifold of dimension $n$, define the *holomorphic Euler characteristic* of $E$ by

$$\chi(X, E) := \sum_{k=0}^{n} (-1)^k h^k(X, E),$$

where $h^k(X, E) := \dim_{\mathbb{C}} H^k(X, E)$. The *holomorphic Euler characteristic of* $X$ is simply the holomorphic Euler characteristic of the trivial line bundle $\mathcal{O}_X$. Note that for a compact complex manifold, $h^0(X, \mathcal{O}_X) = 1$.

We are now in a position to state the famous Riemann-Roch-Hirzebruch theorem. Going through its proof is certainly well beyond the scope of this thesis. Nonetheless, we will see this useful result come up time and time again in the sections that follow:

**Theorem 2.4.2** (Riemann-Roch-Hirzebruch)**.** *For any holomorphic vector bundle $E \to X$ over a compact complex manifold of dimension $n$, we have*

$$\chi(X, E) = \sum_{i=0}^{n} \langle \mathrm{ch}_i(E) \smile \mathrm{td}_{n-i}(X), [X] \rangle.$$

To conclude this section, let us briefly spend some time discussing line bundles a little further. The space of isomorphism classes of holomorphic line bundles on a complex manifold $X$ plays quite an important role in complex geometry. It forms an abelian group under the natural tensor product operation with identity being the trivial line bundle and inverses being given by the dual:

**Definition 2.4.6.** For a complex manifold $X$, denote by $\mathrm{Pic}(X)$ the abelian group consisting of isomorphism classes of holomorphic line bundles on $X$ under the tensor product operation.

Precisely because the data of a holomorphic line bundle is contained in specifying a Čech 1-cocycle with coefficients in $\mathcal{O}_X^*$, we see that there is a canonical isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$. It turns out that under this isomorphism, the first Chern class can be related to something very familiar from homological algebra:

**Theorem 2.4.3.** *Let $X$ be a complex manifold. Under the canonical isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$*, the first Chern class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ corresponds to the connecting homomorphism $\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ coming from the long exact sequence on cohomology*

$$\ldots \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow \ldots$$

*arising from the exponential sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0.$$

*Proof.* See [17, p. 141]. □

The image of the first Chern class map is usually given a special name:

**Definition 2.4.7.** Let $X$ denote a complex manifold. Then, the *Néron-Severi group* of $X$ is defined to be the image of the first Chern class map in $H^2(X, \mathbb{R})$:

$$\mathrm{NS}(X) := \mathrm{im}(\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R})).$$

Using some standard results from the theory of hermitian bundles, we can provide a more tangible description of the Néron-Severi group. We start with the following definition:

**Definition 2.4.8.** Let $X$ be an arbitrary complex manifold. Denote by $H^{1,1}(X, \mathbb{R})$ the set of de Rham classes $\alpha \in H^2_{\mathrm{dR}}(X) \cong H^2(X, \mathbb{R})$ which can be represented by closed real $(1,1)$-forms. The space of integral such classes is denoted

$$H^{1,1}(X, \mathbb{Z}) := H^{1,1}(X, \mathbb{R}) \cap \mathrm{im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})).$$

**Theorem 2.4.4** (Lefschetz Theorem on $(1,1)$-classes)**.** *For any complex manifold $X$, we have* $\mathrm{NS}(X) = H^{1,1}(X, \mathbb{Z})$.

*Proof.* Given an arbitrary complex line bundle $L \to X$, standard Chern-Weil theory tells us that $c_1(L) = \left[\frac{i}{2\pi} F_\nabla\right] \in H^2(X, \mathbb{R})$, where $F_\nabla \in \Omega^2_{\mathbb{R}}(X)$ is the curvature of an arbitrary connection $\nabla : \Omega^0(L) \to \Omega^1(L)$. If $L$ is holomorphic and equipped with a hermitian metric, then it is standard that the curvature of its Chern connection (i.e. the unique metric-compatible connection whose $(0,1)$-part coincides with $\overline{\partial}_L$) is real and of type $(1,1)$, in which case $c_1(L) \in H^{1,1}(X, \mathbb{Z})$. Conversely, given any class $\alpha \in H^{1,1}(X, \mathbb{Z})$, we can find a hermitian holomorphic line bundle $L \to X$ whose Chern curvature $F_\nabla \in \Omega^{1,1}_{\mathbb{R}}(X)$ satisfies $c_1(L) = \left[\frac{i}{2\pi} F_\nabla\right] = \alpha$ (see, for example, [11, Theorem 13.9 (b)]). □

**Remark 2.4.1.** The above theorem is usually stated in the form where one considers the image of the first Chern class in the set $H^{1,1}(X) \cap \operatorname{im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}))$, but the issue is that a $\overline{\partial}$-closed $(1,1)$-form is not necessary closed. Thus, one generally restricts to the case of compact Kähler $X$, and then the Hodge decomposition can be applied to make sense of this (see Theorem 2.6.2). However, none of this is necessary assuming the above definitions.

We conclude by mentioning an important, but rather elementary result:

**Proposition 2.4.1** (Adjunction Formula)**.** *For a complex submanifold $Y$ of a complex manifold $X$, we have a canonical isomorphism of holomorphic line bundles*

$$K_Y \cong K_X|_Y \otimes \det(\mathcal{N}_{Y|X}),$$

*where $\mathcal{N}_{Y|X}$ denotes the holomorphic normal bundle of $Y$ in $X$.*

*Proof.* The normal bundle is defined by the short exact sequence of holomorphic vector bundles given by

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y|X} \longrightarrow 0.$$

Dualising and taking determinants yields the result. □

## 2.5 Elementary Hodge Theory

The principal goal of this section is to introduce some of the key ideas in the area of harmonic forms and Hodge theory. The main result we will cover is the famous Serre duality theorem concerning the cohomology of a holomorphic vector bundle. We will begin the discussion in the smooth case, and then proceed to the complex setting. So, let $X$ be a smooth oriented manifold of dimension $n$ equipped with a Riemannian metric $g$. The metric and orientation determine a natural volume form $\mathrm{vol}_g$ (or simply vol when the metric is understood), and furthermore it induces metrics on the cotangent bundle, as well as on all $k$-form bundles $\bigwedge^k T^*X$, which we will denote by

$$(\cdot, \cdot) : \Omega^k_{\mathbb{R}}(X) \times \Omega^k_{\mathbb{R}}(X) \to C^\infty(X).$$

We can then define an $L^2$-inner product on the space of smooth $k$-forms $\Omega^k_{\mathbb{R}}(X)$ for $k = 1, \ldots, n$ by setting

$$(\alpha, \beta)_{L^2} := \int_X (\alpha, \beta) \,\mathrm{vol}.$$

The orientation and metric $g$ on $X$ also give us another natural operator, known as the Hodge $*$-operator, which for each $k = 1, \dots, n$ provides us an isomorphism of $C^\infty(X)$-modules

$$* : \Omega^k_{\mathbb{R}}(X) \cong \Omega^{n-k}_{\mathbb{R}}(X),$$

which is determined by the relation

$$\alpha \wedge *\beta = (\alpha, \beta)\,\mathrm{vol},$$

and on $\Omega^k_{\mathbb{R}}(X)$ it satisfies $*^2 = (-1)^{k(n-k)} \cdot \mathrm{id}$. Restricting to the important case of $4k$-manifolds, the Hodge $*$-operator supplies us with the following definition:

**Definition 2.5.1.** Given a smooth oriented Riemannian $4k$-manifold $(X, g)$, we obtain a decomposition

$$\Omega^{2k}_{\mathbb{R}}(X) = \Omega^{2k}_{+}(X) \oplus \Omega^{2k}_{-}(X),$$

where $\Omega^{2k}_{\pm}(X) := \{\alpha \in \Omega^{2k}_{\mathbb{R}}(X) \mid *\alpha = \pm\alpha\}$ are the $\pm 1$-eigenspaces of the involution $* : \Omega^{2k}_{\mathbb{R}}(X) \cong \Omega^{2k}_{\mathbb{R}}(X)$. Forms $\alpha \in \Omega^{2k}_{+}(X)$ are known as *self-dual* $2k$-forms, whereas $\alpha \in \Omega^{2k}_{-}(X)$ is called an *anti-self-dual* $2k$-form.

Now, Stokes' theorem implies that the operator defined by

$$d^* = (-1)^{n(k+1)+1} * d * : \Omega^k_{\mathbb{R}}(X) \to \Omega^{k-1}_{\mathbb{R}}(X)$$

is the formal adjoint of $d : \Omega^{k-1}_{\mathbb{R}}(X) \to \Omega^k_{\mathbb{R}}(X)$ with respect to the $L^2$-inner product. Note that for a manifold of even dimension (e.g. a complex manifold), we have that $d^* = - * d *$. We can then define another key operator, known as the *Laplacian*

$$\Delta := dd^* + d^*d : \Omega^k_{\mathbb{R}}(X) \to \Omega^k_{\mathbb{R}}(X).$$

It is obvious that $\Delta$ is self-adjoint with respect to the $L^2$-inner product defined by the Riemannian metric $g$.

**Definition 2.5.2.** We say that a $k$-form $\alpha \in \Omega^k_{\mathbb{R}}(X)$ is *harmonic* if $\Delta\alpha = 0$. The space of harmonic $k$-forms is denoted $\mathcal{H}^k_{\Delta}(X, g)$. Note that $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$.

Suppose now that $X$ is complex of dimension $n$ and is equipped with a hermitian metric $g$. Viewing $X$ as a smooth manifold with integrable almost complex structure $I$, to say that $g$ is hermitian means precisely that $I$ is an orthogonal transformation of $g$. Extending the Riemannian metric $g$ by $\mathbb{C}$-linearity in the first entry and $\mathbb{C}$-antilinearity in the second yields a hermitian metric on $T_{\mathbb{C}}X$, and hence all complex $k$-form bundles $\bigwedge^k T^*_{\mathbb{C}}X$. Moreover, the $\mathbb{C}$-linear extension of the ordinary Hodge star of $g$ yields isomorphisms $* : \Omega^k_{\mathbb{C}}(X) \cong \Omega^{2n-k}_{\mathbb{C}}(X)$. Given that conjugation gives us $\mathbb{R}$-linear isomorphisms $\Omega^{p,q}(X) \cong \Omega^{q,p}(X)$, the complexified Hodge $*$-operator restricts to isomorphisms $* : \Omega^{p,q}(X) \cong \Omega^{n-q,n-p}(X)$. By repeating the same analysis as for the real case, we can define formal adjoints of the Dolbeault operators $\partial$ and $\overline{\partial}$, and their respective Laplacians $\Delta_\partial$ and $\Delta_{\overline{\partial}}$.

If $E$ denotes a holomorphic vector bundle over $X$ equipped with its own hermitian metric $h$, we obtain hermitian metrics on each of the complex bundles $\bigwedge^{p,q} T^*X \otimes E$. Furthermore, regarding the metric on $E$ as an anti-linear isomorphism $h : E \to E^*$, we can extend the Hodge $*$-operator to $E$-valued forms by setting

$$*_E : \Omega^{p,q}(E) \to \Omega^{n-p,n-q}(E^*), \alpha \otimes s \mapsto *(\overline{\alpha}) \otimes h(s),$$

for $\alpha \in \Omega^{p,q}(X)$ and $s \in \Gamma(X, E)$. Using the natural pairing of sections of the dual of $E$ with the section of $E$, we get an $L^2$-inner product on $\Omega^{p,q}(E)$, and can thus similarly define a formal adjoint for $\overline{\partial}_E$ with its corresponding Laplacian $\Delta_E$. The associated space of $(p,q)$-$\Delta_E$-harmonic forms will be denoted $\mathcal{H}^{p,q}(X, E)$.

All of the Laplacians defined thus far are instances of *elliptic* differential operators, and so we can appeal to results from the theory (see, for example, [52, Section 5.2]). The key result in this connection is usually referred to as the *Hodge decomposition*, and for our purposes it yields the following important consequence when restricting to the case of compact manifolds:

**Theorem 2.5.1.** *Given a compact oriented Riemannian manifold $(X, g)$, the canonical map*

$$\mathcal{H}^k_\Delta(X, g) \to H^k_{\mathrm{dR}}(X) \cong H^k(X, \mathbb{R})$$

*which sends a $g$-harmonic form to its de Rham cohomology class is an isomorphism. If $X$ is assumed to be complex and hermitian and $E \to X$ is a hermitian holomorphic vector bundle, then the canonical map*

$$\mathcal{H}^{p,q}(X, E) \to H^{p,q}(X, E) \cong H^q(X, E \otimes \Omega^p_X)$$

*which sends a $\Delta_E$-harmonic form to its Dolbeault cohomology class is an isomorphism. In particular, all of the Betti numbers $b_k(X)$ are finite, as are the dimensions of each of the cohomology groups $H^q(X, E)$. Moreover, $H^q(X, E) = 0$ for $q > \dim_{\mathbb{C}} X$.*

We conclude this section by stating the famous Serre duality theorem, which is an immediate consequence of the above:

**Theorem 2.5.2** (Serre Duality). *Let $X$ be a compact complex manifold of dimension $n$. For any holomorphic vector bundle $E$ on $X$, the natural pairing*

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \to \mathbb{C}, (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

*is non-degenerate. In particular, we have the isomorphisms of complex vector spaces for* $0 \leq p,q \leq n$ :

$$H^q(X,\Omega_X^p \otimes E) \cong H^{n-q}(X,\Omega_X^{n-p} \otimes E^*)^*,$$

*and so*

$$h^q(X,\Omega_X^p \otimes E) = h^{n-q}(X,\Omega_X^{n-p} \otimes E^*).$$

**Remark 2.5.1.** For the special case of $p = 0$, we have

$$h^q(X,E) = h^{n-q}(X,K_X \otimes E^*),$$

and taking $E = \mathcal{O}_X$ yields

$$h^{p,q}(X) = h^{n-p,n-q}(X),$$

where $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X) = h^q(X,\Omega_X^p)$. The non-negative integers $h^{p,q}(X)$ are usually referred to as the *Hodge numbers* of the compact complex manifold $X$.

## 2.6 Some Important Classes of Metrics

Although we will not be concerning ourselves with any significant amount of Riemannian geometry in this thesis, it is important that we give at least a cursory overview of some of the most important classes of metrics that we will encounter later on. To this end, we will first introduce the notion of a Kähler manifold, which forms an extremely important class of complex manifolds in complex geometry for the fact that they have many desirable properties, an important example of which is the descent of the bidegree decomposition of complex forms to the level of cohomology. We will subsequently touch on the notions of Einstein, Kähler-Einstein and hyperkähler metrics.

Let $(X,I)$ be an almost complex manifold. Recall that $I$ gives $TX$ the structure of a complex vector bundle. Equipping it with a hermitian metric, that is, a Riemannian metric $g$ which is compatible with the almost complex structure, we can define an associated real $(1,1)$-form $\omega \in \Omega_{\mathbb{R}}^{1,1}(X)$ by setting

$$\forall Y,Z \in \Gamma(TX) : \omega(Y,Z) := g(IY,Z).$$

Note that any two of $I, g$ or $\omega$ determine the third.

**Definition 2.6.1.** We say that an almost complex manifold $(X, I)$ equipped with compatible Riemannian metric $g$ is *Kähler* if $I$ is integrable and the associated real $(1,1)$-form is closed. A complex manifold $X$ is called Kähler if there exists a Kähler metric for $X$.

One simple result which is worth explicitly stating is the following:

**Proposition 2.6.1.** *If $(X, g)$ is a Kähler manifold of dimension $n$ with Kähler form $\omega$, then $\omega$ is harmonic with respect to $g$. Moreover, if $X$ is a complex surface, then $\omega$ is self-dual.*

*Proof.* This follows at once by the fact that $*\omega = \frac{(\omega,\omega)}{n!}\omega^{n-1}$ and $(\omega,\omega) = n$. □

Wedging with the associated real $(1,1)$-form on a Kähler manifold yields a very important operator in the theory known as the *Lefschetz operator*. Conjugating by the Hodge $*$-operator yields the *dual Lefschetz operator*. These satisfy a number of commutation relations (see [52, Proposition 6.5]) with the operators $\partial$ and $\overline{\partial}$, as well as their formal adjoints. This yields the central result:

**Theorem 2.6.1.** *Let $(X, g)$ be a Kähler manifold. Then we have the following equalities:*

$$\Delta_{\partial} = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta.$$

As a consequence, it follows that the respective spaces of harmonic forms associated to the operators $\Delta_{\partial}$, $\Delta_{\overline{\partial}}$ and $\Delta$ all coincide. In particular, given that it easily follows that the decomposition of $\overline{\partial}$-harmonic (equally $\partial$-harmonic) forms into bidegree components is satisfied, Hodge theory yields the following central result when restricting to the case of compact $X$:

**Theorem 2.6.2** (Hodge Decomposition)**.** *Let $X$ be a compact Kähler manifold of dimension $n$, then for each $1 \leq k \leq n$, we have*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

*Moreover, the above decomposition does not depend on the choice of Kähler metric and complex conjugation yields the equality $H^{p,q}(X) = \overline{H^{q,p}(X)}$. In particular, $h^{p,q}(X) = h^{q,p}(X)$ and $b_k(X) = \sum_{p+q=k} h^{p,q}(X)$.*

A simple, but important consequence of the Hodge decomposition is:

**Corollary 2.6.1.** *The odd Betti numbers $b_{2k+1}(X)$ of a compact Kähler manifold $X$ are even.*

The existence of three mutually compatible structures, i.e. a symplectic structure, hermitian structure and complex structure make Kähler manifolds particularly nice to work with. However, for the work we will do here, it will not be necessary to delve into all the consequences that a Kähler metric brings.

In order to properly introduce the remaining classes of metrics, let us start by briefly recalling some elementary concepts from Riemannian geometry. If $M$ is a smooth manifold which is equipped with a Riemannian metric $g$, then there exists a unique torsion-free connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which is compatible with $g$ called the *Levi-Civita connection*. The *Riemann curvature tensor* is simply the curvature of the Levi-Civita connection, and it will be denoted $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. By taking the trace of the bundle endomorphism defined by the $C^\infty(M)$-linear map

$$\Gamma(TM) \to \Gamma(TM),\ Z \mapsto R(Z, X)Y$$

for $X, Y \in \Gamma(TM)$, we obtain the *Ricci tensor* $\mathrm{Ric} : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$.

**Definition 2.6.2.** A Riemannian metric $g$ on a smooth manifold is said to be *Einstein* if its Ricci tensor is a constant multiple of $g$, that is, if there exists $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda g$. It is said to be *Ricci-flat* if $\lambda = 0$.

There are many interesting things that can be said or asked about such manifolds. The question of whether or not a smooth manifold admits an Einstein metric $g$ is fully characterised in dimensions $\leq 3$. The question in higher dimensions is slightly more complicated. We will look at the four-dimensional case when we come to discuss K3 surfaces in more detail (Section 4.5).

Now, suppose that $(X, g)$ is a Kähler manifold with integrable almost complex structure $I$ and associated Kähler form $\omega$. Then, we have that

$$\forall X, Y, Z \in \Gamma(TX) : R(X, Y)(IZ) = I(R(X, Y)Z).$$

This follows from the standard fact that the metric $g$ is Kähler if and only if $I$ is parallel with respect to the Levi-Civita connection, i.e. $\nabla I = 0$. We can then define what is usually called the *Ricci form* of the Kähler manifold $(X, g)$ by setting

$$\rho(X, Y) = \mathrm{Ric}(IX, Y).$$

It is a closed, real $(1,1)$-form $\rho \in \Omega^{1,1}_{\mathbb{R}}(X)$. Then, in this case, the Einstein condition is equivalent to

$$\rho = \lambda\omega.$$

**Definition 2.6.3.** A hermitian metric $g$ on a complex manifold $(X, I)$ is said to be *Kähler-Einstein* if it is a Kähler metric which simultaneously satisfies the Einstein equation.

Simple examples of Kähler-Einstein metrics are the Fubini-Study metric on $\mathbb{P}^n$ and flat metrics on complex tori. Note that if $(X, g, \omega)$ is a Kähler manifold of dimension $n$, then the coefficient $\lambda$ in the Einstein equation can be computed explicitly as (see [25, p. 223] for details):

$$\lambda = \frac{2\pi \int_X c_1(X) \wedge \omega^{n-1}}{\int_X \omega^n}.$$

In fact, if $[\omega] \in H^2(X, \mathbb{R})$ denotes the de Rham class of the Kähler form, then the first Chern class of $X$ can be written as:

$$c_1(X) = \left[\frac{\lambda}{2\pi}\omega\right] = \left[\frac{1}{2\pi}\rho\right].$$

In particular, any Kähler-Einstein manifold $X$ with $c_1(X) = 0$ is automatically Ricci-flat. The existence of Kähler-Einstein metrics in non-trivial cases was not clear for a long time. The fundamental result on the existence of such metrics came with Yau's resolution of the Calabi Conjecture (see [54] and [55]). We state the result in the form that is most relevant to us:

**Theorem 2.6.3** (Yau)**.** *Let $X$ be a compact complex manifold with $c_1(X) = 0 \in H^2(X, \mathbb{R})$. If an element $\alpha \in H^2(X, \mathbb{R})$ can be represented by a Kähler-form $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$, then there exists a unique Ricci-flat Kähler-Einstein metric $g$ whose Kähler form $\omega_g$ represents $\alpha$, i.e. $\alpha = [\omega] = [\omega_g]$.*

**Remark 2.6.1.** In particular, all *Calabi-Yau* manifolds, that is, compact Kähler manifolds with trivial canonical bundle can be equipped with Ricci-flat Kähler-Einstein metrics (one for each cohomology class defined by a Kähler metric, i.e. *Kähler class*).

We conclude this section with a brief discussion of one final class of metric:

**Definition 2.6.4.** We say that a Riemannian manifold $(X, g)$ is *hyperkähler* if there exist two distinct integrable almost complex structures $I$ and $J$ for $X$ which are both Kähler with respect to $g$ and $IJ = -JI$.

In particular, the almost complex structure $K = IJ$ is convariantly constant with respect to the Levi-Civita connection, i.e. it is Kähler with respect to $g$. Moreover, given a point $(x, y, z) \in S^2 \subset \mathbb{R}^3$, we have that $xI + yJ + zK$ defines another integrable Kähler structure with respect to $g$, and so every hyperkähler metric $g$ defines a 2-sphere of complex structures which are Kähler with respect to $g$. Clearly, the corresponding Kähler form is given by $x\omega_I + y\omega_J + z\omega_K$, where $\omega_I$ is the Kähler form associated to $(X, g, I)$, etc. One elementary result we will explicitly need later is the following:

**Proposition 2.6.2.** *If $(X, g, I, J, K)$ is a hyperkähler manifold of dimension $\dim_{\mathbb{C}} X = 2n$, then the complex 2-form*

$$\kappa := \omega_J + i\omega_K \in \Omega^2_{\mathbb{C}}(X)$$

*defines a holomorphic symplectic form for $(X, I)$. Hence, $\kappa^n$ is nowhere vanishing and thus trivialises the canonical bundle of $(X, I)$. In particular, all compact hyperkähler manifolds are Calabi-Yau.*

*Proof.* The closed condition and non-degeneracy are immediate. Thus, it suffices to show $\kappa$ is of type $(2, 0)$ with respect to the decomposition induced by $I$. We show that $\kappa$ vanishes on complex vector fields where one entry is of type $(0, 1)$. Let $Y, Z \in \Gamma(T_{\mathbb{C}}X)$ be arbitrary complex vector fields. By making the appropriate substitutions and using the defining relation $IJ = K$, we easily find:

$$\kappa(Y + iIY, Z + iIZ) = 0,$$
$$\kappa(Y + iIY, Z - iIZ) = 0.$$

And so $\kappa$ is of type $(2, 0)$ with respect to $I$, closed and hence holomorphic. □

## 2.7 Divisors and Line Bundles

The theory of divisors plays a central role in the world of complex algebraic geometry, and as we shall see, in the classification of compact complex surfaces. Their theory is vast and there exists a lot of literature on the subject. In this section, we introduce some elementary concepts from the theory, but do not venture too far, as it will not be necessary for our analysis of K3 surfaces and Enriques surfaces that comes later.

Understanding divisors relies on understanding subvarieties of a complex manifold:

**Definition 2.7.1.** Let $X$ be a complex manifold. An *analytic subvariety* is a closed subset $V \subseteq X$ such that each point $x \in X$ has an open neighbourhood $U$ with

$$V \cap U = f_1^{-1}(0) \cap \ldots \cap f_k^{-1}(0)$$

for some local holomorphic functions $f_1, \ldots, f_k : U \to \mathbb{C}$. A *hypersurface* is an analytic subvariety of codimension 1, i.e. it is locally given as the zero set of a single non-trivial holomorphic function.

Furthermore, we say that an analytic subvariety $V$ is *irreducible* if it cannot be written as the union of two proper analytic subvarieties. Any reducible subvariety can be written as the union of its irreducible components. We also make the following definition:

**Definition 2.7.2.** We say that a point $x$ of an analytic subvariety $V$ in a complex manifold $X$ is *smooth* if $V$ is a submanifold of $X$ near $x$, that is, if there are local defining equations $f_1, \dots, f_k : U \to \mathbb{C}$ for $V$ near $x$ such that $f = (f_1, \dots, f_k) : U \to \mathbb{C}^k$ has rank $k$ at $x$. The *smooth locus* of $V$ is the collection of smooth points of $V$. A point $x \in V$ is *singular* if it is not smooth. The *singular locus* of $V$ is the collection of singular points of $V$. We say that $V$ is *smooth* if its singular locus is empty, i.e. if $V$ is complex submanifold of $X$.

In order to quantify how singular a point on a hypersurface is, one introduces the notion of *multiplicity:*

**Definition 2.7.3.** A point $x \in V$ of a hypersurface in a complex manifold $X$ is said to have *multiplicity* $m$, and we denote this number by $\mathrm{mult}_x(V)$, if when $V$ is given in local holomorphic coordinates $z = (z_1, \dots, z_n) : U \to \mathbb{C}^n$ by $f : U \to \mathbb{C}$, $m$ is the order of vanishing of $f$ at $x$, that is, the largest integer such that for all $k \leq m-1$, we have

$$\left. \frac{\partial^k f}{\partial z_{i_1} \dots \partial z_{i_k}} \right|_x = 0.$$

In particular, $\mathrm{mult}_x(V) = 1$ when $x$ is smooth.

We now state the definition of a divisor:

**Definition 2.7.4.** A *divisor* on a complex manifold $X$ is a formal linear combination

$$D = \sum_i a_i V_i,$$

where each $a_i \in \mathbb{Z}$ and each $V_i$ is an irreducible hypersurface. The sum is taken to be locally finite, but it is actually finite if $X$ is compact. The abelian group generated by all irreducible hypersurfaces of $X$ is denoted $\mathrm{Div}(X)$. We say that a divisor $D = \sum_i a_i V_i$ is *effective*, and write $D \geq 0$, if $a_i \geq 0$ for all $i$.

Every hypersurface defines a divisor in a natural way by considering the sum of its irreducible components. Moreover, given a divisor $D = \sum_i a_i V_i$, we can construct natural transition functions for a line bundle. Indeed, we can find an open cover $X = \bigcup_j U_j$ so that in each $U_j$, the hypersurface $V_i$ is defined by a holomorphic function $g_{ij} : U_j \to \mathbb{C}$. Then the functions

$$f_j := \prod_i g_{ij}^{a_i} : U_j \to \mathbb{C}$$

define invertible meromorphic functions on $U_j$, and on overlaps $U_j \cap U_k$, the functions $f_j/f_k : U_j \cap U_k \to \mathbb{C}$ are nowhere vanishing and holomorphic. The resulting line bundle does not depend on the choice of defining functions, and we denote it by $\mathcal{O}_X(D)$ or $\mathcal{O}(D)$. With a small amount of extra work, one obtains:

**Proposition 2.7.1.** *On a complex manifold $X$, there exists a canonical homomorphism*

$$\mathrm{Div}(X) \to \mathrm{Pic}(X),\ D \mapsto \mathcal{O}_X(D).$$

*In particular, $\mathcal{O}_X(0) = \mathcal{O}_X$ and $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^*$.*

**Remark 2.7.1.** It can be shown (see [25, Corollary 5.3.7]) that the above homomorphism is surjective when $X$ is projective. Moreover, the image consists precisely of those line bundles which admit non-trivial *meromorphic* sections (i.e. which are locally defined by meromorphic, as opposed to holomorphic functions in any trivialisation). When restricting to the space of effective divisors, the image consists of those $L \in \mathrm{Pic}(X)$ with $h^0(X, L) > 0$.

As is standard with most geometric operations, we can pull back divisors. If $f : X \to Y$ is a holomorphic map between connected complex manifolds with dense image, then we can define

$$f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X),$$

which sends a divisor $D$ defined by the meromorphic functions $f_i : U_i \to \mathbb{C}$ over $Y = \bigcup_i U_i$ to the divisor defined by the meromorphic functions $f_i \circ f : f^{-1}(U_i) \to \mathbb{C}$. It follows immediately that for such a map, pulling back divisors respects pulling back line bundles, i.e. for every divisor $D \in \mathrm{Div}(Y)$, we have

$$f^*(\mathcal{O}_Y(D)) \cong \mathcal{O}_X(f^*D).$$

We conclude this section by stating two key results in the theory. However, in order for the first of these to be fully grasped, we must make a slight technical detour. Given a compact analytic subvariety on a complex manifold, we wish to construct a fundamental class for it. Naively, one would like to take the class defined by the smooth locus, however the latter is in general not compact. For our purposes, we will just assert that there exists one, but for the details one should consult [52, Chapter 11] for a variety of approaches:

**Theorem 2.7.1.** *Given a compact analytic subvariety $V$ of dimension $k$ (i.e. whose smooth locus is of dimension $k$ as a complex manifold) on a complex manifold $X$, there exists a well-defined fundamental class $[V] \in H_{2k}(X, \mathbb{Z})$ for $V$.*

We can then talk about the fundamental class of hypersurfaces, and extend to the case of divisors by linearity. The first major result of interest states:

**Theorem 2.7.2.** *If $X$ is a compact complex manifold of dimension $n$ and $D \in \mathrm{Div}(X)$ is any divisor, then*

$$\mathscr{P}_X[c_1(\mathcal{O}(D))] = [D] \in H_{2n-2}(X,\mathbb{Z}),$$

*where $[D] \in H_{2n-2}(X,\mathbb{Z})$ is the fundamental class of $D$ and $\mathscr{P}_X : H^k(X,\mathbb{Z}) \cong H_{2n-k}(X,\mathbb{Z})$ denotes Poincaré duality.*

*Proof.* See [17, p. 141-2] for a proof in the case $D$ is smooth. □

Finally, recalling the adjunction formula from Section 2.4 (Proposition 2.4.1), one can show that:

**Proposition 2.7.2.** *If $V$ is a smooth hypersurface in a compact complex manifold $X$ and $\mathcal{N}_{V|X}$ denotes its holomorphic normal bundle in $X$, then we have the following isomorphism of line bundles*

$$\mathcal{N}_{V|X} \cong \mathcal{O}(V)|_V.$$

*In particular, the adjunction formula reads:*

$$K_V \cong (K_X \otimes \mathcal{O}(V))|_V.$$

## 2.8 Important Invariants

In this final preliminary section, we briefly introduce some of the key quantities that play a critical role in the classification of compact complex surfaces. We start by defining the algebraic dimension of a compact connected complex manifold.

**Definition 2.8.1.** A subset $S$ of a field $F$ is *algebraically independent* over a subfield $E \subseteq F$ if the elements of $S$ do not satisfy any non-trivial polynomial equation with coefficients in $E$. A *transcendence basis* of the field extension $F/E$ is a maximal algebraically independent set of $F$ over $E$.

It can be shown that every transcendence basis of a field extension $F/E$ has the same cardinality, which we will denote $\mathrm{trdeg}_E F$. In particular, note that on a connected compact complex manifold $X$, the space of global meromorphic functions $\mathcal{M}(X)$ is indeed a field, and $\mathbb{C}$ (consisting of the constant holomorphic maps) is naturally a subfield. We then make the following definition:

**Definition 2.8.2.** On a compact connected complex manifold $X$, its *algebraic dimension*, denoted $a(X)$, is defined as

$$a(X) := \mathrm{trdeg}_{\mathbb{C}}\mathcal{M}(X).$$

We have the following key proposition:

**Theorem 2.8.1.** *If $X$ is a compact connected complex manifold of dimension $n$, then*

$$a(X) \leq n.$$

*Proof.* See [25, p. 54-55]. □

The following quantities also play a very important role in classifying compact complex manifolds:

**Definition 2.8.3.** If $X$ is a compact complex manifold, we define the $n^{th}$ *plurigenus of* $X$ to be the non-negative integer

$$P_n(X) := h^0(X, K_X^{\otimes n}),$$

where, as usual, $K_X$ denotes the canonical bundle of $X$.

Closely related to the plurigenera of a compact complex manifold is the Kodaira dimension of a connected compact complex manifold $X$. Observe that the space

$$R(X) := \bigoplus_{k \geq 0} H^0(X, K_X^{\otimes k})$$

with $K_X^0 := \mathcal{O}_X$ has a natural ring structure. In fact, since $X$ is connected, it is an integral domain and hence we can define its field of quotients, denoted $Q(X)$.

**Definition 2.8.4.** The *Kodaira dimension* of a connected compact complex manifold $X$ is the quantity

$$\operatorname{kod}(X) = \begin{cases} -\infty \text{ if } R(X) \cong \mathbb{C}, \\ \operatorname{trdeg}_{\mathbb{C}} Q(X) - 1 \text{ otherwise.} \end{cases}$$

It satisfies the relation:

**Proposition 2.8.1.** *For a connected compact complex manifold $X$, we have*

$$\operatorname{kod}(X) \leq a(X).$$

*Proof.* See [25, p. 74] □

The Kodaira dimension is also related to the plurigenera in the following non-trivial way:

**Theorem 2.8.2.** *If $X$ is a connected compact complex manifold, then*

$$\begin{aligned} \mathrm{kod}(X) = -\infty &\iff P_n(X) = 0 \textit{ for all } n \geq 1. \\ \mathrm{kod}(X) = 0 &\iff P_n(X) = 0 \textit{ or } 1 \textit{ but not all } 0. \\ \mathrm{kod}(X) = k &\iff P_n(X) \textit{ grows like } n^k. \end{aligned}$$

*Proof.* Consult [51, p. 86]. □

This form of the Kodaira dimension is more explicit and will be the way we calculate it in this thesis. Nevertheless, the first definition makes the relationship between the Kodaira dimension and algebraic dimension of a compact connected complex manifold much clearer. One important property of the Kodaira dimension is the following:

**Theorem 2.8.3.** *If $X$ and $Y$ are both compact connected complex manifolds, then*

$$\mathrm{kod}(X \times Y) = \mathrm{kod}(X) + \mathrm{kod}(Y).$$

We conclude this section and chapter by mentioning three other important quantities historically used in the context of classifying compact complex surfaces:

**Definition 2.8.5.** Let $X$ be a compact complex manifold of dimension $n$. Its *geometric genus* is the quantity $p_g(X) := h^{0,n}(X) = h^{n,0}(X) = P_1(X)$. The *irregularity* of $X$ is $q(X) := h^{0,1}(X)$. Finally, its *arithmetic genus* is the quantity

$$p_a(X) := (-1)^n(\chi(\mathcal{O}_X) - 1),$$

where we recall $\chi(\mathcal{O}_X)$ is the holomorphic Euler characteristic of $X$.

# Chapter 3

# On Compact Complex Surfaces

Having covered many of the fundamental concepts in differential geometry that will be of continual use to us, we can restrict our focus to compact complex surfaces, that is, two-dimensional compact complex manifolds. The principal goal of this chapter is to acquaint the reader with the main classes of complex surfaces that are to be found in the Enriques-Kodaira classification of minimal compact complex surfaces, and, of course, to introduce the two objects of focus which comprise this work: K3 and Enriques surfaces. However, given the vast nature of the theory of complex surfaces, we will not have the luxury of exploring all the interesting algebraic, differential-geometric and complex analytic aspects of these objects. Moreover, many of the more technical aspects of the theory, for example, on the classification and resolutions of singularities, or considerations on analytic fibrations will only be touched on or otherwise completely skipped over. For a comprehensive overview of the theory, one can refer to the standard reference text by Barth *et al.* (see [43] and [44]). For the algebraic aspects of the theory, [2] offers quite a good account. Griffiths and Harris [17] also have a chapter (Chapter 4) dedicated to the subject in their text.

In order to properly grasp the contents of the Enriques-Kodaira classification, one needs to obtain at least a superficial understanding of the concept of *minimality.* The first three sections of this chapter are devoted solely to guiding the reader through this process, and in the final section, we finally introduce the classification.

## 3.1 Intersections of Curves

Understanding the geometry of embedded curves in complex surfaces and how they intersect plays an extremely important role in obtaining global information of the surface.

In this short section, we will refine the definitions of Section 2.2 to the case of divisors on a compact complex surface and introduce some standard notation in the theory. A much more in-depth analysis can be found in Chapter II of [44]; there, they deal with the added complexity that comes with concerning oneself with potentially singular curves in a surface. The first definition we make is:

**Definition 3.1.1.** By a *curve* on a compact complex surface $X$ we will mean an effective divisor (cf. Definition 2.7.4). A curve is *smooth* if all of its connected components are smooth. It is *irreducible* if it cannot be written as the sum of two non-trivial curves. In particular, a smooth irreducible curve is just an embedded Riemann surface. A *rational* curve is a smooth irreducible curve biholomorphic to $\mathbb{P}^1$. An *elliptic* curve is a smooth irreducible curve with the structure of an one-dimensional complex torus, i.e. biholomorphic to some $\mathbb{C}/\Lambda$, where $\Lambda$ is an arbitrary lattice in $\mathbb{C}$.

Now, recall that for a compact complex manifold, any analytic subvariety $V$ is compact and hence has an associated fundamental class (Theorem 2.7.1). In particular, we can then consider intersections of analytic subvarieties of complementary dimension. Given two analytic subvarieties of complementary dimension, it must be the case that they intersect positively at any point of transverse intersection. This is because the canonical orientation on their tangent spaces is always compatible with that of the ambient manifold in which they are embedded. In particular, two hypersurfaces on a compact complex surface $X$ have complementary dimension, so we can certainly extend the intersection form to divisors by setting

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}, \left(\sum_i a_i D_i, \sum_j b_j D'_j\right) \mapsto \sum_{i,j} a_i b_j (D_i \cdot D'_j).$$

By considering the first Chern class of a line bundle, we can just as well talk about their intersections:

**Definition 3.1.2.** If $L$ is a holomorphic line bundle on a compact complex surface $X$ and $D \in \mathrm{Div}(X)$, then set

$$L \cdot D := c_1(L)[D] \in \mathbb{Z}.$$

If $L' \in \mathrm{Pic}(X)$ is another line bundle, then define

$$L \cdot L' := c_1(L) \cdot c_1(L') = (c_1(L) \smile c_1(L'))[X] \in \mathbb{Z}.$$

**Remark 3.1.1.** Recall that Theorem 2.7.2 tells us that $c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{Z})$ is Poincaré dual to the fundamental class of $D \in \mathrm{Div}(X)$, and thus for any line bundle $L \in \mathrm{Pic}(X)$, we have

$$L \cdot D = L \cdot \mathcal{O}_X(D).$$

Moreover, it is useful to write out the Riemann-Roch formula (Theorem 2.4.2) for two important cases. Given that $\mathrm{td}_2(X) = \frac{1}{12}(c_1^2(X) + c_2(X))$ and $c_2(X)[X] = \chi(X)$, we obtain *Noether's formula:*

**Theorem 3.1.1** (Noether's formula)**.** *If $X$ is a compact complex surface, then*

$$\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) = \mathrm{td}_2(X)[X] = \frac{K_X \cdot K_X + \chi(X)}{12}.$$

Using the expansion for the Chern character, we also obtain:

**Theorem 3.1.2.** *If $L$ is a holomorphic line bundle over a compact complex surface $X$, then*

$$\chi(X, L) = \chi(\mathcal{O}_X) + \frac{L \cdot L - L \cdot K_X}{2}.$$

Finally, we have the standard definition:

**Definition 3.1.3.** If $L \to C$ is a line bundle over a compact Riemann surface, then its *degree* is the integer $\deg L := c_1(L)[C] \in \mathbb{Z}$.

The Riemann-Roch formula for a line bundle over a compact Riemann surface then becomes:

**Theorem 3.1.3.** *If $L$ is a holomorphic line bundle over a compact Riemann surface $X$ of genus $g$, then*

$$\chi(X, L) = h^0(X, L) - h^1(X, L) = \deg L + (1 - g).$$

*Proof.* We have $\mathrm{td}_1(X)[X] = \frac{1}{2}c_1(X)[X] = \frac{1}{2}\chi(X) = 1 - g$ and $\mathrm{ch}_1(L) = c_1(L)$. □

Thus, if $C$ is a smooth irreducible curve of genus $g(C) = h^1(C, \mathcal{O}_C) = 1 - \chi(\mathcal{O}_C)$, then applying the Riemann-Roch formula and Serre duality to its canonical bundle $K_C$, we obtain:

$$h^0(C, K_C) - h^1(C, K_C) = g(C) - 1 = 1 - g(C) + \deg K_C,$$

that is,

$$g(C) = \frac{1}{2}\deg K_C + 1.$$

Now, if $C$ is embedded in a compact complex surface $X$, Proposition 2.7.2 tells us that

$$K_C = (K_X \otimes \mathcal{O}(C))|_C,$$

and so the genus of $C$ can be written as

$$g(C) = \frac{K_X \cdot C + C \cdot C}{2} + 1.$$

If $C$ is *not* smooth, then we will *define* its genus by the above formula. In the next section, we will see how we can concretely construct the *desingularisation* of an irreducible curve $C$, that is, a compact Riemann surface $\hat{C}$ and a map $f : \hat{C} \to C$ which is a biholomorphism away from the singular points of $C$. In the process, we will be able to show that a curve of genus zero must be smooth (and hence rational).

We have skipped over many of the details of singular curves on surfaces as they will not be entirely necessary for the purposes of what will come later. A proper treatment requires looking at sheaves on complex spaces. For now, we move on to discussing the fundamental construction of the blowup of a complex surface.

## 3.2 Blowing Up and Minimality

Blowing up complex manifolds is an important operation in complex geometry. In essence, one could say that it is a form of complex surgery in which one attaches copies of projective space along points of a complex submanifold, and in this way one separates points out based on their tangent directions. Its central position in the theory can be seen by the fact that the proof of the famous *Kodaira embedding theorem* relies on this operation. But, as we shall see, this operation is also fundamental to obtaining a classification of complex surfaces. Given that blowing up points is the only thing one is concerned with in the case of surfaces, we restrict our discussion in this section to this particular case. For the more general discussion, one can refer to any of the standard resources mentioned in the previous chapter.

As is standard, we begin by constructing the blowup of a point in $\mathbb{C}^2$. We can then transport the construction to a complex surface through the use of a holomorphic chart. So, let $U$ be a connected open neighbourhood of the origin $0 \in \mathbb{C}^2$. We will denote coordinates of $\mathbb{C}^2$ by $x = (x_1, x_2)$ and those of $\mathbb{P}^1$ by $[z] = [z_1 : z_2]$. The blowup at 0 is then

$$\widetilde{U} := \{(x, [z]) \in U \times \mathbb{P}^1 \,|\, x_1 z_2 = x_2 z_1\} \subseteq \mathbb{C}^2 \times \mathbb{P}^1.$$

We have that $\widetilde{U}$ is a submanifold of $\mathbb{C}^2 \times \mathbb{P}^1$, and the projection $\pi : \widetilde{U} \to U$, $(x, [z]) \mapsto x$ is holomorphic. Given that $(z_1, z_2) \neq 0$, it is not hard to see that that the condition $x_1 z_2 = x_2 z_1$ is equivalent to stipulating $(x_1, x_2) = \lambda(z_1, z_2)$ for some $\lambda \in \mathbb{C}$, that is, that

$(x_1, x_2)$ lies on the line defined by $(z_1, z_2)$. Since any non-zero $x \in U$ will lie on a unique line through the origin, we have that the restriction

$$\pi|_{\widetilde{U}\setminus\pi^{-1}(0)} : \widetilde{U} \setminus \pi^{-1}(0) \to U \setminus \{0\}$$

is a biholomorphism. However, the preimage $\pi^{-1}(0)$ contains all lines through the origin, and thus $\pi^{-1}(0) \cong \mathbb{P}^1$. We call $E := \pi^{-1}(0)$ the *exceptional divisor* of the blowup $\pi : \widetilde{U} \to U$, and by considering the open cover $\widetilde{U} = V_1 \cup V_2$, where $V_i = \{(x, [z]) \in \widetilde{U} \mid z_i \neq 0\}$, we see that

$$\mathcal{O}(E)|_E \cong \mathcal{O}_{\mathbb{P}^1}(-1).$$

Now, suppose $X$ is an arbitrary complex surface and $x \in X$ is any point. Choose a holomorphic chart $\phi : V \to U \subseteq \mathbb{C}^2$ centred at $x$. Using the same notation as above, we glue the manifold $X \setminus \{x\}$ to $\widetilde{U}$ by identifying the subset $V \setminus \{x\}$ with $\widetilde{U} \setminus E$ via the biholomorphism

$$V \setminus \{x\} \xrightarrow{\phi} U \setminus \{0\} \xrightarrow{\pi^{-1}} \widetilde{U} \setminus E.$$

We will often denote this new surface by $\widetilde{X}$. The obvious projection is still denoted by $\pi : \widetilde{X} \to X$, and as before it restricts to a biholomorphism

$$\pi|_{\widetilde{X}\setminus E} : \widetilde{X} \setminus E \to X \setminus \{x\},$$

where $E = \pi^{-1}(x)$ is still called the exceptional divisor. It can be shown (for example, see [17, p. 183-4]) that $\widetilde{X}$ does not depend on the choice of chart up to biholomorphism. Moreover, given local holomorphic coordinates, we get a biholomorphism $E \cong \mathbb{P}(T_xX) \cong \mathbb{P}^1$, which also does not depend on the choice of chart. Furthermore, the isomorphism $\mathcal{O}(E)|_E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ still holds and implies:

**Proposition 3.2.1.** *If $\pi : \widetilde{X} \to X$ denotes the blowup of a complex surface $X$ at $x \in X$, and $E = \pi^{-1}(x)$ is the exceptional divisor, then*

$$E \cdot E = -1.$$

*Proof.* By Corollary 2.2.2 and the fact that the Euler class and first Chern class of a complex line bundle coincide (Theorem 2.1.1), we see that

$$E \cdot E = \deg \mathcal{O}(E)|_E = \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -1,$$

where the final equality follows by the fact that $c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = [\omega_{\mathrm{FS}}] \in H^2(\mathbb{P}^1, \mathbb{R})$, where $\omega_{\mathrm{FS}}$ is the Fubini-Study metric for $\mathbb{P}^1$ (or simply by (iv) of Proposition 2.1.4). $\square$

For the sake of completeness, we mention some important properties of the blowup. Proofs of them can be found in any of the texts mentioned in this chapter. We state the result in the case of surfaces:

**Theorem 3.2.1** (Properties of the Blowup). *If $X$ is a complex surface and $\pi : \widetilde{X} \to X$ is the blowup of $X$ at a point $x \in X$ with exceptional divisor $E = \pi^{-1}(x)$, then we have:*

*(i) There is a diffeomorphism $\widetilde{X} \cong X \# \overline{\mathbb{P}^2}$, where $\overline{\mathbb{P}^2}$ denotes $\mathbb{P}^2$ with canonical orientation reversed. In particular, $\pi_1(\widetilde{X}) \cong \pi_1(X)$ and $b_1(\widetilde{X}) = b_1(X)$.*

*(ii) There is an isomorphism $H^2(\widetilde{X}, \mathbb{Z}) \cong \pi^* H^2(X, \mathbb{Z}) \oplus \mathbb{Z}\{c_1(\mathcal{O}(E))\}$. In particular, $b_2(\widetilde{X}) = b_2(X) + 1$.*

*(iii) We have an isomorphism $\operatorname{Div}(\widetilde{X}) \cong \pi^*\operatorname{Div}(X) \oplus \mathbb{Z}\{E\}$, and the map $\operatorname{Pic}(X) \oplus \mathbb{Z} \to \operatorname{Pic}(\widetilde{X})$, $(L, m) \mapsto \pi^* L \otimes \mathcal{O}(mE)$ is an isomorphism. In particular, $K_{\widetilde{X}} \cong \pi^* K_X \otimes \mathcal{O}(E)$.*

**Remark 3.2.1.** It is also true that the blowup of a Kähler surface is again Kähler, and that in the case $X$ is compact, that $a(X) = a(\widetilde{X})$ and $P_n(X) = P_n(\widetilde{X})$ for all $n \neq 1$. Hence, $\operatorname{kod}(X) = \operatorname{kod}(\widetilde{X})$. Moreover (cf. (ii) above), $h^{1,1}$ is the only Hodge number which increases by 1 when blowing up at a point on a complex surface. These properties will be important when we come to look at the effects of bimeromorphic maps.

Let us get a bit more intuition for the geometry of blowups. We start by making the following definition:

**Definition 3.2.1.** Let $\pi : \widetilde{X} \to X$ denote the blowup of a complex surface $X$ at a point $x \in X$ with exceptional divisor $E = \pi^{-1}(x)$. If $V$ is a hypersurface, we define the *proper transform* of $V$ by

$$\widetilde{V} := \overline{\pi^{-1}(V \setminus \{x\})} = \overline{\pi^{-1}(V) \setminus E} \subseteq \widetilde{X}.$$

We define the proper transform $\widetilde{D} \in \operatorname{Div}(\widetilde{X})$ of a divisor $D \in \operatorname{Div}(X)$ in the obvious way.

In particular, it is immediate that

$$\pi|_{\widetilde{V} \setminus E} : \widetilde{V} \setminus E \to V \setminus \{x\}$$

is a biholomorphism. By choosing local holomorphic coordinates $z = (z_1, z_2) : U \to \mathbb{C}^2$ for $X$, we have that $\widetilde{X}$ is locally given by the set

$$\{(x, [y]) \in U \times \mathbb{P}^1 \mid z_1(x) y_2 = y_1 z_2(x)\}.$$

Using this, one can consider a local description of the proper transform to show:

**Proposition 3.2.2.** *Let $C$ be an irreducible curve on a compact complex surface $X$ which has multiplicity $m$ through a point $x \in X$. If $\pi : \widetilde{X} \to X$ denotes the blowup of $X$ at $x$, then*

$$\widetilde{C} = \pi^* C - mE.$$

*In particular, if $C$ is smooth (i.e. has multiplicity 1) at $x$, then $\widetilde{C} = \pi^* C - E$.*

We are now in a position to construct the desingularisation. Suppose that $C$ is an irreducible curve on a compact complex surface $X$ with multiplicity $m$ through a point $x \in X$. Let $\pi : \widetilde{X} \to X$ be the blowup of $X$ at $x$ and $E = \pi^{-1}(x)$. Since the fundamental class of $E$ is in the kernel of $\pi_* : H_2(\widetilde{X}, \mathbb{Z}) \to H_2(X, \mathbb{Z})$, for every divisor $D \in \mathrm{Div}(X)$, we have

$$\pi^* D \cdot E = \pi^* c_1(\mathcal{O}_X(D))[E] = c_1(\mathcal{O}_X(D))\pi_*[E] = 0.$$

Since $\pi$ is of degree 1, we also have $\pi^* D_1 \cdot \pi^* D_2 = D_1 \cdot D_2$. Proposition 3.2.2 and Theorem 3.2.1 (iii) tell us:

$$\widetilde{C} \cdot \widetilde{C} = (\pi^* C - mE) \cdot (\pi^* C - mE) = C \cdot C - m^2,$$
$$K_{\widetilde{X}} \cdot \widetilde{C} = (\pi^* K_X + E) \cdot (\pi^* C - mE) = K_X \cdot C + m.$$

In particular, the genus of the proper transform of $C$ becomes

$$g(\widetilde{C}) = \frac{K_{\widetilde{X}} \cdot \widetilde{C} + \widetilde{C} \cdot \widetilde{C}}{2} + 1 = \frac{K_X \cdot C + C \cdot C - m^2 + m}{2} + 1 = g(C) - \frac{m(m-1)}{2}.$$

And so, $g(\widetilde{C}) < g(C)$, if $m > 1$. In particular, if we blow up any singular point of $\widetilde{C}$ and take the proper transform, its genus will be further reduced. Continuing this way, we must eventually arrive at a smooth irreducible curve $\hat{C}$ which is biholomorphic to $C$ away from its singular points. If this was not the case, we would have constructed a strictly decreasing sequence of non-negative integers, which is impossible. A consequence of the above calculation is the following:

**Proposition 3.2.3.** *Let $C$ be an irreducible curve on a compact complex surface. If $\hat{C}$ denotes the desingularisation of $C$ obtained by repeatedly blowing up the singular points $x_1, \ldots, x_n$ of $C$, each with respective multiplicity $m_i$, then*

$$g(\hat{C}) = g(C) - \sum_{i=1}^{n} \frac{m_i(m_i - 1)}{2}.$$

*In particular, $g(\hat{C}) \leq g(C)$ with equality if and only if $C$ is smooth. Moreover, $g(C) = 0$ if and only if $C$ is a smooth rational curve.*

Now, an important question to ask is: when can a surface be obtained by blowing up another surface, or better put, when can it be *blown down*? The Castelnuovo criterion says that the existence of an exceptional curve is not just necessary but even sufficient for being able to blow down a compact complex surface:

**Theorem 3.2.2** (Castelnuovo Criterion). *Let $X$ be a compact complex surface and $C$ a smooth rational curve with $C \cdot C = -1$. Then there exists a compact complex surface $Y$ and a map $\pi : X \to Y$ which is the blowup of $Y$ at some point $y_0 \in Y$ with $C = \pi^{-1}(y_0)$.*

**Remark 3.2.2.** Smooth irreducible curves $C$ which are rational and which have self-intersection $-1$ are often called *exceptional curves of the first kind*, or sometimes more simply $(-1)-$*curves*. By extension, a $(-n)$-*curve* is a smooth rational curve of self-intersection $-n$.

In light of Proposition 3.2.3, we can refine the above to obtain:

**Theorem 3.2.3** (Blow-down Criterion). *An irreducible curve $C$ on a compact complex surface $X$ is a $(-1)$-curve if and only if $C \cdot C < 0$ and $K_X \cdot C < 0$.*

*Proof.* This follows immediately by the fact that

$$g(C) = \frac{C \cdot C + K_X \cdot C}{2} + 1 \geq 0,$$

and the fact that $C$ is smooth and rational if and only if $g(C) = 0$. □

We conclude this section with the following fundamental definition:

**Definition 3.2.2.** A compact complex surface is said to be *minimal* if it cannot be blown down, that is, if it has no $(-1)$-curves.

We are now in a good position to discuss bimeromorphic transformations of complex surfaces as well as the notion of a minimal model. From there, we will be able to state the famous Enriques-Kodaira classification of minimal compact complex surfaces.

## 3.3 Bimeromorphic Maps and Minimal Models

As we will see, the classification of compact complex surfaces restricts itself to the slightly weaker notion of *bimeromorphism*, i.e. the analytic variant of the algebro-geometric notion of birationality, and when restricting to the case of minimal surfaces, this is effectively sufficient to speak of classifying *all* compact complex surfaces. The details of the theory are not particularly relevant for our purposes, so we will only briefly cover some of the key results as they pertain to the study of complex surfaces. We start by defining what a bimeromorphic map is:

**Definition 3.3.1.** A proper holomorphic surjective map $\pi : X \to Y$ between complex surfaces is called *bimeromorphic* if there are proper analytic subvarieties $U \subset X$ and $V \subset Y$ such that $\pi : X \setminus U \to Y \setminus V$ is a biholomorphism.

The most important example of a bimeromorphic map for surfaces is the blowup $\pi : \widetilde{X} \to X$. In some sense, and this is particular to the case of complex dimension two, the blowup is the only bimeromorphic map one needs to consider. To make this precise, we will need to mention some important results from the theory. The first states that any bimeromorphic map factors through another as well as through blowing up a point of indeterminacy:

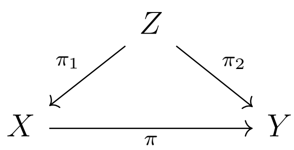
**Proposition 3.3.1** (Factorisation Lemma)**.** *Let $\pi : X \to Y$ be a bimeromorphic map between complex surfaces which is not a biholomorphism over a point $y \in Y$. Then there exists a complex surface $Z$, a bimeromorphism $f : X \to Z$ and $\pi' : Z \to Y$ such that $\pi = \pi' \circ \sigma$, where $\pi' : Z \to Y$ is the blowup of $Y$ at $y$.*

*Proof.* See [44, Lemma 4.3, Chapter III]. □

**Remark 3.3.1.** Recall that a complex surface $X$ is minimal if it has no exceptional curves of the first kind. Using the factorisation lemma above, we see that a complex surface $X$ is minimal if and only if every bimeromorphic map into an arbitrary complex surface $Y$ is a biholomorphism.

Applying an inductive argument to the number of irreducible components of each singular fibre of a bimeromorphic map, we obtain:

**Theorem 3.3.1** (Structure of Bimeromorphic Maps)**.** *Let $\pi : X \to Y$ be a bimeromorphic map between complex surfaces. Then there exists a complex surface $Z$ and maps $\pi_1 : Z \to X$, $\pi_2 : Z \to Y$ such that the following diagram commutes*




*where $\pi_1$ and $\pi_2$ are compositions of blowups and biholomorphisms.*

*Proof.* See [44, Theorem 6.3, Chapter III]. □

**Remark 3.3.2.** In particular, we see that the algebraic dimension, Kodaira dimension, first Betti number, irregularity, etc. of a compact complex surface are all bimeromorphism invariants (Theorem 3.2.1). And, as we alluded in the introduction, these are the main quantities which are used to distinguish the classes of compact complex surfaces.

We conclude by introducing the concept of a *minimal model*:

**Definition 3.3.2.** A complex surface $Y$ is called a *minimal model* for a complex surface $X$ if $Y$ is minimal and bimeromorphic to $X$.

By repeatedly blowing down a non-minimal surface and using the fact that the second Betti number diminishes by 1 with every iteration (Theorem 3.2.1 (ii)), one obtains a quick proof of the important fact that:

**Theorem 3.3.2.** *Every compact complex surface has a minimal model.*

The following result (see [44, Proposition 4.6] or [43]) in conjunction with Remark 3.3.1 tells us that the restriction to minimal surfaces is for the most part sufficient to talk about the classification of compact complex surfaces up to biholomorphism:

**Theorem 3.3.3.** *If $X$ is a compact connected complex surface with* $\mathrm{kod}(X) \geq 0$*, then all minimal models of $X$ are biholomorphic.*

**Remark 3.3.3.** One can take the simple example of $X := \mathbb{P}^2$ to see that the statement is not generally true for the case $\mathrm{kod}(X) = -\infty$. Blowing up any two distinct points on $\mathbb{P}^2$ and taking the proper transform of the line (i.e. a copy of $\mathbb{P}^1$) passing through them yields a $(-1)-$curve. Blowing this curve down gives a surface biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which is not biholomorphic to $\mathbb{P}^2$. Nonetheless, this problem can be overcome by completely classifying those surfaces of Kodaira dimension $-\infty$ which do not have a unique minimal model.

We can now state the central result of this chapter, which will set the stage for the work we will do in the final chapters.

## 3.4 The Enriques-Kodaira Classification

In this section, we will finally cover the celebrated Enriques-Kodaira classification of minimal compact complex surfaces. Given that K3 and Enriques surfaces are the particular classes of the Enriques-Kodaira classification that we will be focusing on, this result contextualises the entirety of the work that will subsequently be undertaken. To go over the classification in full detail is well beyond the scope of this thesis. The sheer amount of auxiliary results required is substantial, and much of the material is itself quite advanced. For our purposes, we will have to content ourselves with a brief description of the classes present in the classification.

As is typical in the study of geometric structures, getting a better grasp of them can be achieved by studying how other better understood objects are related to them. In the case at hand, we have mentioned that much can be learnt about the structure of compact complex surfaces by considering their relationship to their lower dimensional analogue: compact Riemann surfaces and singular versions of these, i.e. singular curves. As a result, it seems instructive to briefly outline the classification theory of compact Riemann surfaces.

As is well known, a topological classification of compact Riemann surfaces relies only on knowing one number associated to the surface – its genus. Of course, this is quite a coarse way to differentiate these objects, and does not make use of the additional complex structure which is present. The famous *uniformisation theorem* of Riemann surfaces distinguishes them based on the structure of their universal covers, there being only three such possibilities: the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$ and the unit disc $\mathbb{D}$. There is only one compact Riemann surface of the first class, namely the Riemann sphere itself. Elliptic curves form the entirety of the second class, and compact Riemann surfaces of genus $g \geq 2$ form the general case. Note that we have $\mathrm{kod}(\mathbb{P}^1) = -\infty$, and given that tori have trivial canonical bundle, we have $\mathrm{kod}(T) = 0$ for any elliptic curve $T$. Now, if $X$ denotes a compact Riemann surface of genus $g \geq 2$, then using the fact that $\deg K_X = -\deg \mathcal{T}_X = -\chi(X) = 2g - 2$ and the Riemann-Roch theorem, we see for $n \geq 2$:

$$h^0(K_X^{\otimes n}) - h^1(K_X^{\otimes n}) = n \deg K_X + (1 - g) = (2n - 1)(g - 1).$$

Serre duality implies

$$h^1(K_X^{\otimes n}) = h^0(\mathcal{T}_X^{\otimes (n-1)}).$$

Since any line bundle of negative degree cannot possess global sections and $\deg \mathcal{T}_X = \chi(X) < 0$, we see $h^0(\mathcal{T}_X) = 0$, and thus $h^1(K_X^{\otimes n}) = 0$. Hence, the plurigenera of $X$ are given by

$$P_n(X) = h^0(K_X^{\otimes n}) = (2n - 1)(g - 1),$$

and so $\mathrm{kod}(X) = 1$. This trichotomy is summarised in Table 3.1.

| **Class of** $X$ | **Universal Covering** | **Genus** $g$ | $\mathrm{kod}(X)$ |
|---|---|---|---|
| Elliptic | $\mathbb{P}^1$ | 0 | $-\infty$ |
| Parabolic | $\mathbb{C}$ | 1 | 0 |
| Hyperbolic | $\mathbb{D}$ | $\geq 2$ | 1 |

Table 3.1: Table outlining the classification of compact Riemann surfaces.

Moving on to the case of compact complex surfaces, it seems reasonable to break up their classes based on the Kodaira dimension, and, indeed, this is what has historically been done. In this case, it can take the values $\mathrm{kod}(X) = -\infty, 0, 1, 2$. However, it is certainly too ambitious to think that the Kodaira dimension can alone give a sufficiently fine, and for that matter, *useful* classification of compact complex surfaces; one should hope that a 'good' classification would organise the classes of objects of interest into subclasses with related properties which are amenable to further study. For that reason, the other bimeromorphism invariants introduced (cf. Remark 3.3.2), such as the plurigenera, first Betti number, algebraic dimension, irregularity, etc. have historically been used to distinguish (minimal) compact complex surfaces.

In his collection of papers *On the Structure of Compact Complex Analytic Surfaces, I - IV* (in particular, I [30] and IV [31]), Kodaira outlines this process and arrives at his own version of the classification as presented below. The key idea is that by placing restrictions on some quantity of interest (Kodaira dimension, algebraic dimension, etc.), one can determine specific characteristics that that kind compact complex surface must possess, and hence meaningfully organise the entire collection of compact complex surfaces.

Following our brief discussion of Riemann surfaces, one could argue that discussing the surfaces in order of increasing Kodaira dimension would make the most sense. However, given that K3 and Enriques surfaces are the central focus of this thesis, it is best to start by defining them. They are both distinct classes in the case where the Kodaira dimension is 0, and as can easily be checked, are minimal by definition:

**Definition 3.4.1.** A *K3 surface* is a compact complex surface $X$ with $K_X \cong \mathcal{O}_X$ and $b_1(X) = 0$.

**Definition 3.4.2.** An *Enriques surface* is a compact complex surface $X$ such that $K_X$ is not trivial, but $K_X^{\otimes 2} \cong \mathcal{O}_X$ and $b_1(X) = 0$.

K3 surfaces have been studied to a great extent, as too have Enriques surfaces. Since Chapters 4 and 5 are respectively dedicated to the classification theory of these surfaces, we will restrict our discussion of them for the moment. We simply note here that examples of them are manifold, and many explicit constructions can be employed in order to form them. Let us briefly consider one of the most well-known K3 surfaces:

**Example 3.4.1.** Consider the following homogeneous polynomial of degree 4:

$$f : \mathbb{C}^4 \to \mathbb{C},\ (z_1, z_2, z_3, z_4) \mapsto z_1^4 + z_2^4 + z_3^4 + z_4^4.$$

It is a standard fact that the holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(k)$, $k > 0$ over $\mathbb{P}^n$ are in bijective correspondence with homogeneous polynomials of degree $k$ in $n+1$ variables. Then, if we define $X \subseteq \mathbb{P}^3$ to be the smooth irreducible divisor given by the zero set of the section $f \in H^0(\mathbb{P}^3, \mathcal{O}(4))$ (called the *Fermat quartic surface*), we see that $\mathcal{O}(X) \cong \mathcal{O}_{\mathbb{P}^3}(4)$. Moreover, we also have the standard result that $K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ (see, for example, [25, Proposition 2.4.3]). Thus, the adjunction formula (Proposition 2.7.2) tells us

$$K_X \cong (K_{\mathbb{P}^3} \otimes \mathcal{O}(X))|_X \cong (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X \cong \mathcal{O}_X.$$

Using a version of the Lefschetz hyperplane theorem (e.g. see [4]), one can show that $X$ is simply connected, and hence $b_1(X) = 0$.

As for the case of Enriques surfaces, we could equally well go through some of their constructions, but the following result, which is shown below, will not only be sufficient, but central to our discussion of Enriques surfaces that we will pursue in Chapter 5. It is the thing which establishes the link between the study of K3 and Enriques surfaces, and, as mentioned, will allow us to transport the classification theory of K3 surfaces to that of Enriques surfaces:

**Theorem 3.4.1.** *If $X$ is a K3 surface and $\sigma : X \to X$ is a free holomorphic involution, then the quotient manifold $E := X/\langle\sigma\rangle$ is an Enriques surface. Conversely, every Enriques surface $E$ is double covered by a K3 surface. In particular, Enriques surfaces are in one-to-one correspondence with K3 surfaces equipped with free holomorphic involutions.*

*Proof.* For the proof of the first statement, we will have to rely on one lemma:

**Lemma 3.4.1** (Lemma 16.2, [44]). *Let $X$ and $Y$ be compact connected complex manifolds and $f : X \to Y$ a covering of degree $d$. If $L$ is a line bundle on $Y$ with $f^*L \cong \mathcal{O}_X$, then $L^{\otimes d} \cong \mathcal{O}_Y$.*

Then, if $E = X/\langle\sigma\rangle$, we have by standard covering space theory that $b_k(E) \leq b_k(X)$ for each $k$, in particular, $b_1(E) \leq b_1(X) = 0$, so $b_1(E) = 0$. Since the projection $\pi : X \to E$ is a local biholomorphism, we obtain an isomorphism of line bundles $\pi^*K_E \cong K_X \cong \mathcal{O}_X$, and Lemma 3.4.1 tells us $K_E$ is 2-torsion in $\mathrm{Pic}(E)$. Moreover, Noether's formula tells us

$$\chi(\mathcal{O}_X) = \frac{K_X \cdot K_X + \chi(X)}{12} = \frac{\pi^*K_E \cdot \pi^*K_E + 2\chi(E)}{12} = 2\chi(\mathcal{O}_E).$$

Thus, $\chi(\mathcal{O}_E) = 1 - q(E) + p_g(E) = \frac{1}{2}\chi(\mathcal{O}_X) = 1$. But since $b_1(E) = 0$, we must have $q(E) = 0$, and hence $p_g(E) = 0$. Thus, $K_E$ is not trivial, and so $E$ is an Enriques surface by definition.

For the converse statement, let $E$ be an Enriques surface. As before, $b_1(E) = 0$ implies $q(E) = 0$, and thus $H^1(E, \mathcal{O}_E) = 0$. The long exact sequence on cohomology arising from the exponential sequence says that the first Chern class map

$$c_1 : \mathrm{Pic}(E) \to H^2(E, \mathbb{Z})$$

is injective (Theorem 2.4.3). Consider the long exact sequence on cohomology

$$\ldots \longrightarrow H^1(E, \mathbb{Z}_2) \xrightarrow{\delta} H^2(E, \mathbb{Z}) \xrightarrow{\times 2} H^2(E, \mathbb{Z}) \longrightarrow \ldots$$

arising from the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}_2 \longrightarrow 0.$$

Since $K_E^{\otimes 2} \cong \mathcal{O}_E$, we have that $c_1(K_E)$ is 2-torsion, and since $K_E$ is non-trivial, so is $c_1(K_E)$. Then, by exactness, there exists $c \in H^1(E, \mathbb{Z}_2)$ such that $\delta(c) = c_1(K_E)$. By standard covering space theory (see, for example, [20, p. 70]), double covers of $E$ are in bijective correspondence with $\mathrm{Hom}(\pi_1(E), \mathbb{Z}_2) \cong H^1(E, \mathbb{Z}_2)$, and so the class $c$ defines a double cover $\pi : X \to E$ uniquely characterised by the fact that $X$ is connected and $\pi^* c_1(K_E) = 0$. Pulling back the complex structure on $E$, we see that $X$ is a compact complex surface. Since $c_1$ is injective, we have that $\pi^* c_1(K_E) = c_1(\pi^* K_E) = c_1(K_X) = 0$ implies $K_X \cong \mathcal{O}_X$. Again, we have

$$\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) = 2 - h^{0,1}(X) = 2\chi(\mathcal{O}_E) = 2.$$

Hence, $h^{0,1}(X) = 0$. Given that $h^{1,0}(X) \leq h^{0,1}(X)$ and $b_1(X) = h^{1,0}(X) + h^{0,1}(X)$ for every compact complex surface $X$ (see [44, Theorem 2.7]), we must have $b_1(X) = 0$. So, $X$ is a K3 surface. Moreover, given that every double covering is normal, we have that the group of deck transformations of $\pi : X \to E$ is $\mathbb{Z}_2$, so there is a unique free holomorphic involution $\sigma : X \to X$ swapping the sheets of $X$. □

Let us briefly remark on the remaining classes of surface. Sticking to the case of Kodaira dimension 0, we have four other classes which are related in pairs in the same way as K3 and Enriques surfaces. We have complex tori of dimension two and also *bielliptic* surfaces, which all turn out to be biholomorphic to the quotient of the product of elliptic curves by free actions of finite order (see [44, Section 5, Chapter V]). We also have *primary* and *secondary Kodaira surfaces*, the former being defined as compact complex surfaces $X$ with $b_1(X) = 3$ and which are equipped with the structure of elliptic fibre bundles over elliptic curves, and the latter being finitely covered by these.

Taking a step back, we have the case $\text{kod}(X) = -\infty$. This collection of surfaces consists of *rational* surfaces (i.e. those bimeromorphic to $\mathbb{P}^2$), *ruled surfaces of genus* $g \geq 1$ (those equipped with the structure of $\mathbb{P}^1$-bundles over smooth irreducible curves of genus $g$) and *surfaces of class VII* (those simply defined by the condition $b_1 = 1$). There is a lot that we could say about all of these classes of surface, particularly the lattermost class, which at the the time of writing this work is still attracting interest for the fact that surfaces of class VII have not completely been classified yet. For the case of minimal rational surfaces, it turns out that we can obtain an enumerable list of them (consisting of $\mathbb{P}^2$ and the so-called *Hirzebruch surfaces* $\Sigma_n$, $n \neq 1$).

Moving on to the case of positive Kodaira dimension, we have that all surfaces of Kodaira dimension 1 are *elliptic* (i.e. which fibre over smooth irreducible curves with generic fibre an elliptic curve), and in this case one usually refers to them as *properly* elliptic surfaces, since many classes of surface with $\text{kod}(X) \neq 1$ are themselves elliptic. Given this quite general description, a complete classification of them is a lot to ask for. Nonetheless, Kodaira in [29, Theorem 6.2] classifies all the possible singular fibres that can occur for a *minimal* elliptic fibration, that is, one in which no fibre has a $(-1)$-curve as a component.

The final class of compact complex surface naturally consists of those of Kodaira dimension 2, and are referred to as *surfaces of general type*. These are the higher-dimensional analogue of the compact Riemann surfaces of genus $g \geq 2$ (i.e. those with Kodaira dimension 1), which represented the 'general' case in dimension 1. Despite their general nature, a great deal can be said about them (for instance, they are all algebraic), and moreover, there are still many open problems surrounding them (the interested reader can consult Chapter VII of the standard reference [43] or [44], which is dedicated solely to this class of surface).

Having elaborated (albeit, extremely briefly) on the different types of complex surfaces that exist, it is beneficial to conclude this chapter by organising them into a table (exactly as we did for compact Riemann surfaces in Table 3.1) in which one can compare and contrast their invariants. The statement of the celebrated Enriques-Kodaira classification is then:

**Theorem 3.4.2** (Enriques-Kodaira Classification of Minimal Compact Complex Surfaces)**.** *Every compact complex surface $X$ has a minimal model in exactly one of the ten classes of Table* 3.2. *This model is unique up to biholomorphism except for those surfaces in class (1) and (3).*

*Proof (Sketch).* As we have seen, the Kodaira dimension is the coarsest invariant that is used to differentiate compact complex surfaces, and moreover, for a compact, connected, complex surface $X$ we always have $\text{kod}(X) \leq a(X)$. In particular, by enumerating all the possible values of $\text{kod}(X) \in \{-\infty, 0, 1, 2\}$ and $a(X) \in \{0, 1, 2\}$, and using the relationships between the standard topological and bimeromorphic invariants we have covered, one can break the analysis into a finite set of cases. Upon establishing results on structures like fibrations over curves, among (many) other things, one can then determine the structure of the resulting surface and show that it has to fall into one of the classes listed in Table 3.2. Chapter VI of [43] or [44] covers the proof in full detail. Kodaira in his fundamental papers (see [30, Theorem 22] and [31, Theorem 55]) elaborates the classification of all compact complex surfaces for the first time in 1966. A classical reference for the algebraic case is by Enriques in [15]. □

| **Class of $X$** | $\text{kod}(X)$ | **Smallest $n > 0$ with $K_X^{\otimes n} \cong \mathcal{O}_X$** | $b_1(X)$ | $a(X)$ | $K_X \cdot K_X$ | $\chi(X)$ |
|---|---|---|---|---|---|---|
| 1) Minimal rational surfaces | | | 0 | 2 | 8 or 9 | 4 or 3 |
| 2) Minimal surfaces of class VII | $-\infty$ | | 1 | 0,1 | $\leq 0$ | $\geq 0$ |
| 3) Ruled surfaces of genus $g \geq 1$ | | | $2g$ | 2 | $8(1-g)$ | $4(1-g)$ |
| 4) Tori | | 1 | 4 | 0, 1, 2 | 0 | 0 |
| 5) Bi-elliptic surfaces | | 2, 3, 4, 6 | 2 | 2 | 0 | 0 |
| 6) K3 surfaces | | 1 | 0 | 0, 1, 2 | 0 | 24 |
| 7) Enriques surfaces | | 2 | 0 | 2 | 0 | 12 |
| 8) Kodaira surfaces | 0 | | | | | |
| a) primary | | 1 | 3 | 1 | 0 | 0 |
| b) secondary | | 2, 3, 4, 6 | 1 | 1 | 0 | 0 |
| 9) Minimal properly elliptic surfaces | 1 | | | 1, 2 | 0 | $\geq 0$ |
| 10) Minimal surfaces of general type | 2 | | even | 2 | $> 0$ | $> 0$ |

Table 3.2: Table showing the ten distinct classes of minimal compact complex surface that exist. Slightly modified version of the one in [44, p. 244].

Before closing this chapter off, let us briefly mention one last important result concerning Kähler surfaces, which in particular has important consequences for the class of K3 surfaces we will be considering in the next chapter. In light of Table 3.2 and excluding the classes with odd first Betti number (i.e. Kodaira surfaces and surfaces of class VII), one immediately observes that all other surfaces besides K3 surfaces and properly elliptic surfaces (i.e. surfaces belonging to the classes (1), (3), (4), (5), (7) and (10)) are Kähler. This led Kodaira to ask the question: *are all compact complex surfaces with even first Betti number Kähler?* Remarkably, this is indeed true. This was proved in the case of properly elliptic surfaces by Miyaoka in [38], and sometime later, Siu in [48] gave a proof for the case of K3 surfaces, which should be noted was highly non-trivial and made use of Yau's resolution of the Calabi conjecture. Thus, we have the wonderful result:

**Theorem 3.4.3.** *A compact complex surface is Kähler if and only if its first Betti number is even.*

A proof which does not rely on the Enriques-Kodaira classification remained lacking for a long time, but in 1999 Buchdahl [6] and Lamari [33] independently gave proofs of Theorem 3.4.3, both making use of the smoothing of $(1,1)$-currents.

The well-versed reader will certainly have observed that we have glossed over many important details in the theory of complex surfaces. Given the vast amount of work that has been done in this area, it is inconceivable to think that all of it could have been covered. Nonetheless, with this background in mind, we can now move on to look at K3 surfaces and their classification in more detail. The results that we cover there will be crucial for our analysis of Enriques surfaces that will subsequently follow.

# Chapter 4

# On K3 Surfaces

Having covered the Enriques-Kodaira classification of minimal compact complex surfaces, we restrict our attention to the theory of K3 surfaces, one of the most interesting classes of minimal compact complex surface of Kodaira dimension 0. The key aim of this chapter is to introduce the most important concepts and results related to the classification of K3 surfaces, which as explained in the introduction, are summarised in the conjectures of André Weil outlined in [53]. We will then see how these ideas can be extended to the case of classifying Enriques surfaces in the next chapter. Here, we will only discuss the results in the form that we will need them, with the discussion being based on the paper [7].

## 4.1 Basics of Lattices

The study of lattices has become an important part of modern mathematics. The material introduced here will be of a different (and naturally simpler) flavour to all that we have discussed until now, but as we will see, obtaining a better understanding of lattices will be indispensable for grasping the structures of K3 surfaces, and in particular, Enriques surfaces. This section can be seen as a natural extension of the considerations of Section 2.2. We begin by making some standard definitions from the theory, and we will then expand on this material as necessary in later sections (see Section 5.2.1):

**Definition 4.1.1.** A *lattice* is a free $\mathbb{Z}$-module $L$ of finite rank equipped with a symmetric integral bilinear form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$. We may sometimes just refer to $L$ as being the lattice, as opposed to the pair $(L, \langle \cdot, \cdot \rangle)$. As usual, for $x \in L$, we set $x^2 := \langle x, x \rangle$. The *dual* of $L$ will be denoted $L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. We shall denote the rank of a lattice $L$ by $\mathrm{rk}(L)$.

**Definition 4.1.2.** We say that a lattice $(L, \langle \cdot, \cdot \rangle)$ is *non-degenerate* if the canonical homomorphism $L \to L^*$, $x \mapsto \langle x, \cdot \rangle$ is injective, and *unimodular* if it is an isomorphism.

**Remark 4.1.1.** If $\mathrm{rk}(L) = n$ and $e_1, \ldots, e_n$ is a basis for $L$, then it is clear that the determinant of the matrix $(\langle e_i, e_j \rangle)_{i,j=1,\ldots,n} \in M_{n\times n}(\mathbb{Z})$ does not depend on this choice up to sign. We will denote this number by $d(L)$. Thus, the lattice $L$ is non-degenerate if and only if $d(L) \neq 0$, and is unimodular if and only if $d(L) = \pm 1$.

**Example 4.1.1.** We have already come across an example of a unimodular lattice: the free part of $H^{2k}(X, \mathbb{Z})$ equipped with its intersection form on a closed oriented topological $4k$-manifold $X$ (cf. Definition 2.2.2). The fact it is unimodular follows by Poincaré duality and the universal coefficient theorem.

**Definition 4.1.3.** We say that a lattice $L$ is *even* if $x^2$ is even for all $x \in L \setminus \{0\}$, otherwise it is *odd*. Moreover, $L$ is called *positive definite* if $x^2 > 0$ for all $x \in L \setminus \{0\}$, and similarly for *negative definite*. We say $L$ is *definite* if it is either positive definite or negative definite. The lattice is *indefinite* if it is not definite.

In exactly the same as the case of the intersection form, we can make the following definition (cf. Definition 2.2.3):

**Definition 4.1.4.** If $L$ is a lattice, we define its *signature* $\sigma(L)$ to be

$$\sigma(L) = b^+(L) - b^-(L),$$

where $b^{\pm}(L)$ is the number of positive/negative entries in some (hence any) diagonalisation of the $\mathbb{R}$-linear extension of its bilinear form. We will often say that the lattice $L$ is of signature $(b^+(L), b^-(L))$. For a non-degenerate lattice $L$, we clearly have $\mathrm{rk}(L) = b^+(L) + b^-(L)$.

**Remark 4.1.2.** It is immediate to see that the signature is additive: given lattices $L_1$ and $L_2$, we have

$$\sigma(L_1 \oplus L_2) = \sigma(L_1) + \sigma(L_2).$$

Other than the intersection form, there are two other explicit examples of lattices that we will need to be aware of for the purposes of this thesis:

**Example 4.1.2.** The first is the standard hyperbolic lattice on $\mathbb{Z}^2$. If $e_1, e_2$ is the standard basis for $\mathbb{Z}^2$, then the matrix of its bilinear form is given by

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clearly even, unimodular and indefinite. Its signature is given by $\sigma(H) = 0$. The second important example is the $E_8$-lattice. As a $\mathbb{Z}$-module, it is $\mathbb{Z}^8$ with bilinear form defined by the matrix

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

It is not hard to show that this is an even, unimodular and positive definite lattice with $\sigma(E_8) = 8$. Flipping all signs gives a negative definite lattice $-E_8$.

Whenever one introduces a new kind of structure, it is natural to ask what the structure-preserving maps should be. In this case, they are known as *isometries:*

**Definition 4.1.5.** Given lattices $(L_1, \langle\cdot,\cdot\rangle_1)$ and $(L_2, \langle\cdot,\cdot\rangle_2)$, an *isometry* between them is an isomorphism of abelian groups $\phi : L_1 \to L_2$ such that for all $x, y \in L_1$, we have

$$\langle \phi(x), \phi(y) \rangle_2 = \langle x, y \rangle_1.$$

In such a case, we say that the lattices $L_1$ and $L_2$ are *isometric.*

The following is an important classification result for indefinite unimodular lattices (see [47]):

**Theorem 4.1.1.** *Any indefinite unimodular lattice is, up to isometry, determined by its rank, signature and its parity.*

In later sections (see Section 5.2), we will look at some classification results relating to non-degenerate lattices. For now, let us conclude with the following standard definition:

**Definition 4.1.6.** If $(L, \langle\cdot,\cdot\rangle)$ is a lattice and $K \subseteq L$ is any sublattice, then its *orthogonal complement* is the sublattice defined as

$$K^\perp = \{x \in L \mid \forall y \in K : \langle x, y \rangle = 0\}.$$

## 4.2 The K3 Lattice

A central fact about K3 surfaces is that their intersection form has a fixed structure. This will play an important role in all that we do from here on. Using some of the results from Section 2.2, we provide a quick proof of this fact. The precise statement reads:

**Proposition 4.2.1.** *If $X$ is a K3 surface, there exists an isometry*

$$H^2(X,\mathbb{Z}) \cong 3H \oplus -2E_8.$$

*Proof.* By definition, we have $b_1(X) = 0$ and $K_X$ is trivial. Then $c_1(X) = -c_1(K_X) = 0$. Noether's formula (Theorem 3.1.1) then reads

$$\chi(\mathcal{O}_X) = 2 = \frac{K_X \cdot K_X + \chi(X)}{12},$$

that is, $\chi(X) = 24$. Since $b_1(X) = b_3(X) = 0$, we find $b_2(X) = 22$. The Hirzebruch signature theorem (Theorem 2.3.1) gives

$$\sigma(X) = \frac{1}{3}(K_X \cdot K_X - 2\chi(X)) = -\frac{2}{3} \times 24 = -16.$$

In particular, $b^+(X) = 3$ and $b^-(X) = 19$, and so the intersection form of $X$ is indefinite. Moreover, $H_1(X,\mathbb{Z})$ has no torsion (which follows immediately by the fact that the holomorphic Euler characteristic is multiplicative under unramified coverings), and thus $H^2(X,\mathbb{Z})$ is torsion-free. By Proposition 2.1.6, we have $w_2(X) = c_1(X) \bmod 2 = 0$, and thus $X$ is spin. By Corollary 2.2.5, $X$ must have even intersection form. Thus, the intersection form of $X$ is an even indefinite unimodular lattice of rank $b_2(X) = 22$ and signature $\sigma(X) = -16$. By the classification of indefinite unimodular lattices (Theorem 4.1.1), there must exist an isometry $H^2(X,\mathbb{Z}) \cong 3H \oplus -2E_8$. □

**Definition 4.2.1.** Once and for all (and where it is clear from context), fix the lattice

$$L := 3H \oplus -2E_8.$$

We will often refer to $L$ as the *K3 lattice*. The real and complex extensions of $L$ will be denoted $L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ and $L_\mathbb{C} := L \otimes_\mathbb{Z} \mathbb{C}$, respectively. We then have $L_\mathbb{R} \cong H^2(X,\mathbb{R})$ and $L_\mathbb{C} \cong H^2(X,\mathbb{C})$ for an arbitrary K3 surface $X$.

In Chapter 5, we will study involutions of the K3 lattice in detail when we come to look at Enriques surfaces. For now, let us move on to investigate the classification results on K3 surfaces.

## 4.3 Kähler Cones of K3 Surfaces

Given that the notion of a Kähler cone plays an important role in the classification theory of K3 surfaces, and will briefly make its appearance in this work, it is certainly worth making a short detour to study its structure in the case of interest. We start with a general definition:

**Definition 4.3.1.** Let $X$ be a compact Kähler manifold. The *Kähler cone* of $X$ is the set of all Kähler classes associated to any Kähler structure on $X$. It is an (open) convex cone in the space $H^{1,1}(X, \mathbb{R})$.

Importantly, since $b_1(X) = 0$ for a K3 surface $X$, Theorem 3.4.3 tells us that:

**Theorem 4.3.1.** *All K3 surfaces are Kähler.*

Though the structure of the Kähler cone for K3 surfaces is well known (see [44, Corollary 3.9, Chapter VIII]), in the process of establishing Theorem 3.4.3, Buchdahl in [6] actually provides a general description of the Kähler cone for any Kähler compact complex surface. He does so by proving a real generalisation of the famous *Nakai-Moishezon criterion:*

**Theorem 4.3.2** (Corollary 6.4, Chapter IV, [44])**.** *A holomorphic line bundle $L$ on a compact complex surface $X$ is positive (i.e. $c_1(L)$ is represented by a Kähler form) if and only if $L \cdot L > 0$ and $L \cdot D > 0$ for every effective divisor $D \in \mathrm{Div}(X)$.*

The precise theorem gives necessary and sufficient conditions for a real $(1, 1)$-class to be represented by a Kähler form:

**Theorem 4.3.3** (Corollary 15, [6])**.** *Let $X$ be a compact Kähler surface. Then a real $(1, 1)$-class $x \in H^{1,1}(X, \mathbb{R})$ is represented by a Kähler form if and only if $x \cdot x > 0$, $x \cdot [\omega] > 0$ for some arbitrary Kähler form $\omega$, and $x \cdot [D] > 0$ for every effective divisor $D \in \mathrm{Div}(X)$.*

Note that the effective divisor $D \in \mathrm{Div}(X)$ in Theorem 4.3.3 can be assumed to be irreducible. Moreover, an application of Proposition 5 of [6] tells us that the condition need only be assumed to hold for those effective irreducible $D \in \mathrm{Div}(X)$ with $D \cdot D < 0$.

Let us frame all of this in slightly different, but equivalent terms. By considering the restriction of the intersection form to the subspace $H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$, we can determine the following, which is a simple consequence of the Hirzebruch signature theorem and Noether's formula:

**Theorem 4.3.4.** *Let $X$ be a compact Kähler surface. Then the restriction of its intersection form to the subspace $H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$ has signature $(1, h^{1,1}(X)-1)$.*

In particular, if $h^{1,1}(X) \geq 2$ (as in the case of K3 surfaces, for which $h^{1,1}(X) = 20$), then the subspace $\{x \in H^{1,1}(X,\mathbb{R}) \,|\, x \cdot x > 0\}$ consists of two disjoint cones. Given that the Kähler cone is connected and a subcone of the above space, it must belong to exactly one of its components, which is usually called the *positive cone* and denoted by $\mathscr{C}_X$. Observe that $x \in \mathscr{C}_X$ if and only if $x \cdot [\omega] > 0$ for some Kähler class, as the sign of the intersection $x \cdot y$ depends continuously on $x$ and $y$.

Now, for an arbitrary complex manifold $X$, if $\mathrm{NS}(X)$ denotes as usual its Néron-Severi group (cf. Definition 2.4.7), we will say that an element $d \in \mathrm{NS}(X)$ is *effective* if $d = c_1(\mathcal{O}_X(D))$ for some effective divisor $D \in \mathrm{Div}(X)$. Moreover, $d$ is called *irreducible* if $D$ is irreducible. Naturally, $d \in \mathrm{NS}(X)$ is called *positive* if it comes from a positive line bundle. If $X$ is assumed to be a K3 surface, and $x \in H^{1,1}(X,\mathbb{R})$ defines a Kähler class, then we have $x \cdot d > 0$ for every irreducible effective $d \in \mathrm{NS}(X)$ with $d \cdot d < 0$. However, applying the genus formula (cf. Section 3.1) to $D$ satisfying $d = c_1(\mathcal{O}_X(D))$, we see it necessarily satisfies $D \cdot D = -2$ and $g(D) = 0$, so Proposition 3.2.3 tells us $D$ is a $(-2)$-curve. Thus, if we temporarily define

$$\Delta_{\mathrm{Eff}} := \{d \in \mathrm{NS}(X) \,|\, d \cdot d = -2 \text{ and } d \text{ is effective}\} \subseteq H^{1,1}(X,\mathbb{R}),$$

then we deduce:

**Proposition 4.3.1.** *For a K3 surface $X$, the Kähler cone of $X$ coincides exactly with the space $\{x \in \mathscr{C}_X \,|\, \forall d \in \Delta_{\mathrm{Eff}} : x \cdot d > 0\} \subset H^{1,1}(X,\mathbb{R})$.*

**Remark 4.3.1.** The above result is exactly Corollary 3.9 of [44, Chapter VIII], which is easily proved using some basic algebraic analysis. Nonetheless, the above elaboration was far more general in nature, which is why we opted for it.

Having pinned down the structure of the Kähler cone of a K3 surface, we conclude by mentioning one important property of isometries between the lattices of K3 surfaces for the purposes of later discussions. For the (simple) proof, one can refer to [44, Proposition 3.11, Chapter VIII]:

**Proposition 4.3.2.** *Let $X_1$ and $X_2$ be any two K3 surfaces, and $\phi : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ be any isometry whose complex-linear extension preserves the Hodge decompositions of $X_1$ and $X_2$ in degree two. Then, $\phi$ maps the Kähler cone of $X_1$ to that of $X_2$ if and only if it maps one element of the Kähler cone of $X_1$ into the Kähler cone of $X_2$.*

Finally, we offer a concrete example of an isometry satisfying the hypothesis of Proposition 4.3.2. For our purposes, we only need to be aware of the following:

**Definition 4.3.2.** Let $X$ be a K3 surface and $d \in H^2(X, \mathbb{Z})$ with $d^2 = -2$ arbitrary. The *Picard-Lefschetz reflection* associated to $d$ is the map

$$L_d : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}),\ x \mapsto x + (x \cdot d)d.$$

It is elementary to see that if $d \in \mathrm{NS}(X)$ (i.e. the image of $d$ in $H^2(X, \mathbb{R})$ is represented by a closed, real $(1,1)$-form, cf. Theorem 2.4.4), then $L_d$ preserves the Hodge decomposition of $X$, and in particular, in such a case, its real extension restricts to a map $L_d \in \mathrm{Aut}(H^{1,1}(X, \mathbb{R}))$.

## 4.4 On the Period Domains of K3 Surfaces

Having determined the structure of the intersection form and Kähler cone of K3 surfaces, we can finally make some headway on classifying them. To this end, we will construct two important spaces called the *(classical) period domain* and *refined period domain* (of K3 surfaces) whose points will correspond to K3 surfaces (resp. K3 surfaces equipped with Kähler metrics) and will be used to determine the isomorphism classes of K3 surfaces. We will make these statements precise by stating (a version of) a theorem known as *Torelli's theorem*. Much of this section is based on the exposition found in [7]. We begin our discussion by stating some important facts concerning K3 surfaces.

Firstly, recall that all K3 surfaces are Kähler (which was also one of the conjectures of Weil). Thus (cf. Remark 2.6.1), since K3 surfaces are instances of Calabi-Yau manifolds (and along with tori, they represent the entire collection of Calabi-Yau manifolds of dimension two), Yau's resolution of the Calabi conjecture (cf. Theorem 2.6.3) then tells us that if $\omega \in \Omega^2_{\mathbb{R}}(X)$ is a Kähler form for some K3 surface $X$, then there exists a unique Ricci-flat Kähler-Einstein metric $g$ whose Kähler form $\omega_g$ satisfies $[\omega_g] = [\omega] \in H^2(X, \mathbb{R})$. We will make use of this (highly non-trivial) result throughout, and as we proceed, we will see that K3 surfaces possess many interesting metric-related properties.

Another extremely well-known and important fact about K3 surfaces is that they form one connected family of complex surfaces, in other words, they can all be deformed into one another (cf. Definition 2.3.9). Recall this was another one of the four conjectures of Weil concerning K3 surfaces:

**Theorem 4.4.1.** *All K3 surfaces are deformation equivalent, and hence are all diffeomorphic.*

Given that we constructed an example of a simply-connected K3 surface in Example 3.4.1, we have the following non-trivial fact as a consequence of the above:

**Corollary 4.4.1.** *All K3 surfaces are simply connected.*

**Remark 4.4.1.** Theorem 4.4.1 was originally proven by Kodaira in his original work on the classification of compact complex surfaces (see [30, Theorem 13]); he showed that every K3 surface can be deformed into a K3 surface which is biholomorphic to a non-singular quartic in $\mathbb{P}^3$. This result historically represents an important step towards obtaining a classification of K3 surfaces. Moreover, note that it is not at all obvious if one can prove Corollary 4.4.1 without relying on this fact, that is, just by using the definition of a K3 surface as in Definition 3.4.1.

Now, given that a key defining feature of a K3 surface $X$ is the fact that its canonical bundle is trivial, observe that if $\kappa \in H^0(X, \Omega^2_X)$ is the nowhere-vanishing holomorphic 2-form trivialising $K_X$, then $\kappa$ is closed, as it is in particular a $(2,0)$-form annihilated by the Dolbeault operator $\overline{\partial}$. Moreover, it immediately follows that $\kappa \wedge \kappa = 0$, and the real 4-form $\kappa \wedge \overline{\kappa} \in \Omega^4_{\mathbb{R}}(X)$ must also be nowhere vanishing. With these elementary observations in mind, we can state the following key result which underpins the study of K3 surfaces:

**Proposition 4.4.1.** *Let $X_0$ be an oriented four-manifold and $\kappa \in \Omega^2_{\mathbb{C}}(X_0)$ be a closed complex 2-form such that $\kappa \wedge \kappa = 0$ and $\kappa \wedge \overline{\kappa} > 0$, then there exists a unique integrable complex structure $I$ on $X_0$ such that $\kappa$ is a holomorphic with respect to $I$. In particular, $\kappa$ is nowhere vanishing and hence trivialises the canonical bundle of $(X_0, I)$.*

*Proof.* First note that an almost complex structure on $X_0$ is equivalent to a decomposition of the complexified tangent bundle $T_{\mathbb{C}}X_0 = T^{1,0}X_0 \oplus T^{0,1}X_0$, where $T^{0,1}X_0 = \overline{T^{1,0}X_0}$. We have that $\kappa$ defines a $\Omega^0_{\mathbb{C}}(X_0)$-linear map

$$\kappa : \Gamma(T_{\mathbb{C}}X_0) \to \Gamma(T^*_{\mathbb{C}}X_0),\ X \mapsto (Y \mapsto \kappa(X, Y)).$$

In particular, it defines a bundle map $\kappa : T_{\mathbb{C}}X_0 \to T^*_{\mathbb{C}}X_0$. Then we set

$$T^{0,1}X_0 := \ker(\kappa) \subseteq T_{\mathbb{C}}X_0.$$

Firstly, we immediately see that the kernel of $\kappa$ is non-trivial. If it were trivial, then $\kappa$ would define an injective map, and hence isomorphism, between the complexified tangent and cotangent space at every point. This is exactly the same as saying $\kappa$ is non-degenerate, which would imply $\kappa \wedge \kappa \neq 0$. Moreover, the kernel of $\kappa$ is not all of $T_{\mathbb{C}}X_0$, for otherwise $\kappa \wedge \overline{\kappa}$ is not nowhere vanishing. Since $\kappa$ is skew-symmetric, its rank at each point is even, and hence constant (equal to 2, being greater than 0 and less than 4), and so $T^{0,1}X_0$ is a proper, non-trivial subbundle of $T_{\mathbb{C}}X_0$ of rank 2. Furthermore, observe that we trivially have the relation

$$\overline{\ker(\kappa)} = \ker(\overline{\kappa}),$$

and so if we set $T^{1,0}X_0 := \ker(\overline{\kappa})$, we have $T^{1,0}X_0 = \overline{T^{0,1}X_0}$. Then, on each fibre, the intersection $T^{1,0}X_0 \cap T^{0,1}X_0$ is trivial, since $\kappa \wedge \overline{\kappa}$ is nowhere vanishing. This gives us our desired decomposition, and thus $\kappa$ defines an almost complex structure $I$.

We just have to check integrability. Given $X, Y \in \ker(\kappa)$, we want to show that $[X, Y] \in \ker(\kappa)$. So, let $Z \in \Gamma(T_{\mathbb{C}}X_0)$ be arbitrary. Using the standard formula for the exterior derivative, we obtain

$$\begin{aligned} 0 = d\kappa(X, Y, Z) =& X\kappa(Y, Z) - Y\kappa(X, Z) + Z\kappa(X, Y) \\ &- \kappa([X, Y], Z) + \kappa([X, Z], Y) - \kappa([Y, Z], X), \end{aligned}$$

where the first equality follows by the fact that $\kappa$ is closed, and the rest of the terms vanish since $X, Y \in \ker(\kappa)$. Thus, $[X, Y] \in \ker(\kappa)$, and hence the almost complex structure $I$ is integrable (Theorem 2.3.2). It immediately follows that $\kappa$ is holomorphic. Uniqueness follows by the fact that any holomorphic 2-form must vanish on $T^{0,1}X_0$. The fact that $\kappa$ is nowhere vanishing follows immediately by the fact that $\kappa \wedge \overline{\kappa}$ is nowhere vanishing. □

In particular, if $X_0$ is the oriented four-manifold underlying a K3 surface $X$, then any form $\kappa \in \Omega^2_{\mathbb{C}}(X_0)$ satisfying the hypotheses of Proposition 4.4.1 defines a complex structure $I$ such that $(X_0, I)$ is another (potentially non-biholomorphic) K3 surface. Given this, we make the following important definition:

**Definition 4.4.1.** Let $X_0$ be the smooth oriented four-manifold underlying some arbitrary K3 surface, fixed once and for all, and define the space

$$\mathcal{K}(X_0) := \{\kappa \in \Omega^2_{\mathbb{C}}(X_0) \,|\, d\kappa = 0, \kappa \wedge \kappa = 0 \text{ and } \kappa \wedge \overline{\kappa} > 0\}.$$

For brevity, the K3 surface defined by a form $\kappa \in \mathcal{K}(X_0)$ will often be denoted by the pair $(X_0, \kappa)$.

**Remark 4.4.2.** Note that any non-zero multiple of a form $\kappa \in \mathcal{K}(X_0)$ defines exactly the same complex structure for $X_0$, and in particular, two forms $\kappa_1, \kappa_2 \in \mathcal{K}(X_0)$ define biholomorphic complex structures for $X_0$ if and only if there exists a diffeomorphism

$f : X_0 \to X_0$ such that $f^*\kappa_2 = \lambda\kappa_1$ for some constant $\lambda \in \mathbb{C}^*$. Given that all K3 surfaces are diffeomorphic (Theorem 4.4.1), the space $\mathcal{K}(X_0)$ properly represents the space of *all* K3 surfaces, that is to say, every K3 surface can be obtained by making a choice of complex structure defined by $\kappa \in \mathcal{K}(X_0)$. Explicitly, if $\kappa \in H^0(X, \Omega_X^2)$ trivialises the canonical bundle of some K3 surface $X$, then choosing any diffeomorphism $f : X_0 \to X$ implies $f^*\kappa \in \mathcal{K}(X_0)$, and $f$ defines a biholomorphism between the K3 surfaces $X$ and $(X_0, f^*\kappa)$.

Given that the complex structure defined by a form $\kappa \in \mathcal{K}(X_0)$ is unchanged by scaling, we are led to make the following definition:

**Definition 4.4.2.** The *(classical) period domain (of K3 surfaces)* is the space:

$$\Omega := \{[c] \in \mathbb{P}(L_{\mathbb{C}}) \mid c \cdot c = 0, c \cdot \bar{c} > 0\},$$

where we recall $L := 3H \oplus -2E_8$. It is a 20-dimensional open subset of a quadric. Fixing an isometry $H^2(X_0, \mathbb{Z}) \cong L$, we define the natural map:

$$\Pi : \mathcal{K}(X_0) \to \Omega,\ \kappa \mapsto [\kappa].$$

This map is known as the *period map (of K3 surfaces)*, and the image of a form $\kappa \in \mathcal{K}(X_0)$ is the *period point* of the corresponding K3 surface $(X_0, \kappa)$. Note, from time to time we may drop the brackets from $[c] \in \Omega$ and just write $c \in \Omega$.

An important fact about the period domain is that it is connected:

**Proposition 4.4.2.** *If* $\mathrm{Gr}_2^+(L_{\mathbb{R}})$ *denotes the Grassmannian of positive definite oriented two-dimensional subspaces of* $L_{\mathbb{R}}$*, then the natural map*

$$\Omega \to \mathrm{Gr}_2^+(L_{\mathbb{R}}),\ [c] \mapsto \langle \mathrm{Re}(c), \mathrm{Im}(c) \rangle$$

*is a diffeomorphism. In particular,* $\Omega$ *is connected.*

*Proof.* Note that the oriented two-plane $\langle \mathrm{Re}(c), \mathrm{Im}(c) \rangle$ is unchanged if we scale $c$ by $\lambda \in \mathbb{C}^*$. The map is obviously injective. Given an arbitrary 2-plane $H \in \mathrm{Gr}_2^+(L_{\mathbb{R}})$ which is spanned by two orthonormal vectors $v_1, v_2 \in L_{\mathbb{R}}$, the vector $c = v_1 + iv_2 \in L_{\mathbb{C}}$ defines a point $[c] \in \Omega$. Following Huybrechts (see [26, Chapter 6]) and setting $b^{\pm} := b^{\pm}(L)$, there exists a natural diffeomorphism

$$\mathrm{Gr}_2^+(L_{\mathbb{R}}) \cong O(b^+, b^-)/\mathrm{SO}(2) \times O(b^+ - 2, b^-).$$

Given that $b^+ = 3 > 2$, using the associated homotopy sequence of the fibre bundle $\mathrm{SO}(2) \times O(1, 19) \to O(3, 19) \to O(3, 19)/\mathrm{SO}(2) \times O(1, 19)$, we see that $\pi_0(\mathrm{Gr}_2^+(L_{\mathbb{R}})) = 0$, i.e. $\mathrm{Gr}_2^+(L_{\mathbb{R}})$, and hence $\Omega$, is connected. $\square$

**Remark 4.4.3.** By considering the homotopy sequence further, one actually sees that $\Omega$ is simply connected. Also note that if we take $b^+ = 2$, then $O(b^+ - 2, b^-) \cong O(b^-)$ and $\pi_0(O(b^-)) \cong \mathbb{Z}_2$. We then get $\pi_0(\mathrm{Gr}_2^+(L_\mathbb{R})) \cong \mathbb{Z}_2$, i.e. $\mathrm{Gr}_2^+(L_\mathbb{R})$ has two connected components in the case $b^+ = 2$.

It turns out that the period map defined above is surjective, and this answers another one of the questions addressed by Weil on the structure of K3 surfaces:

**Theorem 4.4.2** (Surjectivity of the Period Map)**.** *Fix an isometry $H^2(X_0, \mathbb{Z}) \cong L$. Then, the period map $\Pi : \mathcal{K}(X_0) \to \Omega$ is surjective. In particular, every element of $\Omega$ arises as the period point of some K3 surface.*

This non-trivial result was first proved by Todorov in [50], but has subsequently been reproved many times by various authors. We now wish to address Weil's final conjecture, that is, that K3 surfaces can be biholomorphically distinguished by the period points they define. In order to elaborate on this, we will need to introduce more structure, which leads us to consider *Kähler* K3 surfaces. In this vein, we make the following definition:

**Definition 4.4.3.** Given a form $\kappa \in \mathcal{K}(X_0)$ which defines a complex structure for a K3 surface, set

$$\Omega_\kappa(X_0) := \{\omega \in \Omega^2_\mathbb{R}(X_0) \,|\, d\omega = 0 \text{ and } \omega \text{ is Kähler with respect to } \kappa\},$$

and by the latter condition, we mean, of course, that $\omega$ should be Kähler with respect to the complex structure defined by $\kappa$. Given that all K3 surfaces are Kähler, for any $\kappa \in \mathcal{K}(X_0)$, we must have $\Omega_\kappa(X_0) \neq \emptyset$. Moreover, let the space $\mathcal{R}(X_0)$ denote the collection of pairs $(\omega, \kappa)$, where $\kappa \in \mathcal{K}(X_0)$ and $\omega \in \Omega_\kappa(X_0)$.

**Remark 4.4.4.** Exactly as in Remark 4.4.2, the space $\Omega_\kappa(X_0)$ represents the space of all possible Kähler metrics for the given K3 surface $(X_0, \kappa)$, and $\mathcal{R}(X_0)$ is the space of all possible Kähler structures with compatible complex structures for $X_0$.

The final conjecture of Weil to be mentioned will follow from a stronger result concerning the structure of Kähler K3 surfaces. To this end, we will define an associated period domain. In order to determine its structure, we begin by first making some elementary observations, the contents of which are summarised in the following proposition and its proof:

**Proposition 4.4.3.** *Suppose $X$ is a Kähler K3 surface defined by the pair $(\omega, \kappa) \in \mathcal{R}(X_0)$. Then the classes $[\omega], [\mathrm{Re}(\kappa)], [\mathrm{Im}(\kappa)] \in H^2(X, \mathbb{R})$ define an (oriented) basis for a maximal positive definite subspace of $H^2(X, \mathbb{R})$.*

*Proof.* The proof is a simple calculation. We have $[\omega] \cdot [\omega] = \int_X \omega \wedge \omega = 2 \int_X \mathrm{vol} > 0$. Since the class $[\kappa]$ spans $H^{2,0}(X)$ and $[\omega] \in H^{1,1}(X, \mathbb{R})$, we have $[\omega] \cdot [\kappa] = 0$. Writing $[\kappa] = a + ib$, we have by definition

$$[\kappa] \cdot [\kappa] = (a \cdot a - b \cdot b) + 2i(a \cdot b) = 0,$$
$$[\kappa] \cdot [\overline{\kappa}] = a \cdot a + b \cdot b > 0.$$

In particular, $a \cdot a = b \cdot b > 0$ and $a \cdot b = 0$. The result follows from the fact that $b^+(X) = 3$. □

Let us now compare the orientations of the subspaces defined by points in $\mathcal{R}(X_0)$. It turns out that they are all 'positively' oriented with respect to one another. In order to make this precise, we consider the following spaces, which will play a central role for the remainder of this thesis:

**Definition 4.4.4.** The Grassmannian consisting of maximal positive definite subspaces of $L_{\mathbb{R}}$ will be denoted $\mathrm{Gr}(L_{\mathbb{R}})$. The Grassmannian of maximal *oriented* positive definite subspaces of $L_{\mathbb{R}}$ will be denoted $\mathrm{Gr}^0(L_{\mathbb{R}})$. In particular, forgetting the orientation gives us a canonical double covering $\mathrm{Gr}^0(L_{\mathbb{R}}) \to \mathrm{Gr}(L_{\mathbb{R}})$.

**Remark 4.4.5.** If we are given a Kähler K3 surface $X$ defined by $(\omega, \kappa) \in \mathcal{R}(X_0)$, then under an isometry $L \cong H^2(X, \mathbb{Z})$, Proposition 4.4.3 yields $\langle [\omega], [\mathrm{Re}(\kappa)], [\mathrm{Im}(\kappa)] \rangle \in \mathrm{Gr}^0(L_{\mathbb{R}})$, where $\langle \cdot \rangle$ denotes the subspace spanned by the specified elements. We may sometimes also denote this subspace by $S(\omega, \kappa) := \langle [\omega], [\mathrm{Re}(\kappa)], [\mathrm{Im}(\kappa)] \rangle \in \mathrm{Gr}^0(L_{\mathbb{R}})$.

It is not too difficult to obtain some important topological information about these Grassmannians:

**Proposition 4.4.4.** *The space* $\mathrm{Gr}(L_{\mathbb{R}})$ *is connected and simply connected. In particular, the double covering* $\mathrm{Gr}^0(L_{\mathbb{R}}) \to \mathrm{Gr}(L_{\mathbb{R}})$ *is trivial, and so* $\mathrm{Gr}^0(L_{\mathbb{R}})$ *consists of two copies of* $\mathrm{Gr}(L_{\mathbb{R}})$.

*Proof.* By choosing an orthogonal basis, we obtain an identification $L_{\mathbb{R}} \cong \mathbb{R}^{3,19}$. Then, $\mathrm{Gr}(L_{\mathbb{R}})$ is simply the space of orthogonal bases of $\mathbb{R}^{3,19}$, related by actions which preserve the positive definite part, that is, we can think of $\mathrm{Gr}(L_{\mathbb{R}})$ as the quotient

$$\mathrm{Gr}(L_{\mathbb{R}}) \cong \frac{O(3,19)}{O(3) \times O(19)}.$$

Then, we have that $O(3) \times O(19) \to O(3,19) \to \mathrm{Gr}(L_{\mathbb{R}})$ defines a fibre bundle. Moreover, it is well known that $O(p) \times O(q)$ is a maximal compact subgroup of $O(p,q)$, and thus they are homotopy equivalent. The result follows by considering the resulting long exact sequence on homotopy groups. □

It turns out that all the subspaces defined by points $(\omega, \kappa) \in \mathcal{R}(X_0)$ always define the same orientation for the positive-definite 'part' of $H^2(X_0, \mathbb{R})$. This important result follows by the work of Donaldson in [12]:

**Theorem 4.4.3.** *Suppose $(\omega_1, \kappa_1), (\omega_2, \kappa_2) \in \mathcal{R}(X_0)$ are arbitrary. Then, the subspaces $S(\omega_1, \kappa_1)$ and $S(\omega_2, \kappa_2)$ lie in the same connected component of $\mathrm{Gr}^0(L_\mathbb{R})$ under any isometry $H^2(X_0, \mathbb{Z}) \cong L$. This component will be denoted by $\mathrm{Gr}^0_+(L_\mathbb{R})$, the other by $\mathrm{Gr}^0_-(L_\mathbb{R})$.*

**Definition 4.4.5.** A subspace $H \in \mathrm{Gr}^0(L_\mathbb{R})$ is called *positively oriented* if $H \in \mathrm{Gr}^0_+(L_\mathbb{R})$.

At this point, one may wonder if there are certain positively oriented subspaces which cannot correspond to Kähler structures $(\omega, \kappa) \in \mathcal{R}(X_0)$. As we saw in Section 4.3, the presence of $(-2)$-curves on a K3 surface is one of the things which can obstruct a real $(1,1)$-class from being able to be represented by a Kähler form. With this in mind, we make the following definition:

**Definition 4.4.6.** The space of *roots* of the K3 lattice $L = 3H \oplus -2E_8$ is given by

$$\Delta := \{d \in L \mid d \cdot d = -2\}.$$

Moreover, associate to any lattice element $d \in L$, is the following codimension 3 submanifold of the Grassmannian $\mathrm{Gr}^0_+(L_\mathbb{R})$:

$$A_d := \{H \in \mathrm{Gr}^0_+(L_\mathbb{R}) \mid H \perp d\}.$$

**Proposition 4.4.5.** *Fix an isometry $H^2(X_0, \mathbb{Z}) \cong L$. Then, for any $(\omega, \kappa) \in \mathcal{R}(X_0)$, we have*

$$\forall d \in \Delta : S(\omega, \kappa) \notin A_d.$$

*Proof.* Let $X$ be the Kähler K3 surface corresponding to $(\omega, \kappa) \in \mathcal{R}(X_0)$. For the sake of contradiction, suppose $S(\omega, \kappa) \in A_d$ for some $d \in \Delta$. This is the same as saying $d \cdot [\omega] = 0$ and $d \cdot [\kappa] = 0$. Given that complex conjugation yields the equality $H^{2,0}(X) = \overline{H^{0,2}(X)}$, the latter condition implies that the image of $d$ in $H^2(X_0, \mathbb{R})$ is a $(1,1)$-class. Thus, by the Lefschetz theorem on $(1,1)$-classes (Theorem 2.4.4), we have that there exists a holomorphic line bundle $L \in \mathrm{Pic}(X)$ such that $c_1(L) = d$. Since $\chi(\mathcal{O}_X) = 2$, $L \cdot L = -2$, the Riemann-Roch formula for $L$ and Serre Duality give us

$$h^0(L) + h^0(L^*) \geq \frac{L \cdot L + L \cdot K_X}{2} + \chi(\mathcal{O}_X) = 1.$$

Hence, either $L$ or $L^*$ has a global holomorphic section, and so either $d$ or $-d$ corresponds to the Poincaré dual of (the fundamental class of) an effective divisor $D$ (recall Remark 2.7.1 and Theorem 2.7.2). Since $\omega$ is a Kähler form, we have that for every effective divisor $D$, $[\omega] \cdot [D] > 0$. But by assumption, we have $d \cdot [\omega] = \int_D \omega = 0$, which is absurd. ☐

With these results in mind, we can finally define our next period domain:

**Definition 4.4.7.** The *refined period domain (of K3 surfaces)* is given by the following subset of $L_{\mathbb{R}} \times \Omega$:

$$\mathcal{P} := \{(a, [c]) \in L_{\mathbb{R}} \times \Omega \,|\, a^2 > 0, a \cdot c = 0, S(a, c) \in \mathrm{Gr}_+^0(L_{\mathbb{R}}) \text{ and } \forall d \in \Delta : S(a, c) \notin A_d\}.$$

Under an isometry $H^2(X_0, \mathbb{Z}) \cong L$, the *refined period map* is given by

$$\hat{\Pi} : \mathcal{R}(X_0) \to \mathcal{P},\ (\omega, \kappa) \mapsto ([\omega], [\kappa]),$$

which is well defined by Theorem 4.4.3 and Proposition 4.4.5

It turns out that the amount of conditions imposed on a pair $(a, [c]) \in \mathcal{P}$ is exactly right: in directly analogy to Theorem 4.4.2, we have the following central result concerning the refined period map:

**Theorem 4.4.4** (Surjectivity of the Refined Period Map)**.** *Given a fixed isometry $H^2(X_0, \mathbb{Z}) \cong L$, the refined period map $\hat{\Pi} : \mathcal{R}(X_0) \to \mathcal{P}$ is surjective. In particular, every element of $\mathcal{P}$ arises as the period point of a K3 surface equipped with some Kähler metric.*

In [7], Buchdahl establishes this result by first proving the surjectivity of the period map. In Section 2.4 and having proved Theorem 4.4.4, he then goes on to show that two Kähler(-Einstein) K3 surfaces are uniquely distinguished by the period points they define in the refined period domain:

**Theorem 4.4.5** (Torelli's Theorem)**.** *Fix an isometry $H^2(X_0, \mathbb{Z}) \cong L$. If $\hat{\Pi}(\omega_1, \kappa_1) = \hat{\Pi}(\omega_2, \kappa_2)$ for arbitrary $(\omega_1, \kappa_1), (\omega_2, \kappa_2) \in \mathcal{R}(X_0)$, then there exists a unique diffeomorphism $f : X_0 \to X_0$ homotopic to the identity (through homeomorphisms) such that $f^*\kappa_2 = \kappa_1$. Moreover, if $\omega_1$ and $\omega_2$ are Kähler-Einstein, then $f^*\omega_2 = \omega_1$.*

**Remark 4.4.6.** There are many versions of the above theorem, which have been proved by various authors under various assumptions. The name 'Torelli's theorem' comes from its analogy to Torelli's theorem for algebraic curves which more or less states that two compact Riemann surfaces can be distinguished by their Jacobian (or, equivalently, Albanese) varieties.

An important corollary of Theorem 4.4.5 is the so-called *weak Torelli theorem*, which gives an answer to the final question posed by Weil regarding whether or not the period point of a K3 surface in $\Omega$ determines its biholomorphism class:

**Corollary 4.4.2** (Weak Torelli Theorem)**.** *Fix an isometry $H^2(X_0, \mathbb{Z}) \cong L$. If $\Pi(\kappa_1) = \Pi(\kappa_2)$ for arbitrary $\kappa_1, \kappa_2 \in \mathcal{K}(X_0)$, then there exists a diffeomorphism $f : X_0 \to X_0$ such that $f^*\kappa_2 = \kappa_1$. In particular, two K3 surfaces are biholomorphic if and only if they define the same period point in $\Omega$.*

**Remark 4.4.7.** Although Buchdahl does not explicitly mention this result in his paper [7], it indeed follows by Torelli's theorem as stated in Theorem 4.4.5. Note though that the implication is certainly not immediate. The problem is that the Kähler cones of the complex structures defined by $\kappa_1, \kappa_2 \in \mathcal{K}(X_0)$ with $[\kappa_1] = [\kappa_2]$ may not be the same. However, after pulling back $\kappa_1$ or $\kappa_2$ by a finite number of diffeomorphisms which induce Picard-Lefschetz reflections, we can make the Kähler cones agree. The result then follows by applying Theorem 4.4.5.

In view of the surjectivity of the period map (Theorem 4.4.2) and the weak Torelli theorem (Corollary 4.4.2), understanding the space of K3 surfaces requires understanding the induced action of the diffeomorphism group of a K3 surface on the lattice $L_{\mathbb{C}}$. To this end, let us conclude this section by mentioning some of the work that has been done in this area. We begin with the following definition:

**Definition 4.4.8.** Given a smooth manifold $X$, denote its group of diffeomorphisms by $\mathrm{Diff}(X)$. If $X$ is also compact, oriented and of dimension 4, let $\Gamma \leq \mathrm{Aut}(H^2(X, \mathbb{Z}))$ denote the subgroup of automorphisms which are realised by diffeomorphisms, that is, $\Gamma$ is the image of the group homomorphism

$$\mathrm{Diff}(X) \to \mathrm{Aut}(H^2(X, \mathbb{Z})),\ f \mapsto f^*.$$

The kernel of the above map will be denoted by $\mathrm{Diff}_T(X)$, and so $\Gamma \cong \mathrm{Diff}(X)/\mathrm{Diff}_T(X)$.

We then wish to determine the structure of the subgroup $\Gamma$ in the case of a K3 surface, and in particular, to ask the question if the above homomorphism is surjective, i.e. if $\Gamma = \mathrm{Aut}(H^2(X_0, \mathbb{Z}))$. The answer is well known in the topological setting under the assumption of simple-connectedness. The following was proved by Freedman in his fundamental work on topological four-manifolds in [16]:

**Theorem 4.4.6.** *Let $X$ be a simply-connected compact topological four-manifold. Denote its group of homeomorphisms by* $\mathrm{Homeo}(X)$*. Then, the natural map* $\mathrm{Homeo}(X) \to \mathrm{Aut}(H^2(X, \mathbb{Z}))$, $f \mapsto f^*$ *is surjective.*

Also note that the work of Quinn in [45] gives the following useful result for determining the structure of the kernel:

**Theorem 4.4.7.** *If $X$ is a smooth closed oriented simply-connected four-manifold, then the group* $\mathrm{Diff}_T(X)$ *is given by the set of diffeomorphisms which are continuously isotopic to the identity through homeomorphisms, that is, a diffeomorphism $f \in \mathrm{Diff}(X)$ acts trivially on the lattice $H^2(X, \mathbb{Z})$ if and only if $f$ is continuously isotopic to the identity.*

**Remark 4.4.8.** In actual fact, if $\pi_0(\mathrm{Homeo}(X))$ denotes the space of isotopy classes of homeomorphisms of a topological manifold $X$, then Freedman showed that in the case of a closed simply-connected four-manifold $X$, the natural map $\pi_0(\mathrm{Homeo}(X)) \to \mathrm{Aut}(H^2(X, \mathbb{Z}))$, $[f] \mapsto f^*$ is surjective. Quinn in [45] proved that this map is injective, so we have an isomorphism $\pi_0(\mathrm{Homeo}(X)) \cong \mathrm{Aut}(H^2(X, \mathbb{Z}))$.

Now, for the smooth setting, Matumoto in [36] showed that all the isometries of a certain index 2 subgroup $\Gamma' \leq \mathrm{Aut}(L)$ can be realised by diffeomorphisms of a K3 surface. The group of these special isometries is defined as follows:

**Definition 4.4.9.** The subgroup $\Gamma' \leq \mathrm{Aut}(L)$ is given by the set of all isometries $\phi \in \mathrm{Aut}(L)$ which *preserve orientation on maximal positive definite subspaces*, that is, the space of isometries $\phi$ whose natural action on $\mathrm{Gr}^0(L)$ preserves its two components $\mathrm{Gr}^0_\pm(L_\mathbb{R})$.

**Remark 4.4.9.** A restatement of Matumoto's result is then $\Gamma' \subseteq \Gamma$. Furthermore, note that an isometry $\phi \in \Gamma'$ if and only if for any orthonormal vectors $v_1, v_2, v_3 \in L_\mathbb{R}$, we have $\det A > 0$, where $A$ is the $3 \times 3$ matrix whose entries are given by $A_{ij} = v_i \cdot \phi(v_j)$ for $i, j = 1, 2, 3$. As Buchdahl explains in [7, p. 196], in order to show that $\phi \in \mathrm{Aut}(L)$ belongs to $\Gamma'$, it suffices to show that it preserves orientation on *a* maximal positive definite subspace, as any two such subspaces are canonically isomorphic.

A highly non-trivial fact is that the other inclusion $\Gamma \subseteq \Gamma'$ is true, that is, that any isometry of the lattice $L$ which is induced by a diffeomorphism of a K3 surface must preserve orientation on maximal positive definite subspaces. This was first proved by Donaldson in [12] using gauge theory. The results of Donaldson and Matumoto can then be summarised in the following theorem:

**Theorem 4.4.8.** *If $X$ is a K3 surface, and an isometry $H^2(X, \mathbb{Z}) \cong L$ is given, the two subgroups $\Gamma'$ and $\Gamma$ of* $\mathrm{Aut}(L)$ *coincide. In particular, the natural map* $\mathrm{Diff}(X) \to \mathrm{Aut}(H^2(X, \mathbb{Z}))$, *$f \mapsto f^*$ is not surjective.*

**Remark 4.4.10.** Given what we said in the paragraph following Remark 4.4.7, the moduli space of K3 surfaces can properly be said to be the quotient space $\Omega/\Gamma$. It is well-known that this quotient space is not Hausdorff. Furthermore, the refined period domain $\mathcal{P}$

can legitimately be called the moduli space of Kähler-Einstein structures for K3 surfaces. Indeed, if we set for the moment

$$\mathcal{R}_{\mathrm{KE}}(X_0) := \{(\omega, \kappa) \in \mathcal{R}(X_0) \,|\, \omega \text{ is Kähler-Einstein with respect to } \kappa\},$$

and consider the natural diagonal action of $\mathrm{Diff}_T(X_0)$ on $\mathcal{R}(K_0)$, then the surjectivity of the refined period map (Theorem 4.4.4) and Torelli's theorem (Theorem 4.4.5) tell us there is a bijective correspondence between the quotient space $\mathcal{R}_{\mathrm{KE}}(X_0)/\mathrm{Diff}_T(X_0)$ and $\mathcal{P}$.

Having covered all of the main results concerning the classification of K3 surfaces, we can move on to investigate some of the metric-related properties of K3 surfaces which will be useful in the work we do in Chapters 5 and 6.

## 4.5 The Teichmüller Space for K3 Surfaces

In this section, we introduce one last important moduli space associated to K3 surfaces: the moduli space of Einstein metrics on a K3 surface. As we will see, it is intimately related to the refined period domain we defined in Section 4.4, and will play an important role in the final two chapters of this work. Given the technical nature of this area of geometry, we will restrict our attention only to those results which directly pertain to obtaining a deeper understanding of K3 surfaces, the other aspects of the theory will unfortunately have to be left out. Most of the exposition contained in this section can be found in the detailed text by Besse on Einstein manifolds [3]. As usual, $X_0$ will always denote the smooth oriented four-manifold underlying some fixed K3 surface.

As we have seen, Calabi-Yau manifolds provide a vast array of examples of Einstein manifolds (given that each such manifold is Kähler-Einstein, thanks to Yau [cf. Theorem 2.6.3]). In particular, K3 surfaces yield an important class of compact Einstein four-manifolds. As we alluded in Section 2.6, the general question of the existence of Einstein manifolds in dimension 4 is more delicate than in the lower-dimensional setting (those being completely determined). Nigel Hitchin in [21] made significant progress in this direction by showing that there is a topological obstruction for the existence of an Einstein metric on a closed four-manifold:

**Theorem 4.5.1** (Hitchin)**.** *Let $X$ be a closed oriented four-dimensional Einstein manifold, then*

$$|\sigma(X)| \leq \frac{2}{3}\chi(X).$$

*Furthermore, if equality occurs, then the Ricci curvature vanishes and $\pm X$ (i.e. $X$ with its normal or reversed orientation) is either flat or its universal cover is a K3 surface. If the universal covering of $X$ is a K3 surface, then $X$ is a K3 surface, an Enriques surface or the quotient of an Enriques surface by a free anti-holomorphic involution.*

**Remark 4.5.1.** The inequality above is usually referred to as the *Hitchin-Thorpe inequality.* Note that a theorem of Bieberbach implies that $X$ is finitely covered by a torus in the case it is flat. Moreover, if $I$ is an integrable almost complex structure for a smooth manifold $X$, then so too is $-I$. The complex manifold $(X, -I)$ is usually referred to as the *conjugate* (often denoted $\overline{X}$), and a map $f : X \to Y$ between complex manifolds is called *anti-holomorphic* if it defines a holomorphic map $f : X \to \overline{Y}$. The quotient of an Enriques surface by a free anti-holomorphic involution is usually referred to as an *Enriques-Einstein-Hitchin manifold.* We will touch on these manifolds in Chapter 6 of this work, after first having thoroughly investigated Enriques surfaces in Chapter 5.

Moreover, Hitchin also proved the following result regarding the correspondence between the different kinds of metrics on a K3 surface:

**Theorem 4.5.2** (Besse [3], Theorem 6.40)**.** *If $X$ is a smooth four-manifold which is homotopy equivalent to a K3 surface, and $g$ is any Riemannian metric on $X$, then the following properties are equivalent:*

*(i) The scalar curvature of $g$ is non-negative;*

*(ii) The metric $g$ is hyperkähler;*

*(iii) The Ricci curvature of $g$ is zero.*

As an immediate corollary of the above and Theorem 4.5.1, we can conclude:

**Corollary 4.5.1.** *Every Einstein metric on a K3 surface is hyperkähler.*

Let us now come to constructing the moduli space of Einstein metrics. It is beyond the scope of this thesis to fully delve into the intricacies underpinning its construction, however, given that it will play an important role for the work we do later on, it cannot be neglected. Obtaining a proper understanding of these concepts relies on being familiar with the theory of infinite-dimensional manifolds (for example, for the basics one can consult [34]).

If $E \to X$ is a smooth vector bundle over a compact base, then its space of smooth sections $\Gamma(X, E)$ can be equipped with the so-called *$C^\infty$-topology*; this is a natural choice of topology for the space of smooth maps $C^\infty(M,N)$ between two smooth manifolds $M$ and $N$, whose definition usually requires the introduction of jet bundles (which we bypass for the purposes of our discussion here). In fact, the topological vector space $\Gamma(X,E)$ can be equipped with a family of *seminorms* (i.e. norms which are not necessarily positive-definite), which then give it the structure of a *Fréchet space* (a special generalisation of the notion of a Banach space). Given that a Riemannian metric $g$ is just a section of the bundle $S^2T^*X$ which induces an inner product on each tangent space, we make the following important definition:

**Definition 4.5.1.** Given a smooth closed oriented manifold $X$, the associated space of Riemannian metrics will be denoted $\mathrm{Riem}(X) \subseteq \Gamma(X, S^2T^*X)$, which we equip with the subspace topology.

In particular, given that we can add Riemannian metrics together and also scale them by positive real numbers, the space $\mathrm{Riem}(X)$ is convex. We can actually say a bit more. The following is standard:

**Proposition 4.5.1.** *Let $X$ be a closed oriented smooth manifold. Then, with respect to the $C^\infty$-topology, the space* $\mathrm{Riem}(X)$ *defines an open positive convex cone in* $\Gamma(X, S^2T^*X)$. *In particular,* $\mathrm{Riem}(X)$ *is contractible.*

Given that we are in a Riemannian context, we also introduce the following notation:

**Definition 4.5.2.** The group of isometries of a Riemannian manifold $(X,g)$ will be denoted

$$\mathrm{Isom}(X,g) := I_g := \{f \in \mathrm{Diff}(X) \mid f^*g = g\} \subseteq \mathrm{Diff}(X).$$

**Remark 4.5.2.** We can rephrase this in slightly different terms. Observe that $\mathrm{Diff}(X)$ (also equipped with the $C^\infty$-topology) acts naturally on $\mathrm{Riem}(X)$ by pulling back:

$$\mathrm{Diff}(X) \times \mathrm{Riem}(X) \to \mathrm{Riem}(X), (f,g) \mapsto f^*g.$$

Given that $(f \circ g)^* = g^* \circ f^*$, this defines a *right* action. Then, the isometry group of $g \in \mathrm{Riem}(X)$ is precisely the associated isotropy subgroup of this action.

In the light of the above remark, one might ask what is known about the structure of the quotient space $\mathrm{Riem}(X)/\mathrm{Diff}(X)$. Ebin in his central article [13] analyses this action, and proves a fundamental result known in the theory as the *slice theorem*. In the course of his work, he actually shows that $\mathrm{Riem}(X)$ can be equipped with a natural Riemannian metric. We omit the details of that, but given that the slice theorem will appear in what we do later, we state it here for completeness. Roughly put, it states that a certain kind of subset exists which is locally transverse to the orbits of the action of $\mathrm{Diff}(X)$ through any point $g \in \mathrm{Riem}(X)$:

**Theorem 4.5.3** (Ebin's Slice Theorem)**.** *Let $X$ be a closed orientable smooth manifold. Then, for any $g \in \mathrm{Riem}(X)$, there exists a submanifold $S \subseteq \mathrm{Riem}(X)$ containing $g$ such that:*

*(i) The action of any isometry of $g$ on $S$ leaves it fixed, that is, for any $f \in I_g$, we have $f^*S = S$.*

*(ii) If $f \in \mathrm{Diff}(X)$ and $f^*S \cap S \neq \emptyset$, then $f \in I_g$.*

*(iii) There is a local cross-section $\chi : U \to \mathrm{Diff}(X)$ defined on a neighbourhood $U \subseteq \mathrm{Diff}(X)/I_g$ of the identity coset such that the map defined by*

$$F : U \times S \to \mathrm{Riem}(X),\ (u, s) \mapsto \chi(u)^* s$$

*is a diffeomorphism onto a neighbourhood of $g$.*

Let us now specialise to the case of Einstein manifolds. We begin with some notation:

**Definition 4.5.3.** A Riemannian metric $g$ on a smooth manifold $X$ is said to have *unit volume* if $\int_X \mathrm{vol}_g = 1$. The space of unit-volume Einstein metrics on a closed oriented smooth manifold $X$ is denoted $\mathrm{Ein}(X) \subseteq \mathrm{Riem}(X)$.

**Remark 4.5.3.** Given that the Einstein equation $\mathrm{Ric} = \lambda g$ is a non-linear differential equation depending on the first and second derivatives of $g$, the space $\mathrm{Ein}(X)$ is a closed subset of $\mathrm{Riem}(X)$ (in the $C^\infty$-topology). Given $f \in \mathrm{Diff}(X)$ and $g \in \mathrm{Ein}(X)$, we first of all have the basic fact that $\mathrm{vol}_{f^*g} = f^*\mathrm{vol}_g$, and so $f^*g$ has unit volume. Secondly, if $\nabla$ denotes the Levi-Cevita connection for $g$, then $f^*\nabla$ will be the Levi-Civita connection of $f^*g$, which implies that $f^*g$ is again Einstein. Thus, $f^*g \in \mathrm{Ein}(X)$, and so in particular, the action of $\mathrm{Diff}(X)$ on $\mathrm{Riem}(X)$ restricts to a right action on $\mathrm{Ein}(X)$.

In light of the slice theorem, one can show that the space of Einstein metrics on a K3 surface is particularly special:

**Theorem 4.5.4.** *The space* $\mathrm{Ein}(X_0)$ *defines a smooth (Fréchet) submanifold of* $\mathrm{Riem}(X_0)$.

*Proof.* Let $g \in \mathrm{Ein}(X_0)$ be arbitrary. Theorem 4.5.3 yields a slice $S \subseteq \mathrm{Riem}(X_0)$ through $g$ which is invariant under $I_g$. If we can show that $\mathrm{Ein}(X_0) \cap S$ is a submanifold of $S$, then $\mathrm{Ein}(X_0)$ must be a submanifold of $\mathrm{Riem}(X_0)$, since every point of $\mathrm{Riem}(X_0)$ has a slice through it. Since $g$ is hyperkähler, and in particular Ricci-flat Kähler-Einstein, the result follows by [3, Theorem 12.88]. □

Ideally, we would then wish to construct the space of equivalent unit-volume, Einstein metrics $\mathrm{Ein}(X)/\mathrm{Diff}(X)$ on a K3 surface $X$. However, it is not at all obvious what the structure of this quotient is. It turns out that the best choice is to instead consider:

**Definition 4.5.4.** Define the quotient space

$$\mathcal{T}(X_0) := \mathrm{Ein}(X_0)/\mathrm{Diff}_T(X_0),$$

where, as usual, $\mathrm{Diff}_T(X_0) = \{f \in \mathrm{Diff}(X_0) \mid f^* = \mathrm{id}_{H^2(X_0,\mathbb{Z})}\}$. We will call $\mathcal{T}(X_0)$ the *Teichmüller space* of the smooth oriented four-manifold $X_0$.

Using the following simple result, we will be able to show that the Teichmüller space of a K3 surface is smooth. It is a direct consequence of Torelli's theorem from the previous section:

**Proposition 4.5.2.** *If* $g \in \mathrm{Ein}(X_0)$ *is arbitrary, the homomorphism*

$$\mathrm{Isom}(X_0, g) \to \mathrm{Aut}(H^2(X_0, \mathbb{Z})),\ f \mapsto f^*$$

*is injective, that is to say, the intersection* $\mathrm{Isom}(X_0, g) \cap \mathrm{Diff}_T(X_0)$ *is trivial.*

*Proof.* Suppose $f \in \mathrm{Diff}(X_0)$ is an arbitrary isometry of a metric $g \in \mathrm{Ein}(X_0)$ which acts trivially on $H^2(X_0, \mathbb{Z})$. By Corollary 4.5.1, the metric $g$ is hyperkähler. If the complex structures $I, J, K$ denote the associated hyperkähler triple, then the form $\kappa := \omega_J + i\omega_K$ is holomorphic with respect to $I$ and nowhere vanishing by Proposition 2.6.2. Then $(\omega_I, \kappa) \in \mathcal{R}(X_0)$, and given that $f^* = \mathrm{id}$ on $H^2(X_0, \mathbb{R}) \cong \mathcal{H}^2_\Delta(X_0, g)$, we must have $f^*\kappa = \kappa$ and $f^*\omega_I = \omega_I$ (by the uniqueness of harmonic representatives). The uniqueness part of Torelli's theorem (Theorem 4.4.5) implies $f = \mathrm{id}_{X_0}$. □

**Corollary 4.5.2.** *The group* $\mathrm{Diff}_T(X_0)$ *acts freely and properly discontinuously on* $\mathrm{Ein}(X_0)$. *In particular, the Teichmüller space* $\mathcal{T}(X_0) = \mathrm{Ein}(X_0)/\mathrm{Diff}_T(X_0)$ *is a smooth manifold.*

*Proof.* Freeness follows immediately by the fact that $I_g \cap \mathrm{Diff}_T(X_0) = \{\mathrm{id}_{X_0}\}$. Now, let $g \in \mathrm{Ein}(X_0) \subseteq \mathrm{Riem}(X_0)$ be arbitrary. The Ebin slice theorem (Theorem 4.5.3) tells us that there is a submanifold $S$ through $g$ such that if $f^*S \cap S \neq \emptyset$, then $f \in I_g$. Hence, if $f^*S \cap S \neq \emptyset$ for $f \in \mathrm{Diff}_T(X_0)$, then Proposition 4.5.2 implies $f = \mathrm{id}_{X_0}$, and so $\mathrm{Diff}_T(X_0)$ acts properly discontinuously on $\mathrm{Ein}(X_0)$. □

By constructing a final 'period map', we will be able to extract more information about the space $\mathcal{T}(X_0)$. We begin with the following definition:

**Definition 4.5.5.** If $(X, g)$ is an oriented Riemannian four-manifold, denote its space of self-dual harmonic 2-forms by $\mathcal{H}_g^+(X) \subseteq \mathcal{H}_\Delta^2(X, g)$.

**Remark 4.5.4.** If we denote the space of anti-self-dual harmonic 2-forms by $\mathcal{H}_g^-(X)$, note that we have $\dim \mathcal{H}_g^\pm(X) = b^\pm(X)$ in the case $X$ is compact. In particular, if $X$ is a K3 surface, then the space $\mathcal{H}_g^+(X)$ is three dimensional. Proposition 2.6.1 shows that every Kähler form $\omega \in \Omega_\mathbb{R}^2(X)$ is harmonic, and even self-dual with respect to the Kähler metric $g$ which defines it. Now, if $g$ is any Einstein metric for $X_0$, then it is hyperkähler. If $I, J, K$ denotes the quarternionic triple of Kähler structures for $g$, then each of the associated Kähler forms $\omega_I, \omega_J$ and $\omega_K$ is self-dual, harmonic with respect to $g$, and they are all linearly independent. In particular, under the isomorphism $H^2(X_0, \mathbb{R}) \cong \mathcal{H}_\Delta^2(X_0, g)$, we have $\mathcal{H}_g^+(X_0) = \langle [\omega_I], [\omega_J], [\omega_K] \rangle$. Moreover, applying Proposition 2.6.2, we see $(\omega_I, \omega_J + i\omega_K) \in \mathcal{R}(X_0)$, and hence $\mathcal{H}_g^+(X_0) \in \mathrm{Gr}_+^0(L_\mathbb{R})$ under an isometry $L \cong H^2(X_0, \mathbb{Z})$ by Theorem 4.4.3.

In light of the above remark, we can then define our final 'period map', however, this time on the space of Einstein structures:

**Definition 4.5.6.** Fix an isometry $H^2(X_0, \mathbb{Z}) \cong L$. Then the period map on Einstein metrics is defined as

$$\tau : \mathrm{Ein}(X_0) \to \mathrm{Gr}_+^0(L_\mathbb{R}),\ g \mapsto \mathcal{H}_g^+(X_0),$$

where the orientation of the subspace $\mathcal{H}_g^+(X_0)$ is determined by the basis $\{[\omega_I], [\omega_J], [\omega_K]\}$, where $I, J, K$ form a hyperkähler triple of complex structures for $g$. Since the hyperkähler metric $g$ determines a two-sphere of complex structures, the orientation on $\mathcal{H}_g^+(X_0)$ only depends on the metric $g$ and not any particular choice of hyperkähler triple, since $S^2$ is connected.

The following simple proposition will tell us that $\tau$ descends to a map on $\mathcal{T}(X_0)$:

**Proposition 4.5.3.** *Let $(X, g)$ be a compact oriented Riemannian manifold and $f$ be an immersion. Let $*$ and $*'$ denote the Hodge $*$-operators induced by $g$ and $f^*g$, respectively. Then, $*' \circ f^* = f^* \circ *$. If, furthermore, $f \in \mathrm{Diff}(X)$, then pulling back gives us a surjective map*

$$f^* : \mathcal{H}_\Delta^k(X, g) \to \mathcal{H}_\Delta^k(X, f^*g)$$

*for all $0 \leq k \leq \dim X$. In particular, if $f \in \mathrm{Isom}(X, g)$, then the pullback of a $g$-harmonic form is again $g$-harmonic.*

*Proof.* This follows simply by the fact that the Hodge star operator is defined by the equation $\alpha \wedge *\beta = (\alpha, \beta)_g \,\mathrm{vol}_g$, as well as $\mathrm{vol}_{f^*g} = f^*\mathrm{vol}_g$ and $f^*(\alpha, \beta)_g = (f^*\alpha, f^*\beta)_{f^*g}$. □

**Corollary 4.5.3.** *Suppose an isometry $L \cong H^2(X_0, \mathbb{Z})$ is given, then for any $g \in \mathrm{Ein}(X_0)$ and $f \in \mathrm{Diff}_T(X_0)$, we have $\mathcal{H}^+_g(X_0) = \mathcal{H}^+_{f^*g}(X_0)$, and so the period map $\tau$ descends to the Teichmüller space:*

$$\tau : \mathcal{T}(X_0) \to \mathrm{Gr}^0_+(L_{\mathbb{R}}),\ [g] \mapsto \mathcal{H}^+_g(X_0).$$

*Proof.* Under an isomorphism $\mathcal{H}^2_\Delta(X_0, f^*g) \cong H^2(X_0, \mathbb{R}) \cong \mathcal{H}^2_\Delta(X_0, g)$, a diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ acts trivially on harmonic 2-forms. By Proposition 4.5.3, we see that a closed 2-form $\alpha$ is $d^*$-closed if and only if it is $d^{*'}$-closed, and so $\mathcal{H}^2_\Delta(X_0, g) = \mathcal{H}^2_\Delta(X_0, f^*g)$. The result follows. □

Before stating our main result for this section, let us also make the following simple observation:

**Proposition 4.5.4.** *Suppose an isometry $L \cong H^2(X_0, \mathbb{Z})$ is fixed. Then, under the isomorphism $\Gamma \cong \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$, the period map $\tau : \mathcal{T}(X_0) \to \mathrm{Gr}^+_0(L_{\mathbb{R}})$ is $\Gamma$-equivariant.*

*Proof.* Suppose $[g] \in \mathcal{T}(X_0)$ and $[f] \in \Gamma$ for $f \in \mathrm{Diff}(X_0)$ are arbitrary, and denote the induced action of $[f]$ on $[g]$ by $[f] \cdot [g] := [f^*g]$. By definition, the action of $[f] \in \Gamma$ on a subspace $H \in \mathrm{Gr}^0_+(L_{\mathbb{R}})$ is given by $[f] \cdot H := f^*H$. Then, by definition and Proposition 4.5.3:

$$\tau([f] \cdot [g]) = \tau([f^*g]) = \mathcal{H}^+_{f^*g}(X_0) = f^*\mathcal{H}^+_g(X_0) = [f] \cdot \mathcal{H}^+_g(X_0) = [f] \cdot \tau([g]).$$

□

The main result of this section is a Torelli-like theorem for Einstein metrics on K3 surfaces:

**Theorem 4.5.5** (Torelli Theorem for Einstein Metrics)**.** *If an isometry $H^2(X_0, \mathbb{Z}) \cong L$ is fixed, then the period map $\tau : \mathcal{T}(X_0) \to \mathrm{Gr}^0_+(L_{\mathbb{R}})$ maps the Teichmüller space bijectively onto the set*

$$\mathcal{E} := \mathrm{Gr}^0_+(L_{\mathbb{R}}) \setminus \bigcup_{d \in \Delta} A_d.$$

*Proof.* For the injectivity of this map, suppose $\tau([g_1]) = \tau([g_2])$, i.e. $\mathcal{H}^+_{g_1}(X_0) = \mathcal{H}^+_{g_2}(X_0)$ as oriented subspaces of $H^2(X_0, \mathbb{R}) \cong L_{\mathbb{R}}$ under their respective identifications $\mathcal{H}^2_\Delta(X_0, g_1) \cong H^2(X_0, \mathbb{R}) \cong \mathcal{H}^2_\Delta(X_0, g_2)$. Then if $\mathcal{H}^+_{g_1}(X_0) = \langle e_1, e_2, e_3 \rangle$ for some orthonormal vectors $e_1, e_2, e_3 \in L_{\mathbb{R}}$, let $\kappa_1$ be the $g_1$-harmonic 2-form corresponding to $e_2 + ie_3$. Then $\kappa_1 \cdot \kappa_1 = 0$ and $\kappa_1 \cdot \overline{\kappa_1} > 0$, and so $\kappa_1$ defines a complex structure for $X_0$, and we can assume that

$e_1$ corresponds to a Kähler-Einstein metric $\omega_1$, defined with respect to $\kappa_1$. We can similarly define $\kappa_2$ and $\omega_2$ for $\mathcal{H}^+_{g_2}(X_0)$. Thus, for $i = 1, 2$ we have $(\omega_i, \kappa_i) \in \mathcal{R}(X_0)$, and by assumption $\hat{\Pi}(\omega_1, \kappa_1) = \hat{\Pi}(\omega_2, \kappa_2)$. It follows by Torelli's theorem (Theorem 4.4.5) that $f^*\kappa_2 = \kappa_1$ and $f^*\omega_2 = \omega_1$ for some unique $f \in \mathrm{Diff}_T(X_0)$. Given that any two of $I, \omega$ and $g$ determine the third for a Kähler structure on a complex manifold, we have $f^*g_2 = g_1$, and so $[g_1] = [g_2]$.

The fact that the image of $\tau$ does not meet any $A_d$ for $d \in \Delta$ follows using exactly the same reasoning as Proposition 4.4.5. Let us show that it surjects onto the subset $\mathcal{E}$. Let $H \in \mathrm{Gr}^+_0(L_\mathbb{R})$ be arbitrary such that $H \notin A_d$ for every $d \in \Delta$. If $H = \langle e_1, e_2, e_3\rangle$ for some orthonormal vectors $e_i \in L_\mathbb{R}$, then by assumption for every $d \in \Delta$, we have $e_i \cdot d \neq 0$ for some $i = 1, 2, 3$, and so $(e_1, [c]) \in \mathcal{P}$, where $c := e_2 + ie_3$. By the surjectivity of the refined period map (Theorem 4.4.4), there exists $(\omega, \kappa) \in \mathcal{R}(X_0)$ such that $[\omega] = e_1$ and $[\kappa] = c$. By Yau's theorem, there exists a unique Kähler-Einstein metric $g$ whose associated Kähler form $\omega_g$ satisfies $[\omega_g] = [\omega]$. Then, the classes $[\omega_g], [\mathrm{Re}(\kappa)], [\mathrm{Im}(\kappa)] \in H^2(X_0, \mathbb{R}) \cong L_\mathbb{R}$ form a basis for $\mathcal{H}^+_g(X_0)$, i.e. $H = \mathcal{H}^+_g(X_0) = \tau([g])$.

□

**Remark 4.5.5.** Given that the Teichmüller space is a smooth manifold (Corollary 4.5.2) and $\mathrm{Gr}^0_+(L_\mathbb{R})$ is an open subset of the Grassmanian of oriented positive definite 3-planes (i.e. a smooth manifold of dimension $3(22-3) = 57$), one can actually show that the period map $\tau : \mathcal{T}(X_0) \to \mathrm{Gr}^0_+(L_\mathbb{R})$ is smooth. One can do this by using general results on families of elliptic differential operators (for each $g \in \mathrm{Riem}(X_0)$, the space $\mathcal{H}^+_g(X_0)$ is the kernel of the restriction of the Laplacian $\Delta_g : \bigwedge^2 T^*X_0 \to \bigwedge^2 T^*X_0$ to the bundle of self-dual $g$-harmonic 2-forms $\bigwedge^2_+ T^*X_0$). We omit the proof. In fact, it turns out that $\tau$ actually maps $\mathcal{T}(X_0)$ *diffeomorphically* onto $\mathcal{E}$, and since it is a smooth bijection, one can show this by proving that the derivative of $\tau$ is an isomorphism, i.e. a local diffeomorphism, which can be done with the help of some Hodge theory and the theory of deformations.

In analogy with $\Omega$ and $\mathcal{P}$, we could call $\mathcal{E}$ the period domain of Einstein metrics on a K3 surface. The above result implies:

**Proposition 4.5.5.** *The space $\mathcal{E}$ is connected and simply connected. As a result, the Teichmüller space $\mathcal{T}(X_0)$ is connected and simply connected.*

*Proof.* This follows by the fact that $\mathrm{Gr}^0_+(L_\mathbb{R}) \cong \mathrm{Gr}(L_\mathbb{R})$ is connected and simply connected (Proposition 4.4.4), the simple observation that each $A_d$ has codimension 3 in $\mathrm{Gr}^0_+(L_\mathbb{R})$ and by applying a transversality argument. See [1, p. 4] for more details. Theorem 4.5.5 then gives us a diffeomorphism $\mathcal{T}(X_0) \cong \mathcal{E}$. □

Let us conclude by noting that there is a natural relation between the refined period domain and period domain for Einstein metrics on a K3 surface:

**Proposition 4.5.6.** *The map* $\pi : \mathcal{P} \to \mathcal{E}$, $(a, [c]) \mapsto S(a, c)$ *defines a fibre bundle whose fibres deformation retract onto* $\mathrm{SO}(3)$. *In particular, the refined period domain for K3 surfaces* $\mathcal{P}$ *is connected.*

*Proof.* The fact that $\pi$ defines a fibre bundle is clear. For the second statement, observe that we can always rescale $(a, [c]) \in \mathcal{P}$ so that $a^2 = 1$ and $c \cdot \bar{c} = 2$, and this leaves the subspace $S(a, c)$ unchanged. Given that $\mathrm{SO}(3) \cong \mathbb{RP}^3$ and $\mathcal{E}$ is connected, it immediately follows that $\mathcal{P}$ is connected. □

Having covered all of the main results concerning K3 surfaces that will be important to us moving forward, we can continue our investigations into compact complex surfaces by looking at the classification of Enriques surfaces in more detail.

# Chapter 5

# On Enriques Surfaces

In this chapter, we will extend the concepts and main results of Chapter 4 to the setting of Enriques surfaces. In order to do this, we will make use of the close relationship that exists between Enriques and K3 surfaces. As mentioned in the Introduction, the main results of this chapter were originally investigated in full by Japanese mathematician Eiji Horikawa in 1978 (see *On the periods of Enriques surfaces, I-II:* [23] and [24]). Yukihiko Namikawa in [40] in 1985 offered a more algebraic and less geometric approach in order to arrive at the same results. And as we have mentioned, Barth *et al.* in Chapter VIII of [43] and [44] also state and prove the analogous results concerning the classification of Enriques surfaces; they follow the approach taken by Horikawa, though not exactly. The principal purpose of this part of the thesis is to prove the analogous results using different methods, which directly extend the investigations of Chapter 4. To begin, we will set up all of the necessary notation and introduce the statement of the main results. From there, we will devote the remaining sections to proving these, and along the way we will make remarks to elucidate some of the differences between the approach taken here and in the literature.

Exactly as in Chapter 4 (cf. Definition 4.4.1), the space $X_0$ will always denote the underlying smooth oriented four-manifold of some fixed K3 surface.

## 5.1 Setup and Background

Recall (cf. Definition 3.4.2) that an Enriques surface $E$ is a compact complex surface with $b_1(E) = 0$ and $K_E^{\otimes 2} \cong \mathcal{O}_E$, but whose canonical bundle $K_E$ is non-trivial. As we saw (see Theorem 3.4.1), Enriques surfaces are in one-to-one correspondence with K3 surfaces equipped with free holomorphic $\mathbb{Z}_2$-actions, and this is what we will now use to obtain a detailed study of Enriques surfaces. Given that all K3 surfaces are simply connected (Corollary 4.4.1), elementary covering space theory immediately yields the following fact about Enriques surfaces:

**Proposition 5.1.1.** *If $E$ is an Enriques surface, then $\pi_1(E) \cong \mathbb{Z}_2$.*

In a similar way as before (cf. Theorem 4.2.1), we can easily determine the structure of the intersection form of Enriques surfaces:

**Proposition 5.1.2.** *If $E$ is an Enriques surface and $T_2 \cong \mathbb{Z}_2$ denotes the torsion subgroup of $H^2(E, \mathbb{Z})$, then we have the isomorphism of lattices*

$$H^2(E, \mathbb{Z})/T_2 \cong L_E := H \oplus -E_8.$$

*We will refer to $L_E$ as the* ***Enriques lattice****.*

*Proof.* We can simply use the numerical invariants of an Enriques surface to show this. We have $p_g(E) = h^{2,0}(E) = 0$ since $K_E \ncong \mathcal{O}_E$. A standard result (see [43, Theorem 2.7]) says $b_1(E) = h^{1,0}(E) + h^{0,1}(E) = 0$, and hence $h^{1,0}(E) = h^{0,1}(E) = 0$. Since $K_E$ is 2-torsion, we have $K_E \cdot K_E = 0$, and so Noether's formula reads

$$\chi(\mathcal{O}_E) = 1 - q(E) + p_g(E) = 1 = \frac{K_E \cdot K_E + \chi(E)}{12},$$

that is, $\chi(E) = 12$. Then, $b_2(E) = 10$. Applying the Hirzebruch signature theorem, we have

$$\sigma(E) = \frac{1}{3}(K_E \cdot K_E - 2\chi(E)) = -8.$$

Hence, $b^+(E) = 1$ and $b^-(E) = 9$, and in particular the intersection form of $E$ is indefinite. Let us show it is even. Serre duality yields $h^{0,2}(E) = h^{2,0}(E) = 0$, and given that $h^{0,1}(E) = 0$, the exponential sequence implies $c_1 : \mathrm{Pic}(E) \to H^2(E, \mathbb{Z})$ is an isomorphism (Theorem 2.4.3). Thus, given any $d \in H^2(E, \mathbb{Z})$, there exists $L \in \mathrm{Pic}(E)$ such that $c_1(L) = d$. Applying the Riemann-Roch formula to $L$, we obtain

$$\chi(E, L) = \frac{L \cdot L - L \cdot K_E}{2} + 1.$$

However, again using the fact that $K_E$ is torsion, we have $L \cdot K_E = 0$, and thus

$$L \cdot L = d \cdot d = 2(\chi(E, L) - 1) \in 2\mathbb{Z},$$

and so the intersection form of $E$ is even. The classification of indefinite unimodular lattices (Theorem 4.1.1) yields the result. □

The details of the above proof allow us to conclude a little bit more about Enriques surfaces:

**Proposition 5.1.3.** *Every Enriques surface $E$ is algebraic.*

*Proof.* Given that $b^+(E) = 1$, there exists $d \in H^2(E, \mathbb{Z})$ such that $d^2 > 0$. Since $c_1 : \mathrm{Pic}(E) \to H^2(E, \mathbb{Z})$ is an isomorphism, there exists a line bundle $L$ with $c_1(L) = d$. Then, $L \cdot L = d^2 > 0$. The result follows by [44, Theorem 6.2]. □

Having covered some of the basic properties of Enriques surfaces, let us show how we can extend the considerations of Section 4.4 to Enriques surfaces. Given that $p_g(E) = 0$ for an Enriques surface $E$, we cannot construct the (classical) period domain and map in the way we did for K3 surfaces (cf. Definition 4.4.2). The standard way to circumvent this issue is by passing to the universal covering. Let us begin by stating an elementary, but nonetheless important observation:

**Proposition 5.1.4.** *If $\sigma : X \to X$ is a free holomorphic involution on a K3 surface, then $\sigma^*\kappa = -\kappa$, where $\kappa \in H^0(X, \Omega^2_X)$ is the holomorphic 2-form trivialising $K_X$.*

*Proof.* This is trivial. The quotient space $E := X/\langle\sigma\rangle$ is an Enriques surface. Denote the quotient map by $\pi : X \to E$. Then, since the space of holomorphic 2-forms on $X$ is 1-dimensional, we have $\sigma^*\kappa = \lambda\kappa$ for some $\lambda \in \mathbb{C}^*$. Since $\sigma$ is an involution, we must have $\lambda = \pm 1$, but the former case is excluded, for otherwise $\kappa = \pi^*\alpha$ for some non-trivial holomorphic 2-form $\alpha$ on $E$, which is impossible given that $p_g(E) = 0$. Thus, $\sigma^*\kappa = -\kappa$. □

**Remark 5.1.1.** If $E$ is an Enriques surface which is double covered by a K3 surface $X$ equipped with covering involution $\sigma \in \mathrm{Aut}(X)$, then as we saw in Remark 4.4.2, a choice of diffeomorphism $f : X_0 \to X$ yields a biholomorphism between $X$ and the K3 surface $(X_0, f^*\kappa)$, where $\kappa$ is the 2-form trivialising $K_X$. If we define $\sigma_0 := f^{-1} \circ \sigma \circ f$, then we immediately find that $\sigma_0^*(f^*\kappa) = -f^*\kappa$. Then, $\sigma_0$ defines a free holomorphic involution for the K3 surface $(X_0, f^*\kappa)$, and taking the quotient defines an Enriques surface which is biholomorphic to $E$. As a result, we will think of an Enriques surface $E$ as a triple $(X_0, \kappa, \sigma)$, where $\kappa \in \mathcal{K}(X_0)$ and $\sigma \in \mathrm{Diff}(X_0)$ is a free smooth involution satisfying $\sigma^*\kappa = -\kappa$.

Moreover, basic covering space theory tells us that two Enriques surfaces $E_1$ and $E_2$ are biholomorphic if and only if there exists a biholomorphism $f : X_1 \to X_2$ satisfying $f \circ \sigma_1 = \sigma_2 \circ f$, where $X_i$ is the universal K3 covering of $E_i$ equipped with its covering involution $\sigma_i : X_i \to X_i$ for $i = 1, 2$. In particular, if each $E_i$ corresponds to an appropriate triple $(X_0, \kappa_i, \sigma_i)$, $i = 1, 2$, then $E_1$ and $E_2$ are biholomorphic precisely when there is a diffeomorphism $f : X_0 \to X_0$ satisfying $f^*\kappa_2 = \lambda\kappa_1$ for $\lambda \in \mathbb{C}^*$ and $f \circ \sigma_1 = \sigma_2 \circ f$.

In light of the above, we make the following definition:

**Definition 5.1.1.** Given a free smooth involution $\sigma : X_0 \to X_0$, define the subset:

$$\mathcal{K}^-(X_0, \sigma) := \{\kappa \in \mathcal{K}(X_0) \,|\, \sigma^*\kappa = -\kappa\} \subseteq \mathcal{K}(X_0).$$

If $\mathcal{K}^-(X_0, \sigma) \neq \emptyset$, then $\sigma$ will be called an *Enriques involution*.

**Remark 5.1.2.** Note that the latter definition is not superfluous. Indeed, it is not clear if the space $\mathcal{K}^-(X_0, \sigma)$ is non-empty for an arbitrary choice of free smooth involution $\sigma \in \mathrm{Diff}(X_0)$. If this were true, then given any two free smooth involutions $\sigma_1, \sigma_2 \in \mathrm{Diff}(X_0)$, the quotients $E_i := (X_0, \kappa_i)/\langle\sigma_i\rangle$ with $\kappa_i \in \mathcal{K}^-(X_0, \sigma_i) \neq \emptyset$, $i = 1, 2$ would be Enriques surfaces. As we will see later, all Enriques surfaces are diffeomorphic (see Theorem 5.1.3), and hence $\sigma_1$ and $\sigma_2$ would have to be conjugate to one another by some diffeomorphism $f \in \mathrm{Diff}(X_0)$. However, to the author's knowledge, it is not known whether all free smooth $\mathbb{Z}_2$-actions on a K3 surface are diffeomorphic, that is to say, if there exist *exotic* involutions on K3 surfaces – these are free smooth involutions on K3 surfaces whose quotients are only homeomorphic to an Enriques surface, but not diffeomorphic to one (note that it is always the case that the quotient of a K3 surface by a free smooth $\mathbb{Z}_2$-action is homeomorphic to an Enriques surface. This can be shown using the classification of closed oriented topological four-manifolds with cyclic fundamental group — see [19]. In particular, any two free smooth involutions on a K3 surface are necessarily conjugate to one another by a homeomorphism).

Now, if $E$ is an Enriques surface equipped with a Kähler metric $\omega$ and covered by a K3 surface $\pi : X \to E \cong X/\langle\sigma\rangle$, then certainly $\pi^*\omega$ is a Kähler metric for $X$, since $\pi$ is in particular a holomorphic immersion. Given that $\pi \circ \sigma = \pi$, we see $\sigma$ leaves this form fixed. Conversely, if $\omega$ is a Kähler form for $X$ such that $\sigma^*\omega = \omega$, then it comes from a Kähler form on $E$. The following definition then also comes naturally:

**Definition 5.1.2.** Suppose $\sigma : X_0 \to X_0$ is an Enriques involution. Then, for any $\kappa \in \mathcal{K}^-(X_0, \sigma)$, define:

$$\Omega_\kappa^+(X_0, \sigma) := \{\omega \in \Omega_\kappa(X_0) \,|\, \sigma^*\omega = \omega\} \subseteq \Omega_\kappa(X_0).$$

Given that every Enriques surface is projective, and hence Kähler, we have $\Omega_\kappa^+(X_0, \sigma) \neq \emptyset$. Moreover, let the subset $\mathcal{R}(X_0, \sigma) \subseteq \mathcal{R}(X_0)$ denote the collection of pairs $(\omega, \kappa)$ with $\kappa \in \mathcal{K}^-(X_0, \sigma)$ and $\omega \in \Omega_\kappa^+(X_0, \sigma)$.

**Remark 5.1.3.** Hence, any point of $\mathcal{R}(X_0, \sigma)$ will define an Enriques surface equipped with a Kähler metric, and conversely, any Enriques surface equipped with a Kähler metric will define a point of $\mathcal{R}(X_0, \sigma)$ for some free smooth involution $\sigma \in \mathrm{Diff}(X_0)$. Moreover, given that $c_1(E)$ is 2-torsion in $H^2(E, \mathbb{Z})$ for an Enriques surface $E$, Yau's theorem (Theorem 2.6.3) implies there exists a unique Ricci-flat Kähler-Einstein metric for each choice of Kähler class for $E$. In particular, if $\pi : X \to E$ denotes the universal K3 covering of $E$ with covering involution $\sigma : X \to X$, and $g$ is any Kähler-Einstein metric for $E$, then $\pi^* g$ is Kähler-Einstein for $X$, and we have $\sigma^*(\pi^* g) = \pi^* g$, i.e. $\sigma \in \mathrm{Isom}(X, \pi^* g)$.

Given the above definitions, it is not clear if we will be able to restrict our period maps for K3 surfaces to Enriques surfaces, since we do not know how free holomorphic (or rather, smooth) involutions on K3 surfaces act on the K3 lattice. It turns out that this action is fixed, and this remarkable result is what makes it possible to extend our investigations into the classification of K3 surfaces to that of Enriques surfaces. Following Horikawa (see [23, p. 81]), we make the following definition:

**Definition 5.1.3.** Fix the following distinguished involution $\rho \in \mathrm{Aut}(L)$ on the K3 lattice $L = 3H \oplus -2E_8$ defined by

$$\rho : L \to L,\ (x, y, z, u, v) \mapsto (-x, z, y, v, u).$$

One of the central results of this chapter is the following:

**Theorem 5.1.1.** *If $\sigma : X \to X$ is a free smooth involution on a K3 surface $X$, then there exists an isometry*

$$\varphi : H^2(X, \mathbb{Z}) \to L$$

*such that*

$$\varphi \circ \sigma^* = \rho \circ \varphi.$$

*In particular, any two free smooth involutions on a K3 surface induce conjugate actions on $L \cong H^2(X, \mathbb{Z})$.*

**Remark 5.1.4.** We will give a proof of this statement in the next section of this chapter, and do so by making use of the work of Nikulin on the classification of non-degenerate lattices. Namikawa in [40] does something similar, though the present exposition differs from his and was arrived at independently. Horikawa (as well as Barth *et al.*) establish this result in the holomorphic setting by making use of the fact that all Enriques surfaces are deformation equivalent, thus they only need to prove it for a particular choice of Enriques surface. By establishing Theorem 5.1.1 by purely algebraic means, we will be able to give a different proof of the deformation equivalence of Enriques surfaces.

**Definition 5.1.4.** If $\sigma \in \mathrm{Diff}(X)$ is a free smooth involution on a K3 surface $X$, then the isometry $L \cong H^2(X, \mathbb{Z})$ guaranteed by Theorem 5.1.1 will be called an *Enriques isometry of* $\sigma$.

The above theorem allows us to make the following important definition:

**Definition 5.1.5.** Denote the *restricted* period domains by

$$\Omega^- := \{[c] \in \Omega \,|\, \rho(c) = -c\} \subseteq \Omega,$$

$$\mathcal{P}^- := \{(a, [c]) \in \mathcal{P} \,|\, \rho(a) = a, \rho(c) = -c\} \subseteq \mathcal{P}.$$

If $\sigma \in \mathrm{Diff}(X_0)$ is an Enriques involution, then the period maps for K3 surfaces (cf. Definition 4.4.2 and Definition 4.4.7) restrict to maps

$$\Pi : \mathcal{K}^-(X_0, \sigma) \to \Omega^-,\ \kappa \mapsto [\kappa],\ \ \hat{\Pi} : \mathcal{R}(X_0, \sigma) \to \mathcal{P}^-,\ (\omega, \kappa) \mapsto ([\omega], [\kappa])$$

under any choice of Enriques isometry $L \cong H^2(X_0, \mathbb{Z})$ of $\sigma$.

Note that if we set for the moment $L^\pm := \{x \in L \,|\, \rho(x) = \pm x\}$, then we can write $\Omega^- = \Omega \cap \mathbb{P}(L^-_\mathbb{C})$ and $\mathcal{P}^- = \mathcal{P} \cap (L^+_\mathbb{R} \times \Omega^-)$. The following elementary proposition will give us some useful topological information about the restricted period domains:

**Proposition 5.1.5.** *We have isometries:*

$$L^+ \cong H(2) \oplus -E_8(2) = L_E(2) \ \text{ and } \ L^- \cong H \oplus H(2) \oplus -E_8(2) = H \oplus L_E(2),$$

*where for a lattice $K$, $K(m)$ denotes the lattice given by scaling the bilinear form by $m \in \mathbb{Z}$.*

*Proof.* This follows immediately by the fact that

$$L^+ = \{(0, x, x, y, y) \in L \,|\, x \in H, y \in -E_8\}$$

and

$$L^- = \{(x, -y, y, -u, u) \in L \,|\, x, y \in H, u \in -E_8\}.$$

□

**Remark 5.1.5.** Given some lattice $(L, \langle\cdot,\cdot\rangle)$, if there exists a rational number $r \in \mathbb{Q}$ such that $r\langle x, y\rangle \in \mathbb{Z}$ for every $x, y \in L$, then we can define the lattice $L(r)$ in the same way as above. In particular, if $L$ is even, then $L(1/2)$ is well defined.

**Proposition 5.1.6.** *The restricted period domain $\Omega^-$ has two components. If $[c] \in \Omega^-$ is in one component, then $[\overline{c}] \in \Omega^-$ is in the other. Hence, the refined period domain for Enriques surfaces $\mathcal{P}^-$ has two components, and the map $\mathcal{P}^- \to \mathcal{P}^-$, $(a,[c]) \mapsto (-a,[\overline{c}])$ interchanges the two components.*

*Proof.* Proposition 5.1.5 shows that $L^-$ has signature $(2,10)$, in particular $b^+(L^-) = 2$. As a result, the space $\Omega^- = \Omega \cap \mathbb{P}(L^-_{\mathbb{C}})$ has two components by the considerations of Remark 4.4.3, and the map $\Omega^- \to \Omega^-$, $[c] \mapsto [\overline{c}]$ swaps these components. Recall that $\mathcal{P}$ is connected (Proposition 4.5.6). Given that we have $\mathcal{P}^- = \mathcal{P} \cap (L^+_{\mathbb{R}} \times \Omega^-)$ and $L^+_{\mathbb{R}}$ is connected, $\mathcal{P}^-$ must have two components. Since $S(a,c) \in \mathrm{Gr}^0_+(L_{\mathbb{R}})$ if and only if $S(-a,\overline{c}) \in \mathrm{Gr}^0_+(L_{\mathbb{R}})$, the map $\mathcal{P}^- \to \mathcal{P}^-$, $(a,[c]) \mapsto (-a,[\overline{c}])$ interchanges the components of $\mathcal{P}^-$. □

Before stating the main results concerning the classification of Enriques surfaces, let us note that points belonging to certain hyperplanes in $\Omega^-$ cannot correspond to the period points of Enriques surfaces. As before (cf. Proposition 4.4.5), the obstruction are certain lattice elements. The precise statement reads:

**Proposition 5.1.7.** *Let $\sigma \in \mathrm{Diff}(X_0)$ be an Enriques involution and fix an Enriques isometry $H^2(X_0,\mathbb{Z}) \cong L$ of $\sigma$. Then, the image of the period map $\Pi : \mathcal{K}^-(X_0,\sigma) \to \Omega^-$ is contained in the subset*

$$\mathcal{D}^- := \Omega^- \setminus \bigcup_{d \in \Delta^-} H_d,$$

*where for $d \in L$, we define $H_d := \{[c] \in \Omega \,|\, c{\cdot}d = 0\}$ and $\Delta^- := \{d \in \Delta \,|\, \rho(d) = -d\} \subseteq \Delta$. In particular, no point of $H_d$ for $d \in \Delta^-$ is the period point of an Enriques surface, and the period map for Enriques surfaces restricts to a map $\Pi : \mathcal{K}^-(X_0,\sigma) \to \mathcal{D}^-$.*

*Proof.* Let $\kappa \in \mathcal{K}^-(X_0,\sigma)$ be arbitrary. We can choose a Kähler form $\omega \in \Omega^+_\kappa(X_0,\sigma)$ so that $(\omega,\kappa) \in \mathcal{R}(X_0,\sigma)$. Let us set $c = [\kappa] \in H^2(X_0,\mathbb{C}) \cong L_{\mathbb{C}}$ and $a = [\omega] \in H^2(X_0,\mathbb{R}) \cong L_{\mathbb{R}}$. Then, as we saw (cf. Proposition 4.4.5), for any $d \in \Delta$, we must have $a \cdot d \neq 0$ or $c \cdot d \neq 0$. Given that the covering transformation $\sigma : X_0 \to X_0$ has degree 1, for any $d \in \Delta^-$, we have

$$a \cdot d = \sigma^*(a) \cdot \sigma^*(d) = a \cdot (-d) = -a \cdot d,$$

that is, $a \cdot d = 0$, and hence $c \cdot d \neq 0$ for every $d \in \Delta^-$. This is exactly the same as saying $c = [\kappa] \notin H_d$ for every $d \in \Delta^-$. □

Let us conclude this section by stating the main results concerning the classification of Enriques surfaces which we will prove in this chapter. They are the direct analogue of the conjectures of Weil on K3 surfaces applied to Enriques surfaces. Using the notation introduced in this section, the precise statement reads:

**Theorem 5.1.2** (Classification Results for Enriques Surfaces)**.** *Let $\sigma : X_0 \to X_0$ be an Enriques involution and fix an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma$. Then:*

*(i) The period map $\Pi : \mathcal{K}^-(X_0, \sigma) \to \mathcal{D}^-$ is surjective. In particular, every element of $\mathcal{D}^-$ is the period point of some Enriques surface.*

*(ii) The refined period map $\hat{\Pi} : \mathcal{R}(X_0, \sigma) \to \mathcal{P}^-$ is surjective.*

*(iii) (Torelli's Theorem) If $\hat{\Pi}(\omega_1, \kappa_1) = \hat{\Pi}(\omega_2, \kappa_2)$ for arbitrary $(\omega_1, \kappa_1), (\omega_2, \kappa_2) \in \mathcal{R}(X_0, \sigma)$, then there exists a unique diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ such that $f \circ \sigma = \sigma \circ f$ and $f^*\kappa_2 = \kappa_1$. In the case where $\omega_1$ and $\omega_2$ are Kähler-Einstein, we also have $f^*\omega_2 = \omega_1$.*

*(iv) (Weak Torelli Theorem) If $\Pi(\kappa_1) = \Pi(\kappa_2)$ for arbitrary $\kappa_1, \kappa_2 \in \mathcal{K}^-(X_0, \sigma)$, then there exists a diffeomorphism $f \in \mathrm{Diff}(X_0)$ such that $f \circ \sigma = \sigma \circ f$ and $f^*\kappa_2 = \kappa_1$. In particular, two Enriques surfaces are biholomorphic if and only if they define the same period point.*

Moreover, exactly as in the case of K3 surfaces, we have:

**Theorem 5.1.3.** *All Enriques surfaces are deformation equivalent. In particular, all Enriques surfaces are diffeomorphic, and hence any two free holomorphic involutions on a K3 surface $X$ are conjugate in $\mathrm{Diff}(X)$, that is to say, all Enriques involutions are conjugate in $\mathrm{Diff}(X_0)$.*

We have already briefly elaborated on the structure of the remainder of this chapter in the Introduction (cf. Section 1.2). Nonetheless, before we move on, let us make some further remarks regarding the approach taken in proving the results listed above. Obviously, establishing Theorem 5.1.1 is of paramount importance, given that the statement of the results to be proved in Theorem 5.1.2 relies on the existence of Enriques isometries. Obtaining an independent proof of this result represents one of the central goals of this chapter. This is done in Section 5.2.3. This then shows that all Enriques involutions on a K3 surface induce conjugate actions on the K3 lattice. The next main problem to be overcome is that of lifting this fact to the level of diffeomorphisms. To this end, Proposition 4.5.2, which was an important consequence of Torelli's theorem for K3 surfaces (Theorem 4.4.5), will play a central role. Recall that it states that any two isometries $f_1, f_2 \in I_g$ of the same Einstein metric $g \in \mathrm{Ein}(X_0)$ on a K3 surface which act the same on the K3 lattice $f_1^* = f_2^* \in \mathrm{Aut}(H^2(X_0, \mathbb{Z}))$ are in fact equal $f_1 = f_2$. Now, given any two Enriques surfaces defined by $(X_0, \kappa_i, \sigma_i)$, $i = 1, 2$, we can equip each of them with Kähler-Einstein metrics $g_i \in \mathrm{Ein}(X_0)$ so that $\sigma_i \in I_{g_i}$. The problem then boils down to finding a diffeomorphism which relates $g_1$ and $g_2$. This is where the Teichmüller space of Section 4.5 comes in. By proving a technical lemma for the Teichmüller space $\mathcal{T}(X_0)$ (Lemma 5.3.1), we will be in good stead to solve this overarching conjugacy problem (cf. Lemma 5.3.2). We will subsequently be able to provide swift proofs of the most difficult parts of the classification results for Enriques surfaces, that is, the surjectivity of the refined period map for Enriques surfaces and the deformation equivalence of Enriques surfaces. The other results will follow either directly from the K3 results or as corollaries of the above.

**Remark 5.1.6.** In light of Theorem 5.1.3, it would suffice to fix some Enriques involution $\sigma_0 : X_0 \to X_0$, so that any other Enriques surface is biholomorphic to an Enriques surface defined by a form $\kappa \in \mathcal{K}^-(X_0, \sigma_0)$. Indeed, if $E$ is an arbitrary Enriques surface defined by a triple $(X_0, \kappa, \sigma)$ for some Enriques involution $\sigma \in \mathrm{Diff}(X_0)$ with $\kappa \in \mathcal{K}^-(X_0, \sigma)$, then Theorem 5.1.3 implies there exists a diffeomorphism $f \in \mathrm{Diff}(X_0)$ such that $f \circ \sigma_0 = \sigma \circ f$. We then have $f^*\kappa \in \mathcal{K}^-(X_0, \sigma_0)$, and the Enriques surface defined by $f^*\kappa$ is biholomorphic to $E$. Thus, the space $\mathcal{K}^-(X_0, \sigma_0)$ truly represents the space of *all* Enriques surfaces, in much the same way $\mathcal{K}(X_0)$ represents the space of all K3 surfaces (cf. Remark 4.4.2).

In analogy to Remark 4.4.10 and in view of the surjectivity of the period map and the weak Torelli theorem for Enriques surfaces (Theorem 5.1.2 (i) and (iv)), we could say that the moduli space of Enriques surfaces is the quotient $\mathcal{D}^-/\Gamma_\rho^-$, where

$$\Gamma_\rho^- := \{\phi|_{L^-} \in \mathrm{Aut}(L^-) \,|\, \phi \in \Gamma, \phi \circ \rho = \rho \circ \phi\} \subseteq \mathrm{Aut}(L^-),$$

and as before $L^- = \{x \in L \,|\, \rho(x) = -x\}$. Barth *et al.* define the above space for $\phi \in \mathrm{Aut}(L)$, but note that for any $\phi \in \mathrm{Aut}(L)$, either $\pm\phi \in \Gamma$. It can be shown (cf. [44, Corollary 20.4, Chapter VIII]) that $\Gamma_\rho^-$ acts properly discontinuously on $\Omega^-$, and so in particular, the space $\mathcal{D}^-/\Gamma_\rho^-$ *is* Hausdorff (we recall the moduli space of K3 surfaces $\Omega/\Gamma$ *is not* Hausdorff). Similarly, the surjectivity of the refined period map and Torelli's theorem for Enriques surfaces (Theorem 5.1.2 (ii) and (iii)) tell us that $\mathcal{P}^-$ is the moduli space of Kähler-Einstein Enriques surfaces. Indeed, the refined period map $\hat{\Pi} : \mathcal{R}(X_0, \sigma_0) \to \mathcal{P}^-$ establishes a one-to-one correspondence between $\mathcal{P}^-$ and the quotient space $\mathcal{R}_{\mathrm{KE}}(X_0, \sigma_0)/\mathrm{Diff}_T(X_0, \sigma_0)$, where

$$\mathrm{Diff}_T(X_0, \sigma_0) := \{f \in \mathrm{Diff}_T(X_0) \,|\, f \circ \sigma_0 = \sigma_0 \circ f\}.$$

**Remark 5.1.7.** Note that Horikawa and Barth *et al.* both prove the corresponding results in a different order. They first establish the deformation equivalence of Enriques surface (which is by far the most complicated part of their argumentation), then proceed to establish Torelli's theorem and conclude by proving the surjectivity of the period map for Enriques surfaces.

## 5.2 Involutions on the K3 Lattice

As mentioned, before we can get to proving the classification results as stated in Theorem 5.1.2 and Theorem 5.1.3, we need to establish the existence of Enriques isometries – a result on which the entirety of this work rests. To this end, we begin this section by investigating unimodular lattices equipped with involutions and their classification. As we shall see, it will also be important for us to prove a kind of converse to Theorem 5.1.1, namely, that any smooth involution $\sigma \in \mathrm{Diff}(X)$ on a K3 surface $X$ which acts as our distinguished involution $\rho \in \mathrm{Aut}(L)$ under a choice of isometry $H^2(X, \mathbb{Z}) \cong L$ must in fact act freely. This will allow us to easily transport the results on K3 surfaces to the case of Enriques surfaces. In this section, we primarily make use of the work of V. Nikulin in [41] and A. Edmonds in [14].

### 5.2.1 Generalities of $\mathbb{Z}_2$-actions on Unimodular Lattices

In order to prove Theorem 5.1.1, we will need to introduce some more concepts and terminology from the theory of non-degenerate lattices. The ideas contained here are not original, but given the vast amount of literature in this area, we will only introduce those results which will directly contribute to proving the existence of Enriques isometries. A large part of the work on the classification of non-degenerate lattices was obtained by Russian mathematician V. Nikulin in his highly influential paper *Integral Symmetric Bilinear Forms and some of their Applications* [41] from 1980. We will mention the results of interest that we will need here. For the most part, however, we will make reference to the corresponding statements which can be found in [27]. This subsection can be regarded as an extension of the basic considerations of Section 4.1.

Recall that a non-degenerate lattice $L$ can be regarded as a subgroup of the dual space $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The following proposition tells us what the order of the quotient space should be:

**Proposition 5.2.1.** *If $L$ is a non-degenerate lattice, then the index of $L$ in $L^*$ under the standard inclusion $L \to L^*$, $x \mapsto \langle x, \cdot \rangle$ is given by $|d(L)|$, where, as usual, $d(L)$ is the determinant of the matrix $(\langle e_i, e_j \rangle)$ for any basis $\{e_i\}$ of $L$. In particular, the quotient space $L^*/L$ is a finite abelian group.*

*Proof.* Suppose the rank of $L$ (and hence $L^*$) is $n$. If $e_1, \dots, e_n$ is a basis for $L$, then the elements $\varepsilon_1, \dots, \varepsilon_n$ in $L^*$ defined by the condition $\varepsilon_i(e_j) = \delta_{ij}$ (and extending by $\mathbb{Z}$-linearity) form a basis for $L^*$. Let us denote the standard homomorphism by $\phi : L \to L^*$, $x \mapsto \langle x, \cdot \rangle$. Then for some uniquely determined coefficients $A_{ij} \in \mathbb{Z}$ we have

$$\phi(e_i) = \sum_{j=1}^{n} A_{ij} \varepsilon_j.$$

It follows that $|\det(A)| = (L^* : \phi(L))$. This follows by considering the Smith normal form of an $n \times n$ matrix with integer entries. Letting $G$ be the matrix with coefficients $G_{ij} = \langle e_i, e_j \rangle$, we see that

$$G_{ik} = \phi(e_i)(e_k) = A_{ij}\delta_{jk} = A_{ik}.$$

Taking determinants, we see $\det(G) = d(L) = \det(A)$, which implies $(L^* : \phi(L)) = |d(L)|$ upon taking absolute values. □

**Definition 5.2.1.** If $L$ is a non-degenerate lattice, then the quotient space $\mathcal{L} := L^*/L$ will be called the *discriminant (group) of* $L$.

We will see that the discriminant of a non-degenerate lattice will play a very important role in what follows. Given that they are finite abelian groups, we make the following definition:

**Definition 5.2.2.** A *finite quadratic form* on a finite abelian group $\mathcal{L}$ is a map $q : \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$ such that $q(nx) = n^2 q(x)$ for $x \in \mathcal{L}$ and $n \in \mathbb{Z}$, and $q(x+y) - q(x) - q(y) = 2b(x, y)$, for some symmetric bilinear form $b : \mathcal{L} \times \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$. We say that a finite quadratic form $q$ is *non-degenerate* if the associated bilinear form $b$ is *non-degenerate*, which means that the homomorphism $\mathcal{L} \to \mathrm{Hom}(\mathcal{L}, \mathbb{Q}/\mathbb{Z})$, $x \mapsto b(x, \cdot)$ is an isomorphism.

Using the bilinear form on $L$, we can actually define a natural bilinear form on $\mathcal{L}$. The contents of this are contained in the following proposition and its proof:

**Proposition 5.2.2.** *If $L$ is a non-degenerate lattice, then $L^*$ can naturally be regarded as a subgroup of $L \otimes_{\mathbb{Z}} \mathbb{Q}$. Under this identification, we obtain a well-defined bilinear form on $\mathcal{L}$ defined by*

$$\langle \cdot, \cdot \rangle : \mathcal{L} \times \mathcal{L} \to \mathbb{Q}/\mathbb{Z},\ ([x], [y]) \mapsto \langle x, y \rangle + \mathbb{Z}.$$

*Moreover, if $L$ is even, then the above bilinear form descends to a well-defined quadratic form $\mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$ on $\mathcal{L}$.*

*Proof.* The natural identification is given by

$$L^* \cong \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \,|\, \forall y \in L : \langle x, y\rangle \in \mathbb{Z}\} =: S.$$

Extending the bilinear form on $L$ by $\mathbb{Q}$-linearity gives a natural map

$$\phi : S \to L^*,\ x \mapsto (\phi_x : y \mapsto \langle x, y\rangle).$$

Injectivity of $\phi$ follows immediately by non-degeneracy. If $f : L \to \mathbb{Z}$ is arbitrary, then extending by $\mathbb{Q}$-linearity gives a map $f \otimes 1 : L \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$. By non-degeneracy, there exists $x \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\langle x, -\rangle = f \otimes 1$. Restricting to $L$, we see $\langle x, -\rangle|_L = f$, and hence $x \in S$, i.e. $\phi_x = f$, and so $\phi$ is surjective. The fact that the induced bilinear form on $\mathcal{L}$ is well defined follows immediately. For the final statement, observe that if $x \in L^* \subseteq L \otimes_{\mathbb{Z}} \mathbb{Q}$ and $y \in L$, then

$$\langle x + y, x + y\rangle = \langle x, x\rangle + \underbrace{2\langle x, y\rangle + \langle y, y\rangle}_{\in 2\mathbb{Z}}.$$

□

The following class of spaces will be critical when we come to consider the classification of non-degenerate lattices:

**Definition 5.2.3.** A finite abelian group $\mathcal{L}$ in which every element is 2-torsion will be said to have *period* 2. Thus, it can naturally be regarded as a $\mathbb{Z}_2$-vector space, and if it is equipped with a quadratic form, it will be called a *quadratic space*. We say that a quadratic space $(\mathcal{L}, q)$ is *even*, if for every $x \in \mathcal{L}$ we have $q(x) = 0 \,(\mathrm{mod}\, \mathbb{Z})$, otherwise it is *odd*. Moreover, its *rank* is simply its dimension as a vector space over $\mathbb{Z}_2$.

With this definition in mind, we can state the following generalisation of the classification of indefinite unimodular lattices (Theorem 4.1.1):

**Theorem 5.2.1** ([41], Theorem 3.6.2)**.** *The isomorphism class of an even indefinite non-degenerate lattice $L$ with period-2 discriminant $\mathcal{L}$ is determined by the rank of $L$, the signature of $L$, the rank of $\mathcal{L}$ and the parity of $\mathcal{L}$.*

We also wish to state a classification result concerning non-degenerate quadratic spaces. In order to understand it, we need to define the following key invariant associated to quadratic spaces:

**Definition 5.2.4.** The *Brown invariant* $\mathrm{Br}(q)$ of a non-degenerate finite quadratic form $q$ on a finite abelian group $\mathcal{L}$ is the (mod 8)-residue implicitly defined by

$$\exp\left(\frac{1}{4} i\pi \mathrm{Br}(q)\right) = \frac{1}{\sqrt{|\mathcal{L}|}} \sum_{x \in \mathcal{L}} \exp(i\pi q(x)),$$

where $|\mathcal{L}|$ denotes the order of $\mathcal{L}$. If the quadratic form is understood, we denote the corresponding Brown invariant as $\mathrm{Br}(\mathcal{L})$.

It can be shown that the Brown invariant is well defined. Much can be said about the Brown invariant, however, for our purposes, the Brown invariant will be computed using the following:

**Theorem 5.2.2** (Van der Blij Formula)**.** *If $L$ is a non-degenerate, even lattice and $\mathcal{L}$ denotes its discriminant, then we have*

$$\mathrm{Br}(\mathcal{L}) = \sigma(L) \bmod 8,$$

*where $\mathrm{Br}(\mathcal{L})$ is the Brown invariant of the canonically induced quadratic form coming from $L$.*

The classification of non-degenerate quadratic spaces then reads:

**Theorem 5.2.3** ([27], 3.4.3)**.** *Two non-degenerate quadratic spaces $(\mathcal{L}_1, q_1)$ and $(\mathcal{L}_2, q_2)$ are isomorphic if and only if they have the same rank, parity and Brown invariant.*

Let us now move on to consider lattices equipped with involutions. In order to classify them, we need to talk about *gluings*:

**Definition 5.2.5.** A sublattice $K \subset L$ is said to be *primitive* if the quotient $L/K$ is torsion-free.

**Definition 5.2.6.** Given an even non-degenerate lattice $L$, an extension $\overline{L}$ is an even lattice of the same rank such that $L \subset \overline{L}$. A *gluing* of two non-degenerate even lattices $L_1$ and $L_2$ is an extension $\overline{L}$ of $L_1 \oplus L_2$, in which both $L_1 \cong L_1 \oplus \{0\}$ and $L_2 \cong \{0\} \oplus L_2$ are primitive.

For our purposes, we will be more interested in *unimodular* gluings. It turns out that there are necessary and sufficient conditions for the existence and uniqueness of gluings of non-degenerate lattices:

**Proposition 5.2.3** ([27], Corollary 4.1.6)**.** *Two even non-degenerate lattices $L_1$, $L_2$ admit a unimodular gluing if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are anti-isomorphic. If for at least one $i = 1, 2$ the canonical homomorphism $\mathrm{Aut}(L_i) \to \mathrm{Aut}(\mathcal{L}_i)$ is onto, all the unimodular gluings of $L_1$ and $L_2$ are isomorphic.*

**Remark 5.2.1.** To say that $(\mathcal{L}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathcal{L}_2, \langle\cdot,\cdot\rangle_2)$ are *anti-isomorphic* is to say that $(\mathcal{L}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathcal{L}_2, -\langle\cdot,\cdot\rangle_2)$ are isomorphic. Moreover, the canonical homomorphism $\mathrm{Aut}(L) \to \mathrm{Aut}(\mathcal{L})$ in question is given simply as:

$$\phi \ni \mathrm{Aut}(L) \mapsto (\phi^* : [f] \mapsto [f \circ \phi]) \in \mathrm{Aut}(\mathcal{L}),$$

where $f \in L^*$ is an arbitrary functional.

Let us now equip our lattices with $\mathbb{Z}_2$-actions:

**Definition 5.2.7.** Given an involution $\phi \in \mathrm{Aut}(L)$ on a unimodular lattice $L$, define the following non-degenerate sublattices

$$L_\phi^\pm := \{x \in L \mid \phi(x) = \pm x\}.$$

Note that $(L_\phi^\pm)^\perp = L_\phi^\mp$.

**Remark 5.2.2.** Observe that if $L$ is also even, then it can be regarded as a gluing of the sublattices $L_\phi^\pm$. Indeed, it is immediate that both $L_\phi^\pm$ are primitive in $L$. Moreover, their intersection is trivial.

The following proposition tells us more about the structure of the discriminants of the eigensublattices:

**Proposition 5.2.4.** *If $L$ is an even unimodular lattice equipped with an involution $\phi : L \to L$, then $\mathcal{L}_\phi^\pm$ are anti-isometric and of period* 2.

*Proof.* Proposition 5.2.3 tells us that $\mathcal{L}_\phi^+$ and $\mathcal{L}_\phi^-$ are anti-isometric. Since $L$ is unimodular, the map given by $L \to L^*$, $x \mapsto \langle x, \cdot\rangle$ is an isomorphism. Moreover, the natural inclusion $L_\phi^+ \hookrightarrow L$ yields a surjection $L^* \to (L_\phi^+)^*$ given by restricting functionals, since $L_\phi^+$ is primitive in $L$. Composing these two maps with the quotient map $(L_\phi^+)^* \to \mathcal{L}_\phi^+$, we get a surjection:

$$L \to \mathcal{L}_\phi^+,\ x \mapsto \left[\langle x, \cdot\rangle\big|_{L^+}\right].$$

Note that the kernel of this map is precisely $L_\phi^+ \oplus (L_\phi^+)^\perp = L_\phi^+ \oplus L_\phi^-$, and hence we get the isomorphism of abelian groups $\mathcal{L}_\phi^+ \cong L/(L_\phi^+ \oplus L_\phi^-)$. Defining the same map, but instead using $L_\phi^-$, we obtain $\mathcal{L}_\phi^- \cong L/(L_\phi^+ \oplus L_\phi^-)$. It is easy to see that $2L \subseteq L_\phi^+ \oplus L_\phi^-$, as any $2x \in L$ can be written as:

$$2x = \underbrace{(x + \phi(x))}_{\in L_\phi^+} + \underbrace{(x - \phi(x))}_{\in L_\phi^-}.$$

Thus, we have $\mathcal{L}_\phi^\pm \cong L/(L_\phi^+ \oplus L_\phi^-)$, and this is a quotient of $L/2L$ which is of period 2. □

Thus, an involution $\phi$ on an even unimodular lattice $L$ yields a gluing of $L_\phi^\pm$ such that $\mathcal{L}_\phi^+$ and $\mathcal{L}_\phi^-$ are anti-isometric and of period 2. An important corollary of Proposition 5.2.3 as well as a key result proved by Nikulin [41, Theorem 1.14.2] is that the process can be reversed, and thus under these circumstances the gluing arising from an involution is unique:

**Theorem 5.2.4** ([27], Corollary 5.1.3)**.** *Let $L_1$, $L_2$ be even lattices such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are both of period 2 and anti-isometric. If at least one of $L_1$ or $L_2$ is indefinite, there is a unique even unimodular lattice $L$ and involution $\phi : L \to L$ such that $L_\phi^+ \cong L_1$ and $L_\phi^- \cong L_2$.*

**Remark 5.2.3.** Note that 'unique' in the above result means 'unique up to conjugation by an isometry'. That is to say, if $(L', \phi')$ is another even unimodular lattice as in Theorem 5.2.4, then there exists an isometry $\beta : L \to L'$ such that $\beta \circ \phi = \phi' \circ \beta$.

With this background in mind, we can quickly come to prove Theorem 5.1.1. However, before we do that, we take a brief detour to prove the other central result of this section. Some of the results we prove in the following section will be useful in establishing the existence of Enriques isometries.

### 5.2.2 Involutory Lifts of $\rho$

An important feature of the isometry $\rho$ introduced in Section 5.1 (recall Definition 5.1.3) is the fact that it preserves orientation on maximal positive definite subspaces:

**Proposition 5.2.5.** *The distinguished isometry $\rho : L \to L$ preserves orientation on maximal positive definite subspaces, that is, $\rho \in \Gamma$. In particular, given an isometry $H^2(X, \mathbb{Z}) \cong L$, where $X$ is any K3 surface, there exists a diffeomorphism $f \in \mathrm{Diff}(X)$ inducing $\rho$.*

*Proof.* Recall (see Remark 4.4.9) that it suffices to check that $\rho$ preserves orientation on any particular instance of maximal positive definite subspaces. So, consider the maximal positive definite subspace defined by the following orthonormal vectors in $L_\mathbb{R}$:

$$a_1 := \frac{1}{\sqrt{2}}(e_1 + e_2, 0, 0, 0, 0), \quad a_2 := \frac{1}{\sqrt{2}}(0, e_1 + e_2, 0, 0, 0), \quad a_3 := \frac{1}{\sqrt{2}}(0, 0, e_1 + e_2, 0, 0),$$

where $e_1, e_2$ are the standard basis elements for $\mathbb{Z}^2$ defining the lattice $H$. We have

$$\rho(a_1) = -a_1, \quad \rho(a_2) = a_3, \quad \rho(a_3) = a_2,$$

and thus

$$(a_i \cdot \rho(a_j))_{i,j=1,2,3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which has determinant 1. Thus, $\rho \in \Gamma$. □

Our goal now is to prove that any smooth involution on a K3 surface which induces an action on the K3 lattice conjugate to $\rho$ must have no fixed points. In order to establish this result, we will make use of the work of A. Edmonds in [14]. In this paper, Edmonds studies cyclic actions of prime order $p$ on closed, simply-connected, topological 4-manifolds given by orientation-preserving, locally linear maps. A locally linear map on a topological manifold is one which is conjugate to a linear map between vector spaces. The advantage of considering these kinds of maps over simple continuous actions is that their fixed-point sets will consist of embedded submanifolds, whereas the fixed-point set of a continuous action on a topological manifold can be quite pathological. For our purposes, we will just need to know that every diffeomorphism on a smooth manifold is locally linear:

**Proposition 5.2.6.** *Suppose $\sigma : X \to X$ is a diffeomorphism of finite order $n$ on a smooth manifold $X$. Then $\sigma$ is locally linear.*

*Proof.* The simplest way to see this is by making use of some elementary Riemannian geometry. Choose an arbitrary Riemannian metric $g$ for $X$. Averaging over the action of $\sigma$ yields a metric

$$\widetilde{g} := g + \sigma^* g + \ldots + (\sigma^{n-1})^* g$$

such that $\sigma \in I_{\widetilde{g}}$. Given any point $p \in X$, we can define the associated exponential map

$$\exp_p : T_pX \to X,\ v \mapsto \gamma_v(1),$$

where $\gamma_v : [0,1] \to X$ is the unique geodesic (defined with respect to $\widetilde{g}$) satisfying $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. It is well known that $\exp_p$ is a diffeomorphism on a neighbourhood of $0 \in T_pX$. Given that $\sigma$ is an isometry of $\widetilde{g}$, for every $p \in X$, there is a commutative diagram:

$$\begin{array}{ccc} T_pX & \xrightarrow{d_p\sigma} & T_{\sigma(p)}X \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{\sigma(p)}} \\ X & \xrightarrow[\sigma]{} & X \end{array}$$

Restricting to the appropriate neighbourhoods for which $\exp_p$ and $\exp_{\sigma(p)}$ are diffeomorphisms, we see that $\sigma$ is conjugate to $d_p\sigma$, and so $\sigma$ is locally linear. □

**Remark 5.2.4.** Note that this kind of an argument can also be used to give a quick proof of the fact that the fixed-point set of a finite group action on a smooth manifold is a submanifold (which may have components of differing dimensions).

Following Edmonds, let $G = \langle \sigma \rangle$ denote a finite cyclic group of prime order $p$ generated by an orientation-preserving, locally linear map $\sigma : X \to X$ acting on a closed, simply-connected, topological 4-manifold $X$. The action of $G$ on $X$ induces an action of $G$ on integral cohomology $H^k(X, \mathbb{Z})$ for each $k = 0, \dots, 4$. The first key result of the paper is the following:

**Proposition 5.2.7** ([14], Proposition 1.1)**.** *The representation of $G$ on $H^2(X, \mathbb{Z})$ is equivalent over the integers to a block sum of indecomposable representations of the following three types:*

1. *one-dimensional, of trivial type: the integers $\mathbb{Z}$ with trivial $G$ action, i.e. acting as multiplication by* 1*.*
2. $(p-1)$*-dimensional, of cyclotomic type:* $\mathbb{Z}[\lambda]$*, the integers $\mathbb{Z}$ with a primitive $p^{th}$ root of unity $\lambda$ adjoined, where $G$ acts by multiplication by $\lambda$.*
3. *$p$-dimensional of regular type: $\mathbb{Z}[G]$, the regular representation, in which $G$ acts by permuting basis vectors.*

**Remark 5.2.5.** In particular, if $\sigma$ is an involution, then its $\mathbb{Z}_2$-action on the intersection form decomposes into $t$ trivial summands (acting as 1 on $\mathbb{Z}$), $c$ cyclotomic summands (acting as $-1$ on $\mathbb{Z}$) and $r$ regular summands (acting as $H$ on $\mathbb{Z}^2$). Moreover, we must have $t+c+2r = b_2$. Importantly, in the case that $\sigma$ acts freely, Edmonds shows (see [14, p. 114]) that $t = 0$ and $c = 2$.

Edmonds then proceeds to prove a variety of other useful results. The following tells us that the Euler characteristic can be calculated based on the decomposition of the action on the second cohomology:

**Proposition 5.2.8** ([14], Corollary 1.4)**.** *If $F$ denotes the fixed-point set of a locally linear, orientation-preserving map of prime order $\sigma : X \to X$ on a closed oriented simply-connected four-manifold $X$, then*

$$\chi(F) = t - c + 2.$$

The next result by Edmonds also tells us more information about the Betti numbers of the fixed-point set:

**Proposition 5.2.9** ([14], Proposition 2.4)**.** *If the fixed-point set $F$ of a locally linear, orientation-preserving map of prime order $p$ on a closed oriented simply-connected four-manifold is non-empty, then its* mod $p$ *Betti numbers satisfy:*

*(i)* $b_1(F) = c$, *and;*

*(ii)* $b_0(F) + b_2(F) = t + 2$.

In order to prove the main theorem of this subsection, we will only need one final result by Edmonds. If one considers the action of a locally linear, orientation-preserving involution on a four-manifold, then its fixed-point set will in general consist of codimension 0 and codimension 2 oriented submanifolds. The following result tells us that in the case of simply-connected, spin four-manifolds, the fixed-point set of an involution only consists of one or the other:

**Theorem 5.2.5** ([14], Corollary 3.3)**.** *Let $X$ be a connected, simply-connected spin* 4*-manifold and $\sigma : X \to X$ be a locally linear, orientation-preserving involution. Then the fixed-point set of $\sigma$, if non-empty, consists either of isolated points or of oriented surfaces.*

Given that K3 surfaces are examples of spin, simply-connected four-manifolds, we will be able to apply all of the above results in our analysis. For what follows, we will need two important lemmas. The first is a standard result in differential geometry:

**Lemma 5.2.1.** *Suppose $\sigma : X \to X$ is an orientation-preserving smooth involution acting on a closed oriented manifold $X$ of dimension $n$ whose fixed-point set $F$ consists only of codimension* 2 *components. Then the quotient space $Y := X/\langle\sigma\rangle$ is a closed, oriented manifold of the same dimension as $X$.*

*Proof.* Given that $X$ is Hausdorff, and is equipped with a finite cyclic action, the quotient $Y$ is automatically Hausdorff. Since the projection $\pi : X \to Y$ is an open map, the second countability of $Y$ follows from that of $X$. The compactness of $Y$ is immediate. The quotient manifold theorem tells us that outside of the fixed-point set, the quotient $Y$ has local coordinates. Thus, we only have to construct local coordinates about a fixed point in the quotient. So let $y \in \pi(F) \subseteq Y$ be arbitrary. There is exactly one point $x \in F \subseteq X$ that lies in the fibre $\pi^{-1}(y)$. Proposition 5.2.6 implies that we can find a neighbourhood of $x$ in which $\sigma$ is conjugate to a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. Given that $\sigma$ (and hence $T$) is of order 2, we have that the matrix $T$ is diagonalisable over $\mathbb{R}$, so let us assume it has the form $T = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. However, since $F$ has codimension 2, the number of entries with value $-1$ is 2, and so $\sigma$ in this neighbourhood of $x$ is given by

$$\sigma(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) = (x_1, \ldots, x_{n-2}, -x_{n-1}, -x_n).$$

In particular, if we set $z := x_{n-1} + ix_n$, then $(x_1, \ldots, x_{n-2}, \mathrm{Re}(z^2), \mathrm{Im}(z^2))$ provide local coordinates for the quotient $Y$, giving the map $\pi : X \to Y$ the structure of a branched double cover. The fact that $Y$ is oriented follows immediately by the fact that $\sigma$ acts orientation-preservingly. □

**Lemma 5.2.2.** *If $\sigma : X \to X$ is an orientation-preserving smooth involution acting on an oriented simply-connected four-manifold $X$ whose fixed-point set is empty or consists only of codimension 2 components, then the quotient map $\pi : X \to Y := X/\langle\sigma\rangle$ satisfies*

$$\pi_! \circ \pi^* = 2 \cdot \mathrm{id}_{H^*(Y,\mathbb{Z})}.$$

*Moreover, for $x \in H^2(X, \mathbb{Z})$, we have*

$$\pi^*(\pi_!(x)) = x + \sigma^*(x).$$

*Hence, if the representation of the action of $\sigma^*$ on $H^2(X, \mathbb{Z})$ contains no trivial summands (for example, if $\sigma$ is in fact free), then it follows that*

$$\pi^* H^2(Y, \mathbb{Z}) = L^+_{\sigma^*},$$

*where, as usual, $L^+_{\sigma^*} = \{x \in H^2(X, \mathbb{Z}) \mid \sigma^*(x) = x\} \subseteq H^2(X, \mathbb{Z})$.*

*Proof.* Lemma 5.2.1 tells us that $Y$ is a closed oriented manifold, and note that the degree of $\pi$ is unaffected by the branching. Then, the first equality immediately follows by Proposition 2.2.1. For the second equality, suppose $x \in H^2(X, \mathbb{Z})$, then $x = c_1(L)$ for some complex line bundle $L \to X$ (Proposition 2.1.5). As we have seen (Theorem 2.2.3 and Theorem 2.1.1), we can find a generic section of $L$ such that the zero locus $C := \{x \in X \mid s(x) = 0\}$ is an embedded submanifold whose fundamental class $[C] \in H_2(X, \mathbb{Z})$ is Poincaré dual to $c_1(L) = x$. Then $\pi_*[C] = [\pi(C)]$. Given that $\pi$ is a (branched) double covering, we have $\pi^{-1}(\pi(C)) = C \cup \sigma(C)$, and this represents the Poincaré dual of $\pi^*(\pi_!(x))$. The result follows. For the last statement, observe that $\pi = \pi \circ \sigma$. Then $\pi^* = \sigma^* \circ \pi^*$, and thus $\pi^* H^2(Y, \mathbb{Z}) \subseteq L^+_{\sigma^*}$. Now, if $x \in L^+_{\sigma^*}$, then since the decomposition of the action of $\sigma^*$ contains no trivial summands, we can write $x = x_0 + \sigma^*(x_0)$ for suitably chosen $x_0 \in H^2(X, \mathbb{Z})$. The above formula implies

$$x = x_0 + \sigma^*(x_0) = \pi^*(\pi_!(x_0)),$$

and so $L^+_{\sigma^*} \subseteq \pi^* H^2(Y, \mathbb{Z})$. □

We are now in a position to prove our result:

**Theorem 5.2.6.** *Any smooth involution $\sigma : X \to X$ on a K3 surface which acts as $\rho$ under an isometry $H^2(X, \mathbb{Z}) \cong L$ must act freely.*

*Proof.* For the sake of contradiction, suppose that $\sigma$ does not act freely on $X$, and let $F \neq \emptyset$ denote its fixed-point set. Since $X$ is spin, Theorem 5.2.5 implies that $F$ consists either of a finite number of points or of a disjoint union of oriented surfaces. However, given that $\rho$ is given by the map

$$3H \oplus -2E_8 \to 3H \oplus -2E_8$$
$$(x, y, z, u, v) \mapsto (-x, z, y, v, u),$$

by inspection, we see that the $\mathbb{Z}_2$-representation of the action of $\sigma^*$ (i.e. $\rho$) on $H^2(X, \mathbb{Z})$ consists of $t = 0$ trivial summands, $c = 2$ cyclotomic summands, and $r = 10$ regular summands. Thus, Proposition 5.2.8 implies

$$\chi(F) = 0,$$

and so $F$ must consist of a disjoint union of oriented surfaces $F = S_1 \sqcup \ldots \sqcup S_n$. Proposition 5.2.9 tells us that $2n = 2$, and so $n = 1$, that is, $F$ is connected. Moreover, given that $b_1(F) = c = 2$, $F$ must be a torus. Let us denote the quotient space by $Y := X/\langle\sigma\rangle$. Since $F$ has codimension 2, $Y$ is a closed oriented four-manifold by Lemma 5.2.1. In fact, $Y$ is simply connected. Indeed, choosing a basepoint on the fixed-point set $F \subseteq Y$ (identifying the fixed-point set $F$ with its image on the base), we can assume (by appropriately homotoping) that, away from the basepoint, an arbitrary loop lies entirely in $Y \setminus F$. Using the lifting property for the covering $\pi : \pi^{-1}(Y \setminus F) \to Y \setminus F$, this lifts to a path in $X$, but given that the fibre of any point in $F \subseteq Y$ is a singleton, this lifted path must in fact be a loop. Since $X$ is simply connected, this loop contracts, and pushing it down by the covering map $\pi$ tells us that our original loop was in fact contractible, so $\pi_1(Y) = 0$. Then, $H^2(Y, \mathbb{Z})$ is free of torsion and hence defines a unimodular lattice. Moreover, since the $\mathbb{Z}_2$-representation of the action of $\sigma^*$ has no trivial summands, Lemma 5.2.2 implies $\pi^* H^2(Y, \mathbb{Z}) = L^+_{\sigma^*}$. Given that $\pi$ has degree 2, we must have $\pi^* H^2(Y, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})(2)$, and since $\sigma$ acts like $\rho$ on $L$, Proposition 5.1.5 yields $L^+_{\sigma^*} \cong L^+_\rho \cong L_E(2)$. Hence, $H^2(Y, \mathbb{Z}) \cong L^+_\rho(1/2) \cong L_E$, i.e. the intersection form of $Y$ is isomorphic to the intersection form of an Enriques surface. In particular, the intersection form of $Y$ is even, but since $Y$ is simply connected, Corollary 2.2.5 tells us that $Y$ is spin. However, $\sigma(Y) = \sigma(L_E) = -8 \not\equiv 0 \pmod{16}$, which contradicts Rokhlin's theorem (Theorem 2.2.7), and so $F = \emptyset$, i.e. $\sigma$ acts freely. $\square$

### 5.2.3 Existence of Enriques Isometries

In this subsection, we will finally give a proof of Theorem 5.1.1, and we will do so by making use of the results introduced in Section 5.2.1. The main idea is the following: by showing that the characteristics of the induced action of a free smooth involution on the K3 lattice and $\rho$ are the same, we will be able to use the classification results introduced there to show that they are conjugate. The main result to keep in mind is Proposition 5.1.5. In order to deduce our result, we will need two auxiliary results. The first states that any quotient of a K3 surface by a free smooth $\mathbb{Z}_2$-action must have the intersection form of an Enriques surfaces:

**Theorem 5.2.7.** *Let $\sigma : X \to X$ be a free smooth involution on a K3 surface. Denote the quotient by $Y := X/\langle\sigma\rangle$. Then the intersection form of $Y$ is even, and hence $H^2(Y,\mathbb{Z})/T_2 \cong L_E = H \oplus -E_8$, where $T_2 \cong \mathbb{Z}_2$ is the torsion subgroup of $H^2(Y,\mathbb{Z})$.*

*Proof.* It suffices to construct a characteristic of the intersection form of $Y$ which is torsion (cf. Definition 2.2.5). Recall that a characteristic for the intersection form of a compact $4k$-manifold $Z$ is an element $c \in H^{2k}(Z,\mathbb{Z})$ such that for every $x \in H^{2k}(Z,\mathbb{Z})$:

$$x^2 = c \cdot x \pmod 2.$$

By Corollary 2.2.4, any integral lift of $w_2(Y)$ will be characteristic. Given that spin$^c$-structures always exist on smooth, oriented four-manifolds (Theorem 2.2.5), let $c \in H^2(Y,\mathbb{Z})$ be an integral lift of $w_2(Y)$. Now, denote the projection by $\pi : X \to Y$. Since $\pi$ is a local diffeomorphism, we have that $\pi^* w_2(Y) = w_2(X) = 0$. In particular, the image of the class $\pi^*(c) \in H^2(X,\mathbb{Z})$ under the canonical map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}_2)$ is zero. Recall the standard long exact sequence

$$\ldots \longrightarrow H^2(X,\mathbb{Z}) \xrightarrow{\times 2} H^2(X,\mathbb{Z}) \xrightarrow{\bmod 2} H^2(X,\mathbb{Z}_2) \longrightarrow \ldots$$

coming from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}_2 \longrightarrow 0.$$

Since $\pi^*(c) \bmod 2 = \pi^* w_2(Y) = 0$, we must have $\pi^*(c) = 2x$ for some $x \in H^2(X,\mathbb{Z})$. Using Lemma 5.2.2, we have $L^+_{\sigma^*} = \pi^* H^2(Y,\mathbb{Z})$, and so $\sigma^*(2x) = 2\sigma^*(x) = 2x$, but since $H^2(X,\mathbb{Z})$ has no 2-torsion, we must have $x \in L^+_{\sigma^*}$. Hence, $x = \pi^*(y)$ for some $y \in H^2(Y,\mathbb{Z})$. Then the element $c' := c - 2y \in H^2(Y,\mathbb{Z})$ is again characteristic for the intersection form of $Y$, and moreover we have

$$\pi^*(c') = \pi^*(c) - 2\pi^*(y) = 2x - 2x = 0.$$

Lemma 5.2.2 then implies,

$$2c' = \pi_!(\pi^*(c)) = 0,$$

that is, $c'$ is 2-torsion, and hence the intersection form of $Y$ is even. To conclude the proof, we just have to determine the rank and signature of $H^2(Y,\mathbb{Z})/T_2$. Given that $\pi : X \to Y$ is a double covering, we see $\chi(Y) = \frac{1}{2}\chi(X) = 12$ and $\sigma(Y) = \frac{1}{2}\sigma(X) = -8$ (Corollary 2.2.6). Since $b_1(Y) \leq b_1(X) = 0$, we find $b_2(Y) = 10$. By the classification of indefinite unimodular lattices (Theorem 4.1.1), we must have $H^2(Y,\mathbb{Z})/T_2 \cong H \oplus -E_8$. □

**Remark 5.2.6.** The observant reader will note that we alluded to this result in Remark 5.1.2, where we mentioned the fact that the quotient of a K3 surface by a free smooth $\mathbb{Z}_2$-action is always homeomorphic to an Enriques surface. Establishing Theorem 5.2.7 is one of the key ingredients to obtaining this result.

The other result we need states that the associated eigensublattices of a free smooth involution on a K3 surface are identical to those of our distinguished involution $\rho \in \mathrm{Aut}(L)$:

**Proposition 5.2.10.** *If $\sigma : X \to X$ is a free smooth involution on a K3 surface, then*

$$L^+_{\sigma^*} \cong L_E(2) \text{ and } L^-_{\sigma^*} \cong H \oplus L_E(2).$$

*Proof.* If $\pi : X \to Y := X/\langle\sigma\rangle$ denotes the quotient map, then Lemma 5.2.2 and Theorem 5.2.7 yield $L^+_{\sigma^*} \cong L_E(2)$. Now, since $L^-_{\sigma^*} = (L^+_{\sigma^*})^\perp$, we see that the signature of $L^-_{\sigma^*}$ is $(2,10)$. Proposition 5.2.1 implies that the discriminant of $L_E(2)$ is of order $2^{10}$, and since $\mathcal{L}^\pm_{\sigma^*}$ are anti-isometric and of period 2 (Proposition 5.2.4), we have $\mathcal{L}^+_{\sigma^*} \cong \mathcal{L}^-_{\sigma^*} \cong \mathbb{Z}_2^{10}$ (as abelian groups), i.e. $\mathrm{rk}(\mathcal{L}^\pm_{\sigma^*}) = 10$. Now, a symmetric bilinear form on a finite-dimensional $\mathbb{Z}_2$-vector space is valued in $\mathbb{Z}_2 \cong (\frac{1}{2}\mathbb{Z})/\mathbb{Z}$, and so any finite quadratic form will be valued in $(\frac{1}{2}\mathbb{Z})/(2\mathbb{Z}) \cong \mathbb{Z}_4$. In particular, since $L^+_{\sigma^*} \cong L_E(2)$ and $L_E$ is even, we have that $x^2 \equiv 0 \bmod 4$ for $x \in L^+_{\sigma^*}$, and so the parity of $\mathcal{L}^+_{\sigma^*}$ (and hence $\mathcal{L}^-_{\sigma^*}$) is even (when equipped with their canonically induced quadratic forms of Proposition 5.2.2). The classification of non-degenerate indefinite even lattices (Theorem 5.2.1) tells us that $L^-_{\sigma^*} \cong H \oplus H(2) \oplus -E_8(2)$. □

We finally deduce:

**Theorem 5.1.1.** *If $\sigma : X \to X$ is a free smooth involution on a K3 surface $X$, then there exists an isometry*

$$\varphi : H^2(X,\mathbb{Z}) \to L$$

*such that*

$$\varphi \circ \sigma^* = \rho \circ \varphi.$$

*In particular, any two free smooth involutions on a K3 surface induce conjugate actions on* $L \cong H^2(X, \mathbb{Z})$*.*

*Proof.* Proposition 5.1.5 and Proposition 5.2.10 together imply $L^{\pm}_{\sigma^*} \cong L^{\pm}_{\rho}$. To deduce the result, we will show that $\mathcal{L}^{\pm}_{\sigma^*} \cong \mathcal{L}^{\pm}_{\rho}$ (as quadratic spaces). As we saw, $\mathrm{rk}(\mathcal{L}^{\pm}_{\sigma^*}) = \mathrm{rk}(\mathcal{L}^{\pm}_{\rho}) = 10$ and $\sigma(L^{\pm}_{\sigma^*}) = \sigma(L^{\pm}_{\rho}) = -8$. The Van der Blij formula (Theorem 5.2.2) implies

$$\mathrm{Br}(\mathcal{L}^{\pm}_{\sigma^*}) = 0 = \mathrm{Br}(\mathcal{L}^{\pm}_{\rho}).$$

The proof of Proposition 5.2.10 also showed that $\mathcal{L}^{\pm}_{\sigma^*}$ and $\mathcal{L}^{\pm}_{\rho}$ are all even. The classification of non-degenerate quadratic spaces (Theorem 5.2.3) implies $\mathcal{L}^{\pm}_{\sigma^*} \cong \mathcal{L}^{\pm}_{\rho}$. The uniqueness of unimodular gluings arising from involutions (Theorem 5.2.4) then yields an isometry $\varphi : H^2(X, \mathbb{Z}) \to L$ such that $\varphi \circ \sigma^* = \rho \circ \varphi$. □

Let us now move on to see how we can deduce the classification results concerning Enriques surfaces as stated in Theorem 5.1.2 and Theorem 5.1.3 using what we have concluded in this section.

## 5.3 Surjectivity of the Period Maps

Having finally established the existence of Enriques isometries, in this section we will focus on deducing the surjectivity of the two period maps for Enriques surfaces defined in Section 5.1. As we will see, the surjectivity of the period map will follow from the surjectivity of the refined period map. Deducing the surjectivity of the latter directly from the surjectivity of the refined period map for K3 surfaces is somewhat tricky. Certainly, if we are given an arbitrary point $(a, c) \in \mathcal{P}^- \subset \mathcal{P}$, there will exist a Kähler K3 surface (which we can take to be Kähler-Einstein) defined by $(\omega, \kappa) \in \mathcal{R}(X_0)$ such that $[\omega] = a$ and $[\kappa] = c$ under an appropriate Enriques isometry of an arbitrary Enriques involution $\sigma \in \mathrm{Diff}(X_0)$ by Theorem 4.4.4. However, there is nothing whatsoever which tells us that this choice of Kähler structure should descend to one for an Enriques surface defined by $\sigma$, that is, that the pair $(\omega, \kappa)$ should satisfy $\sigma^*\kappa = -\kappa$ and $\sigma^*\omega = \omega$. The resolution of this difficulty is where the Teichmüller space $\mathcal{T}(X_0)$, which has hitherto played no role in our discussion of Enriques surfaces, will turn out to be useful. Before stating the two main technical results we will need for the proof of the surjectivity of the refined period map, let us establish some notation.

**Definition 5.3.1.** Under any isometry $H^2(X_0,\mathbb{Z}) \cong L$, denote the image of the distinguished isometry $\rho \in \Gamma$ under the isomorphism $\Gamma \cong \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$ by $[\rho] \in \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$. Then, $[\rho]$ acts on $\mathcal{T}(X_0) = \mathrm{Ein}(X_0)/\mathrm{Diff}_T(X_0)$. Denote its fixed-point set by

$$\mathcal{T}(X_0)^\rho := \{t \in \mathcal{T}(X_0) \,|\, \rho^* t = t\}.$$

Recall the period domain of Einstein metrics $\mathcal{E} = \mathrm{Gr}^0_+(L_\mathbb{R}) \setminus \bigcup_{d\in\Delta} A_d$. If we set $\mathcal{E}^+ := \{H \in \mathcal{E} \,|\, \rho(H) = H\}$, then the period map $\tau : \mathcal{T}(X_0) \to \mathcal{E}$, $[g] \mapsto \mathcal{H}^+_g(X_0)$ restricts to a diffeomorphism $\tau : \mathcal{T}(X_0)^\rho \to \mathcal{E}^+$. Also, the fibre bundle $\pi : \mathcal{P} \to \mathcal{E}, (a,c) \to S(a,c)$ of Proposition 4.5.6 restricts to a map $\pi : \mathcal{P}^- \to \mathcal{E}^+$. We fix the composition

$$\widetilde{\pi} := \tau^{-1} \circ \pi : \mathcal{P}^- \to \mathcal{T}(X_0)^\rho.$$

The following technical lemma, which is primarily a consequence of the Ebin slice theorem (Theorem 4.5.3), tells us that any map in $\Gamma \cong \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$ which fixes a point in the Teichmüller space will be able to be lifted to a genuine isometry of an Einstein metric and that we can construct a local continuous inverse of the projection $\mathrm{Ein}(X_0) \to \mathcal{T}(X_0)$. It represents the final key result, from which in essence the remainder of the classification results will follow. The precise statement reads:

**Lemma 5.3.1.** *For arbitrary $t = [g] \in \mathcal{T}(X_0)$, let $\Gamma_t \subseteq \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$ denote the stabiliser of $t$, i.e.*

$$\Gamma_t := \{[f] \in \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0) \,|\, [f^* g] = [g]\}.$$

*Then, there exists a $\Gamma_t$-invariant open neighbourhood $U$ of $t$ in $\mathcal{T}(X_0)$ and a continuous map $\psi : U \to \mathrm{Ein}(X_0)$ such that $\psi(t') = g'$ for $t' = [g'] \in U$. Moreover, the map $\phi : \Gamma_t \to I_g$, $[f] \mapsto f$ is well defined and defines an isomorphism, and $\psi$ is $\Gamma_t$-equivariant with respect to this, that is, for every $t' \in U$ and $[f] \in \Gamma_t$, we have $\psi(f^* t') = \phi([f])^* \psi(t')$.*

*Proof.* Let $t = [g] \in \mathcal{T}(X_0)$ be any point in the Teichmüller space, with $g \in \mathrm{Ein}(X_0) \subset \mathrm{Riem}(X_0)$. By Theorem 4.5.3, there exists a slice $S \subset \mathrm{Riem}(X_0)$ through $g$ which is $I_g$-invariant. Recall (Theorem 4.5.4) that the intersection $\widetilde{S} := \mathrm{Ein}(X_0) \cap S$ is a smooth submanifold of $S$. Then the composition $\widetilde{S} \hookrightarrow S \hookrightarrow \mathrm{Riem}(X_0)$ is smooth with image contained in $\mathrm{Ein}(X_0)$. If we denote the composition of this map with the projection $\pi : \mathrm{Ein}(X_0) \to \mathcal{T}(X_0)$ (which is smooth) by $\beta : \widetilde{S} \to \mathcal{T}(X_0)$, $g \mapsto [g]$, then $\beta$ is smooth and injective. Indeed, if $g_1 = f^* g_2$ for $g_1, g_2 \in \widetilde{S}$ and $f \in \mathrm{Diff}_T(X_0)$, then $f^* S \cap S \neq \emptyset$, and hence $f \in I_g$ also by Theorem 4.5.3. Proposition 4.5.2 implies $f = \mathrm{id}_{X_0}$, that is, $g_1 = g_2$. By invariance of domain, the image of $\beta$, denote it by $U := \beta(\widetilde{S}) \subseteq \mathcal{T}(X_0)$, must be open and $\beta$ restricts to a homeomorphism $\beta : \widetilde{S} \to U$. Denote the continuous inverse by $\psi : U \to \widetilde{S} \subset \mathrm{Ein}(X_0)$, $[g] \mapsto g$. To see that $U$ is $\Gamma_t$-invariant, observe that if $[f] \in \Gamma_t$ and $t' = [g']$ where $g' \in \widetilde{S}$, then $f^*t' = [f^*g']$. Since $[f] \in \Gamma_t$, there exists $h \in \mathrm{Diff}_T(X_0)$ such that $h^*f^*g = g$, i.e. $f \circ h \in I_g$. Given that $S$ is $I_g$-invariant, we have that $\widetilde{S}$ is invariant under the action of $f \circ h$, and so $h^*(f^*g') \in \widetilde{S}$. Since $h \in \mathrm{Diff}_T(X_0)$, we have $[f^*g'] = [h^*f^*g'] = \beta(h^*f^*g') \in U$, and so $U$ is $\Gamma_t$-invariant.

For the homomorphism, define its inverse as follows: let $\alpha$ be the composition $I_g \hookrightarrow \mathrm{Diff}(X_0) \to \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$, $f \mapsto [f]$. The image is clearly a subset of $\Gamma_t$, and so $\alpha$ defines a homomorphism $\alpha : I_g \to \Gamma_t$. Moreover, Proposition 4.5.2 immediately implies $\alpha$ is injective. If $[f] \in \Gamma_t$ is arbitrary, then as above, there exists $h \in \mathrm{Diff}_T(X_0)$ such that $f \circ h \in I_g$, and we have $[f \circ h] = [f]$, so $\alpha$ is surjective. Thus, $\alpha$ is an isomorphism. Let us denote its inverse by $\phi : \Gamma_t \to I_g$, $[f] \to f$. The $\Gamma_t$-equivariance of $\psi$ follows simply: for any $t' = [g'] \in U$ and $[f] \in \Gamma_t$, we must have $\psi(f^*t') = \psi([f^*g']) = f^*g' = f^*\psi(t') = \phi([f])^*\psi(t')$. □

We can now apply Lemma 5.3.1 to our distinguished involution $\rho \in \Gamma$. Observe that for any $(a, c) \in \mathcal{P}^-$, we have $\rho(S(a, c)) = S(a, c)$. As a result, to any such point in the refined period domain we will be able to associate an involution of an Einstein metric which acts like $\rho$ under some isometry $H^2(X_0, \mathbb{Z}) \cong L$. The work of Edmonds (Theorem 5.2.6) then tells us that this involution must act freely. The following central lemma formalises this, and moreover yields the important fact that the resulting involutions of Einstein metrics must be conjugate, which is more or less the statement that all Enriques surfaces are diffeomorphic (cf. Theorem 5.5.1):

**Lemma 5.3.2.** *Fix an isometry $H^2(X_0, \mathbb{Z}) \cong L$ and let $(a, c) \in \mathcal{P}^-$ be arbitrary. Then for any Einstein metric $g \in \mathrm{Ein}(X_0)$ such that $\mathcal{H}^+_g(X_0) = S(a, c)$, there exists a unique smooth involution $\lambda \in \mathrm{Diff}(X_0)$ which preserves $g$, acts freely on $X_0$ and whose induced action on $L$ coincides with that of $\rho$. Moreover, for any $(a_1, c_1), (a_2, c_2) \in \mathcal{P}^-$, $g_1, g_2 \in \mathrm{Ein}(X_0)$ and $\lambda_1 \in \mathrm{Isom}(X_0, g_1)$, $\lambda_2 \in \mathrm{Isom}(X_0, g_2)$ as above, there exists a diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ such that $\lambda_1 \circ f = f \circ \lambda_2$.*

*Proof.* For the first part, let $(a_1, c_1) \in \mathcal{P}^-$ be arbitrary and suppose $\mathcal{H}^+_{g_1}(X_0) = S(a_1, c_1)$ for some Einstein metric $g_1 \in \mathrm{Ein}(X_0)$. Let $t_1 = [g_1] \in \mathcal{T}(X_0)$ be the associated point in the Teichmüller space. By definition, our isometry $\rho \in \Gamma$ preserves $S(a_1, c_1)$ and hence $\mathcal{H}^+_{g_1}(X_0)$ under the isomorphism $\Gamma \cong \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$. By the equivariance of $\tau$ (Proposition 4.5.4), we have

$$\rho^*\mathcal{H}^+_{g_1}(X_0) = [\rho] \cdot \tau([g_1]) = \tau(\rho^*[g_1]) = \tau([g_1]) \iff \rho^*t_1 = t_1.$$

Thus, we see $[\rho] \in \Gamma_{t_1}$. If $\phi_1 : \Gamma_{t_1} \to I_{g_1}$, $[f] \to f$ is the isomorphism of Lemma 5.3.1, then $\lambda_1 := \phi_1([\rho])$ is the unique isometry of $g_1$ such that $\lambda_1^* = \rho$ on $L$. Indeed, by definition we have $[\lambda_1] = [\rho]$, and under the identification $\Gamma \cong \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$, we get $\lambda_1^* = \rho$. Theorem 5.2.6 shows that $\lambda_1$ must have no fixed points.

For the second part, suppose we are a given a second arbitrary point $(a_2, c_2) \in \mathcal{P}^-$. Recall that $\mathcal{P}^-$ has two connected components (Proposition 5.1.6). So, if $(a_1, c_1)$ and $(a_2, c_2)$ are in different components, then replacing $(a_1, c_1)$ by $(-a_1, \overline{c_1})$ has no effect on the choice of $g_1$ and hence $\lambda_1$, given that $S(a_1, c_1) = S(-a_1, \overline{c_1})$. Thus, we can without loss of generality assume $(a_1, c_1)$ and $(a_2, c_2)$ lie in the same connected component of $\mathcal{P}^-$. To complete the proof, we will show that the conclusion holds in a neighbourhood of $(a_1, c_1)$.

So, let $\psi : U \to \mathrm{Ein}(X_0)$, $[g] \mapsto g$ be the local section defined in a neighbourhood of $t_1 = [g_1]$ guaranteed by Lemma 5.3.1. Then, by the $\Gamma_{t_1}$-equivariance of $\psi$, for any $t_2 = [g_2] \in U \cap \mathcal{T}(X_0)^\rho$, we obtain

$$\phi_1([\rho])^* \psi(t_2) = \lambda_1^* g_2 = \psi(\rho^* t_2) = \psi(t_2) = g_2,$$

that is, $\lambda_1$ is an isometry of all metrics in the set $\psi(U \cap \mathcal{T}(X_0)^\rho)$. Note that for any $t_2 \in \mathcal{T}(X_0)^\rho$, we have by definition $[\rho] \in \Gamma_{t_2}$ and also that $V := \widetilde{\pi}^{-1}(U \cap \mathcal{T}(X_0)^\rho) \subseteq \mathcal{P}^-$ is a neighbourhood of our arbitrary point $(a_1, c_1) \in \mathcal{P}^-$. So, let $(a_2, c_2) \in V$ be any other point in the same neighbourhood, and $g_2 \in \mathrm{Ein}(X_0)$ be any metric such that $\mathcal{H}^+_{g_2}(X_0) = S(a_2, c_2)$. Considering the map $\phi_2 : \Gamma_{t_2} \to I_{g_2}$, where $t_2 := [g_2] \in \mathcal{T}(X_0)$, the free involution $\lambda_2 := \phi_2([\rho])$ is the unique isometry of $g_2$ such that $\lambda_2^* = \rho$ on $L$. Now, if $g_2 \in \psi(U \cap \mathcal{T}(X_0)^\rho)$, then we have shown that $\lambda_1 \in I_{g_2}$, and given that $\lambda_1^* = \rho = \lambda_2^*$, Proposition 4.5.2 shows $\lambda_1 = \lambda_2$, i.e. we can take $f = \mathrm{id}_{X_0}$. On the other hand, if $g_2 \notin \psi(U \cap \mathcal{T}(X_0)^\rho)$, then since $(a_2, c_2) \in V$, we have $\mathcal{H}^+_{g_2}(X_0) = S(a_2, c_2) = \tau(t_2') = \mathcal{H}^+_{g_2'}(X_0)$ for unique $t_2' := [g_2'] \in U \cap \mathcal{T}(X_0)^\rho$. The injectivity of the map $\tau$ tells us that $g_2' = f^* g_2$ for some $f \in \mathrm{Diff}_T(X_0)$. By assumption, $\lambda_1$ is an isometry of $g_2'$, and hence $f \circ \lambda_1 \circ f^{-1}$ is an isometry of $g_2$. Since $(f \circ \lambda_1 \circ f^{-1})^* = \rho = \lambda_2^*$, Proposition 4.5.2 tells us that $f \circ \lambda_1 \circ f^{-1} = \lambda_2$, which completes the proof. □

We are finally in a position to prove the surjectivity of the refined period map. Given any point in the refined period domain for Enriques surfaces $\mathcal{P}^-$, Lemma 5.3.2 will allow us to construct a free smooth involution of a Kähler-Einstein structure mapping to this point. As the following proof shows, such involutions must necessarily yield Enriques surfaces as quotients, and the conjugacy fact above ensures that we can restrict to the case of a fixed Enriques involution:

**Theorem 5.3.1.** *Suppose $\sigma : X_0 \to X_0$ is an Enriques involution and an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma$ is fixed. Then, the refined period map $\hat{\Pi} : \mathcal{R}(X_0, \sigma) \to \mathcal{P}^-$ is surjective.*

*Proof.* Suppose $(a, c) \in \mathcal{P}^-$ is arbitrary. By the surjectivity of the refined period map for K3 surfaces (Theorem 4.4.4), there exists a Kähler K3 surface defined by the pair $(\omega, \kappa) \in \mathcal{R}(X_0)$ such that $a = [\omega]$ and $c = [\kappa]$ under $H^2(X_0, \mathbb{Z}) \cong L$. Yau's theorem (Theorem 2.6.3) implies that there exists a unique Ricci-flat Kähler-Einstein metric $g$ (Kähler with respect to the complex structure defined by $\kappa$, call it $I$) whose Kähler form $\omega_g$ satisfies $[\omega_g] = [\omega] = a$. Then, $\mathcal{H}^+_g(X_0) = S(a, c)$. By Lemma 5.3.2, there exists a unique free smooth involution $\lambda \in \mathrm{Diff}(X_0)$ preserving $g$ and which satisfies $\lambda^* = \rho = \sigma^*$ on $L \cong H^2(X_0, \mathbb{Z})$. Since $\lambda^* g = g$, Proposition 4.5.3 tells us that pulling back by $\lambda$ gives an isomorphism $\lambda^* : \mathcal{H}^2_\Delta(X_0, g) \to \mathcal{H}^2_\Delta(X_0, g)$. Given that $\lambda^*[\omega_g] = [\omega_g]$ and $\omega_g$ is harmonic with respect to $g$ (Proposition 2.6.1), we must have $\lambda^*\omega_g = \omega_g$ by the uniqueness of harmonic representatives. Furthermore, it must be true that $\lambda^* I = I$, since any two of $I, g$ and $\omega_g$ determine the third. As we have seen (Proposition 5.1.4), since $\lambda$ is a free holomorphic involution of $(X_0, I)$, we have $\lambda^*\kappa = -\kappa$, i.e. $(\omega_g, \kappa) \in \mathcal{R}(X_0, \lambda)$. Now, $\sigma$ is an Enriques involution, so there exists $\kappa' \in \mathcal{K}(X_0)$ such that $\sigma^*\kappa' = -\kappa'$. We can equip the Enriques surface defined by the triple $(X_0, \kappa', \sigma)$ with a Kähler-Einstein metric $g'$ with associated Kähler form $\omega' \in \Omega^+_{\kappa'}(X_0, \sigma)$. In particular, $\sigma^* g' = g'$ and Lemma 5.3.2 implies there exists a diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ with $\lambda \circ f = f \circ \sigma$, as $\sigma$ acts as $\rho$ under the fixed Enriques isometry of $\sigma$. It immediately follows that $\sigma^*(f^*\kappa) = -f^*\kappa$ and $\sigma^*(f^*\omega_g) = f^*\omega_g$, that is to say, $(f^*\omega_g, f^*\kappa) \in \mathcal{R}(X_0, \sigma)$. But since $f^*$ acts trivially on $H^2(X_0, \mathbb{Z})$, we see $\hat{\Pi}(f^*\omega_g, f^*\kappa) = (a, c)$, and so the refined period map for Enriques surfaces is surjective. □

To conclude this section, we deduce the surjectivity of the period map for Enriques surfaces. No particularly technical tools are required for its proof; we apply a simple genericity argument.

**Corollary 5.3.1.** *Suppose $\sigma : X_0 \to X_0$ is an Enriques involution and an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma$ is fixed. Then, the period map $\Pi : \mathcal{K}^-(X_0, \sigma) \to \mathcal{D}^-$ is surjective.*

*Proof.* Suppose $c \in \mathcal{D}^-$ is arbitrary. Recall this means that $c \in \Omega^-$ and $c \cdot d \neq 0$ for every $d \in \Delta^-$. Let us define the set

$$S_c := \{a \in L_{\mathbb{R}} \mid a \cdot c = 0, \rho(a) = a, a \cdot a > 0\},$$

which is an open subset of the non-empty linear subspace $\{a \in L_{\mathbb{R}} \mid a \cdot c = 0, \rho(a) = a\} \subseteq L_{\mathbb{R}}$. Since vector spaces are examples of Baire spaces (by the Baire category theorem) and open subsets of Baire spaces are themselves Baire, $S_c$ is a Baire space. Given a lattice element $d \in L$, define the following hyperplane

$$C_d := \{a \in S_c \mid a \cdot d = 0\}.$$

Note that the space $\Delta \backslash \Delta^-$ is countable, and hence the set $\bigcup_{d \in \Delta \backslash \Delta^-} C_d$ is a countable union of nowhere-dense subsets of a Baire space, and thus has empty interior. In particular, the hyperplanes $C_d$ for $d \in \Delta \setminus \Delta^-$ cannot cover $S_c$, and so the space

$$S_c \setminus \bigcup_{d \in \Delta \backslash \Delta^-} C_d$$

is non-empty. If $a \in L_{\mathbb{R}}$ is an arbitrary member of the above set, then by construction $a \cdot d \neq 0$ or $c \cdot d \neq 0$ for all $d \in \Delta$, and either $S(-a, c) \in \mathrm{Gr}^0_+(L_{\mathbb{R}})$ or $S(a, c) \in \mathrm{Gr}^0_+(L_{\mathbb{R}})$, that is to say, either $(-a, c) \in \mathcal{P}^-$ or $(a, c) \in \mathcal{P}^-$. The surjectivity of the refined period map for Enriques surfaces (Theorem 5.3.1) gives $\kappa \in \mathcal{K}^-(X_0, \sigma)$ with $[\kappa] = c$. □

## 5.4 Torelli Theorems

Now that we have proved the surjectivity of the respective period maps, we come to establish the Torelli theorems for both the classical and refined period maps for Enriques surfaces. These tell us that the period point of an Enriques surface (or, one with a Kähler structure) determines its isomorphism class. This section is arguably the simplest in our investigations into Enriques surfaces. We begin by restating and proving Torelli's theorem for Enriques surfaces, which is a simple consequence of the corresponding result for K3 surfaces. We will then see how the weak Torelli theorem follows from this.

**Theorem 5.4.1.** *Let $\sigma : X_0 \to X_0$ be an Enriques involution and fix an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma$. Then, if $\hat{\Pi}(\omega_1, \kappa_1) = \hat{\Pi}(\omega_2, \kappa_2)$ for arbitrary $(\omega_1, \kappa_1), (\omega_2, \kappa_2) \in \mathcal{R}(X_0, \sigma)$, there exists a unique diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ such that $f^*\kappa_2 = \kappa_1$ and $\sigma \circ f = f \circ \sigma$. Moreover, if $\omega_1$ and $\omega_2$ are Kähler-Einstein, then $f^*\omega_2 = \omega_1$.*

*Proof.* We might as well assume that $\omega_i \in \Omega^+_{\kappa_i}(X_0, \sigma)$ is Kähler-Einstein with respect to the structure defined by $\kappa_i \in \mathcal{K}^-(X_0, \sigma)$, $i = 1, 2$. By Torelli's theorem for K3 surfaces (Theorem 4.4.5), there exists a unique diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ such that $f^*\kappa_2 = \kappa_1$ and $f^*\omega_2 = \omega_1$. Since $(\omega_i, \kappa_i) \in \mathcal{R}(X_0, \sigma)$ for $i = 1, 2$, we have

$$(\sigma \circ f \circ \sigma)^*\kappa_2 = -\sigma^*(f^*\kappa_2) = -\sigma^*\kappa_1 = \kappa_1,$$
$$(\sigma \circ f \circ \sigma)^*\omega_2 = \sigma^*(f^*\omega_2) = \sigma^*\omega_1 = \omega_1.$$

The uniqueness part of Theorem 4.4.5 implies $f = \sigma \circ f \circ \sigma$, given that the composition $\sigma \circ f \circ \sigma \in \mathrm{Diff}_T(X_0)$. $\square$

**Remark 5.4.1.** Note that the proof of Torelli's theorem for K3 surfaces in [7, Section 2.4] was the most complicated part of Buchdahl's paper, whereas in our case the Torelli theorem for Enriques surfaces follows trivially from this.

As in the case of K3 surfaces (cf. Remark 4.4.7), we can use Picard-Lefschetz reflections to deduce the weak Torelli theorem for Enriques surfaces from Torelli's theorem above. The proof shown below is based on that of [44, Proposition 21.1] (note that an ample class in the Néron-Severi group of a compact complex manifold is exactly the same as a positive class, thanks to the Kodaira embedding theorem).

**Corollary 5.4.1.** *Let $\sigma : X_0 \to X_0$ be an Enriques involution and fix an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma$. Then, if $\Pi(\kappa_1) = \Pi(\kappa_2)$ for arbitrary $\kappa_1, \kappa_2 \in \mathcal{K}^-(X_0, \sigma)$, there exists a diffeomorphism $f : X_0 \to X_0$ such that $f^*\kappa_2 = \kappa_1$ and $\sigma \circ f = f \circ \sigma$.*

*Proof.* Suppose $c := [\kappa_1] = [\kappa_2] \in H^2(X_0, \mathbb{C})$ for arbitrary $\kappa_1, \kappa_2 \in \mathcal{K}^-(X_0, \sigma)$. Let $X_i$ be the K3 surface defined by $\kappa_i$, and $E_i$ correspond to the triple $(X_0, \kappa_i, \sigma)$ for $i = 1, 2$. Let us also denote the subset of roots which define $(1,1)$-classes for $X_1$ and $X_2$ by

$$\Delta_c := \{d \in \Delta \,|\, d \cdot c = 0\}.$$

Now, since Enriques surfaces are algebraic (Proposition 5.1.3), choose a positive line bundle for $E_2$, pull it back to $X_2$ and denote the resulting positive line bundle by $L \to X_2$. Then, by assumption, the class $a := c_1(L) \in H^2(X_0, \mathbb{R})$ is represented by a Kähler form $\omega \in \Omega^+_{\kappa_2}(X_0, \sigma)$, and defines a positive element $a \in \mathrm{NS}(X_2)$ of the Néron-Severi group for $X_2$. We have $(a, c) \in \mathcal{P}^-$, and moreover, $a \cdot c = a \cdot [\kappa_1] = 0$ implies that $a$ is also a $(1,1)$-class for $X_1$, and hence defines an element $a \in \mathrm{NS}(X_1)$ by the Lefschetz theorem on $(1,1)$-classes (Theorem 2.4.4). Note that $a$ must at least lie in the positive cone of $X_1$. Indeed, if $a'$ is any Kähler class for $X_1$, then $a \cdot a' > 0$, since both subspaces $S(a, c)$ and $S(a', c)$ lie in $\mathrm{Gr}^0_+(L_{\mathbb{R}})$.

Now, if $a$ happens to also define a positive class in $\mathrm{NS}(X_1)$, then by definition, it can be represented by a Kähler form $a = [\omega']$ for some $\omega' \in \Omega^+_{\kappa_1}(X_0, \sigma)$. Then, $\hat{\Pi}(\omega', \kappa_1) = \hat{\Pi}(\omega, \kappa_2)$ and Torelli's theorem for Enriques surfaces (Theorem 5.4.1) yields the result.

Suppose then that the Kähler cones of $X_1$ and $X_2$ do not agree, and that $a$ is not positive for $\mathrm{NS}(X_1)$. Since $(a, c) \in \mathcal{P}^-$, we have that $a \cdot d \neq 0$ for every $d \in \Delta_c$, and so we can proceed as Barth *et al.* do. Denote the quotient map by $\pi : X_1 \to E_1$. Given that $\sigma^*(a) = a$, the class $\pi_*(a) \in \mathrm{NS}(E_1)$ is also not positive. The Nakai-Moishezon criterion (Theorem 4.3.2) and the fact that all line bundles are divisorial on an algebraic manifold (Remark 2.7.1) implies there exists an irreducible divisor $C_1 \in \mathrm{Div}(E_1)$ such that $\pi_*(a) \cdot e_1 \leq 0$, where $e_1 := c_1(\mathcal{O}(C_1)) \in H^2(E_1, \mathbb{Z})$. Barth *et al.* then show that the curve $\pi^{-1}(C_1)$ consists of two disjoint irreducible $(-2)$-curves $D_1$ and $D_2 := \sigma(D_1)$. If the Poincaré duals of their associated classes are denoted $d_1, d_2 = \sigma^*(d_1) \in \Delta_c$, then $d_1 \cdot d_2 = 0$. As usual (cf. Definition 4.3.2), given any $d \in \Delta$, denote its associated Picard-Lefschetz reflection by

$$L_d : L \to L,\ x \mapsto x + (x \cdot d)d.$$

Note that if we are given any $(1,1)$-class $d \in \Delta_c$, then $L_d(c) = c$. Since $d_1 \cdot d_2 = 0$, a simple calculation shows that $L_{d_1} \circ L_{d_2} = L_{d_2} \circ L_{d_1}$. Moreover, given that $d_2 = \sigma^*(d_1)$, the product $L_{d_1} \circ L_{d_2}$ commutes with $\sigma$, and in fact, $\sigma^* \circ L_{d_1} \circ \sigma^* = L_{d_2}$. Now, $D_1$ and $D_2$ define embedded $(-2)$-spheres in $X_0$; following Matumoto [36, Lemma 2.1], there exists a self-diffeomorphism $f_1 \in \mathrm{Diff}(X_0)$ supported in a neighbourhood of $D_1$, that is, $f_1$ is the identity outside of a neighbourhood $U$ of $D_1$, such that $f_1^* = L_{d_1}$ on $L \cong H^2(X_0, \mathbb{Z})$. Setting $f_2 := \sigma \circ f_1 \circ \sigma \in \mathrm{Diff}(X_0)$, we find that $f_2$ is supported in a neighbourhood of $\sigma(D_1) = D_2$ with $\sigma(U) \cap U = \emptyset$, and so $f_1$ and $f_2$ commute. In particular, one easily sees that $f_1 \circ f_2$ commutes with $\sigma$. Also, note that by definition we have $f_2^* = \sigma^* \circ L_{d_1} \circ \sigma^* = L_{d_2}$. Barth *et al.* show that repeating this process a finite number of times yields a finite ordered set of classes $d_1, d_2 = \sigma^*(d_1), \ldots, d_{2k-1}, d_{2k} = \sigma^*(d_{2k-1}) \in \Delta_c$ such that the product of their reflections $\phi := L_{d_{2k}} \circ \ldots \circ L_{d_1}$ commutes with $\sigma^*$ and $\phi(a) \in \mathrm{NS}(X_1)$ is positive for $X_1$. Moreover, by repeating the above construction, we get a collection of diffeomorphisms $f_1, \ldots, f_{2k} \in \mathrm{Diff}(X_0)$ such that $g := f_1 \circ \ldots \circ f_{2k}$ commutes with $\sigma$ and $f_i^* = L_{d_i}$, $i = 1, \ldots, 2k$. In particular, $g^* = \phi$ and $\phi(c) = c$, i.e. $[g^*\kappa_2] = [\kappa_1]$, since $d_i \in \Delta_c$ for each $i$. By construction, $g$ takes the Kähler cone of $X_2$ to the Kähler cone of $X_1$ (Proposition 4.3.2), and hence there exists $[\omega_1] = [\omega_2] \in H^2(X_0, \mathbb{R})$ where $\omega_1 \in \Omega^+_{\kappa_1}(X_0, \sigma)$ and $\omega_2 \in \Omega^+_{g^*\kappa_2}(X_0, \sigma)$. Then, $\hat{\Pi}(\omega_1, \kappa_1) = \hat{\Pi}(\omega_2, g^*\kappa_2)$, and Torelli's theorem for Enriques surfaces (Theorem 5.4.1) yields $f \in \mathrm{Diff}_T(X_0)$ such that $f \circ \sigma = \sigma \circ f$ and $f^*(g^*\kappa_2) = (g \circ f)^*\kappa_2 = \kappa_1$. Since $f$ and $g$ commute with $\sigma$, so too does $g \circ f$, which concludes the proof. □

In Section 4.5 of Chapter 4, we investigated the moduli space of Einstein metrics on a K3 surface and stated a Torelli-like theorem. Before closing this section off, let us briefly extend these considerations to the case of Enriques surfaces. For the remainder of this section, we fix an Enriques surface $E$ with its universal K3 covering $\pi : X \to E$ and covering involution $\sigma : X \to X$. As usual, we fix an Enriques isometry $H^2(X, \mathbb{Z}) \cong L$ of $\sigma$

so that $\sigma^* = \rho$. Note that since $X$ is simply connected, every diffeomorphism $f \in \mathrm{Diff}(E)$ automatically lifts to a diffeomorphism $\widetilde{f} \in \mathrm{Diff}(X)$. Moreover, if $f$ is homotopic to the identity, then certainly $\widetilde{f} \in \mathrm{Diff}_T(X)$. We make the following definition:

**Definition 5.4.1.** Denote by $\mathrm{Diff}_T(X, E) \subset \mathrm{Diff}(E)$ the subgroup consisting of those diffeomorphisms $f \in \mathrm{Diff}(E)$ which admit a lift $\widetilde{f} : X \to X$ such that $\widetilde{f} \in \mathrm{Diff}_T(X)$.

Using some simple algebraic topology, we can describe the group $\mathrm{Diff}_T(X, E)$ in a different but equivalent way:

**Proposition 5.4.1.** *The subgroup* $\mathrm{Diff}_T(X, E)$ *consists precisely of those diffeomorphisms* $f : E \to E$ *which act trivially on* $\pi_2(E)$*.*

*Proof.* Since $\pi : X \to E$ is a covering, it induces an isomorphism $\pi_* : \pi_2(X) \to \pi_2(E)$. Moreover, given that $X$ is simply connected, Hurewicz' theorem tells us that the canonical map

$$h_2 : \pi_2(X) \to H_2(X, \mathbb{Z}),\ \gamma \mapsto \gamma_*[S^2]$$

is an isomorphism, where $\gamma \in \pi_2(X)$ is thought of as a (homotopy class of a) map $\gamma : S^2 \to X$. Thus, if $\widetilde{f} \in \mathrm{Diff}(X)$ denotes the lift of $f \in \mathrm{Diff}(E)$, we see that $f_* = \mathrm{id}_{\pi_2(E)}$ if and only if $\widetilde{f}_* = \mathrm{id}_{\pi_2(X)}$. Observe that we also have a commutative diagram:

$$\begin{array}{ccccc}
\pi_2(X) & \xrightarrow{h_2} & H_2(X,\mathbb{Z}) & \xrightarrow{\mathscr{P}_X^{-1}} & H^2(X,\mathbb{Z}) \\
\widetilde{f}_* \uparrow & & \widetilde{f}_* \uparrow \ \uparrow (\widetilde{f}^!)^{-1} & & \downarrow \widetilde{f}^* \\
\pi_2(X) & \xrightarrow[h_2]{} & H_2(X,\mathbb{Z}) & \xrightarrow[\mathscr{P}_X^{-1}]{} & H^2(X,\mathbb{Z})
\end{array}$$

The fact that both squares of the diagram commute follows by definition. We just have to establish the equality $\widetilde{f}^! = (\widetilde{f}^{-1})_*$ on homology (that is, the two middle vertical arrows are really the same). However, this follows immediately by the naturality of the cap product:

$$\widetilde{f}_*(\widetilde{f}^*(\alpha) \frown c) = \alpha \frown \widetilde{f}_*(c),$$

and the fact that $\widetilde{f}_*[X] = [X]$. Thus, we see that $\widetilde{f}^* = \mathrm{id}_{H^2(X,\mathbb{Z})}$ if and only if $\widetilde{f}_* = \mathrm{id}_{\pi_2(X)}$. Hence, $f \in \mathrm{Diff}_T(X, E)$ if and only if $f_* = \mathrm{id}_{\pi_2(E)}$, and we are done. □

We now wish to construct a Teichmüller space for $E$ which is related to the usual Teichmüller space $\mathcal{T}(X) = \mathrm{Ein}(X)/\mathrm{Diff}_T(X)$ of the K3 surface $X$. For simplicity, we opt to consider the following:

**Definition 5.4.2.** Define the Teichmüller space of the Enriques surface $E$ by

$$\mathcal{T}(E) := \mathrm{Ein}(E)/\mathrm{Diff}_T(X, E).$$

**Remark 5.4.2.** It is not at all obvious what the structure of the quotient space above is (beyond that of a topological space). For our purposes, we will not be concerned with this, although it would certainly be interesting to investigate. Equally, it is not clear what the structure of the quotient $\mathrm{Ein}(E)/\mathrm{Diff}_T(E)$ would be, where as usual $\mathrm{Diff}_T(E) = \{f \in \mathrm{Diff}(E) \mid f^* = \mathrm{id}_{H^2(E,\mathbb{Z})}\}$.

Given that the covering $\pi : X \to E$ has degree 2, pulling back a metric $g \in \mathrm{Riem}(E)$ yields a metric $\pi^* g$ on $X$ with twice the volume, and which is invariant under $\sigma$. Thus, after rescaling, pulling back gives us a natural map:

$$\alpha := \frac{1}{\sqrt{2}}\pi^* : \mathrm{Ein}(E) \to \mathrm{Ein}(X),\ g \mapsto \widetilde{g} := \frac{1}{\sqrt{2}}\pi^* g.$$

It is easy to see that it descends to a map $\alpha : \mathcal{T}(E) \to \mathcal{T}(X)$. Indeed, if $g \in \mathrm{Ein}(E)$ and $f \in \mathrm{Diff}_T(X, E)$, then since $\pi \circ \widetilde{f} = f \circ \pi$ for the lift $\widetilde{f} \in \mathrm{Diff}_T(X)$, we have

$$\alpha(f^* g) = \frac{1}{\sqrt{2}}\pi^*(f^* g) = \frac{1}{\sqrt{2}}(f \circ \pi)^* g = \frac{1}{\sqrt{2}}(\pi \circ \widetilde{f})^* g = \widetilde{f}^*(\alpha(g)).$$

Moreover, we have that pulling back by $\sigma$ gives us a map $\sigma^* : \mathrm{Ein}(X) \to \mathrm{Ein}(X)$, which descends to a map $\sigma^* : \mathcal{T}(X) \to \mathcal{T}(X)$, since $f \circ \sigma \circ f \in \mathrm{Diff}_T(X)$ for $f \in \mathrm{Diff}_T(X)$. Furthermore, any $\sigma$-invariant Einstein metric of volume 2 on $X$ is certainly the pullback of an element of $\mathrm{Ein}(E)$. Thus, we obtain:

**Proposition 5.4.2.** *The image of the canonical map* $\alpha : \mathcal{T}(E) \to \mathcal{T}(X)$, $[g] \mapsto \left[\frac{1}{\sqrt{2}}\pi^* g\right]$ *is the fixed-point set of the action of* $\sigma$ *on* $\mathcal{T}(X)$*, which is precisely* $\mathcal{T}(X)^\rho$ *under the fixed Enriques isometry of* $\sigma$*.*

Using what we know about the period map, it is not difficult to show that the map also has to be injective:

**Theorem 5.4.2.** *The map* $\alpha : \mathcal{T}(E) \to \mathcal{T}(X)$ *is a bijection onto its image* $\mathcal{T}(X)^\rho$*.*

*Proof.* Given $[g_1], [g_2] \in \mathcal{T}(E)$, suppose $\alpha([g_1]) = \alpha([g_2])$. Thus, there exists $\widetilde{f} \in \mathrm{Diff}_T(X)$ such that $\widetilde{f}^*\widetilde{g}_2 = \widetilde{g}_1$ for $\widetilde{g}_i := \frac{1}{\sqrt{2}}\pi^* g_i$, $i = 1, 2$. Given that $\sigma^*\widetilde{g}_i = \widetilde{g}_i \in \mathrm{Ein}(X)$ for $i = 1, 2$, we have:

$$\begin{aligned} &\widetilde{f}^*\widetilde{g}_2 = \widetilde{g}_1 \\ \iff &\widetilde{f}^*(\sigma^*\widetilde{g}_2) = \sigma^*\widetilde{g}_1 \\ \iff &\sigma^*(\widetilde{f}^*(\sigma^*\widetilde{g}_2)) = \widetilde{g}_1 = \widetilde{f}^*\widetilde{g}_2 \\ \iff &(\sigma \circ \widetilde{f} \circ \sigma \circ \widetilde{f}^{-1})^*\widetilde{g}_2 = \widetilde{g}_2. \end{aligned}$$

So, $\sigma \circ \widetilde{f} \circ \sigma \circ \widetilde{f}^{-1} \in \mathrm{Diff}_T(X)$ is an isometry of $\widetilde{g}_2$, and hence $\sigma \circ \widetilde{f} \circ \sigma = \widetilde{f}$ because $I_{\widetilde{g}_2} \cap \mathrm{Diff}_T(X) = \{\mathrm{id}_X\}$ by Proposition 4.5.2. Thus, $\widetilde{f}$ descends to $f \in \mathrm{Diff}_T(X, E)$. As a result,

$$\pi^*g_1 = \widetilde{f}^*(\pi^*g_2) = \pi^*(f^*g_2).$$

However, since $\pi : X \to E$ is a covering space, the map

$$\pi^* : \mathrm{Riem}(E) \to \mathrm{Riem}(X),\ g \mapsto \pi^*g$$

is injective, and so $f^*g_2 = g_1$, which means $[g_1] = [g_2] \in \mathcal{T}(E)$. Thus, $\alpha$ is injective and hence a bijection onto its image. □

**Remark 5.4.3.** Note that pulling back certainly yields a bijection on the level of Einstein metrics $\alpha : \mathrm{Ein}(E) \to \mathrm{Ein}(X)^\sigma$. Although we do not show the details of this, it can be shown that pulling back is continuous (with respect to the $C^\infty$-topology). Moreover, given a sequence of $\sigma$-invariant metrics converging to a $\sigma$-invariant metric on the total space, this yields a convergent sequence of induced metrics on the base, and so pulling back Einstein metrics defines a homeomorphism onto its image, which must descend to a homeomorphism $\mathcal{T}(E) \to \mathcal{T}(X)^\rho$.

## 5.5 Deformation Equivalence of Enriques Surfaces

To conclude our investigations into Enriques surfaces, in this final section we will prove their deformation equivalence. For the proof, we will begin by showing that all Enriques surfaces are diffeomorphic, which more or less follows directly from Lemma 5.3.2. This will allow us to fix one Enriques involution on $X_0$, and moreover, as a result of Remark 2.3.1, we then simply have to show that any two integrable almost complex structures for a fixed smooth Enriques surface can be connected by a continuous path. Using Lemma 5.3.1 and the refined period map for Enriques surfaces, we will be able to explicitly construct a deformation between any two (Kähler) Enriques surfaces defining period points in the same connected component of $\mathcal{P}^-$. The only problem is that $\mathcal{P}^-$ has two connected components! This latter point makes things slightly more awkward, but to overcome this difficulty, we will use the fact that *real* Enriques surfaces exist, that is to say, Enriques surfaces with anti-holomorphic involutions (see Remark 4.5.1). Why this is useful is for the following reason: if $E$ is an Enriques surface which corresponds to some period point $c \in \mathcal{D}^- \subset \Omega^-$, then its conjugate $\overline{E}$ will correspond to the period point $\overline{c}$, which will not lie in the same component as $c$ (Proposition 5.1.6). Before establishing their existence, we begin by showing:

**Theorem 5.5.1.** *All Enriques surfaces are diffeomorphic.*

*Proof.* Suppose $E_1$ and $E_2$ are any two Enriques surfaces. As in Remark 5.1.1, each $E_i$ corresponds to a triple $(X_0, \kappa_i, \sigma_i)$, where $\sigma_i \in \mathrm{Diff}(X_0)$ is an Enriques involution and $\kappa_i \in \mathcal{K}^-(X_0, \sigma_i)$ for $i = 1, 2$. Let us fix Enriques isometries $\varphi_i : H^2(X_0, \mathbb{Z}) \cong L$ for both $\sigma_1$ and $\sigma_2$. We can equip $E_i$ with Kähler-Einstein metrics $g_i \in \mathrm{Ein}(X_0)$ with corresponding Kähler forms $\omega_i \in \Omega^+_{\kappa_i}(X_0, \sigma_i)$. In particular, $\sigma_i \in \mathrm{Isom}(X_0, g_i)$. Define $a_i := \varphi_i([\omega_i]) \in L_{\mathbb{R}}$ and $c_i := \varphi_i([\kappa_i]) \in L_{\mathbb{C}}$. We then have $\mathcal{H}^+_{g_i}(X_0) = \langle a_i, \mathrm{Re}(c_i), \mathrm{Im}(c_i)\rangle$. Then, Lemma 5.3.2 tells us there exist free smooth involutions $\lambda_i \in \mathrm{Isom}(X_0, g_i)$ such that $\lambda_i^* = \sigma_i^*$ (given that each $\sigma_i$ acts as $\rho$ under their respective Enriques isometries). But, by the uniqueness of such isometries, we have $\lambda_i = \sigma_i$. Lemma 5.3.2 then shows that $f \circ \sigma_1 = \sigma_2 \circ f$ for some diffeomorphism $f \in \mathrm{Diff}_T(X_0)$, that is, $E_1$ and $E_2$ are diffeomorphic. □

As in Remark 5.1.6, we can then fix an Enriques involution $\sigma_0 : X_0 \to X_0$ so that any Enriques surface is defined by a structure $\kappa \in \mathcal{K}^-(X_0, \sigma_0)$. We then also fix an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma_0$ once and for all. Now, let us establish the existence of real Enriques surfaces using the results of previous sections:

**Lemma 5.5.1.** *Real Enriques surfaces exists, that is, there exists an Enriques surface $E$ which has an anti-holomorphic involution $\theta : E \to \overline{E}$.*

*Proof.* Since every Enriques surface is double-covered by a K3 surface, we can convert the problem of finding a real structure on an Enriques surface to finding a free holomorphic involution on a K3 surface which commutes with a real structure. That is, if $E$ is given by $\kappa \in \mathcal{K}^-(X_0, \sigma_0)$, then a smooth involution $\mu : X_0 \to X_0$ such that $\mu \circ \sigma_0 = \sigma_0 \circ \mu$ and $\mu^* \kappa = \overline{\kappa}$ will define a real structure on the Enriques surface.

To begin, let us find a suitable period point $c \in \mathcal{D}^-$ for an Enriques surface. To simplify matters, we will set the $E_8$ components to zero. If $c = (x, y, z, 0, 0) \in L_{\mathbb{C}}$ satisfies $\rho(c) = -c$, then $c = (x, y, -y, 0, 0)$ for some $x, y \in H_{\mathbb{C}}$. We need $c^2 = x^2 + 2y^2 = 0$ and $c \cdot \overline{c} = x \cdot \overline{x} + 2(y \cdot \overline{y}) > 0$. One possibility is $x = i\beta = i(2e_1 + e_2)$ and $y = e_1 + e_2$, where $e_1, e_2$ form the standard basis for the lattice $H$. Then

$$c^2 = -(2e_1 + e_2)^2 + 2(e_1 + e_2)^2 = 0,$$

since $e_1 \cdot e_2 = 1$ and $e_i^2 = 0$ for $i = 1, 2$. Moreover, we have $c \cdot \bar{c} = 8 > 0$. If $c$ is to correspond to an Enriques surface, then for any $d \in \Delta^-$ we must have $c \cdot d \neq 0$. So let $d \in L$ be arbitrary. If $\rho(d) = -d$, then $d = (a, b, -b, u, -u)$ for some $a, b \in H$ and $u \in E_8$. Moreover, $d$ must satisfy $d^2 = -2 = a^2 + 2b^2 + 2u^2$. Then

$$c \cdot d = i(\beta \cdot a) + 2(y \cdot b).$$

We need this to be non-zero. If it were zero, then $\beta \cdot a = y \cdot b = 0$. If we set $a = a_i e_i$ and $b = b_i e_i$ for $a_i, b_i \in \mathbb{Z}$, $i = 1, 2$, then

$$\beta \cdot a = 0 \iff 2a_2 + a_1 = 0,$$
$$y \cdot b = 0 \iff b_1 + b_2 = 0.$$

In particular, exactly one of $a_1$ and $a_2$ must be negative, and similarly for $b_1$ and $b_2$. Since $d^2 = -2$, $a^2 = 2a_1a_2$ and $b^2 = 2b_1b_2$, we have

$$a^2 + 2b^2 + 2u^2 = -2 \iff u^2 = -1 - 2b_1b_2 - a_1a_2.$$

Since $E_8$ is negative definite, it must be the case that $u^2 \leq 0$. However, $-a_1a_2 \geq 0$ and $-b_1b_2 \geq 0$, so $u^2 \geq -1$. But $u^2 \neq -1$, as $E_8$ is even. It also cannot be zero, for then we have

$$a_1^2 + b_1^2 = \frac{1}{2},$$

which is impossible since $a_i, b_i \in \mathbb{Z}$. Thus, $c = (i\beta, y, -y, 0, 0) \in \mathcal{D}^-$ corresponds to an Enriques surface by the surjectivity of the period map $\Pi : \mathcal{K}^-(X_0, \sigma_0) \to \mathcal{D}^-$ (Corollary 5.3.1).

Now, in order to construct an anti-holomorphic involution, we will first find an involution $\phi \in \mathrm{Aut}(L)$ of the K3 lattice which commutes with $\rho$, and then lift it to a diffeomorphism defining an anti-holomorphic involution for the complex structure associated to $c$ above. There are many involutions $\phi$ that we could make work, but notice that if we define

$$\phi : L \to L,\ (x, y, z, u, v) \mapsto (-x, -z, -y, v, u),$$

then

$$\phi(c) = \phi(i\beta, y, -y, 0, 0) = \phi(-i\beta, y, -y, 0, 0) = \bar{c},$$

and it is obvious that $\phi$ commutes with $\rho$. Moreover, to see that $\phi$ preserves orientation on maximal positive definite subspaces, we can use exactly the same argument as in Proposition 5.2.5. Thus, $\phi \in \Gamma$.

Now, applying the same genericity argument as in the proof of Corollary 5.3.1, we can find $a \in L_{\mathbb{R}}$ such that $(a,c) \in \mathcal{P}^-$ and $\phi(a) = -a$, since $L_\phi^- \cap L_\rho^+ \neq \{0\}$. The surjectivity of the refined period map $\hat{\Pi} : \mathcal{R}(X_0, \sigma_0) \to \mathcal{P}^-$ yields a Kähler Enriques surface defined by a complex structure $\kappa \in \mathcal{K}^-(X_0, \sigma_0)$ with associated Kähler form $\omega \in \Omega_\kappa^+(X_0, \sigma_0)$ such that $[\omega] = a$ and $[\kappa] = c$. Using Yau's theorem, we can assume that $\omega$ is Ricci-flat Kähler-Einstein with associated metric $g \in \mathrm{Ein}(X_0)$ such that $\sigma_0 \in I_g$. Then we have $\mathcal{H}_g^+(X_0) = S(a,c) \in \mathcal{E}$, and note that both $\rho$ and $\phi$ preserve $S(a,c)$. Hence, if $t = [g]$ is the unique point in the Teichmüller space with $\tau(t) = \mathcal{H}_g^+(X_0)$, then under the isomorphism $\Gamma_t \cong I_g$ guaranteed by Lemma 5.3.1, $\rho$ lifts to $\sigma_0 \in I_g$ (since $\sigma_0$ acts as $\rho$ under the fixed Enriques isometry) and $\phi$ lifts to some $\mu \in I_g$. Since $\rho \circ \phi = (\mu \circ \sigma_0)^* = \phi \circ \rho = (\sigma_0 \circ \mu)^*$, Proposition 4.5.2 tells us $\mu \circ \sigma_0 = \sigma_0 \circ \mu$. Moreover, we have $[\mu^*\omega] = [-\omega]$ and $[\mu^*\kappa] = [\overline{\kappa}]$, and since $\mu$ defines an involution $\mu^* : \mathcal{H}^2_\Delta(X_0, g) \to \mathcal{H}^2_\Delta(X_0, g)$, we find $\mu^*\kappa = \overline{\kappa}$. □

We can now prove the main result of this section. We chiefly make use of Lemma 5.3.1 and the fact that Einstein metrics on K3 surfaces are hyperkähler. One useful observation we make to this end is the following: if we are given a continuous path of Einstein metrics $g_s : [0,1] \to \mathrm{Ein}(X_0)$ on a K3 surface, then we certainly obtain a continuously-varying path of hyperkähler structures $(I_t, J_t, K_t)$ for $X_0$ (i.e. a continuous section of the associated principal SO(3)-bundle $P \to [0,1]$).

**Theorem 5.5.2.** *All Enriques surfaces are deformation equivalent.*

*Proof.* As before, fix an Enriques isometry $H^2(X_0, \mathbb{Z}) \cong L$ of $\sigma_0$ and consider the associated refined period map $\hat{\Pi} : \mathcal{R}(X_0, \sigma_0) \to \mathcal{P}^-$. To begin, let $E_1$ be an arbitrary Enriques surface equipped with a Kähler-Einstein structure defined by $(\omega_1, \kappa_1) \in \mathcal{R}(X_0, \sigma_0)$ and $(a_1, c_1) := \hat{\Pi}(\omega_1, \kappa_1) \in \mathcal{P}^-$ be the associated point in the refined period domain. We will show that all Enriques surfaces defining period points in some neighbourhood of $(a_1, c_1)$ are deformation equivalent to $E_1$. This is certainly sufficient, given that Torelli's theorem (Theorem 5.4.1) states that the fibres of the refined period map $\hat{\Pi}$ consist of biholomorphic Enriques surfaces (equipped with Kähler structures), and so all of the Enriques surfaces defining period points in each of the two connected components of $\mathcal{P}^-$ (cf. Proposition 5.1.6) are deformation equivalent. In order to deduce that all Enriques surfaces are deformation equivalent from this, note that by Lemma 5.5.1, there exists a real (Kähler-Einstein) Enriques surface $E$ which defines a point $(a,c) \in \mathcal{P}^-$, and hence its conjugate $\overline{E}$ is defined by the point $(-a, \overline{c})$, which must lie in the other component of $\mathcal{P}^-$ by Proposition 5.1.6, and by assumption, $E$ and $\overline{E}$ are deformation equivalent, since the constructed anti-holomorphic involution $\theta : E \to \overline{E}$ satisfies $\theta^* I = -I$, where $I$ is the associated complex structure for $E$.

Now, denote the associated Einstein metric for $E_1$ by $g_1 \in \mathrm{Ein}(X_0)$, which is hyperkähler with structure $(I_1, J_1, K_1)$. After normalising (i.e. $a_1^2 = 1$ and $c_1 \cdot \overline{c_1} = 2$), we can assume $\omega_1 = \omega_{I_1}$ and $\kappa_1 = \omega_{J_1} + i\omega_{K_1}$ (this is a general fact for Einstein metrics on K3 surfaces: self-dual harmonic 2-forms of unit norm are in one-to-one correspondence with complex structures that are compatible with the metric and orientation, see [3, p. 366]). As usual, we have $\sigma_0^* g_1 = g_1$. Recall the map $\widetilde{\pi} : \mathcal{P}^- \to \mathcal{T}(X_0)^\rho$ (cf. Definition 5.3.1). If $t_1 := [g_1] \in \mathcal{T}(X_0)$ is the associated point in the Teichmüller space with $\tau(t_1) = S(a_1, c_1)$, then we have by definition $\sigma_0^* \in \Gamma_{t_1}$. Let $\psi : U \subseteq \mathcal{T}(X_0) \to \mathrm{Ein}(X_0)$, $[g] \mapsto g$ denote the local continuous section of the projection $\mathrm{Ein}(X_0) \to \mathcal{T}(X_0)$ guaranteed by Lemma 5.3.1. If $\phi : \Gamma_{t_1} \to I_{g_1}$ denotes the isomorphism $[f] \to f$, then we have $\phi([\sigma_0]) = \rho$, as $\sigma_0$ acts as $\rho$ under the fixed Enriques isometry of $\sigma_0$. Then as we have seen (cf. proof of Lemma 5.3.2), the map $\sigma_0$ is an isometry of all the Einstein metrics in $\psi(U \cap \mathcal{T}(X_0)^\rho)$. Define $V := \widetilde{\pi}^{-1}(U \cap \mathcal{T}(X_0)^\rho)$, and by restricting to the connected component which contains $(a_1, c_1) \in \mathcal{P}^-$, we can assume $V$ and $U \cap \mathcal{T}(X_0)^\rho$ are connected.

Let $E_2$ be an arbitrary Kähler-Einstein Enriques surface defined by $(\omega_2, \kappa_2) \in \mathcal{R}(X_0, \sigma_0)$ and $g_2 \in \mathrm{Ein}(X_0)$ such that $(a_2, c_2) := \hat{\Pi}(\omega_2, \kappa_2)$ which lies in the neighbourhood $V$ of $(a_1, c_1)$. Exactly as above, we can assume that the hyperkähler triple $(I_2, J_2, K_2)$ associated to $g_2$ is defined by $\omega_2 = \omega_{I_2}$ and $\kappa_2 = \omega_{J_2} + i\omega_{K_2}$. Since $V$ is connected, there exists a path $(a_s, c_s) : [1, 2] \to V$ from $(a_1, c_1)$ to $(a_2, c_2)$. The image $t_s := \widetilde{\pi}(a_s, c_s) \in U \cap \mathcal{T}(X_0)^\rho$ defines a path in the Teichmüller space, and composing once more with the local section $\psi$ yields a continuous path $g_s := \psi(t_s) \in \mathrm{Ein}(X_0)$ of Einstein metrics from $g_1$ to $g_2$ such that $\sigma_0^* g_s = g_s$ for every $s$. If we set $\omega_s := \omega_{I_s}$ and $\kappa_s := \omega_{J_s} + i\omega_{K_s}$, where $(I_s, J_s, K_s)$ is the associated hyperkähler triple of the Einstein metric $g_s$, then the remarks preceding this proof tell us that we have a path $(\omega_s, \kappa_s) : [1, 2] \to \mathcal{R}(X_0)$. However, given that $\sigma_0$ is a free smooth involution of the hyperkähler metric $g_s$ for every $s \in [1, 2]$, the same argument used in the surjectivity of the refined period map for Enriques surfaces (Theorem 5.3.1) shows that in fact $(\omega_s, \kappa_s) \in \mathcal{R}(X_0, \sigma_0)$ for every $s \in [1, 2]$, that is, we have constructed a deformation of the Kähler-Einstein Enriques surfaces $E_1$ and $E_2$. $\square$

# Chapter 6

# Extension to Enriques-Einstein-Hitchin Manifolds

In this short chapter, we further the investigations of Chapter 5 and see how some of the ideas there can be used to obtain information about quotients of Enriques surfaces by free anti-holomorphic involutions, that is, Enriques-Einstein-Hitchin manifolds. The main fact we wish to prove about these manifolds is that they are all diffeomorphic, or more generally of the same deformation type. We begin by taking a look at some elementary properties of these four-manifolds.

As in Chapters 4 and 5, $X_0$ will always refer to the smooth oriented four-manifold underlying some K3 surface. Moreover, $\sigma_0 : X_0 \to X_0$ will refer to a fixed Enriques involution so that every Enriques surface is defined by a form $\kappa \in \mathcal{K}^-(X_0, \sigma_0)$.

## 6.1 Background

We introduced the notion of an Enriques-Einstein-Hitchin manifold way back in Chapter 4 (see Remark 4.5.1). From now on, we will refer to such manifolds as *EEH* manifolds. One of the first questions that one may rightly ask themselves and which we have not yet dealt with is the existence of these objects. That they do exist is indeed true (for example, Hitchin provides an example of one in his paper [21]). Following a similar argument as in Lemma 5.5.1, one could in principle construct a *free* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on a K3 surface to establish the existence of EEH manifolds, but note that constructing an *explicit* description of a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on the K3 lattice $L = 3H \oplus -2E_8$ which should correspond to the lifted action of a free anti-holomorphic involution on an Enriques surface is not straightforward. As we shall see in the next section, having an explicit description of the action will not be necessary for our purposes.

As the above suggests, one way of obtaining an EEH manifold is by finding commuting free holomorphic and anti-holomorphic involutions on a K3 surface. But it is not immediately obvious that the converse is true, that is to say, that any free anti-holomorphic involution on an Enriques surface lifts to an empty real structure, i.e. free anti-holomorphic involution, on the universal K3 covering, necessarily commuting with the covering involution. The main point of difficulty is showing that the lifted structure has to be an involution. Hitchin in [21] proves this fact, but he assumes that the EEH manifold is equipped with an Einstein metric. We show this without this additional assumption:

**Proposition 6.1.1.** *If $\pi : X \to E = X/\langle\sigma\rangle$ denotes the universal K3 double cover of an Enriques surface $E$, then any empty real structure for $E$ lifts to an empty real structure for $X$ commuting with $\sigma$. In particular, EEH manifolds are in one-to-one correspondence with K3 surfaces equipped with a free $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action, where one factor acts holomorphically and the other factor anti-holomorphically.*

*Proof.* Let $\theta : E \to \overline{E}$ be an arbitrary free anti-holomorphic involution of $E$. Since $X$ is simply connected, elementary covering space theory tells us that the diffeomorphism $\theta \circ \pi : X \to E$ necessarily lifts to a diffeomorphism $\mu : X \to X$ such that $\mu$ commutes with $\sigma$. Note that $\mu$ acts freely since $\theta$ acts freely. If the complex structure for $E$ is denoted by $J$, then the equalities $d\theta \circ J = -J \circ d\theta$ and $d\theta \circ d\pi = d\pi \circ d\mu$ imply that $\mu$ is anti-holomorphic. Since $\mu^2$ covers the identity, we must either have $\mu^2 = \mathrm{id}_X$ or $\mu^2 = \sigma$. Let $G = \langle \mu, \sigma \rangle \subseteq \mathrm{Diff}(X)$ denote the group generated by $\mu$ and $\sigma$. If $\mu^2 = \sigma$, then $G \cong \mathbb{Z}_4$. We will show that this is impossible, so that $\mu^2 = \mathrm{id}_X$, and hence $\mu$ defines an empty real structure for $X$.

Denote the quotient space by $M = X/G$ and let $\pi : X \to M$ be the quotient map. Then we have $\chi(M) = \frac{1}{4}\chi(X) = 6$ and $\sigma(M) = \frac{1}{4}\sigma(X) = -4$. Given that $b_0(M) = b_4(M) = 1$, we see $b_2(M) = 4$, and so $M$ is negative definite, i.e. $b^+(M) = 0$. A standard result (see, for example, [20, Proposition 3G.1]) says that $\pi^* : H^2(M, \mathbb{R}) \to H^2(X, \mathbb{R})$ is injective with image $\pi^* H^2(M, \mathbb{R}) = H^2(X, \mathbb{R})^G := \{x \in H^2(X, \mathbb{R}) \,|\, \forall f \in G : f^*(x) = x\}$. Under a choice of Enriques isometry $H^2(X, \mathbb{Z}) \cong L$ for $\sigma$, we have $\sigma^* = \rho$. Choosing an arbitrary metric $\widetilde{g} \in \mathrm{Riem}(X)$ and averaging $g := \sum_{f \in G} f^* \widetilde{g}$ defines a metric $g'$ for $M$. Identifying harmonic 2-forms with real cohomology classes for both $(X, g)$ and $(M, g')$, we obtain an isomorphism $0 = \mathcal{H}^+_{g'}(M) \cong \mathcal{H}^+_g(X)^G$ under pulling back $\pi^* : \mathcal{H}^2_\Delta(M, g') \to \mathcal{H}^2_\Delta(X, g)$. Given that the restriction $\rho$ to $\mathcal{H}^+_g(X)$ has eigenvalues $1, -1, -1$ (recall $L^+_\rho$ has signature

(1,9) and $L_\rho^-$ has signature (2,10)), we see that $\mu^*$ must have eigenvalues $1, i, -i$ or $-1, i, -i$, since $(\mu^*)^2 = \rho$. However, the latter case is excluded, since then $\mu^*$ would not preserve orientation on maximal positive definite subspaces, and so 1 is also an eigenvalue of $\mu^*$. Thus, there exists a one-dimensional subspace of $\mathcal{H}_g^+(X)$ which is preserved by $G$, i.e. $\mathcal{H}_g^+(X)^G \neq 0$, contradicting $b^+(M) = 0$. □

Now, given that EEH manifolds are oriented smooth four-manifolds, we can certainly ask if they admit the structure of a compact complex surface. Suppose that this were true. Since $b_1(M) = 0$ for any EEH manifold $M$, Theorem 3.4.3 tells us that $M$ would have to be Kähler. However, Kähler forms have positive self-intersection in $H^2(M, \mathbb{R})$, which contradicts the fact that $b^+(M) = 0$. So, EEH manifolds cannot be given a complex structure. One is then naturally led to wonder if such manifolds at least admit almost complex structures. It turns out this is also false:

**Proposition 6.1.2.** *No EEH manifold admits an almost complex structure.*

*Proof.* A standard result (often credited to Ehresmann and Wu) states that a closed smooth oriented four-manifold $X$ admits an almost complex structure if and only if there exists a class $c \in H^2(X, \mathbb{Z})$ such that $c^2 = 3\sigma(X) + 2\chi(X)$ and $w_2(X) = c \bmod 2$ (then $c$ is the first Chern class of the associated almost complex structure). Using this and Wu's theorem, Hongzhu and Ren in [22, Lemma 1] prove that a closed smooth oriented four-manifold $X$ with $b_1(X) = 0$ admits an almost complex structure if and only if $b^+(X) \equiv 1 \pmod 2$. Given that $b^+(M) = 0$ for any EEH manifold $M$, we see that $M$ cannot be equipped with an almost complex structure. □

Moreover, let us also note the fact that $b^+(M) = 0$ for an EEH manifold $M$ means that the lattice structure of $H^2(M, \mathbb{Z})/T_2$ is completely determined and rather dull. This is thanks to a (highly non-trivial!) result of Donaldson:

**Theorem 6.1.1** (Donaldson's Theorem)**.** *The intersection form of a smooth compact oriented four-manifold which is definite is diagonalisable (over $\mathbb{Z}$).*

Given that EEH manifolds do not admit complex structures, there is no hope of constructing period domains or maps for them as we did in Chapters 4 and 5. However, since the covers of an EEH manifold, that is, K3 and Enriques surfaces, are diffeomorphic and in fact deformation equivalent, one should hope that the same holds true for this class of four-manifold. Remarkably, this is true! The algebraic geometers A. Degtyarev, I. Itenberg and V. Kharlamov in [27] prove that empty real Enriques surfaces are in the same deformation class (i.e. can be put in a continuous one-parameter family). They actually go a great deal further than this and enumerate all the possible deformation classes of real Enriques surfaces, the classes being distinguished by the topology of the fixed-point set of the real structure. Their analysis was to a large part based on a systematic study of $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-actions on the K3 lattice $L = 3H \oplus -2E_8$. Degtyarev and Kharlamov in [10] provide an alternative proof of the fact that EEH manifolds are of the same deformation type by studying elliptic fiberings of Enriques surfaces over the projective line. The general problem of classifying real Enriques surfaces based on the topological type of their real parts was started by Nikulin in [42].

For the remainder of this chapter, we will be focused on proving the above observations about EEH manifolds using some of the ideas found in Chapter 5. To this end, the Teichmüller space of Section 4.5 will again prove very useful. Given that EEH manifolds are examples of compact Ricci-flat Einstein four-manifolds, we will make use of the following notion of 'deformation equivalence' in this context:

**Definition 6.1.1.** Two compact oriented Einstein manifolds $(M_1, g_1)$ and $(M_2, g_2)$ will be called *deformation equivalent* if there exists a diffeomorphism $f : M_1 \to M_2$ such that $f^*g_2$ and $g_1$ can be joined by a continuous path in $\mathrm{Ein}(M_1)$.

As in the case of Enriques surfaces, we will begin by proving that all EEH manifolds are diffeomorphic. We will then be able to fix one such manifold, or rather, exactly one K3 surface equipped with free commuting holomorphic and anti-holomorphic involutions and proceed to prove that any two Einstein metrics are deformation equivalent on this fixed manifold.

Before we elaborate on the proofs, we think it would interesting for the reader to be aware of some results related to the structure of diffeomorphisms on an Enriques surface, and compare them to the case of K3 surfaces. As we saw, the fact that all free smooth involutions on a K3 surface induce conjugate actions on the K3 lattice (cf. Theorem 5.1.1) was the thing that made the investigations of Chapter 5 possible. It turns out something similar is true in the case at hand:

**Proposition 6.1.3.** *Any two free anti-holomorphic involutions* $\theta_1, \theta_2 : E \to \overline{E}$ *on an Enriques surface induce conjugate actions on the Enriques lattice* $H^2(E, \mathbb{Z})/T_2 \cong L_E$.

*Proof.* Using the classification of even non-degenerate sublattices, a relatively simple calculation shows that

$$L^+_{\theta_1^*} \cong D_4 \cong L^+_{\theta_2^*},$$

where $D_4$ is the negative definite root lattice (i.e. defined as the sublattice of $\oplus_{i=1}^4 \mathbb{Z}(-1)$ given by generators $x, y, z, w \in \mathbb{Z}(-1)$ satisfying $x + y + z + w \in 2\mathbb{Z}$). For example, see [10, Lemma 5.1.1]. One then finds in a similar way as in the proof of Proposition 5.2.10 that $L^-_{\theta_1^*} \cong H \oplus D_4 \cong L^-_{\theta_2^*}$. The uniqueness of unimodular gluings (Theorem 5.2.4) then tells us that $\theta_1^*$ and $\theta_2^*$ must be conjugate. □

Interestingly enough, the above result gives us a little more information. Recall that Lemma 5.2.2 tells us that any free smooth involution $\sigma : X \to X$ on a compact oriented simply-connected four-manifold satisfies $L^+_{\sigma^*} = \pi^* H^2(Y, \mathbb{Z})$, where $\pi : X \to Y := X/\langle\sigma\rangle$ is the quotient map. The simple-connectedness assumption cannot be done away with. Indeed, if $\theta : E \to \overline{E}$ is an empty real structure on an Enriques surface, and $M = E/\langle\theta\rangle$ is the EEH quotient, then the inclusion $\pi^* H^2(M, \mathbb{Z}) \subseteq L^+_{\theta^*}$ certainly still holds as $\pi \circ \theta = \pi$, where $\pi : E \to M$ is the quotient map. However, given that Donaldson's theorem (Theorem 6.1.1) tells us that $H^2(M, \mathbb{Z})/T_2 \cong 4\langle -1\rangle$, we find $\pi^* H^2(M, \mathbb{Z})/T_2 \cong 4\langle -2\rangle$, i.e. $\pi^* H^2(M, \mathbb{Z}) \neq L^+_{\theta^*} \cong D_4$.

Moreover, recall that at the end of Section 4.4, we found that for a K3 surface $X$, the image of the natural map

$$\mathrm{Diff}(X) \to \mathrm{Aut}(H^2(X, \mathbb{Z})),\ f \mapsto f^*$$

consists precisely of those automorphisms of $H^2(X, \mathbb{Z})$ preserving orientation on maximal positive definite subspaces (Theorem 4.4.8). Now, to say that an isometry $\gamma \in \mathrm{Aut}(L_E)$ preserves orientation on maximal positive definite subspaces is simply to say that given any $v \in L_E$ in the real extension of $L_E$ with $v \cdot v > 0$, we must have $v \cdot \gamma(v) > 0$. It is certainly not true that every diffeomorphism of an Enriques surface induces an action of the Enriques lattice which preserves orientation on maximal positive definite subspaces. Indeed, if $E$ is a Kähler real Enriques surface with anti-holomorphic involution $\theta : E \to \overline{E}$ and Kähler form $\omega \in \Omega^2_{\mathbb{R}}(E)$, then $\theta^*\omega = -\omega$ and so $[\theta^*\omega]\cdot[\omega] = -[\omega]\cdot[\omega] < 0$. Nonetheless, with the help of some Seiberg-Witten theory, one can prove the following noteworthy fact:

**Theorem 6.1.2** (Theorem 7, [35])**.** *If $E$ is an Enriques surface, then any isometry of $H^2(E, \mathbb{Z})$ preserving orientation on maximal positive definite subspaces is induced by a diffeomorphism of $E$.*

Let us now get to proving that all EEH manifolds are of the same deformation type.

## 6.2 Deformation Equivalence

Thanks to Proposition 6.1.1, we now just have to study free $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-actions on a K3 surface in order to get a better grasp of EEH manifolds. Specifically, we should hope that the induced $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on the K3 lattice arising from the lifted empty real structure on an Enriques surfaces should be fixed (up to isomorphism), exactly as in Theorem 5.1.1 and Proposition 6.1.3. This is in fact true, and will be the thing which will help us establish the deformation equivalence of EEH manifolds. The proof relies on classifying $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-actions on unimodular lattices, which is a natural extension to the considerations of Section 5.2.1, although the calculations are a great deal more tedious:

**Theorem 6.2.1.** *There is (up to isomorphism) exactly one $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on the K3 lattice which corresponds to an empty real Enriques surface. In particular, given a K3 surface $X$ equipped with a free holomorphic involution $\sigma : X \to X$ commuting with a free anti-holomorphic involution $\mu : X \to \overline{X}$, there exists an isometry $\varphi : H^2(X,\mathbb{Z}) \to L = 3H \oplus -2E_8$ such that $\varphi \circ \sigma^* = \rho \circ \varphi$ and $\varphi \circ \mu^* = \phi \circ \varphi$, where $\phi \in \Gamma$ is some fixed conjugate of the usual involution*

$$\rho : L \to L,\ (x, y, z, u, v) \mapsto (-x, z, y, v, u).$$

*We will call such a choice of isometry an **EEH isometry** of the pair $(\sigma, \mu)$.*

*Proof.* Nikulin proves this in [42, Theorem 5], as do Degtyarev *et al.* in [27, p. 189-190]. □

Now, since Enriques surfaces are parameterised by the space $\mathcal{K}^-(X_0, \sigma_0)$, we can think of an EEH manifold as a quadruple $M = (X_0, \kappa, \sigma_0, \mu)$, where $\kappa \in \mathcal{K}^-(X_0, \sigma_0)$ and $\mu : X_0 \to X_0$ is any free smooth involution satisfying $\mu \circ \sigma_0 = \sigma_0 \circ \mu$ and $\mu^*\kappa = \overline{\kappa}$. Following similar reasoning as in the proof of Lemma 5.3.2 and Theorem 5.5.1, we will easily be able to deduce the fact that EEH manifolds are diffeomorphic. Note that, exactly as in the case of $\rho$ (cf. Definition 5.3.1), the fixed-point set of the action of $\phi \in \Gamma \cong \mathrm{Diff}(X_0)/\mathrm{Diff}_T(X_0)$ on the Teichmüller space will be denoted $\mathcal{T}(X_0)^\phi$.

**Lemma 6.2.1.** *Given $t_1 \in \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ with $[g_1] = t_1$ for some $g_1 \in \mathrm{Ein}(X_0)$, there exists a unique pair of commuting involutions $\sigma_1, \mu_1 : X_0 \to X_0$ which each act freely, preserve $g_1$ and respectively act as $\rho$ and $\phi$ under an arbitrary choice of isometry $H^2(X_0, \mathbb{Z}) \cong L$. If $t_2 \in \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ is any other such point, with $g_2$, $\sigma_2$ and $\mu_2$ as above, then there exists a diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ such that $\sigma_1 \circ f = f \circ \sigma_2$ and $\mu_1 \circ f = f \circ \mu_2$.*

*Proof.* The proof is quite similar to that of Lemma 5.3.2. Let $\psi : U \subseteq \mathcal{T}(X_0) \to \mathrm{Ein}(X_0)$, $[g] \mapsto g$ and $\alpha_1 : \Gamma_{t_1} \to I_{g_1}$, $[f] \mapsto f$, respectively denote the continuous section of the projection $\mathrm{Ein}(X_0) \to \mathcal{T}(X_0)$, which is defined in a neighbourhood of $t_1$, and isomorphism guaranteed by Lemma 5.3.1. Since $t_1 \in \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$, by definition $[\rho], [\phi] \in \Gamma_{t_1}$. Then, exactly as in Lemma 5.3.2, $\sigma_1 := \alpha_1([\rho])$ and $\mu_1 := \alpha_1([\phi])$ are the unique smooth freely acting involutions of $X_0$ preserving $g_1$ and satisfying $\sigma_1^* = \rho$ and $\mu_1^* = \phi$. Given that $\rho$ and $\phi$ commute, and $\sigma$ and $\mu$ are isometries of the same Einstein metric $g_1 \in \mathrm{Ein}(X_0)$, Proposition 4.5.2 tells us that $\sigma_1$ and $\mu_1$ must commute.

Now, let $t_2 \in \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ be any other such point, defined by $[g_2] = t_2$, with $\sigma_2 \in \mathrm{Diff}(X_0)$ and $\mu_2 \in \mathrm{Diff}(X_0)$ as above, this time defined by the map $\alpha_2 : \Gamma_{t_2} \to I_{g_2}$, $[f] \mapsto f$. As we have seen, if $g_2 \in \psi(U \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi)$, say $g_2 = \psi(t_2')$ for $t_2' \in U \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$, then

$$\sigma_1^* g_2 = \alpha_1([\rho])^* \psi(t_2') = \psi(\rho^* t_2') = \psi(t_2') = g_2,$$
$$\mu_1^* g_2 = \alpha_1([\mu])^* \psi(t_2') = \psi(\phi^* t_2') = \psi(t_2') = g_2,$$

that is, $\sigma_1, \mu_1 \in I_{g_2}$, in which case we can take $f = \mathrm{id}_{X_0}$. If, however, $g_2 \notin \psi(U \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi)$, then it will be related to a point in this space by a diffeomorphism $f \in \mathrm{Diff}_T(X_0)$. After conjugating and another application of Proposition 4.5.2 (cf. end of proof of Lemma 5.3.2), we then see $f \circ \sigma_1 \circ f^{-1} = \sigma_2$ and $f \circ \mu_1 \circ f^{-1} = \mu_2$. □

Exactly as before, the above lemma is a restatement of the fact that:

**Theorem 6.2.2.** *All EEH manifolds are diffeomorphic.*

*Proof.* Let $(M_1, \widetilde{g_1})$ and $(M_2, \widetilde{g_2})$ be any two EEH manifolds, each equipped with some arbitrary Ricci-flat Einstein metric. As above, they are respectively defined by quadruples $(X_0, \kappa_1, \sigma_0, \mu_1)$ and $(X_0, \kappa_2, \sigma_0, \mu_2)$, where each K3 cover $(X_0, \kappa_i)$ is equipped with a Kähler-Einstein structure $g_i \in \mathrm{Ein}(X_0)$ lifting $\widetilde{g_i}$ and necessarily satisfying $\sigma_0^* g_i = g_i$ and $\mu_i^* g_i = g_i$, and so under some fixed EEH isometries $\varphi_i : H^2(X_0, \mathbb{Z}) \cong L$ of each of the pairs $(\sigma_0, \mu_i)$, $i = 1, 2$, we have $t_i := [g_i] \in \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$. By uniqueness, each of the pairs $(\sigma_0, \mu_i)$ correspond exactly to the ones guaranteed by Lemma 6.2.1, and given that each respective pair acts as $\rho$ and $\phi$ under the associated EEH isometry, Lemma 6.2.1 also gives a diffeomorphism $f \in \mathrm{Diff}_T(X_0)$ such that $\sigma_0 \circ f = f \circ \sigma_0$ and $\mu_1 \circ f = f \circ \mu_2$, that is, $f$ descends to a diffeomorphism $M_1 \cong M_2$. □

We now fix an arbitrary EEH manifold $M_0$ defined by a quadruple $(X_0, \kappa_0, \sigma_0, \mu_0)$ and an EEH isometry $H^2(X_0, \mathbb{Z}) \cong L$ of the pair $(\sigma_0, \mu_0)$. In order to establish deformation equivalence, we need one key lemma:

**Lemma 6.2.2.** *The space $\mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ is connected.*

*Proof.* Recall that $\mathcal{T}(X_0) \cong \mathcal{E} = \mathrm{Gr}^0_+(L_\mathbb{R}) \setminus \bigcup_{d\in\Delta} A_d$ where $A_d = \{H \in \mathrm{Gr}^0_+(L_\mathbb{R}) \,|\, H \perp d\}$. To simplify matters, note that any maximal positive definite subspace can be uniquely oriented so that it is positively oriented, that is, we can work over $\mathrm{Gr}(L_\mathbb{R})$ instead of $\mathrm{Gr}^0_+(L_\mathbb{R})$ (recall $\mathrm{Gr}(L_\mathbb{R})$ is the Grassmannian of maximal positive definite subspaces of $L_\mathbb{R}$, which is diffeomorphic to each of the connected components $\mathrm{Gr}^0_\pm(L_\mathbb{R})$ of the oriented Grassmannian $\mathrm{Gr}^0(L_\mathbb{R})$). Under the diffeomorphism $\mathcal{T}(X_0) \cong \mathcal{E}$, the space $\mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ corresponds to

$$(\mathrm{Gr}(L_\mathbb{R})^\rho \cap \mathrm{Gr}(L_\mathbb{R})^\phi) \setminus \bigcup_{d\in\Delta} A_d,$$

where the superscripts denote the fixed-point set of the actions of $\rho$ and $\phi$ on $\mathrm{Gr}(L_\mathbb{R})$. We show that this space is connected. So, consider the intersections

$$L^{\pm\pm} := L^\pm_\rho \cap L^\pm_\phi = \{x \in L \,|\, \rho(x) = \pm x, \phi(x) = \pm x\}.$$

Then, denoting their real extensions by $L^{\pm\pm}_\mathbb{R}$, we get a decomposition of $L_\mathbb{R}$ into four eigenspaces:

$$L_\mathbb{R} = L^{++}_\mathbb{R} \oplus L^{+-}_\mathbb{R} \oplus L^{-+}_\mathbb{R} \oplus L^{--}_\mathbb{R}.$$

It is known (see [27, p. 170]) that $L^{++}$ has signature $(0,4)$ (in fact, $L^{++} \cong D_4(2)$) and the remaining subspaces have signature $(1,5)$. In particular, any direct sum of the form $H^{+-} \oplus H^{-+} \oplus H^{--}$, where $H^{\pm\pm}$ is a maximal positive definite subspace of $L^{\pm\pm}_\mathbb{R}$, necessarily defines a maximal positive definite subspace of $L_\mathbb{R}$. Moreover, it is not difficult to see that the above sum is preserved by $\rho$ and $\phi$, and so $H^{+-} \oplus H^{-+} \oplus H^{--} \in \mathrm{Gr}(L_\mathbb{R})^\rho \cap \mathrm{Gr}(L_\mathbb{R})^\phi$. Conversely, if $H \in \mathrm{Gr}(L_\mathbb{R})^\rho \cap \mathrm{Gr}(L_\mathbb{R})^\phi$, then $H^{\pm\pm} := H \cap L^{\pm\pm}_\mathbb{R}$ is a maximal positive definite subspace of $L^{\pm\pm}_\mathbb{R}$ and $H = H^{+-} \oplus H^{-+} \oplus H^{--}$, as each of the subspaces $L^{\pm\pm}$ have trivial intersection with each other. So, if $\mathrm{Gr}(L^{\pm\pm}_\mathbb{R})$ denotes the Grassmannian of maximal positive definite subspaces of $L^{\pm\pm}_\mathbb{R}$, then we have the identification:

$$\mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi \cong \mathrm{Gr}(L^{+-}_\mathbb{R}) \times \mathrm{Gr}(L^{-+}_\mathbb{R}) \times \mathrm{Gr}(L^{--}_\mathbb{R}) \setminus \bigcup_{d\in\Delta} B_d,$$

where $B_d := \{(H_1, H_2, H_3) \in \mathrm{Gr}(L^{+-}_\mathbb{R}) \times \mathrm{Gr}(L^{-+}_\mathbb{R}) \times \mathrm{Gr}(L^{--}_\mathbb{R}) \,|\, H_1 \oplus H_2 \oplus H_3 \perp d\}$. Given that $\mathrm{Gr}(L^{\pm\pm}_\mathbb{R}) \cong O(1,5)/O(1) \times O(5)$, an identical argument as in Proposition 4.4.4 shows that each of $\mathrm{Gr}(L^{\pm\pm}_\mathbb{R})$ are connected. To complete the proof, we show that the union $\bigcup_{d\in\Delta} B_d$ does not disconnect the above product of Grassmannians $X := \mathrm{Gr}(L^{+-}_\mathbb{R}) \times \mathrm{Gr}(L^{-+}_\mathbb{R}) \times \mathrm{Gr}(L^{--}_\mathbb{R})$. Given that any one-dimensional subspace is uniquely defined (up to sign) by a unit vector, we will identify a maximal positive definite subspace of $L^{\pm\pm}_\mathbb{R}$ with such a vector. Then, in an arbitrarily small neighbourhood $U$ of a point $v :=$

$(v_1, v_2, v_3) \in B_d$, any other point in the product $X$ will be defined by $v + w$ for some unique $w = (w_1, w_2, w_3) \in X$ with $w_i \cdot v_i = 0$, $i = 1, 2, 3$. Such $w$ define local coordinates for $X$. Then, for any $d \in \Delta$, the intersection $B_d \cap U$ is given precisely by the zero-set of the linear function

$$f_d : U \to \mathbb{R}^3,\ (w_1, w_2, w_3) \mapsto (w_1 \cdot d, w_2 \cdot d, w_3 \cdot d),$$

which shows that $B_d$ is a submanifold of $X$ near the arbitrary point $v \in B_d$. Note that the codimension of $B_d$ is precisely the dimension of the image of $f_d$. We show that $f_d$ cannot have zero or one-dimensional image, and so $B_d$ has codimension 2 or 3. This will prove our result, as a transversality argument (cf. Proposition 4.5.5) shows that a submanifold of codimension at least 2 will not disconnect the ambient manifold upon being removed. Moreover, given that the collection $\{B_d\}_{d\in\Delta}$ is locally finite, any path in $X$ will meet only finitely many of the submanifolds $B_d$. The path can then be deformed so that it meets none of them, and so the space $X \setminus \bigcup_{d\in\Delta} B_d$ is connected.

Now, note that $f_d$ is the zero map if and only if $d \perp L_{\mathbb{R}}^{+-} \oplus L_{\mathbb{R}}^{-+} \oplus L_{\mathbb{R}}^{--}$ if and only if $d \in L^{++} \cong D_4(2)$. However, this is impossible, as $d^2 = -2$ and $D_4$ is even, i.e. any element $x \in D_4(2)$ will satisfy $x^2 \equiv 0 \pmod 4$. Similarly, $f_d$ has one-dimensional image if and only if $d$ is orthogonal to any two of $L_{\mathbb{R}}^{+-}, L_{\mathbb{R}}^{-+}, L_{\mathbb{R}}^{--}$. Assume that $d$ is orthogonal to $L_{\mathbb{R}}^{-+}$ and $L_{\mathbb{R}}^{--}$ (the other cases are analogous). Then $d \in L_\rho^+ \cong L_E(2)$ (recall Proposition 5.1.5), which is impossible using the same reasoning as above, i.e. the Enriques lattice $L_E = H \oplus -E_8$ is even. □

**Theorem 6.2.3.** *All Einstein structures on $M_0$ are deformation equivalent.*

*Proof.* Let $\widetilde{g_0}$ and $\widetilde{g_1}$ be any two Einstein structures for $M_0 = (X_0, \kappa_0)/\langle\sigma_0, \mu_0\rangle$. Denote the lifts to $X_0$ by $g_i \in \mathrm{Ein}(X_0)$ with $\sigma_0, \mu_0 \in \mathrm{Isom}(X_0, g_i)$, $i = 0, 1$. The respective points of the Teichmüller space are denoted $t_i := [g_i] \in \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$. Lemma 6.2.2 tells us there exists a path $t : [0, 1] \to \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ with $t(0) = t_0$ and $t(1) = t_1$. Since $t([0, 1]) \subseteq \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$ is compact, there exists a finite collection of connected open subsets $U_1, \ldots, U_n \subseteq \mathcal{T}(X_0)$ associated to the local sections $\psi_i : U_i \to \mathrm{Ein}(X_0)$, $[g] \mapsto g$ guaranteed by Lemma 5.3.1, where each $U_i$ covers $t([s_{i-1}, s_i])$ for some $0 =: s_0 < s_1 < \ldots < s_n := 1$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$. Let us say

$$t((r_i, r_{i+1})) \subset U_i \cap U_{i+1} \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi$$

for an increasing sequence $r_1 < r_2 < \ldots < r_{2n-3} < r_{2n-2}$ where $r_{2i-1} < s_i < r_{2i}$ for each $i = 1, \ldots, n-1$. Then, as we have seen numerous times, all the metrics in the image of a path $\psi_i \circ t : [s_{i-1}, s_i] \to \psi_i(U_i \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi)$ will be invariant under $\sigma_0$ and $\mu_0$, i.e. descend to a path of Einstein structures on $M_0$. Moreover, it is easy to see that metrics in the image $\psi_i(U_i \cap U_{i+1})$, but not in $\psi_{i+1}(U_i \cap U_{i+1})$ will be related by an element of $\mathrm{Diff}_T(X_0)$, that is to say, the local sections in general differ by an element of $\mathrm{Diff}_T(X_0)$ on overlaps. Combining this with Lemma 6.2.1, we see that a metric $g$ in the image of $\psi_i \circ t : (r_{2i-1}, r_{2i}) \to \psi_i(U_i \cap U_{i+1} \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi)$ and $g'$ in the image of $\psi_{i+1} \circ t : (r_{2i-1}, r_{2i}) \to \psi_{i+1}(U_i \cap U_{i+1} \cap \mathcal{T}(X_0)^\rho \cap \mathcal{T}(X_0)^\phi)$ will satisfy $g' = f^*g$ for some $f \in \mathrm{Diff}_T(X_0)$ with $f \circ \sigma_0 = \sigma_0 \circ f$ and $f \circ \mu_0 = \mu_0 \circ f$, that is, the induced metrics on $M_0$ are deformation equivalent. Thus, we have constructed a finite sequence of deformations connecting $\widetilde{g_0}$ and $\widetilde{g_1}$, which completes the proof. □

We close this section off by making some observations similar to those introduced at the end of Section 5.4. If $M = X/\langle \sigma, \mu \rangle$ is an arbitrary EEH manifold with its universal K3 covering $\pi : X \to M$, then pulling metrics back gives us exactly in the case of Enriques surfaces a natural map, which we denote here by

$$\beta := \frac{1}{2}\pi^* : \mathrm{Ein}(M) \to \mathrm{Ein}(X),$$

where the scaling accounts for the fact that $\pi$ has degree four. If we momentarily define $\mathrm{Diff}_T(X, M)$ to be the subgroup of diffeomorphisms of $M$ whose lift to $X$ acts trivially on $H^2(X, \mathbb{Z})$, then identical reasoning as in Section 5.4 tells us that $\beta$ descends to a map

$$\beta : \mathcal{T}(M) \to \mathcal{T}(X),$$

where $\mathcal{T}(M) := \mathrm{Ein}(M)/\mathrm{Diff}_T(X, M)$ is the associated 'Teichmüller space' for $M$. If an EEH isometry of the pair $(\sigma, \mu)$ is given, we have that $\beta$ surjects onto the set $\mathcal{T}(X)^\rho \cap \mathcal{T}(X)^\phi$. Using more or less identical reasoning as in Theorem 5.4.2 (and Remark 5.4.3), we obtain:

**Theorem 6.2.4.** *The canonical map $\beta : \mathcal{T}(M) \to \mathcal{T}(X)$ is a homeomorphism onto its image $\mathcal{T}(X)^\rho \cap \mathcal{T}(X)^\phi$.*

**Remark 6.2.1.** For an alternative and more detailed analysis of the various moduli spaces associated to Ricci-flat Einstein metrics on Enriques surfaces and EEH manifolds, one can consult the work of Itoh in [28] where he obtains descriptions of them by studying anti-self-dual conformal structures on each of these four-manifolds.

# Appendix A

# Notation

For the convenience of the reader, we include a list of the most important notation that is used throughout this thesis. In the following list and unless otherwise stated, $M$ refers to an arbitrary smooth manifold and $X$ denotes an arbitrary complex manifold:

- $TM$ and $T^*M$ − the tangent and cotangent bundles of $M$, respectively. If $M$ is complex, then these represent the bundles of the underlying smooth manifold.
- $T_{\mathbb{C}}M$ and $T^*_{\mathbb{C}}M$ − the complexified tangent and cotangent bundles of $M$, respectively.
- $\Omega^k_{\mathbb{R}}(M)$ − the space of smooth global sections of $\bigwedge^k T^*M$, $k > 0$.
- $\Omega^k_{\mathbb{C}}(M)$ − the space of smooth global sections of $\bigwedge^k T^*_{\mathbb{C}}M$, $k > 0$.
- $C^\infty(M) =: \Omega^0_{\mathbb{R}}(M)$ − the $\mathbb{R}$-algebra of smooth $\mathbb{R}$-valued functions on $M$.
- $\Omega^0_{\mathbb{C}}(M)$ − the $\mathbb{C}$-algebra of smooth $\mathbb{C}$-valued functions on $M$.
- $\mathcal{T}_X$ and $\mathcal{T}^*_X$ − the holomorphic tangent and cotangent bundles of $X$.
- $K_X := \det(\mathcal{T}^*_X) = \bigwedge^n \mathcal{T}^*_X$ − the canonical bundle of $X$, where $n = \dim_{\mathbb{C}} X$.
- $\Omega^p_X$ − the sheaf of germs of holomorphic sections of the holomorphic vector bundle $\bigwedge^p \mathcal{T}^*_X$. Its space of global sections is denoted by $H^0(X, \Omega^p_X)$.
- $\Omega^{p,q}(X)$ − the space of global $(p, q)$-forms of $X$.
- $\mathcal{O}_X$ − the sheaf of germs of holomorphic functions on $X$.
- $\mathcal{O}^*_X$ − the sheaf of germs of nowhere-vanishing holomorphic functions on $X$.

- $\mathrm{kod}(X)$ – the Kodaira dimension of $X$, when it is compact and connected.
- $\mathrm{Div}(X)$ – the space of divisors on $X$.
- $\mathcal{O}_X(D) = \mathcal{O}(D)$ – the line bundle associated to a divisor $D \in \mathrm{Div}(X)$.
- $\mathrm{NS}(X)$ – the Néron-Severi group of $X$ (i.e. the image of the first Chern class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{R})$).
- $\mathscr{P}_X : H^k(X, \mathbb{Z}) \to H_{n-k}(X, \mathbb{Z})$ – the Poincaré duality isomorphism on a closed oriented $n$-dimensional topological manifold $X$.
- $\chi(X)$ – the topological Euler characteristic of a topological space $X$, that is, the alternating sum of its Betti numbers.
- $\mathcal{H}^k_\Delta(M, g)$ – the space of $g$-harmonic $k$-forms for an oriented Riemannian manifold $(M, g)$.
- $\mathcal{H}^+_g(M)$ – the space of self-dual harmonic 2-forms for an oriented Riemannian four-manifold $(M, g)$.
- $\mathrm{Diff}(M)$ – the group of self-diffeomorphisms of $M$ under composition.
- $\mathrm{Diff}_T(M)$ – the group of self-diffeomorphisms of a compact, oriented four-manifold $M$ which act trivially on $H^2(M, \mathbb{Z})$.
- $\mathrm{Isom}(M, g) = I_g$ – the isometry group of a Riemannian manifold $(M, g)$.
- $E_8$ – the positive definite $E_8$-lattice.
- $H$ – the standard hyperbolic lattice on $\mathbb{Z}^2$.
- $L := 3H \oplus -2E_8$ – the 'K3 lattice'. The real and complex extensions of $L$ are denoted $L_\mathbb{R}$ and $L_\mathbb{C}$.
- $L_E := H \oplus -E_8$ – the "Enriques lattice".
- $\mathrm{Gr}^0(L_\mathbb{R})$ – the Grassmannian of maximal oriented positive definite subspaces of $L_\mathbb{R}$.
- $\mathrm{Gr}^0_\pm(L_\mathbb{R})$ – the two connected components of $\mathrm{Gr}^0(L_\mathbb{R})$.
- $S(a, c) := \mathrm{span}\{a, \mathrm{Re}(c), \mathrm{Im}(c)\} \subseteq L_\mathbb{R}$ for $a \in L_\mathbb{R}$ and $c \in L_\mathbb{C}$.
- $\Delta := \{d \in L \,|\, d \cdot d = -2\}$ – the space of "roots" of the K3 lattice.

- $A_d := \{H \in \mathrm{Gr}^0_+(L_\mathbb{R}) \,|\, H \perp d\}$ for $d \in L$.
- $\Omega := \{[c] \in \mathbb{P}(L_\mathbb{C}) \,|\, c \cdot c = 0, c \cdot \overline{c} > 0\}$ – the classical period domain of K3 surfaces.
- $\mathcal{P} := \{(a,[c]) \in L_\mathbb{R} \times \Omega \,|\, a^2 > 0, a \cdot c = 0, S(a,c) \in \mathrm{Gr}^0_+(L_\mathbb{R})$ and $\forall d \in \Delta : S(a,c) \notin A_d\}$ – the refined period domain of K3 surfaces.
- $\mathcal{E} := \mathrm{Gr}^0_+(L_\mathbb{R}) \setminus \bigcup_{d\in\Delta} A_d$ – the "period domain" of Einstein metrics for K3 surfaces.
- $\rho : L \to L$ – the distinguished involution on the K3 lattice defined as $\rho(x,y,z,u,v) = (-x,z,y,v,u)$. It extensions to $L_\mathbb{R}$ and $L_\mathbb{C}$ are also denoted by $\rho$.
- $L^\pm_\phi$ – the $\pm1$-eigensublattices of an involution $\phi$ on an arbitrary unimodular lattice.
- $\Delta^- := \Delta \cap L^-_\rho$
- $\Omega^- := \Omega \cap \mathbb{P}(L^-_\mathbb{C})$, where $L^- := L^-_\rho$.
- $\mathcal{D}^- := \Omega^- \setminus \bigcup_{d\in\Delta^-} H_d$ where $H_d := \{[c] \in \Omega \,|\, c \cdot d = 0\}$ – the period domain of Enriques surfaces.
- $\mathcal{P}^- := \mathcal{P} \cap (L^+_\mathbb{R} \times \Omega^-)$, where $L^\pm := L^\pm_\rho$ – the refined period domain of Enriques surfaces.
- $\mathcal{K}(X_0)$ – the space of complex 2-forms $\kappa \in \Omega^2_\mathbb{C}(X_0)$ satisfying $\kappa \wedge \kappa = 0$, $\kappa \wedge \overline{\kappa} > 0$ and $d\kappa = 0$ for some oriented four-manifold $X_0$ underlying a fixed K3 surface.
- $\Omega_\kappa(X_0)$ – the space of Kähler metrics for the K3 surface defined by the structure $\kappa \in \mathcal{K}(X_0)$.
- $\mathcal{R}(X_0)$ – the space of pairs $(\omega,\kappa)$ where $\kappa \in \mathcal{K}(X_0)$ and $\omega \in \Omega_\kappa(X_0)$.
- $\mathcal{K}^-(X_0,\sigma) := \{\kappa \in \mathcal{K}(X_0) \,|\, \sigma^*\kappa = -\kappa\}$ for any free smooth involution $\sigma : X_0 \to X_0$.
- $\Omega^+_\kappa(X_0,\sigma) := \{\omega \in \Omega_\kappa(X_0) \,|\, \sigma^*\omega = \omega\}$ for $\kappa \in \mathcal{K}^-(X_0,\sigma) \neq \emptyset$.
- $\mathcal{R}(X_0,\sigma)$ – the space of pairs $(\omega,\kappa)$ with $\kappa \in \mathcal{K}^-(X_0,\sigma)$ and $\omega \in \Omega^+_\kappa(X_0,\sigma)$.

# Bibliography

[1] D. Baraglia and H. Konno, *A note on the Nielsen realization problem for K3 surfaces*, Proceedings of the American Mathematical Society **151** (2023), no. 9, 4079–4087.

[2] A. Beauville, *Complex Algebraic Surfaces*, 2nd ed., Cambridge University Press, Cambridge, 1996.

[3] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.

[4] R. Bott, *On a Theorem of Lefschetz.*, Michigan Mathematical Journal **6** (1959), no. 3, 211–216.

[5] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer, New York, 1982.

[6] N. Buchdahl, *On Compact Kähler Surfaces*, Annales de l'Institut Fourier **49** (1999), no. 1, 287–302.

[7] ———, *Compact Kähler Surfaces with Trivial Canonical Bundle*, Annals of Global Analysis and Geometry **23** (2003), no. 2, 189–204.

[8] F. Catanese, *A Superficial Working Guide to Deformations and Moduli*, arXiv: (2011), available at `1106.1368`.

[9] F. R. Cossec and I. V. Dolgachev, *Enriques Surfaces I*, Vol. 76, Birkhäuser, 1989.

[10] A. Degtyarev and V. Kharlamov, *Empty real Enriques surfaces and Enriques-Einstein-Hitchin 4-manifolds.* (1997).

[11] J-P Demailly, *Complex Analytic and Differential Geometry*, 2012.

[12] S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), no. 3, 257–315.

[13] D. G. Ebin, *The manifold of Riemannian metrics*, Proc. Symp. AMS, Global Analysis **15** (1970), 11–40.

[14] A. L. Edmonds, *Aspects of group actions on four-manifolds*, Topology and its Applications **31** (1989), no. 2, 109–124.

[15] F. Enriques, *Le superfichie algebriche* (1949).

[16] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982), no. 3, 357–453.

[17] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, 1994.

[18] V. Guillemin and A. Pollack, *Differential Topology*, Pearson, 1974.

[19] I. Hambleton and M. Kreck, *Cancellation, elliptic surfaces and the topology of certain four-manifolds.*, Journal für die reine und angewandte Mathematik **444** (1993), 79–100.

[20] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2009.

[21] N. Hitchin, *Compact four-dimensional Einstein Manifolds*, Journal of Differential Geometry **9** (1974), 435 –441.

[22] G. Hongzhu and G. Ren, *Quotient spaces of anti-holomorphic involutions*, Science in China Series A: Mathematics **48** (2005), no. 12, 1631–1636.

[23] E. Horikawa, *On the periods of Enriques surfaces. I*, Mathematische Annalen **234** (1978), no. 1, 73–88.

[24] ———, *On the periods of Enriques surfaces. II*, Mathematische Annalen **235** (1978), no. 3, 217–246.

[25] D. Huybrechts, *Complex Geometry: An Introduction*, Springer, 2005.

[26] ———, *Lectures on K3 Surfaces*, Cambridge University Press, 2016.

[27] A. Degtyarev, I. Itenberg and V. Kharlamov, *Real Enriques Surfaces*, Springer, 2000.

[28] M. Itoh, *Conformal geometry of Ricci-flat 4-manifolds*, Kodai Math. J. **17** (1994), 179–200.

[29] K. Kodaira, *On Compact Analytic Surfaces, II*, Annals of Mathematics **77** (1963), no. 3, 563–626.

[30] ———, *On the Structure of Compact Complex Analytic Surfaces, I*, American Journal of Mathematics **86** (1964), no. 4, 751–798.

[31] ———, *On the Structure of Complex Analytic Surfaces, IV*, American Journal of Mathematics **90** (1968), no. 4, 1048–1066.

[32] ———, *Complex Manifolds and Deformation of Complex Structures*, Classics in Mathematics, Springer, 2005.

[33] A. Lamari, *Courants kählériens et surfaces compactes*, Annales de l'Institut Fourier **49** (1999), no. 1, 263–285.

[34] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics, Springer, New York, 1999.

[35] M. Lönne, *On the diffeomorphism groups of elliptic surfaces*, Mathematische Annalen **310** (1998), no. 1, 103–117.

[36] T. Matumoto, *On diffeomorphisms of a K3 surface*, Algebraic and topological theories (1986), 616–621.

[37] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, 1974.

[38] Y. Miyaoka, *Kähler Metrics on Elliptic Surfaces*, Proceedings of the Japan Academy **50** (1974), no. 8, 533–536.

[39] J. W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Princeton University Press, 1996.

[40] Y. Namikawa, *Periods of Enriques Surfaces*, Mathematische Annalen **270** (1985), 201–222.

[41] V. V. Nikulin, *Integral Symmetric Bilinear Forms and some of their Applications*, Mathematics of the USSR-Izvestiya **14** (1980), no. 1, 103.

[42] ———, *On the Topological Classification of Real Enriques Surfaces. I*, arXiv: (1993), available at 9311012.

[43] W. Barth, C. Peters. and A. Van de Ven, *Compact Complex Surfaces*, 1st ed., Springer, Berlin, 1984.

[44] W. Barth, K. Hulek, C. Peters. and A. Van de Ven, *Compact Complex Surfaces*, 2nd ed., Springer, Berlin, 2004.

[45] F. Quinn, *Isotopy of 4-manifolds*, J. Differential Geometry **24** (1986), no. 3, 343–372.

[46] J. A. Schafer, *Topological Pontrjagin classes*, Commentarii Mathematici Helvetici **45** (1970), no. 1, 315–332.

[47] J-P Serre, *Course in Arithmetic*, Springer, 1973.

[48] Y. T. Siu, *Every K3 surface is Kähler*, Inventiones mathematicae **73** (1983), no. 1, 139–150.

[49] R. Thom, *Quelques propriétés globales des variétés différentiables*, Commentarii Mathematici Helvetici **28** (1954), no. 1, 17–86.

[50] A. N. Todorov, *Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces*, Inventiones mathematicae **61** (1980), no. 3, 251–265.

[51] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Vol. 439, Springer, 1975.

[52] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, translated by L. Schneps, Cambridge University Press, 2002 (eng).

[53] A. Weil, *Final report on contract AF 18(603)-57*, Oeuvres Scientifiques (1958).

[54] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proceedings of the National Academy of Sciences **74** (1977), no. 5, 1798–1799.

[55] ———, *On the ricci curvature of a compact kähler manifold and the complex monge-ampére equation, I*, Communications on Pure and Applied Mathematics **31** (1978), no. 3, 339–411.