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MATHEMATICAL MODELS FOR COMMUTER TRAFFIC IN CITIES

by

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SUMMARY

The subject of this thesis is the study of urban commuter traffic in a general, mathematical fashion. A simple model of an inhomogeneous city is considered. It is assumed that the city essentially consists of an inner central business district, which contains only work places, surrounded by an annular residential zone, consisting only of homes. Some concepts are explained and discussed in Chapter II, while the remainder of the thesis is concerned mainly with the detailed investigation of several variations of this model.

An idealized circular city is studied in Chapter III, where the effect of growth in the number of persons living in the suburbs of the city and working in a central business district of constant size is investigated. Seven alternative basic routeing systems are examined. It is shown that for some important routeing systems the total city size, relative to the size of its central business district, can only be increased to a certain extent before the routeing system becomes inefficient and non-optimal.

Critical values for the ratio of the density of work places to homes are then obtained. The criteria used to evaluate the routeing systems are average distance travelled, average travel time, and average area of road space required for the journey to work.

This basic model of an inhomogeneous city is further investigated in Chapter IV, where we consider first the average distance travelled from home to work, in cities with rectangular routeing systems but of different geometrical shapes. Then the case of a square city with three modified rectangular routeing systems is examined and the average trip length and average time taken are calculated. It is shown that several interesting conclusions follow from the calculated results.

A simple mathematical model of a satellite town is presented in Chapter V. It is shown that the model contains the basic features of a satellite New Town as well as of a satellite Residential Town. The characteristics of commuter travel in a satellite Residential Town are investigated, and the size of the Central Area and of the satellite town, the distance travelled per unit area, and the average distance travelled, are evaluated. For an initial sample set of data, numerical results are presented and discussed.

Some work trip data for Adelaide are analysed in Chapter VI. The distribution of homes is studied in some detail, and it is shown that the density of homes can be expressed in terms of the distance from the centre of the city by means of a simple formula. Data for the distribution of work places within Adelaide are also given and commented on.

Finally Chapter VII discusses some general aspects of the problem, and offers suggestions for further work.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

(Tony K. Tan)

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CHAPTER I

INTRODUCTION

1.1 General Survey

Traffic congestion in cities is one of the most formidable and pressing problems of modern living. The frustrations, associated with travel in an urban region, are so familiar that it is scarcely necessary to ennumerate them. The motorist to-day has become resigned to the inevitability of traffic jams, parking difficulties, crawling automobiles, accidents and general confusion. Furthermore the seemingly endless flood of vehicles onto inadequate city road networks produces undesirable side-effects such as fumes, noise, smell, and a clutter of unsightly traffic signs and signals, all of which lead to a serious deterioration of the urban environment.

While the present difficulties are severe enough, it is highly probable that even more desperate conditions will prevail in the future. The trend towards increasing urbanisation and motorisation of the world's population seems to be firmly entrenched. Data collected by Smeed [17] shows, for example, that while 35 per cent of the total population of the United States resided in urban areas in 1890, this figure had risen to 63 per cent by 1956. Moreover the number of motor vehicles per head of population had a compound rate of increase of 3.7 per cent per annum for the five years up to 1959, and this in a country which already had 0.40 vehicles per head of population in 1959. There is thus some substance in the claim that the very existence of modern cities is being threatened by the explosive growth in the number of motorised, rubber-tyred vehicles.

It is no wonder that the urban transportation problem has received a tremendous amount of attention in recent years. An enormous literature exists on the subject, and numerous conferences have been organised to discuss various aspects of the problem. An interesting and extremely lucid account of the urban traffic problem is given in the Buchanan report [2]. In this classical work, Buchanan and his committee have analysed the basic nature and causes of traffic congestion, provided a theoretical basis for resolving the problem, and illustrated their conclusions with a series of practical studies.

A central idea in Buchanan's approach is the concept of "environmental areas". An environmental area is a region where civilized human activities are given primary importance, and traffic facilities are limited to the service of these activities. These areas are to be linked by a network of distributory roads, and all inter-area traffic is to be channelled onto these roads without choice.

In contrast to the "bird's-eye" view adopted by Buchanan, there exists a number of very detailed transportation studies centred on individual cities. A typical example is the Chicago Area Transportation Study (CATS) [4]. The main objective in such a study is invariably the formulation of a comprehensive transportation plan for some target year in the future. The standard technique in conducting a transportation study consists of five phases [4] :

(a) Compiling inventories of present transportation facilities, land use, and travel characteristics;

(b) Forecasting of future population, economic activities, and land use;

(c) Forecasting of future travel demand, trip generation, and the distribution of travel between private and public modes;

(d) Preparation of a comprehensive plan involving both highways and public transport facilities;

(e) Testing the plan against future travel requirements, followed by revision and further testing until a satisfactory solution is achieved.

A comprehensive transportation study is usually characterised by the vast amount of detailed information that is collected, and this necessitates the use of a large computer to process the data and to assist in the execution of the study.

The two types of transportation studies, exemplified by the Buchanan and the CATS reports, are markedly different in their methods of analysis. While the first type treats the problem in a general but qualitative fashion, the second type examines the problem quantitatively but limits the analysis to the peculiarities of a specific town or city. It is clear that there is a wide gap between these two approaches, and attempts have been made to bridge them by studying the subject both comprehensively and quantitatively. This third approach is of value because it enables accurate calculation to be made of parameters that could formerly only be discussed qualitatively. Moreover the results obtained are a guide to the ideal transportation network and are therefore useful in the initial selection of desirable comprehensive transportation plans. On the other hand, it must be admitted that only a small number of variables can be considered and various simplifying assumptions have to be made as otherwise the problem becomes mathematically intractable. This thesis is concerned mainly with a number of mathematical investigations in this field.

1.2 Historical Background

The first attempt to attack the problem of traffic congestion in a general mathematical fashion was made by Smeed. In a pioneering paper [17], Smeed introduces the important concept of the area of road required for a journey, and also proposes a theoretical model of a town. He assumes that the town is circular and symmetrical about the centre. All the workers in the town work in the circular central area, and live in the surrounding annular residential zone. Except for the space required for roads, the work places are uniformly distributed in the central area, while the homes are uniformly distributed in the residential zone. Each worker is assumed to follow a simple form of routeing for the journey from his home to his work place. With these assumptions, Smeed is able to calculate the amount of road space required for the journey to work for a given town population. The size of the central area and of the whole town are also derived.

In a second paper [19], the calculations for the central area of a town are extended, and three routeing systems are studied in detail. Two distributions of work places are considered:

(a) A uniform distribution as above;

(b) The density of work places is inversely proportional to the distance from the centre of the town. Some data on the journey to work in London are analysed and compared with the theoretical results. This paper is notable for the empirical results given, and for the effort made to examine critically the applicability of the theoretical model.

Smeed and Jeffcoate [23] have calculated the distances travelled in the central area when people use a rectangular grid of roads, and compared the results with those for other forms of routeing. In these papers it has been assumed that the distribution of work places is unaffected by the existence of the road network and full account was not taken of the space required for the roads in some forms of routeing. This approximation is not made and a more exact treatment given by Smeed in a fourth paper [21]. Various aspects of the model have also been discussed in other papers [18,20,22,24,]. As Smeed's work is fundamental and forms the basis for the analysis given in this thesis, a fuller discussion of his results and assumptions is given in Chapter 2.

Smeed's basic model of a circular city has been further investigated by Haight [12], who derives the probability distribution for trip length, and the probability distribution for traffic weight at each point in the city, under the assumption of six different routeing systems. For any particular routeing, the traffic weight at a point is defined to be the measure of all destinations requiring travel through that point. Fairthorne [10] discusses the concept of accessibility within a city, and calculates the distances between pairs of points in towns of simple geometrical shapes. He considers four types of networks : direct, rectangular, radial-circular, and triangular.

Holroyd [13] classifies each journey that takes place in a particular area as (i) internal, (ii) cross-cordon, or (iii) through, according to whether the origin and destination of the journey are (i) both inside, (ii) one inside and one outside, or (iii) both outside the area. The average distance travelled within the area by such a journey is obtained by finding the average length of a route between (i) two random interior points (ii) a random interior point and a random peripheral point, and (iii) two random peripheral points. These three quantities are calculated for a circular area with twelve different routeing systems. For those routeing systems which make use of an inner circular ring-road, concentric with the area considered, Holroyd determines the positions of the ring-road which minimizes the average length of internal, cross-cordon, and through journeys.

Miller and Holroyd [14] point out that a large part of the congestion experienced by traffic in towns arises from the fact that vehicles have to cross each other's paths. Unless these crossings take place at a flyover, some delay at the intersection is almost inevitable. Pursueing this idea they discuss how the number of route-crossings can be used as one characterisation of the traffic movement in an area, and derive the expected number of crossings for two randomly selected routes in a simple circular city, where the origins and destinations of journeys are uniformly and independently distributed. Six types of routeing systems are considered. The problem of selecting a set of routes which minimizes the number of route-crossings on a given set of journeys is then studied. Finally methods are given for calculating the travel intensity (the total vehicle distance travelled per unit time per unit area) and the crossings intensity (the number of crossings per unit time per unit area) for every point in a city.

The optimal location of expressways comprise a group of problems, which are obviously related to those described above. As expressways are a prime ingredient of any transportation plan, it is not surprising that the problems, connected with them, are of great interest and have been intensively studied.

Creighton et al [7,8] have determined the optimal spacing of expressways and arterials on the basis of the following simple assumptions:

(a) There are only three types of streets - local, arterial, and expressways - with driving speeds in the ratio of 1 : 2 : 4.

(b) All streets exist in a square grid pattern only.

(c) The spacing of local streets is fixed at 0.125 miles.

(d) There is a uniform density of trip generation at all places in the urban terrain considered.

(e) There are ramps to expressways at the same spacing as arterial streets.

Other factors taken into account include the frequency distribution of trip length, and the usage made of the different types of roads by trips of different lengths. Expressions for the total cost of travel and the total cost of construction are given, and these are used to find the spacings of arterials and expressways which minimize the overall cost.

A more general model has been studied by Tanner [28], who assumes that a basic road system exists which permits travel in a straight line between any two points. Tanner's paper is concerned with the effect on the average travel time of a journey of superimposing an additional network of motorways (British for expressways) with higher travel speeds onto the basic road system. His other assumptions are:

(a) Trips are generated at random points on an infinite plane;

(b) Each trip has the same straight-line distance from origin to destination;

(c) All directions of travel from origin to destination are equally likely;

(d) A motorway may be joined or left at any point.

Tanner gives results for parallel, square grid, rectangular grid, and multi-directional motorways.

Friedrich [11] has described some interesting applications of the calculus of variations in the solution of town planning problems, particularly the optimal location of roads. The basic integral for the variational problem is obtained by summing up the total person - distance travelled in the urban region under consideration. The person - distance associated with a journey is defined to be the length of the journey multiplied by the number of persons going on the journey. The form of the road layout for which this basic integral is an extremum is calculated by solving the usual Euler differential equation. For practical cases, it turns out that the extremal solution is also the minimal solution, and Friedrich regards this as his main objective. The minimization of total person - distance usually means that the zones with the heaviest inter-zonal traffic are routed as directly as possible.

The distribution of traffic on a given transportation network is another topic, which has been studied by many authors including Wardrop [29]. The emphasis here is on the relationship between the demand for travel between the various zones of an urban area, and the 'cost' of travel along prescribed routes. The term 'cost' is often generalized to include such factors as time, price of fuel, or wear of the vehicle. Although the traffic aspects are not neglected, it appears that the major interest at present is on the derivation of some formula that will represent accurately the manner in which a given zone attracts traffic from other zones.

While theoretical work in the field of urban traffic networks has been fairly substantial, it is unfortunate that empirical analyses have not kept pace due to the difficulty of obtaining the required information. This is often only available in such a summarised form as to render it impossible for any general analysis to be made.

Of the few studies available, the most relevant appears to be the papers by Smeed [19] and Clark [5,6]. Smeed gives some results for the journey-to-work and for the distribution of homes and work places in London. One of his significant conclusions is that, if people travel directly from their homes to their work places, then a considerable proportion of unnecessary travel is bound to occur in the central city area. Clark has studied, in detail, the distribution of population in twenty cities at various times, and shown that the population density decreases more-or-less exponentially with increasing distance from the city centre. Another of his interesting conclusions is that in most cities, as time progresses, the density tends to fall in the most crowded inner suburbs and to rise in the outer suburbs so that the whole city tends to expand. Weiss [30] has reviewed Clark's results, developed some extensions, and provided an application for a servicing problem where there are installation as well as coverage costs. This topic has also been investigated by Sherratt [16].

An analysis of some work trip data for Adelaide is given in Chapter VI of the thesis, and it is shown that Clark's formula gives a good fit for the distribution of homes within Adelaide. The distribution of work places is also discussed although no simple pattern is apparent in this case.

It is obvious from the above discussion that the study of urban traffic in a general, mathematical fashion is still in a very primitive state. A realistic statement of the problem will involve so many complications that no solution appears to be possible. Hence the use of simplified models is inevitable. This thesis considers several such models and shows that a detailed study of these cases can provide new and important, even though limited, insights into the overall problem.

The main parameters investigated are the average distance travelled, average travel time, and average area of road space required for the journey to work in various types of cities. The concept of the area of road required for a journey is defined and explained in Chapter II, which also includes a summary of the later chapters. However the two most important results, presented in this thesis and previously discussed by Tan [25,26,27] in a series of papers, can be summarized as follows:

(a) For some important routeing systems the total city size, relative to the size of its central business district, can only be increased to a certain extent before the routeing system becomes inefficient and non-optimal;

(b) A simple modification of Smeed's basic model can be used to study the traffic pattern of commuter travel in satellite towns. This modified model is surprisingly comprehensive and enables a large number of parameters to be studied.

CHAPTER II

THE ROAD SPACE REQUIREMENTS FOR TRAFFIC IN CITIES

2.1 The area of road required for a journey

The concept of the area of road required for a journey was introduced by Smeed [17], and is essential for the calculation of the amount of road space required for travel within cities. This concept is meaningful whenever a network of roads is occupied to capacity for a definite period of time, and hence is particularly suitable for the study of commuter traffic in the morning and evening peak periods.

Consider a road of width W feet and capacity Q vehicles per hour. If this road is used to capacity for a period of T hours and the number of persons to a vehicle is c, then the width of road required for a person journey is W/QcT feet. It is convenient to denote W/QcT by λ. If each person travels a distance of one mile along this road, then

Area of road required per person for a journey of one mile $= 5280\,\lambda$ square feet.

The following points should be noted:

(a) The value of λ can be decreased either by lengthening the duration of the peak travel period or by increasing the number of persons to a vehicle;

(b) Since the capacity of a road depends on the speed of travel, it follows that the travel speed has been implicity assumed in the above calculation. The road capacity decreases as the travel speed increases within a certain range, and thus the value of λ will increase with increasing speed.

(c) Although it has been assumed that the road under consideration is used to capacity it is obvious that if this is not the case, then some road space is wasted and the area of road required for a journey will increase. Hence the above procedure can be considered as giving, in a sense, the minimum area of road required for a journey.

Some very useful data, reproduced in TABLE I, have been published by the Road Research Laboratory [15] on the road space required by a person to go a journey of one mile with different modes of transport and under different conditions. The assumptions behind the table are that the peak travel period lasts for two hours, a car carries 1.5 persons on the average, a bus carries 32 passengers (these are considered to be average figures for the peak period in Central London),

TABLE I

ROAD SPACE REQUIRED BY A PERSON

FOR A JOURNEY OF ONE MILE

	Speed (mile/hour)	Capacity (p.c.u./hour)	Road space required per person-mile (square feet)	
			Car	Bus
Urban Street 24ft. wide				
In Central London	15	610	69	10
In Central London	10	990	42	6
With few pedestrians or intersections	30	1400	30	
Urban street 44ft. wide				
In Central London	15	1970	39	6
In Central London	10	2790	28	4
With few pedestrians or intersections	30	4400	17	
Urban motorway	40	2000 per 12 ft. lane	11	

and that a bus is equivalent to three passenger car units or automobiles. It is interesting to see from Table I that the road space required by a person for a one-mile journey by car varies from 69 square feet on narrow Central London streets at a travel speed of 15 m.p.h. to only 11 square feet on urban motorways at 40 m.p.h.

2.2 The road space required in a city centre

The usefulness of the above concept of the area of road required for a journey is best illustrated by describing some of Smeed's results for the road space requirements of a city centre.

A simple theoretical model of a city is considered, and the analysis is confined to the journeys to and from work in the peak travel period. It is assumed that the central area is circular, and that all the people who work in the central area live outside it. The other relevant assumptions are:

(a) The number of workers entering the central area is known and is the same at all parts of the circumference;

(b) The point at which a worker enters the central area is not correlated with the position of his place of work;

(c) The density of work-places is inversely proportional to the distance from the city centre.

Since actual road networks and travel routes are too complicated for mathematical analysis, a simplifying assumption has to be made and only ideal routeing systems are considered. Three types of routes are studied:

(a) Direct: The route follows a straight line from the edge of the central area to the destination;

(b) Radial: The route follows a radial to the city centre and then follows another radial to the destination;

(c) Ring: The route follows the circumference of the central area to the radial on which the destination lies, and finally goes straight to the destination.

The average distances travelled in the central area by means of the above three routes can be calculated, and then the road space required per person for the journey to and from work is easily found. The details are given in Smeed [19]. The results, assuming that all the workers travel by car, are summarized in an interesting figure reproduced in Fig. 1. This shows the amount of road space required per person plotted against the number of persons working in the central area.

The data used for this figure relate to conditions prevailing in Central London, and it is assumed that the streets are of width 44 feet with traffic moving at 10 miles per hour, 1.5 persons to the car, and a peak period of

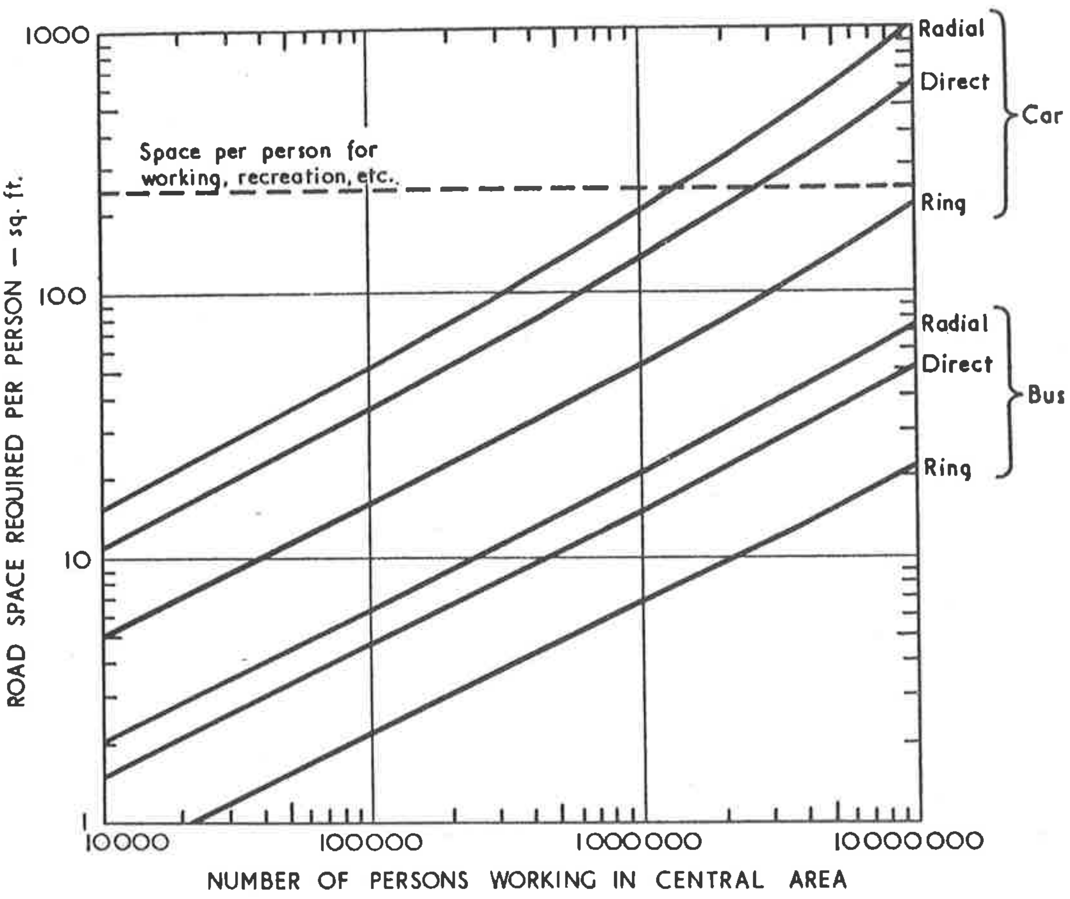

FIGURE 1. ROAD SPACE REQUIRED PER PERSON FOR TRAVEL BY CAR AND BUS WITH VARIOUS ROUTING SYSTEMS

(Ring routeing excludes the area of the ring road.)

2 hours. The road space required per person for a one-mile journey is taken to be 28 square feet, while the space required per person for working and for parking are assumed to be 245 square feet and 25 square feet respectively.

The results, representing travel by bus, are also included in Fig. 1. In this case it is assumed that the space required per person for a one-mile journey is 4 square feet, and no parking space is required.

It is immediately seen, from Fig. 1, that the road space required per person for the home-work journey increases rapidly with the number of persons working in the central area. This is an important result, and shows clearly that each new worker requires an additional amount of road space which increases in relation to the total number of workers already present.

2.3 Summary of later chapters

The effect of growth in the number of persons living in the suburbs of an idealized circular city and working in a central business district of constant size is investigated in Chapter III. Seven alternative basic routeing systems are examined. Critical values for the size of the whole city, relative to the size of its central business district, and for the ratio of the density of work-places to homes are obtained. The criteria used to evaluate the routeing systems are average distance travelled, average travel time, and average area of road space required for the journey to work.

This basic model of an inhomogeneous city is further investigated in Chapter IV, where we consider first the average distance travelled from home to work, in cities with rectangular routeing systems but of different geometrical shapes. Then the case of a square city with three modified rectangular routeing systems is examined and the average trip length and average time taken are calculated. It is shown that several interesting conclusions follow from the calculated results.

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Chapter VI. The distribution of homes is studied in some detail, and it is shown that the density of homes can be expressed in terms of the distance from the centre of the city by means of a simple formula. Data for the distribution of work-places within Adelaide are also given and commented on.

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CHAPTER III

AN EXPANDING CIRCULAR CITY

3.1 The 'rush-hour' problem

The traffic problem, in many cities, is essentially the problem of catering for the very large number of workers who commute from their homes in the suburbs to a densely packed city area or central business district. This rush of workers in the morning and evening takes place within a relatively short period of time, usually one or two hours, and seriously overloads the available road network at these periods. As a result excessive delays and congestion occur. Efforts to alleviate these hazards have often been frustrated due to the difficulty of regulating the vehicle owners. Ring roads, for example, which are designed to divert through-traffic from the city centre, have frequently been heavily utilized by those going into the central business district.

An idea of the magnitude of the problem may be gauged from the fact that travel to work accounts for more than 40 per cent of all home-based trips in cities of over one million population. Some transportation planners have even stated that as much as 90 per cent of their time and effort is directed towards finding means of catering for the 'home-to-work' commuter trip.

One of the most perplexing aspects of the rush-hour problem is that any solution, which is to be really effective, must take into account not only the present state of the commuter traffic but also the fact that this traffic is likely to become heavier and more prevalent in the future. Thus the town planner has to plan for an expanding as well as for a congested city. This expansion is difficult to plan for because a feature, common in urban growth, is that the central business district, with a high concentration of workers, grows very slowly in size whereas the surrounding suburban region, which is mainly residential, spreads rapidly and constitutes an expanding 'suburban sprawl'. In many cases the difficulties of the town planner are compounded by the fact that he is saddled with a basic road system, which has grown with the city, so that he is limited to only minor changes.

This chapter studies this aspect of the 'rush-hour' problem and deals with the effect of some basic routeing systems on the average distance travelled, average time taken, and average area of road space required for the journey from home to work in a city where the residential area is growing rapidly. It is based on an extension of a model, due originally to Smeed [17].

3.2 Formulation

The city is assumed to be in the form of a circle of radius R, within which is enclosed an inner concentric circle of radius r. Work places are assumed to be uniformly distributed over the inner circle (the CBD) and homes to be uniformly distributed over the outer concentric zone (the residential suburb). It is assumed that there is no correlation between the positions of homes and work places, i.e. people do not try to live in suburbs which are near their places of work. These simplifying assumptions are necessary in order to make the problem tractable. They can be regarded as reflecting our meagre understanding of urban traffic phenomena. For the purpose of examining the effect of an expanding suburban zone, we shall later keep r constant and increase R.

A person can travel from his home to his work place in any one of seven ways:

(a) Direct: By going in a straight line from his home to his destination. This routeing requires a network of straight roads running in all directions at all points. There is no real road network which can provide even an approximation to direct routeing. However it is of interest as it gives a lower bound for the travel distances.

(b) Rectangular: By going along one of the many shortest routes from his home to his work place, utilizing routes that are either parallel or perpendicular to a particular diameter of the circles. This routeing requires a rectangular grid of roads.

(c) Radial: By going along a radial to the centre of the city and then along another radial to his destination. This routeing requires a network of radial roads running in all directions and connected at the centre.

(d) Ring: By going along a radial until he reaches the inner circle, then along the circumference (clockwise or anticlockwise - whichever is shorter) until he reaches the radial on which his destination lies, and finally straight to his destination. This routeing requires a network of radial roads running in all directions, not connected at the centre, and a ring road at the circumference of the inner circle.

(e) Arc-radial: By going along a circle concentric with his home until he reaches a point on the same radial as his destination and then straight along the radial to his destination. This routeing requires a network of radial roads running in all directions, not connected at the centre, and ring-roads concentric with the circles at all distances in the residential zone.

(f) Radial-arc: By going along a radial until he reaches a circle concentric with his destination and then along this circle (clockwise or anticlockwise - whichever is shorter) to the destination. This routeing requires a network of radial roads running in all directions, which need not be connected at the centre, and ring-roads concentric with the circles at all distances in the central area. Holroyd [13] points out that this is the shortest route consisting of one radial segment and one arc. Moreover it would be difficult to ensure that no person passes through the centre, for although the radial roads may not be directly connected, they are connected to a ring-road very close to the centre.

(g) Polar: By using either a Radial-arc or Radial route - whichever is shorter. This routeing requires the same network of roads as in Radial-arc routeing, but has the radial roads connected at the centre.

3.3 Average trip length

Place a polar co-ordinate system on the city, as shown in Fig. 2. A home is defined by a point $H(h,0)$ and a work place by a point $G(\ell,\theta)$. Because of the symmetry of the figure no generality is lost (except in the case of Rectangular routeing) by assuming that the angular co-ordinate

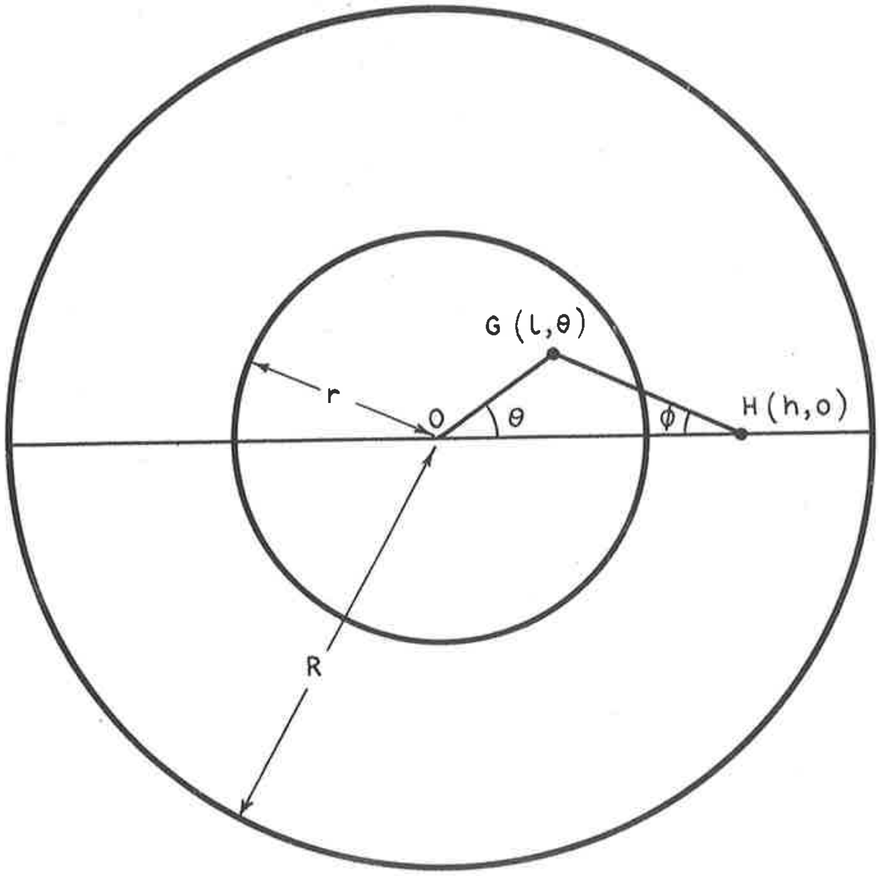

FIGURE 2. THE MODEL OF AN INHOMOGENEOUS CIRCULAR CITY WITH AN INNER CENTRAL BUSINESS DISTRICT AND AN OUTER CONCENTRIC RESIDENTIAL ZONE.
$\sphericalangle GOH = \theta$, $\sphericalangle OHG = \varphi$

of H is zero and by restricting θ to the range $0 \leqslant \theta \leqslant \pi$. The average length of a Rectangular route will later be calculated by a method which is not affected by these restrictions.

Since homes and work places are uniformly distributed, the probability of finding a home or a work place at a point is proportional to the polar element of area e.g. $\ell d\ell d\theta$. The distance d travelled along the first six routes from H to G are:

Direct	$d = \sqrt{[h^2 + \ell^2 - 2h\ell \cos \theta]}$	(3.1)
Rectangular	$d = h - \ell \cos \theta + \ell \sin \theta$	(3.2)
Radial	$d = h + \ell$	(3.3)
Ring	$d = (h - r) + r\theta + (r - \ell)$	(3.4)
Arc-radial	$d = (h - \ell) + h\theta$	(3.5)
Radial-arc	$d = (h - \ell) + \ell\theta$.	(3.6)

For polar routeing the traveller will use the Radial-arc route if $\theta \leqslant 2$, and the Radial route if $\theta > 2$.

The average distances travelled can be obtained by several methods. However the most convenient appears to be that of straight forward calculation.

(a) Direct.

It is convenient for this particular calculation to denote the distance d by η. The method used is to calculate first the expected value of η assuming that h is constant and then to allow for the variation of h.

The expected value $\bar{\eta}$ is thus given by

$$\bar{\eta} = \underset{h}{E} \underset{\ell,\theta}{E} (\eta) = \underset{h}{E} \underset{\eta,\varphi}{E} \quad \text{where} \quad \sin\varphi = \ell \sin\theta/\eta. \tag{3.7}$$

It is seen from Fig. 2 that, for a particular value of h, the parameters η and φ in equation (3.7) are restricted to the ranges $\eta_1 \leqslant \eta \leqslant \eta_2$ and $0 \leqslant \varphi \leqslant \sin^{-1}(r/h)$ where

$$\eta_1 = h\cos\varphi - \surd(r^2 - h^2\sin^2\varphi) \tag{3.8}$$

$$\eta_2 = h\cos\varphi + \surd(r^2 - h^2\sin^2\varphi) \quad . \tag{3.9}$$

The probability that point G is within the polar element of area $\eta d\eta d\varphi$, given that it is in the upper half of the inner circle, is $2\eta d\eta d\varphi/\pi r^2$.

Therefore

$$\underset{\eta,\varphi}{E} (\eta) = \frac{2}{\pi r^2} \int_0^{\sin^{-1}(\frac{r}{h})} \int_{\eta_1}^{\eta_2} \eta^2 \, d\eta \, d\varphi \ ,$$

$$= \frac{2}{3\pi r^2} \int_0^{\sin^{-1}(\frac{r}{h})} [6(h\cos\varphi)^2 (r^2 - h^2\sin^2\varphi)^{1/2} + 2(r^2 - h^2\sin^2\varphi)^{3/2}] \, d\varphi \quad . \tag{3.10}$$

The usual convention has been adopted so that

$$\iint (\ldots) \, d\eta \, d\varphi = \int [\int (\ldots) \, d\eta \,] \, d\varphi \quad . \tag{3.11}$$

Substituting $r \sin\alpha = h \sin\varphi$ in equation (3.10) gives

$$\underset{\eta,\varphi}{E}(\eta) = \frac{4}{\pi}\int_0^{\pi/2} [\cos^2\alpha \sqrt{(h^2 - r^2 \sin^2\alpha)}]\, d\alpha$$

$$+ \frac{4r^2}{3\pi}\int_0^{\pi/2} [\cos^4\alpha/\sqrt{(h^2 - r^2 \sin^2\alpha)}]\, d\alpha \;. \qquad (3.12)$$

The expected value $\bar{\eta}$ is

$$\bar{\eta} = \frac{2\pi}{\pi(R^2 - r^3)}\int_r^R [h \underset{\eta,\phi}{E}(\eta)]\, dh$$

$$= \frac{8R}{3\pi(R^2 - r^2)}\left\{ R^2 \int_0^{\pi/2} [\cos^2\alpha(1 - r^2\sin^2\alpha/R^2)^{3/2}]\, d\alpha \right.$$

$$\left. + r^2 \int_0^{\pi/2} [\cos^4\alpha(1 - r^2\sin^2\alpha/R^2)^{1/2}]\, d\alpha \right\}$$

$$- 128r^3/45\pi(R^2 - r^2) \;. \qquad (3.13)$$

In order to facilitate numerical computation, this expression can be written in terms of elliptic integrals. The average distance $\bar{\eta}$ or $\bar{d}$ comes to

$$\bar{d} = \frac{8R}{45\pi r^2(R^2 - r^2)} [(r^4 + 14r^2R^2 + R^4)E(r/R)$$
$$- (R^2 - r^2)(7r^2 + R^2)K(r/R)]$$
$$- 128r^3/45\pi(R^2 - r^2) \quad . \qquad (3.14)$$

$K(r/R)$ and $E(r/R)$ are the Legendre complete elliptical integrals of the first and second kind respectively. These have been tabulated in numerous books of tables.

(b) Rectangular

The easiest method of obtaining the average distance travelled using rectangular routeing is to use a result of Fairthorne (10), who points out that the average length of a rectangular route can be found by multiplying the average length of the corresponding direct route by the average ratio of rectangular distance to direct distance. The reason for this is that the symmetry of the circle ensures that the orientation, relative to the rectangular grid, of the straight line joining an origin to a destination is independent of its length. This method cannot be used, for example, in a square or rectangular city.

Let the straight line joining a home to a work place be of length η. If this line makes an angle α with a particular axis of the rectangular grid, then the rectangular distance is

$$\eta(\cos\alpha + \sin\alpha) \quad . \qquad (3.15)$$

The average ratio of rectangular distance to direct distance is thus

$$\frac{2}{\pi}\int_0^{\pi/2} (\cos\alpha + \sin\alpha)\, d\alpha = \frac{4}{\pi} \quad . \qquad (3.16)$$

Hence the average distance travelled from home to work is

$$\bar{d} = \frac{32R}{45\pi^2 r^2(R^2 - r^2)}\,[(r^4 + 14r^2R^2 + R^4)E(r/R)$$
$$- (R^2 - r^2)(7r^2 + R^2)K(r/R)]$$
$$- 512r^3/45\pi^2(R^2 - r^2) \quad . \qquad (3.17)$$

The same result can also be obtained by a direct calculation.

(c) Radial

The average trip length is

$$\bar{d} = E(h) + E(\ell)$$
$$= 2R^2/3(R + r) + 4r/3 \quad . \qquad (3.18)$$

(d) Ring

The expected trip length is

$$\bar{d} = E(h - r) + E(r\theta) + E(r - \ell)$$
$$= 2R^2/3(R + r) + \pi r/2 \quad . \qquad (3.19)$$

(e) Arc-radial

The expected trip length is

$$\bar{d} = E(h - \ell) + E(h\theta)$$
$$= E(h - \ell) + E(h)E(\theta) \qquad (3.20)$$

since the variables h and θ are independent. Therefore

$$\bar{d} = (\pi + 2)R^2/3(R + r) + \pi r/3 \quad . \qquad (3.21)$$

(f) Radial-arc

The expected trip length is

$$\bar{d} = E(h - \ell) + E(\ell)E(\theta)$$
$$= 2R^2/3(R + r) + \pi r/3 \quad . \qquad (3.22)$$

(g) Polar

Since polar routeing utilizes a Radial-arc route if $0 \leqslant \theta \leqslant 2$, the expected trip length for this range is

$$2R^2/3(R + r) + 2r/3 \quad . \qquad (3.23)$$

Hence the average trip length for polar routeing over the whole range $(0 \leqslant \theta \leqslant \pi)$ is

$$\bar{d} = \frac{2}{\pi}[2R^2/3(R + r) + 2r/3] + \frac{\pi - 2}{\pi}[2R^2/3(R + r) + 4r/3]$$
$$= 2R^2/3(R + r) + 4r(\pi - 1)/3\pi \quad . \qquad (3.24)$$

It is clear from the above formulae that the last five routes can be considered as a group. These routes can be arranged in the following order: Polar, Radial-arc, Radial, Ring, and Arc-radial. The average distance travelled from home to work becomes progressively longer as one descends down the order. This point has also been noted by Einhorn [9].

Numerical values for $\bar{d}$ with various values of R have been calculated and Fig. 3 shows the variation of mean trip length with increasing R. It is easily seen that the Direct routeing system gives the minimum trip length,

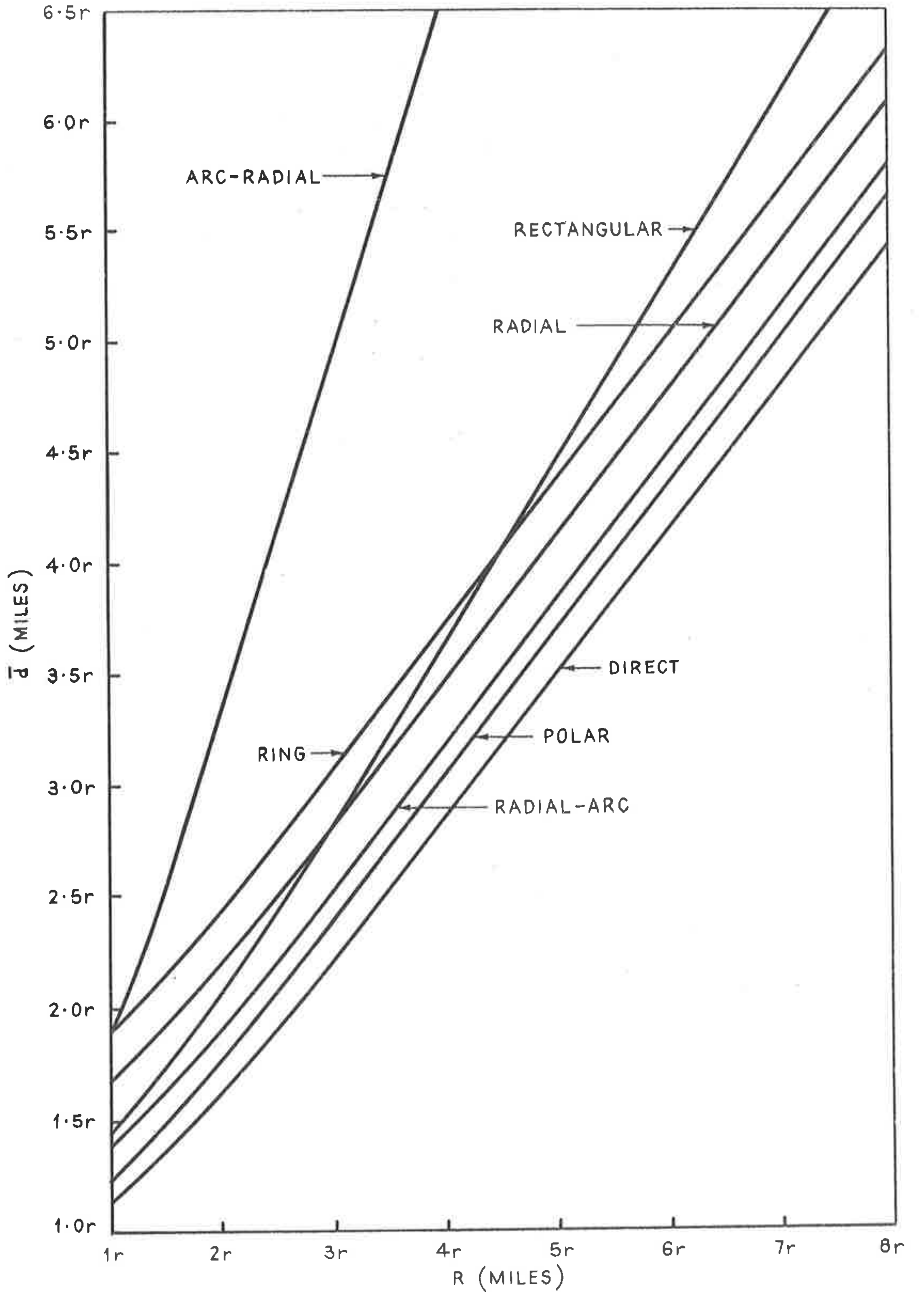

FIGURE 3. GRAPH OF $\bar{d}$ = AVERAGE DISTANCE TRAVELLED, AGAINST R = OVERALL RADIUS OF THE CITY; r IS THE RADIUS OF THE CENTRAL BUSINESS DISTRICT

while the Arc-radial system gives the maximum. However the curve for the Rectangular routeing system, which is the basic routeing system for many cities, intersects the curves for the Radial and Ring systems. This means that the average trip length for the commuter journey, using a Rectangular route, is less than the Radial or the Ring systems only for cities that are limited in size when compared to the size of their central business districts. In this case the total radius of the city has to be less than 3.0r miles and 4.5r miles respectively, where r is the radius of the central business district.

It can be assumed in our model city that the number of homes in the residential zone is equal to the number of work places in the central business district. Then the ratio ψ of the density of work places to homes is given by

$$\psi = (R^2 - r^2)/r^2 \quad . \tag{3.25}$$

From Fig. 3, we see that for all values of $\psi>0$, the Direct routeing system gives the shortest mean commuter trip length and the Arc-radial system gives the longest. The Rectangular routeing system results in a shorter average journey length than the Radial or Ring systems only for $\psi<8.0$ and $\psi<19.3$ respectively.

3.4 Average travel time

Another criterion that can be employed to evaluate the merit of a routeing system is the average time of travel from home to work. While the average distance travelled is important, the travel time is perhaps the factor of greatest direct interest to the motorist. Let V_1 be the speed of travel in the outer residential zone, V_2 the speed in the central business district, and V_3 the speed on the ring-road, if it exists.

Referring to Fig. 2, the time of travel from a home at $H(h,0)$ to a work place at $G(\ell,\theta)$ is given by:

Direct
$$t = [h \cos\varphi - \sqrt{(r^2 - h^2\sin^2\varphi)}]/V_1 + \sqrt{(\ell^2 + h^2 - 2h\ell\cos\theta)}/V_2 - [h\cos\varphi - \sqrt{(r^2 - h^2\sin^2\varphi)}]/V_2 \quad ; \tag{3.26}$$

Rectangular: The travel time depends on the particular route taken and is not unique;

Radial
$$t = (h - r)/V_1 + (r + \ell)/V_2 \quad ; \tag{3.27}$$

Ring
$$t = (h - r)/V_1 + r\theta/V_3 + (r - \ell)/V_2 \quad ; \tag{3.28}$$

Arc-radial
$$t = (h\theta + h - r)/V_1 + (r - \ell)/V_2 \quad ; \tag{3.29}$$

Radial-arc
$$t = (h - r)/V_1 + (r - \ell + \ell\theta)/V_2 \quad ; \tag{3.30}$$

Polar: Same as Radial or Radial-arc - whichever is shorter.

The average travel times are easily calculated, as in the previous section, by the methods of elementary probability theory.

Direct

Let the distance HG (Fig. 2) be denoted by η.

Then the time of travel from H to G can be expressed as

$$t = \eta/V_2 - (1/V_2 - 1/V_1)\eta_1 \tag{3.31}$$

where η_1 is defined in equation (3.8).

Then

$$\bar{t} = (1/V_2)E(\eta) - (1/V_2 - 1/V_1)E(\eta_1) \, , \tag{3.32}$$

and $E(\eta)$ is given in equation (3.14).

The probability that η_1 is of length $[h \cos \varphi - \sqrt{(r^2 - h^2\sin^2\varphi)}]$ is

$$(2/\pi r^2) . [d\varphi(\eta_2^2 - \eta_1^2)/2] = 4h \cos \varphi\sqrt{(r^2 - h^2\sin^2\varphi)}d\varphi/\pi r^2 \; . \tag{3.33}$$

The variable η_2 is defined in equation (3.9).

Therefore

$$\begin{aligned} \underset{\varphi}{E}(\eta_1) &= \frac{1}{\pi r^2}\int_0^{\sin^{-1}(r/h)} \{4h \cos \varphi \sqrt{(r^2 - h^2\sin^2\varphi)} \, [h \cos \varphi \\ &\quad - \sqrt{(r^2 - h^2\sin^2\varphi)}]\} \, d\varphi \\ &= \frac{4h^2}{\pi r^2}\int_0^{\sin^{-1}(r/h)} [\cos^2\varphi \sqrt{(r^2 - h^2\sin^2\varphi)}] \, d\varphi - 8r/3\pi \\ &= \frac{4}{\pi}\int_0^{\pi/2} [\cos^2\alpha \sqrt{(h^2 - r^2\sin^2\alpha)}] \, d\alpha - 8r/3\pi \; , \end{aligned} \tag{3.34}$$

where the substitution $r \sin \alpha = h \sin \varphi$ has been used.

The expression for $E(\eta_1)$ is obtained by averaging $E_{\varphi}(\eta_1)$ over h, and can be reduced to terms containing standard elliptic integrals. Finally $\bar{t}$ is calculated by equation (3.32) and can be expressed as

$$\begin{aligned}\bar{t} = {} & 8R[(3R^4 + 7r^2R^2 - 2r^4)E(r/R)]/45\pi r^2(R^2 - r^2)V_1 \\ & - 8R[(R^2 - r^2)(r^2 + 3R^2)K(r/R)]/45\pi r^2(R^2 - r^2)V_1 \\ & - 64r^3/45\pi(R^2 - r^2)V_1 - 8r/3\pi V_1 \\ & + 8R[(3r^4 + 7r^2R^2 - 2R^4)E(r/R)]/45\pi r^2(R^2 - r^2)V_2 \\ & - 8R[(R^2 - r^2)(6r^2 - 2R^2)K(r/R)]/45\pi r^2(R^2 - r^2)V_2 \\ & - 64r^3/45\pi(R^2 - r^2)V_2 + 8r/3\pi V_2 \quad . \end{aligned} \qquad (3.35)$$

The average travel times for the other routes can be calculated very simply using the same methods as before, and can be written down as:

Radial: $\bar{t} = \bar{t}_1 + 5r/3V_2$; (3.36)

Ring: $\bar{t} = \bar{t}_1 + \pi r/2V_3 + r/3V_2$; (3.37)

Arc-radial: $\bar{t} = \bar{t}_1 + \pi R^2/3(R + r)V_1 + \pi r/3V_1 + r/3V_2$; (3.38)

Radial-arc: $\bar{t} = \bar{t}_1 + r(\pi + 1)/3V_2$; (3.39)

Polar: $\bar{t} = \bar{t}_1 + (5\pi - 4)r/3\pi V_2$; (3.40)

where $\bar{t}_1 = 2R^2/3(R + r)V_1 - r/3V_1$. (3.41)

As a numerical example, we now set $V_1 = 30$ mph, $V_2 = 10$ mph, and $V_3 = 25$ mph, which may be regarded as being typical values for travel speeds on residential, urban, and ring roads respectively.

Numerical values for $\bar{t}$ with various values of R and a constant r have been calculated and Fig. 4 illustrates the manner in which the travel time along the various routes are affected by the expanding suburb (increasing R). As might be expected from the above formulae, the Polar, Radial-arc, and Radial routeing systems can be ranked in increasing order of magnitude of travel time, and this ranking is not affected by the value of R. However the curve for the Arc-radial system intersects the other curves, and there is also an intersection between the curves for the Direct and Ring routeing systems. These intersections of the curves show that the size of the city is an important factor in the design of a suitable routeing system with this criterion.

In our particular example, the minimum average travel time is obtained with the Arc-radial system for the case $R < 1.4r$ miles, the Ring system is the best for the range $1.4r$ miles $< R < 5.7r$ miles, and for $R > 5.7r$ miles we choose the Direct routeing system. The corresponding critical values for the ratio ψ of the density of work places to homes are 1.0 and 31.5 respectively.

3.5 Average area of road space required

To obtain the average area of road space necessary for

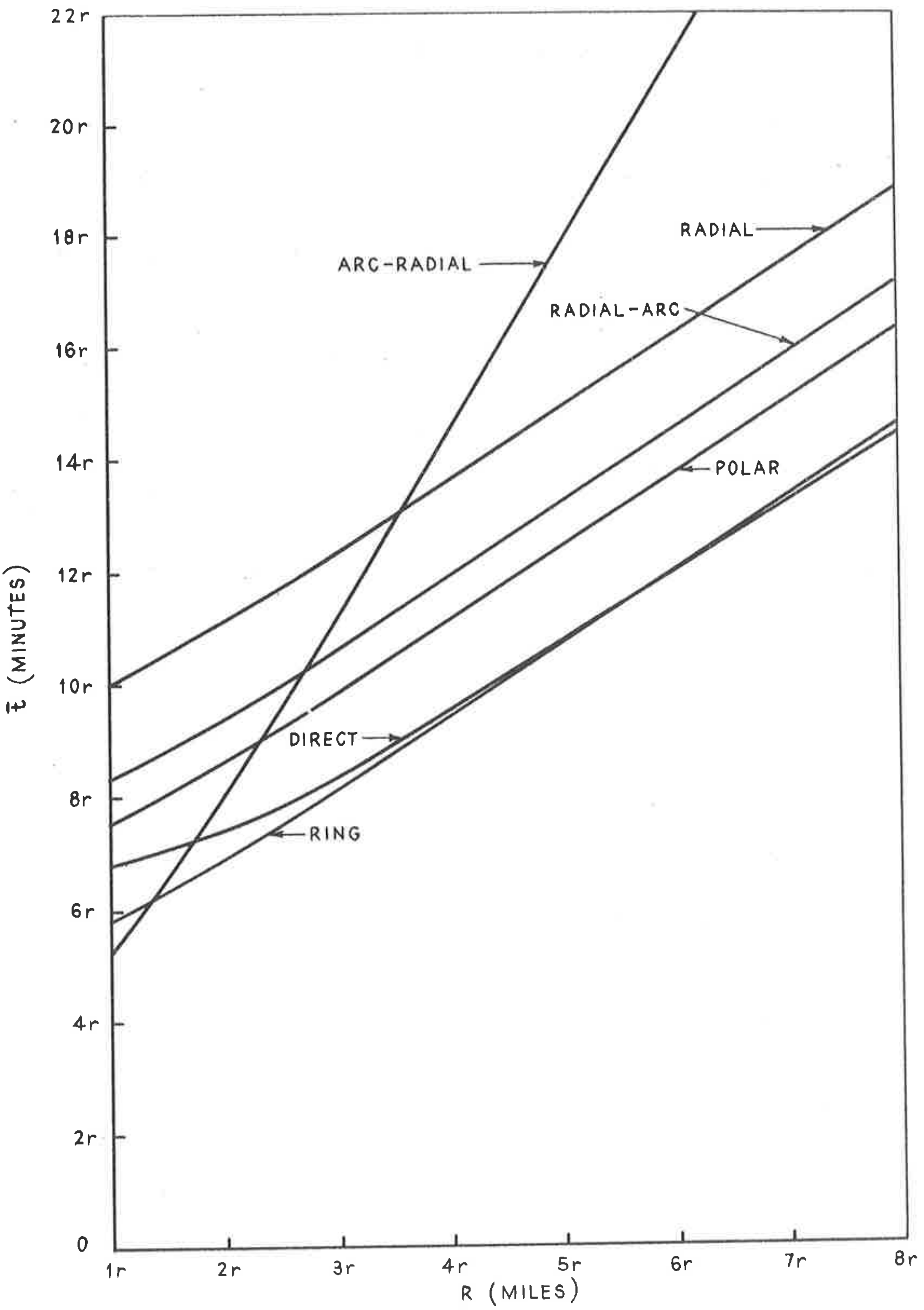

FIGURE 4. GRAPH OF $\bar{t}$ = AVERAGE TRAVEL TIME, AGAINST R = OVERALL RADIUS OF THE CITY; r IS THE RADIUS OF THE CENTRAL BUSINESS DISTRICT

a commuter journey, we multiply the average distances travelled in each region of the city by the appropriate value for the area of road required per person-mile for a journey.

The values in Table I will be taken as an example. It will be assumed that travel is on streets of width 44 feet at a speed of 30 mph in the residential zone and along the Ring-road. In the central business district, it will be assumed that travel is on narrow streets of width 24 feet at a speed of 10 mph. Using Table I, the road spaces required per person-mile for a journey are seen to be 17 and 42 square feet respectively.

The average road space required for the 'home-to-work' journey has been calculated for various values of R and constant r. The final result is a graph, very much like Fig. 4. The curve for the Arc-radial system intersects the curves for the Direct, Polar, Radial-arc, and Radial systems at $1.4r$ miles, $1.8r$ miles, $2.2r$ miles, and $2.9r$ miles respectively, and there is also an intersection between the curves for the Direct and Ring routeing systems at $5.7r$ miles.

This shows that when two routeing systems are compared on the basis of road space requirements, the size of the whole city relative to the size of its central business district is, in many cases, an important factor in deciding which is the better routeing system.

3.6 Discussion of results

The simple model investigated in this chapter has shown that, from the viewpoint of distance travelled, travel time, or road space required, the basic routeing system in a number of cases can cater for the expansion of the city only up to a critical size beyond which major alterations may be desirable. This is of significance to the town planner who has to decide at what point he should think about developing a satellite town to accommodate the increasing urban population as an alternative to enlarging the main city.

The model is, of course, extremely crude and the results obtained should be regarded as valid only in a qualitative sense. There are many other factors, such as the difficulties of having a central junction or of having ring roads in the central business district, which would have to be considered as well. This basic model of a city will be investigated further in Chapter IV, where we shall study some routeing systems which are a little bit more realistic.

CHAPTER IV

RECTANGULAR ROUTEING SYSTEMS

4.1 Introduction and Formulation

A rectangular street network is a common feature in the transportation system of many cities. It is popular because traffic is generally well distributed over the network, and awkward intersections can be avoided. Indeed the rectangular grid has practically become the standard road network in a number of countries particularly the United States. It is thus of interest to make a more thorough study of rectangular routeing systems.

In this chapter the average distance travelled from home to work, in cities with rectangular routeing systems but of different geometrical shapes, is considered first. Then the case of a square city with three modified rectangular routeing systems is examined and the average trip length and average time taken are calculated. It will be shown again that the total size of a city, relative to the size of its central business district, is an important factor in the choice of an optimal routeing system.

The general layout of the city and system of co-ordinates used are shown in Fig. 5. The whole city is assumed to be in the form of a rectangle with sides of

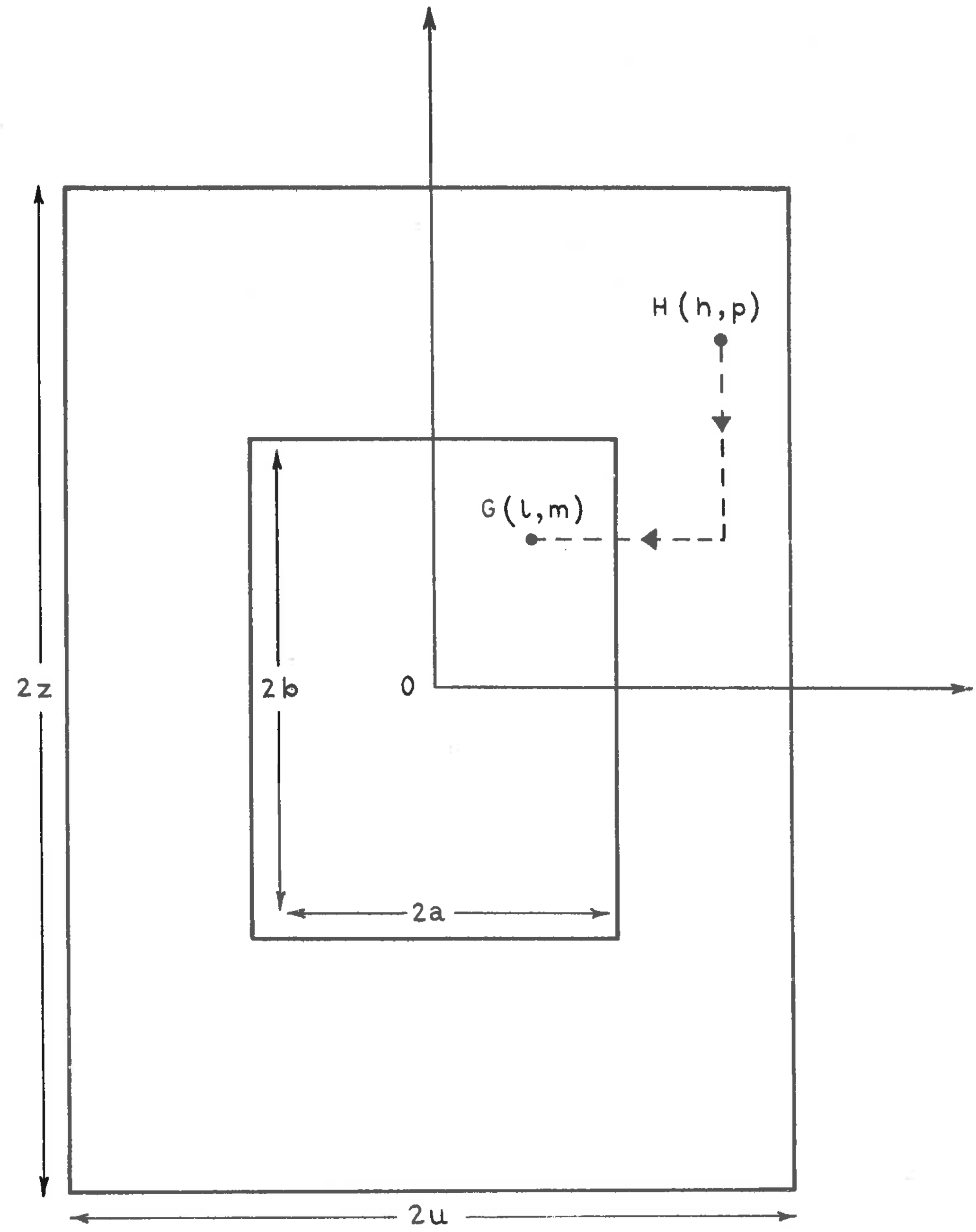

FIGURE 5. THE MODEL OF AN INHOMOGENEOUS RECTANGULAR CITY WITH AN INNER RECTANGULAR BUSINESS DISTRICT AND AN OUTER CONCENTRIC RESIDENTIAL ZONE

length $2z$ and $2u$. The central business district is represented by an inner rectangle, with sides parallel to those of the first but of lengths $2a$ and $2b$ $(0 < a < u, 0 < b < z)$. A home is defined by a point $H(h,p)$ and a work place by a point $G(\ell,m)$. The co-ordinates (h,p) are restricted to the ranges $(|h| < a, b < |p| < z)$ and $(a < |h| < u, |p| < z)$, while the co-ordinates (ℓ,m) are restricted to the range $(|\ell| < a, |m| < b)$.

As before, homes are assumed to be uniformly distributed over the outer annular zone (the residential suburb), and work places to be uniformly distributed over the inner rectangle (the central business district). No correlation is taken to exist between homes and work places.

4.2 Rectangular routeing systems

Average Travel Distance

It will be assumed, in this section, that the commuter follows a rectangular route, which is any one of the many shortest routes from a home to a work place utilizing routes which are parallel to the sides of the rectangles.

The average distance travelled from home to work is most easily calculated by considering various special cases for the position of H. Assume, for example, that H lies within the region $(a < h < u,\ b < p < z)$. The probability of this event occurring is $(u - a)(z - b)/4(zu - ab)$. The distance from home to work can now be written as

$$d = (p - b) + (b - m) + (h - a) + (a - \ell) \ , \qquad (4.1)$$

and the corresponding average travel distance is

$$\bar{d} = (z - b)/2 + b + (u - a)/2 + a \ . \qquad (4.2)$$

These calculations are repeated for the other portions of the outer annular zone, and the final expression for the average trip length can be written as

$$\bar{d} = \frac{a + b + u + z}{2} - \frac{b(u - a)(3z - b) + a(z - b)(3u - a)}{6(uz - ab)} \ . \qquad (4.3)$$

A special case of a rectangular city, whose length $2z$ is k times its breadth $2u$ and whose central business district is a square $(b = a)$, will now be considered. It can be assumed that $k \geqslant 1$ without loss of generality. Equation (4.3) simplifies to

$$\bar{d} = \frac{(k + 1)(3ku^3 + a^2u) - 8a^3}{6(ku^2 - a^2)} \ . \qquad (4.4)$$

If the area of the whole city is taken to be sA, where A is the area of the central business district, then the following dimensionless equation is obtained:

$$\frac{\bar{d}}{\sqrt{A}} = \frac{(k + 1)(3s\sqrt{s} + \sqrt{s}) - 8\sqrt{k}}{12(s - 1)\sqrt{k}} \ . \qquad (4.5)$$

Fig. 6 shows the average commuter distance, for various city sizes, in a square city, a 2 x 1 rectangular city and a 3 x 1 rectangular city, all containing square central business districts. This figure also includes the corresponding curve for the average length of a rectangular route in the inhomogeneous circular city, which was studied in Chapter III. It is seen that the average trip length is a minimum with a circular city, and is progressively lengthened as the shape of the city changes into a square and then into an elongated rectangle.

Comparison with a homogeneous city

These conclusions are in agreement with those of Fairthorne [10], who has calculated the corresponding values for rectangular routeing in a homogeneous city, where the homes and work-places are uniformly distributed all over the city. The average trip length in such a rectangular homogeneous city, with sides of length in the ratio 1 : k, is

$$\frac{\bar{d}}{\sqrt{A}} = \frac{(1 + k)\sqrt{s}}{3\sqrt{k}} , \tag{4.6}$$

where sA is the area of the whole city, and A is the area of the square central business district in the inhomogeneous city.

The ratio α of the average trip length in an

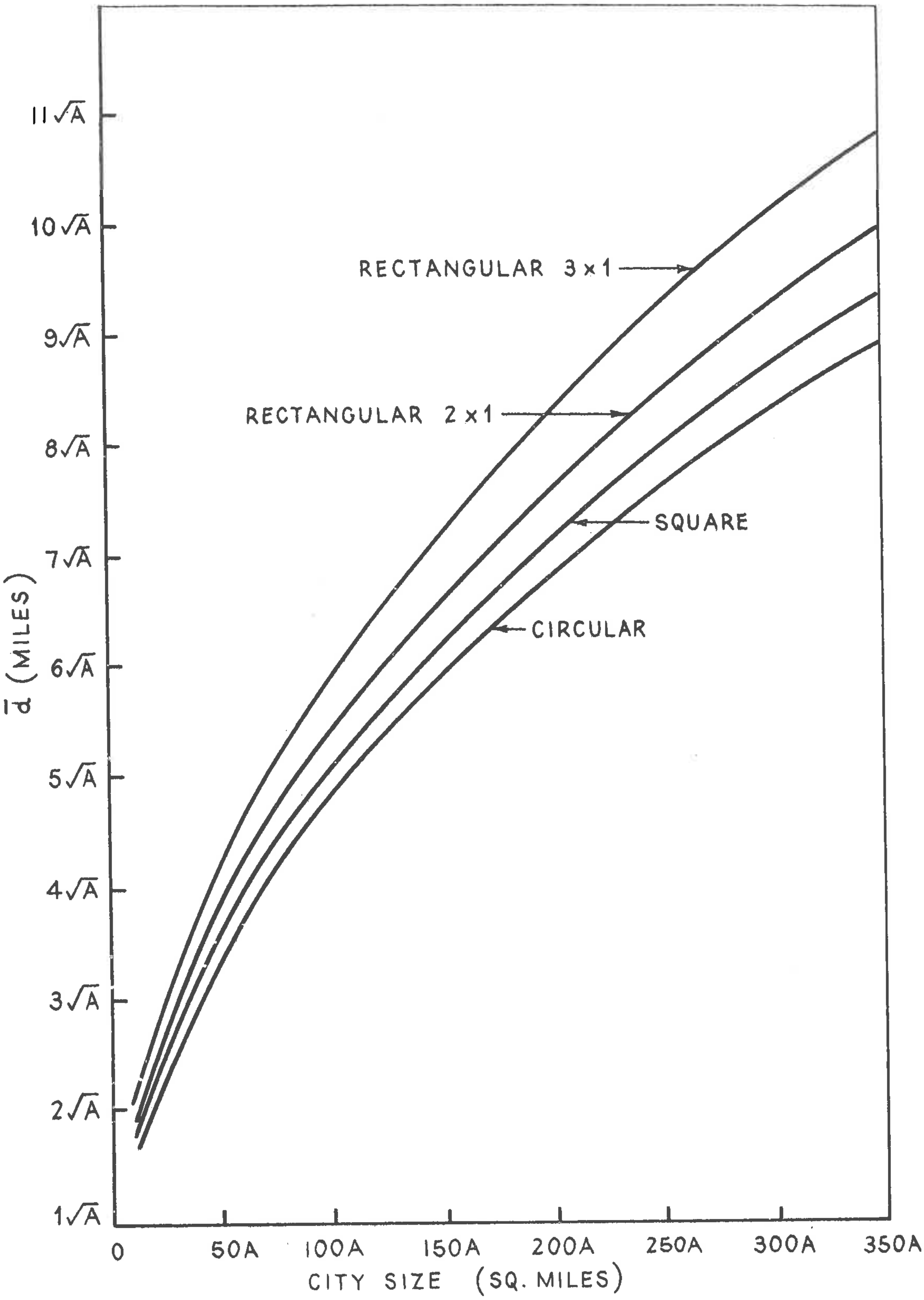

FIGURE 6. GRAPH OF $\bar{d}$ = AVERAGE COMMUTER TRIP LENGTH AGAINST THE TOTAL CITY SIZE FOR FOUR CITIES WITH RECTANGULAR ROUTING SYSTEMS BUT OF DIFFERENT SHAPES. A (SQUARE MILES) IS THE AREA OF THE CENTRAL BUSINESS DISTRICT IN EACH CASE, AND THE UNIT OF LENGTH IS $\sqrt{A}$ MILES

inhomogeneous rectangular city to that in a homogeneous rectangular city is

$$\alpha = \frac{3s + 1}{4s - 4} - \frac{2\sqrt{k}}{(s - 1)(k + 1)\sqrt{s}} \quad . \tag{4.7}$$

For $\alpha \leqslant 1$, the following inequality must hold

$$(5 - s)\sqrt{s} \leqslant \frac{8\sqrt{k}}{(1 + k)} \quad . \tag{4.8}$$

A further restriction $(s \geqslant k)$ is necessary to ensure that the shorter of the two city sides is longer or equal to a side of the square central business district.

It can be easily shown that these restrictions lead to the following bounds for α :

(a) for $k > 3$ and $s \geqslant k$, α is always less than unity, and

(b) for $1 \leqslant k \leqslant 3$ and $s \geqslant k$, the range of values of s, for which $\alpha \geqslant 1$, can only lie in the closed interval $[1,3]$.

A similar result holds for circular cities. The average length of a rectangular route from home to work in an inhomogeneous circular city is always shorter than the corresponding average length in a homogeneous city except for the range $(1 \leqslant s \leqslant 2.3)$.

It is thus possible to state that, for all practical cases, the average commuter trip length is reduced for an inhomogeneous city, as compared with a homogeneous city, even when the total city size is not very large relative to the size of the central business district. This reduction is considerable as α (for rectangular cities) tends to its limiting value of 0.75 rapidly for initial values of s. This is evident from TABLE II, which gives the values of α for rectangular cities of sizes 10A, 100A, and 200A, and of various shapes. The corresponding limiting value for circular cities is $15\pi/64 = 0.7363$ and the ratio for the average trip length in an inhomogeneous circular city to that in a homogeneous circular city takes on the values 0.8275, 0.7453, and 0.7392 for circular cities of sizes 10A, 100A, and 200A respectively. This also indicates a rapid convergence to the limiting value.

As a practical example, consider the city of Adelaide. In this case the central business district may be taken to be a square of area 1 sq. mile (A = 1). The total metropolitan region covers an area of about 160 sq. miles (s = 160), and is roughly rectangular in shape, with its length approximately twice its width (k = 2). Thus, if all the work places were concentrated within the central business district and all the homes scattered outside of it, the average commuter trip length would only be approximately three-quarters of that required if the homes and work

TABLE II

COMPARISON OF HOMOGENEOUS AND INHOMOGENEOUS CITIES

Values of α for various city sizes

Shape parameter k	City Size		
	10A	100A	200A
1	0.8260	0.7591	0.7547
2	0.8280	0.7591	0.7547
3	0.8307	0.7592	0.7547
4	0.8330	0.7593	0.7547
5	0.8349	0.7593	0.7548

places were scattered over the whole urban region.

It must be recognised that many factors have not been considered in the preceding discussion. One example is the area required for homes and work places. If a proportion of the workers had their homes transferred from the suburbs into the central business district, this would enlarge the size of the central business district and thus lengthen the distances travelled within it. This would have the undesirable effect of increasing the amount of traffic in an area which is already highly congested. Some comments on these points are contained in a paper by Smeed [20].

4.3 Modified Rectangular routeing systems

Where there is an important central business district, drawing workers from an extensive region as in our model, highly directional commuter flows occur. Travelling distances may be unnecessarily long with the rectangular street network and it is often supplemented with radial freeways. The effect of three modified rectangular routeing systems on the average trip length and average travel time in a square inhomogeneous city, with a square central business district, is studied in the following sections. Homes and work places are assumed to be distributed as before.

4.4 Travel Distance in a Square City

There are two main assumptions behind the first two modified systems. Firstly a person will, as far as possible, select routes which are not directed towards the central business district, so as to avoid congestion, and secondly, entry to the central business district is only to be effected along special routes.

The first system is called the 'radial-rectangular system'. In addition to the rectangular street network, there now exist eight freeways, placed symmetrically at 45° to each other. The lay-out of the network is shown in Fig. 7. A commuter initially goes along a route, which is parallel to the nearer side of the square city, until a freeway is reached. He then goes along the freeway to the edge of the central business district, and from there, he takes any one of the many shortest rectangular routes to his destination.

Two possible journeys generally exist at each origin (Fig. 7). The distance travelled along route A is

$$d = (h - p) + \sqrt{2}(h - a) + (a - \ell) + (a - m) \qquad (4.9)$$

and along route B is

$$d = p + (h - a) + m + (a - \ell). \qquad (4.10)$$

It is assumed that the particular freeway will be selected so as to minimize the travelling distance.

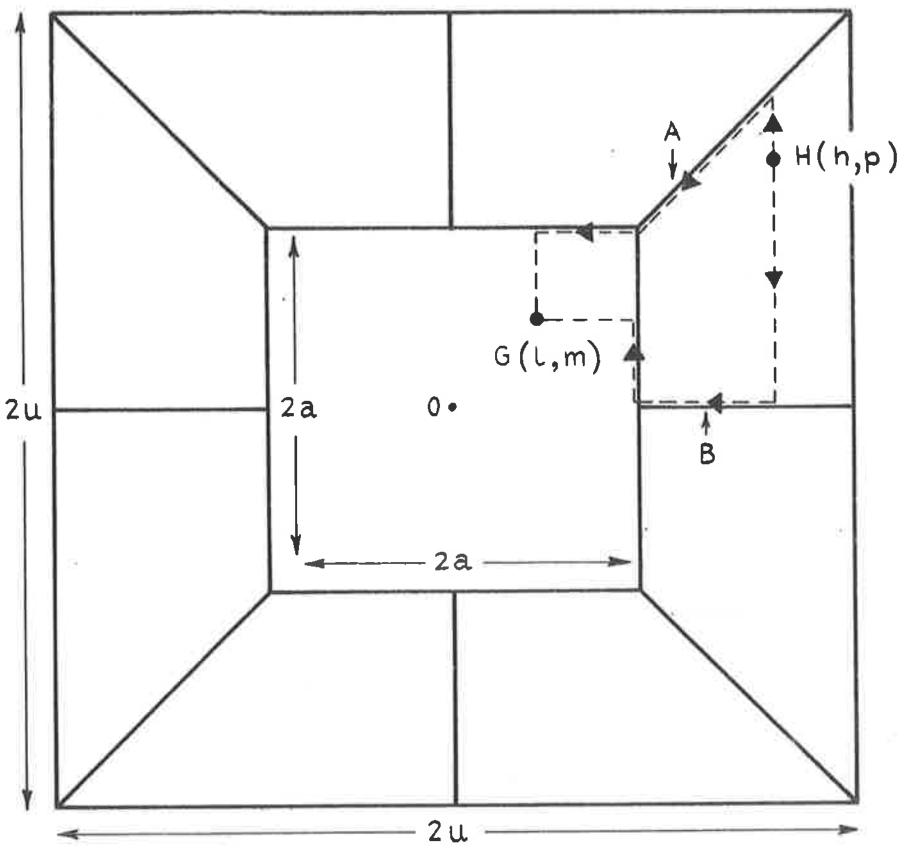

FIGURE 7. THE MODEL OF AN INHOMOGENEOUS SQUARE CITY WITH EIGHT RADIAL ROUTES, PLACED SYMMETRICALLY AT 45° TO EACH OTHER

The second modified system is called the 'mass transit' system. It is similar to the first, but the commuter continues along the radial route to the centre O of the city, and from there takes a rectangular route to his destination. The eight radial routes may now be thought of as underground railways, or surface routes which are specially reserved for mass transportation. As before two routes exist at each origin with travelling distances

$$d = (h - p) + \sqrt{2}h + \ell + m \qquad (4.11)$$

and

$$d = p + h + \ell + m \, . \qquad (4.12)$$

The choice of route is governed by the minimum travelling distance.

The third modified system, which was suggested by Dr. A.J. Miller in a discussion on a paper by Tan [26], may be termed the 'diagonal' system. It consists simply of superimposing a system of four diagonal roads (instead of the eight radials in the radial-rectangular system) onto a basic rectangular grid. The assumption that drivers take the shortest routes implies that those commuters with homes in the corner regions ($a < |h| < u$, $a < |p| < u$) will use a diagonal road to get to their work places, whereas those with homes in the remaining regions of the Residential Zone will follow the usual rectangular route.

A commuter, who uses a diagonal road, is assumed to travel initially perpendicular to the nearer side of the square city until he reaches a diagonal road; then he goes along the diagonal road to the edge of the central business district, and finally takes a rectangular route to his destination. The distance travelled along such a route is

$$d = (h - p) + \sqrt{2}(p - a) + (a - \ell) + (a - m) \ . \qquad (4.13)$$

Average Travel Distance

Consider a commuter, whose home is within the narrow annular zone of width dP bounded by squares with sides of length $2P$ and $2(P + dP)(P > a)$. His average travelling distance to a destination in the central business district is given by:

rectangular $\quad \bar{d} = 3P/2 + a^2/6P \qquad (4.14)$

radial rectangular $\quad \bar{d} = \sqrt{2}P + a(3 - 7\sqrt{2}/4) + (a^2/P)(3\sqrt{2}/4 - 7/6) \qquad (4.15)$

mass transit $\quad \bar{d} = \sqrt{2}P + a \qquad (4.16)$

diagonal $\quad \bar{d} = (1 + \sqrt{2})P/2 + a(2 - \sqrt{2}) + (a^2/P)(\sqrt{2}/2 - 5/6) \ . \qquad (4.17)$

The calculations, leading to the above results, are elementary although a little involved.

When the rectangular system is compared with the radial-rectangular system, it is found that a critical point exists at $P = 5.55a$. The minimum commuter trip length is obtained with the rectangular system for $P < 5.55a$, and with the radial-rectangular system for $P > 5.55a$. Similarly there is a critical point at $P = 11.49a$ in a comparison of the rectangular and mass transit systems. However the radial-rectangular route is always somewhat shorter than the mass transit route for all values of P. The transition points indicate that, on the basis of travel distance, a commuter, living near the central business district, will tend to prefer a rectangular route. If he is living in the outlying suburbs, he will tend to choose a radial-rectangular or mass transit route.

Of the four systems considered, the shortest average trip length is obtained with the diagonal system for all values of P. This is to be expected as this system effectively allows the commuter to choose between a rectangular route and a shortened radial-rectangular route.

As a practical illustration of the results, suppose that the central business district is a square of area 1 sq. mile as in the case of Adelaide. Then $a = 0.5$ mile and the transition points $P = 5.55a$ and $P = 11.49a$ take on the values 2.78 miles and 5.75 miles respectively.

The overall efficiency of commuter travel in a square city of area sA, which contains a square central business district of area A, will now be considered. The average trip lengths, in this case, are easily obtained by an integration of the last three equations, and may be expressed as:

rectangular
$$\frac{\bar{d}}{\sqrt{A}} = \frac{3s + 3\sqrt{s} + 4}{6(\sqrt{s} + 1)} \tag{4.18}$$

radial-rectangular
$$\frac{\bar{d}}{\sqrt{A}} = \frac{8\sqrt{2}s + (36 - 13\sqrt{2})\sqrt{s} + (8 + 5\sqrt{2})}{24(\sqrt{s} + 1)} \tag{4.19}$$

mass transit
$$\frac{\bar{d}}{\sqrt{A}} = \frac{2\sqrt{2}s + (2\sqrt{2} + 3)(\sqrt{s} + 1)}{6(\sqrt{s} + 1)} \tag{4.20}$$

diagonal
$$\frac{\bar{d}}{\sqrt{A}} = \frac{(1 + \sqrt{2})s + (7 - 2\sqrt{2})\sqrt{s} + (2 + \sqrt{2})}{6(\sqrt{s} + 1)}. \tag{4.21}$$

A comparison of the routeing systems shows that the average trip length for the rectangular system is smaller or greater than that for the radial-rectangular system, according as to whether the total area of the city is less or greater than 64.2A. Similarly the rectangular and mass transit systems have a transition point at 292.7A. As before, the mass transit route is always longer than the radial-rectangular route and the diagonal route is the shortest of all the routes considered.

It is interesting to compare these results with those for the limiting case of direct routeing, when the commuter goes straight from his origin to his destination. Direct routeing represents the ideal as far as travel distance is concerned. The straight-line distance between a home $H(h,p)$ and a work place $G(\ell,m)$ is

$$d = \sqrt{[(h - \ell)^2 + (p - m)^2]} \quad , \qquad (4.22)$$

and thus the average distance is

$$\bar{d} = \underset{p\,h\,\ell\,m}{EEEE} \{\sqrt{[(h - \ell)^2 + (p - m)^2]}\} \quad . \qquad (4.23)$$

The average trip length is obtained in exactly the same fashion as in the previous cases, the necessary integrations being performed in the order given in Equation (4.23). The calculation, however, is extremely lengthy in this case. The details are given in Appendix A, and only the final result will be recorded here. The average distance travelled with direct routeing in a square inhomogeneous city of area sA is

$$\begin{aligned}\frac{\bar{d}}{\sqrt{A}} = & \frac{(\sqrt{s} + 1)^3}{96} \ln\left(\frac{\sqrt{s} + 1}{\sqrt{s} - 1 + \sqrt{(2s + 2)}}\right) \\ & + \frac{(\sqrt{s} - 1)^3}{96} \ln\left(\frac{\sqrt{s} - 1}{\sqrt{s} + 1 + \sqrt{(2s + 2)}}\right) \\ & + \frac{[5\ln(1 + \sqrt{2}) + \sqrt{2}](s^2\sqrt{s} + 10s\sqrt{s} + 5\sqrt{s} - 16)}{240(s - 1)} \\ & - \frac{[\sqrt{(2s + 2)}](s^2 - 18s + 1) + 32}{240(s - 1)} \quad . \qquad (4.24)\end{aligned}$$

Table III shows some numerical results for the average trip length $\bar{d}$ for various city sizes. It is clear that there is a large gap between the direct routeing system and the rectangular, radial-rectangular, and mass transit

TABLE III

AVERAGE COMMUTER TRIP LENGTH IN A SQUARE INHOMOGENEOUS CITY

City Size	Routeing Systems*				
	Rectangular	Radial-rectangular	Mass-transit	Direct	Diagonal
10A	1.741	1.841	2.104	1.338	1.632
50A	3.618	3.641	3.892	2.771	3.172
150A	6.174	6.064	6.309	4.726	5.242
250A	7.945	7.738	7.982	6.081	6.671
350A	9.388	9.100	9.343	7.185	7.835

*The unit of length is taken to be $\sqrt{A}$.

systems, whose values are closely grouped together. This gap is the penalty that has to be paid for restricting traffic to specified channels, thus giving the commuter very little choice of routes. The diagonal system allows the commuter a more favourable choice of routes, and as may be expected, the values for this system are much closer to the corresponding direct distances.

4.5 Travel Time in the Square City

Two other parameters, that are of interest in the evaluation of a routeing system, are the average travel time and the average area of road space required for a journey. As shown in the previous chapter, the calculations involved are practically the same for the two cases, and thus only the average travel time will be considered here.

Ideally commuters should be allowed to choose their routes on the basis of minimum travel time. However, this makes the calculations extremely involved, and the necessity of specifying numerous regions makes it difficult to evaluate the network as a whole. To simplify matters, the choice of routes is assumed to be the same as in the previous section.

A few additional conditions have to be imposed, however, to specify the routes exactly:

(a) In cases where the commuter uses rectangular routeing to go from his home to his work place, it is assumed that he endeavours to travel, as far as possible, outside the central business district.

(b) A limited choice of routes, on the basis of travel time, is allowed with the radial-rectangular system. On reaching the central business district in this case, the commuter is assumed to choose, from among the many shortest rectangular routes to his destination, that one which involves the greatest amount of travel on the edge of the central business district. This specialized choice of routes makes it possible to treat the case of a ring road around the central business district, when the speed of travel along the edge of the central business district will not be the same as that within it. Of course, if these two speeds are equal all the shortest rectangular routes, joining any two points in the central business district, will have the same travel time.

Since the diagonal system (unlike the radial-rectangular system) allows free entry into the central business district, it is unlikely that a ring road would be incorporated in this system. Thus it is sufficient to assume a constant speed of travel at all points, including the edge, of the central business district.

Let V_1 be the speed of travel in the outer residential zone, V_2 the speed along the radial routes, V_3 the speed at the edge of the central business district, and V_4 the speed within the central business district. Typical expressions for the time of a journey from a home $H(h,p)$ in the residential suburb to a work place $G(\ell,m)$ in the central business district are:

rectangular
$$t = (p - m)/V_1 + (h - a)/V_1 + (a - \ell)/V_4 \qquad (4.25)$$

radial-rectangular
$$t = (h - p)/V_1 + \sqrt{2}(h - a)/V_2 + (a - \ell)/V_3 + (a - m)/V_4 \qquad (4.26)$$

or
$$t = p/V_1 + (h - a)/V_2 + m/V_3 + (a - \ell)/V_4 \qquad (4.27)$$

mass transit
$$t = (h - p)/V_1 + \sqrt{2}h/V_2 + (\ell + m)/V_4 \qquad (4.28)$$

or
$$t = p/V_1 + h/V_2 + (\ell + m)/V_4 \qquad (4.29)$$

diagonal
$$t = (h - p)/V_1 + \sqrt{2}(p - a)/V_2 + (a - \ell)/V_4 + (a - m)/V_4 \qquad (4.30)$$

or
$$t = (p - m)/V_1 + (h - a)/V_1 + (a - \ell)/V_4 \quad . \qquad (4.31)$$

Average Travel Time

The most significant parameter, in the case of travel time, is the overall average time of travel from home to work. With the same assumptions and similar calculations as before, the average travel time is given by

rectangular
$$\frac{\bar{t}}{\sqrt{A}} = \frac{(3s + \sqrt{s})}{6(\sqrt{s} + 1)V_1} + \frac{(2\sqrt{s} + 4)}{6(\sqrt{s} + 1)V_4} \qquad (4.32)$$

radial-rectangular
$$\frac{\bar{t}}{\sqrt{A}} = \frac{s(8 - 4\sqrt{2}) + \sqrt{s}(11\sqrt{2} - 13) + (15 - 7\sqrt{2})}{24(\sqrt{s} + 1)V_1}$$
$$+ \frac{s(12\sqrt{2} - 8) + \sqrt{s}(25 - 21\sqrt{2}) + (9\sqrt{2} - 17)}{24(\sqrt{s} + 1)V_2}$$
$$+ \frac{\sqrt{s}(64 - 20\sqrt{2}) + (20\sqrt{2} - 7)}{96(\sqrt{s} + 1)V_3}$$
$$+ \frac{\sqrt{s}(32 + 8\sqrt{2}) + (47 - 8\sqrt{2})}{96(\sqrt{s} + 1)V_4} \qquad (4.33)$$

mass transit
$$\frac{\bar{t}}{\sqrt{A}} = \frac{(2 - \sqrt{2})(s + \sqrt{s} + 1)}{6(\sqrt{s} + 1)V_1}$$
$$+ \frac{(3\sqrt{2} - 2)(s + \sqrt{s} + 1)}{6(\sqrt{s} + 1)V_2}$$
$$+ \frac{1}{2V_4} \qquad (4.34)$$

diagonal
$$\frac{\bar{t}}{\sqrt{A}} = \frac{(s + \sqrt{s} + 2)}{6(\sqrt{s} + 1)V_1} + \frac{\sqrt{2}(s - 2\sqrt{s} + 1)}{6(\sqrt{s} + 1)V_2}$$
$$+ \frac{\sqrt{s}}{(\sqrt{s} + 1)V_4} \quad . \qquad (4.35)$$

Certain forms of these expressions are of special interest:

(a) The radial-rectangular system with $V_2 > V_1$ and $V_3 = V_4$, or the diagonal system with $V_2 > V_1$.

This represents the case of a rectangular street network to which has been added a system of fast radial freeways. With the radial-rectangular system, entry into the central business district can only be effected at certain points so that some commuters may have to make long detours to reach these points. Entry into the central business district, however, is freely permitted with the diagonal system.

(b) The radial-rectangular system with $V_2 > V_1$ and $V_3 > V_4$.

In this case a ring road around the central business district has been added to the freeways.

(c) The mass transit system with $V_2 > V_1 > V_4$.

This represents the case of a commuter who drives to a radial route and then utilizes some form of rapid mass transportation to get to the centre 0 of the city. From there he either walks briskly to his destination or else gets into a bus, which has to navigate very slowly within the central business district.

Fig. 8 shows the variation of average travel time with city size. The values for the speed of travel along residential roads, freeways or fast mass transit routes, ring roads, and central business district streets have been set at 30 m.p.h., 40 m.p.h., 30 m.p.h., and 10 m.p.h. respectively. These may be regarded as being typical values. For the mass transit system, the travel speed

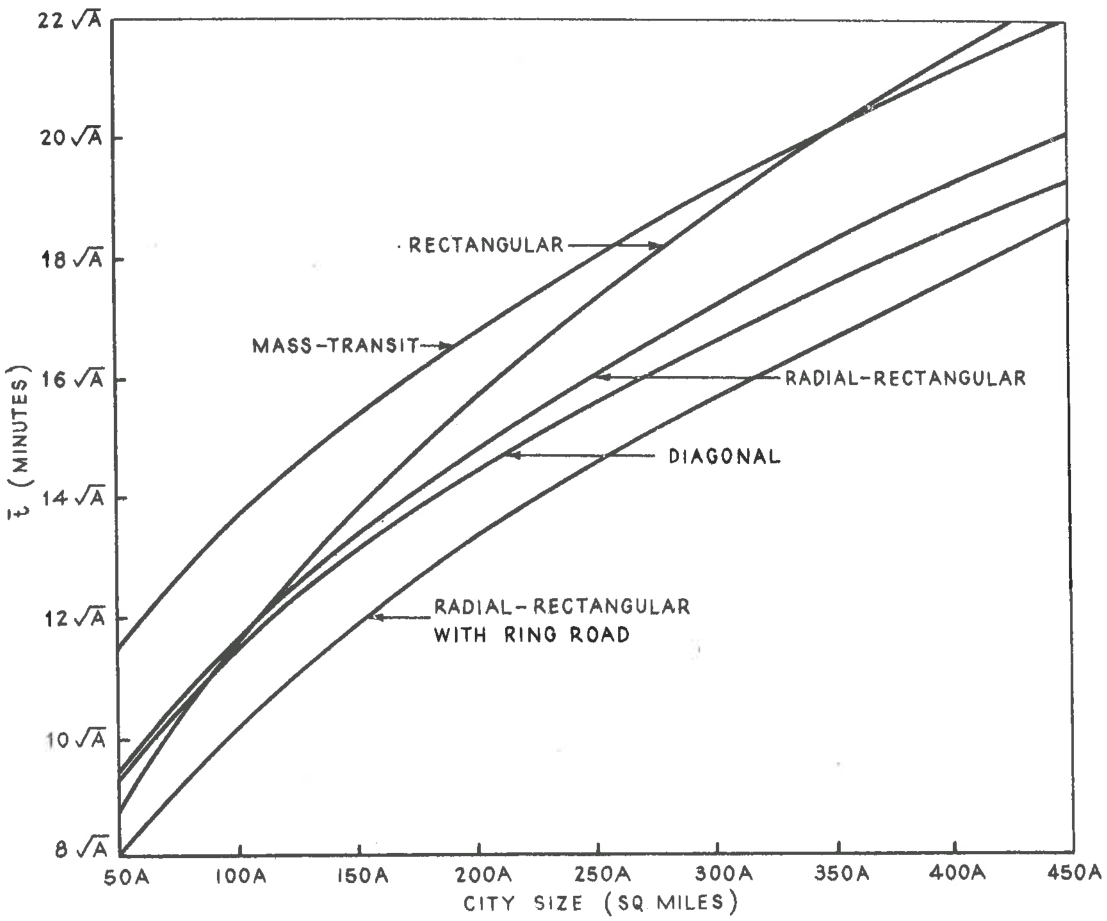

FIGURE 8. GRAPH OF $\bar{t}$ = AVERAGE TRAVEL TIME FOR A COMMUTER AGAINST THE TOTAL CITY SIZE FOR A SQUARE INHOMOGENEOUS CITY WITH FOUR DIFFERENT ROUTING SYSTEMS. A (SQUARE MILES) IS THE AREA OF THE SQUARE CENTRAL BUSINESS DISTRICT IN EACH CASE, AND THE UNIT OF TIME IS $\sqrt{A}$ MINUTES

within the central business district has been taken to be 5 m.p.h., as it may be expected that the commuter will frequently walk during the last section of his journey with this particular system.

The pattern of critical points is observed here again. On the basis of minimum travel time for the commuter journey, it is found that when the rectangular system is compared with the diagonal, radial-rectangular, or mass transit systems, the total city size has critical values at 90A, 105A, and 350A respectively. Another surprising result is the wide gap in travel time between the mass transit and radial-rectangular with ring road systems. Most commuters would therefore prefer the latter route, unless a rather severe penalty, such as a heavy parking cost, is associated with it.

4.6 Conclusions

The problem of evaluating the efficiency of travel along various theoretical routeing systems in a city has been studied by many authors. Previous work, however, has either been focused on the central business district or assumed that the homes and work places are uniformly distributed all over the city.

In Chapters III and IV of this thesis, the interaction between the central business district and the residential suburb is studied by means of a simple model of an inhomogeneous city. The city is assumed to consist of a central business district surrounded by a residential zone. Work places are assumed to be uniformly distributed all over the central business district and homes to be uniformly distributed all over the residential suburb. Circular cities are studied in Chapter III, whereas Chapter IV deals mainly with square and rectangular cities.

In the first part of Chapter IV, the inhomogeneous city is assumed to have a rectangular routeing system. It is shown that the shape of the city has a significant effect on the average distance travelled from home to work. This average commuter trip length is a minimum with a circular city, and is progressively lengthened as the shape of the city changes into a square and then into an elongated rectangle. It is also shown that, when the total city size is more than a hundred times that of the central business district, the average commuter trip length in an inhomogeneous city is about three-quarters of that in the corresponding homogeneous city, where the homes and work places are uniformly distributed all over the city.

The case of a square inhomogeneous city, with a square central business district, is studied in the second part of Chapter IV. Various modified rectangular routeing systems are examined, and it is shown that the total size of the city, relative to the size of its central business district, is an important factor in the choice of an optimal routeing system. The criteria used to evaluate the routeing systems are average distance travelled and average travel time.

The above conclusions are of interest to city planners and transportation engineers, especially with regard to the construction of new cities, when the shape and size of the urban communities can be influenced.

CHAPTER V

SATELLITE TOWNS

5.1 Introduction

One of the consequences of the present world-wide trend towards urban living is a steady increase in the size of cities and towns. These expanding urban communities create a multitude of problems, which include longer and longer 'home-to-work' commuter trips, traffic congestion, and difficulties in finding space for vehicle parking especially within the central area of the city. One particular problem has been studied in the previous chapters. Needless to say, there has been much discussion in recent years on methods to cater for or to halt the seemingly inevitable growth of the world's towns and cities.

Of the many schemes put forward to contain city growth, one of the most popular involves the creation of satellite towns. These towns are planned to be physically separated from the main city by green belts, and are to be strictly limited in size, so as to prevent their gradually merging with the main city to form a continuous urban area. Their function is to siphon off the excess population, which will otherwise be attracted to the main city.

Satellite towns may be divided into two types, New

Towns and Residential Towns. A New Town is designed to be fairly self-contained, with numerous job opportunities so that the majority of the inhabitants of the Town live and work within it. On the other hand, a Residential Town is closely linked with the main city, where most of the town's inhabitants work. It is planned that these residents shall commute daily to-and-from the main city, usually by means of efficient public transport such as an underground railway. The use of private motor vehicles is generally discouraged to prevent additional strain on the road system of the main city.

Although much has been written on satellite towns, the literature on the general quantitative aspects of the scheme is as yet very scanty. The standard technique has been to focus attention on a specific town, and the results obtained cannot usually be generalized.

A simple mathematical model of a satellite town is proposed in this chapter. With this model, it is possible to investigate some general problems concerning the structure of the town. In particular the size of the town and the amount of road space required, for a given town population are calculated.

5.2 Mathematical Model

The basic approach to the problem follows that of

Smeed [17,21], and the model itself is a refinement of the one used previously to study the case of an expanding city. The main assumptions made, concerning the structure of the satellite town, are (Fig. 9):

(a) the town is in the form of a circle of radius R, within which is contained an inner, concentric, circular Central Area of radius r $(r < R)$;

(b) the whole town is surrounded by an outer Ring Road, which effectively delimits the size of the town.

Workers are assumed to live in the annular Residential zone, bounded by the circles of radii r and R. During the morning rush period they commute to work either radially inwards to the Central Area or radially outwards to the Ring Road. These directions of travel are reversed in the evening. It is assumed that, in the morning, the fraction of commuters, living in a region and going inwards, is x, and the fraction going outwards is $y = 1 - x$. These fractions are constant and independent of position. These assumptions are made because of the absence of relevant data, and should only be regarded as crude approximations to reality.

The Ring Road may be regarded as being designed for those residents of the town, who work outside and utilize their cars to convey them to their destinations. As they

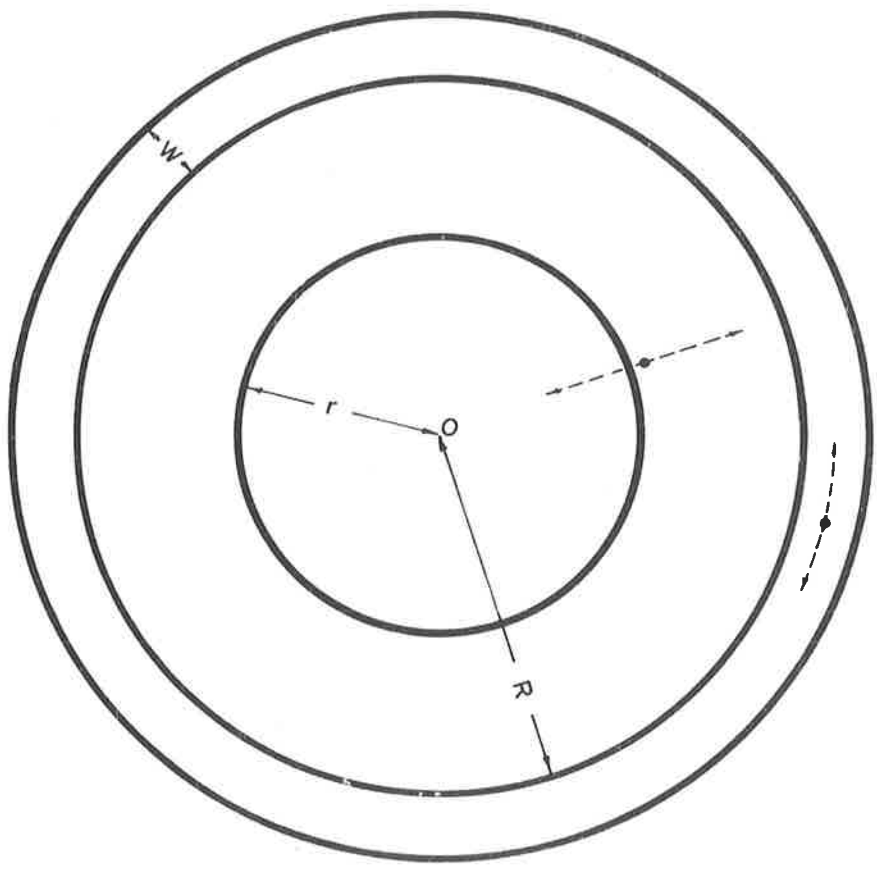

FIGURE 9. THE MODEL OF A SATELLITE TOWN WITH AN INNER CIRCULAR CENTRAL AREA, A CONCENTRIC RESIDENTIAL ZONE, AND AN OUTER RING ROAD. THE ARROWS INDICATE THE POSSIBLE DIRECTIONS OF COMMUTER TRAVEL IN THE PEAK PERIOD

are diverted to the outer limits of the town, their routes need not pass through the Central Area as is the case in many towns. Thus these commuters do not add to the traffic at the centre.

The actual form of the Central Area will depend on the major activity existing there, and this will be discussed now.

5.3 The Central Area

In the case of a satellite New Town, the Central Area is often an important working centre, and a high proportion of the inhabitants of the town find employment there. Thus the function of the Central Area is similar to that in other large towns, and the space there is essentially used up by places of work, roads, and parked vehicles. The detailed analysis, given by Smeed [21], is directly applicable to this case, and many of the characteristic features are known such as the effect of various forms of routeing systems. This type of Central Area will thus not be discussed in this chapter.

For a satellite Residential Town, the form of the Central Area will be quite different. Many planners have suggested that, in this case, it is logical to emphasize the importance of the Central Area, more as a collecting centre than as a working centre. While some of the commuters entering the Central Area will go to work there, large numbers will simply park their cars and then utilize some form of public transport to get out of the satellite town and into the main city. Hence the Central Area becomes the intermediate station in a 'park-and-ride' scheme. This scheme has the advantages that it limits the entry of private vehicles into the main city, and the concentration of commuters allows a great reduction in the costs of public transport. Furthermore the elimination of the necessity of providing a large number of work places in a small region allows the planner to concentrate on making the Central Area a pleasant and attractive place, with pedestrian walks, spacious shops, and green parks.

It is reasonable to expect that the siting of commuter roads and parking will be planned to some extent. This can be accounted for by specifying a planning function g, so that the value of g at a point is the fraction of ground area to be used for commuter roads and parking at that point, and $(1-g)$ is the fraction to be used for other purposes such as public transport facilities, work places, and pedestrian paths. The function g may also be regarded as giving the space available for commuter use per unit area at any point. It will be assumed that the function g is independent of angle and depends only on the distance

ρ or ξ from the town centre O (Fig.10). It is generally desirable to prevent commuters from penetrating too deeply into the Central Area so that the valuable space there can be protected from being used up by vehicles parked the whole day long. Therefore the space available for commuter use per unit area will decrease as the town centre O is approached, and the following form of the function $g(\rho)$ appears to be reasonable:

$$g(\rho) = 1 - e^{-K\rho} \quad (K > 0). \tag{5.1}$$

The parameter K is calculated by specifying a value for $g(\rho)$ at the boundary $(\rho = r)$ of the Central Area.

During the morning (or evening) peak travel period of duration T, it is assumed that n workers enter (or leave) the Central Area, and that each worker requires an average area of ground P for parking. These commuters travel radially and get as near to the town centre O as they can. They then park their cars and proceed on foot to their destinations. It is assumed that most of the commuters will get onto some form of public transport (located usually around the town centre O), so that the space, which is not used for commuter roads and parking, is more than sufficient to provide the pedestrian paths and few places of work that will be necessary.

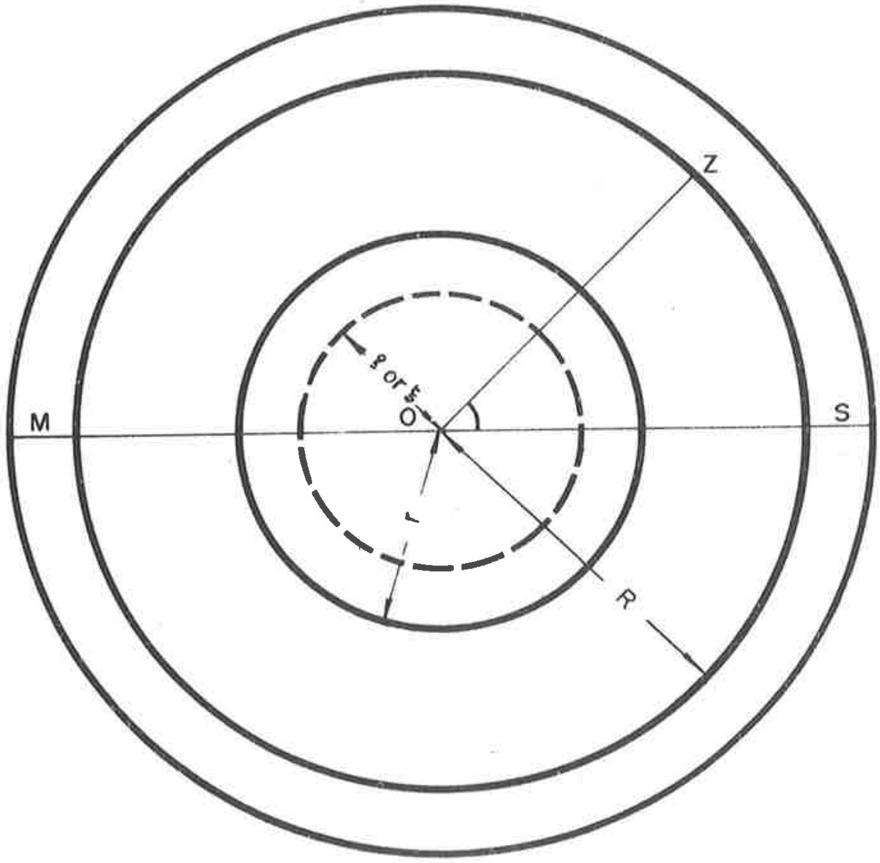

FIGURE 10. THE CO-ORDINATE SYSTEM USED IN THE MODEL OF A SATELLITE TOWN
$\measuredangle$ ZOS = θ

The remainder of this section follows closely on the lines of Smeed [21]. If the roads used by the commuters are of width W with capacity Q vehicles per unit time, and the average number of persons per vehicle is c, then, as shown in Chapter II, a width of road $\lambda = W/QcT$ can be regarded as being required for a journey. To take into account both the morning and evening journeys, each commuter will be assumed to require a width of road $j\lambda_1$ within the Central Area, where $j = 1$ if the roads are reversible and the whole road surface can be used for travel in one direction in the morning and in the reverse direction in the evening, and $j = 2$ if the roads are irreversible and one side only is used in the morning and the other side in the evening.

Assume that at a distance ρ or ξ from the town centre O (Fig.10), the space, set aside for commuter roads and parking, is divided into two parts: a fraction $f(\xi)$ for parking places and a fraction $[1 - f(\xi)]$ for roads. It follows that in an annulus centred at O and bounded by concentric circles of radii ξ and $\xi + d\xi$, the area used for parking is $2\pi\xi f(\xi)g(\xi)d\xi$, and the area used for roads is $2\pi\xi[1 - f(\xi)]g(\xi)d\xi$.

The number of parking spaces within the above annulus is $2\pi\xi f(\xi)g(\xi)d\xi/P$, and thus the number within a circle of radius ρ, centred at O, is $2\pi\int_0^\rho [\xi f(\xi)g(\xi)/P]d\xi$. Since a commuter proceeding to any of these parking places requires an area of road $j\lambda_1 d\rho$ within the annulus, bounded by circles of radii ρ and $\rho + d\rho$, the total area of road required within this annulus is $j\lambda_1 d\rho 2\pi\int_0^\rho [\xi f(\xi)g(\xi)/P]d\xi$. This has to be equal to the space available for roads and so

$$j\lambda_1 d\rho 2\pi\int_0^\rho [\xi f(\xi)g(\xi)/P]d\xi = 2\pi\rho[1-f(\rho)]g(\rho)d\rho. \quad (5.2)$$

The differential equation, corresponding to equation (5.2), is

$$\frac{dF(\rho)}{d\rho} + \frac{j\lambda_1 F(\rho)}{P} = g(\rho) + \rho\,\frac{dg(\rho)}{d\rho}, \quad (5.3)$$

where a new function $F(\rho) = \rho f(\rho)g(\rho)$ has been introduced, The initial condition is $F(0) = 0$. Assuming that the function $g(\rho)$ is as given in equation (5.1), then

$$F(\rho) = \rho f(\rho)[1 - \exp(-K\rho)]. \quad (5.4)$$

If a new constant $A = j\lambda_1/P$ is defined, the solution to equation (5.3) can be written as

$$F(\rho) = [1 - \exp(-A\rho)]/A + A[\exp(-A\rho) - \exp(-K\rho)]/(A-K)^2 + K\rho\exp(-K\rho)/(A-K) \quad (5.5)$$

provided $K \neq A$. For the case $K = A$, equation (5.3) assumes the particular form

$$\frac{dF(\rho)}{d\rho} + AF(\rho) = 1 - e^{-A\rho} + \rho A e^{-A\rho}, \quad (5.6)$$

and the corresponding solution is

$$F(\rho) = [1 - \exp(-A\rho)]/A + A\rho^2\exp(-A\rho)/2 - \rho\exp(-A\rho). \quad (5.7)$$

In order that these solutions should be physically meaningful, it is necessary to ensure that $0 \leqslant f(\rho) \leqslant 1$ or equivalently $0 \leqslant F(\rho) \leqslant \rho g(\rho)$ for values of ρ greater than zero. This is easily shown. Consider, for example, equation (5.5) and assume that $A > K > 0$. Then for $F(\rho) \geqslant 0$ we must have

$$AK\rho \geqslant A^2\{1 - \exp[(K-A)\rho]\}/(A-K) - (A-K)\{\exp(K\rho) - \exp[(K-A)\rho]\} \; . \qquad (5.8)$$

This is true because:

(a) The functions on the Left-Hand-Side and Right-Hand-Side are both 0 for $\rho = 0$, and

(b) The function on the LHS increases faster than the function on the RHS for $\rho > 0$.

The proofs for the other cases are very similar.

Various characteristics of the Central Area are now easily found. The substitution $\rho = r$ in equation (5.2), where r is the radius of the Central Area, yields

$$j\lambda_1 \int_0^r [2\pi\xi f(\xi)g(\xi)/P]d\xi = 2\pi[rg(r) - F(r)]. \qquad (5.9)$$

Since the number n of workers entering the Central Area is $\int_0^r [2\pi\xi f(\xi)g(\xi)/P]d\xi$, this gives the result

$$n = 2\pi[rg(r) - F(r)]/j\lambda_1 \quad , \qquad (5.10)$$

which provides the relation between r and n.

Using equation (5.2), the distance travelled by commuters in the annulus between the concentric circles of radii ρ and $\rho + d\rho$ can be expressed as

$$d\rho\int_0^{\rho}[2\pi\xi f(\xi)g(\xi)/P]d\xi = 2\pi\rho[1 - f(\rho)]g(\rho)d\rho/j\lambda_1 \ . \qquad (5.11)$$

The vehicle distance travelled per unit area is obtained by dividing expression (5.11) by the area of the annulus and is therefore

$$[1 - f(\rho)]g(\rho)/j\lambda_1 = [\rho - \rho e^{-K\rho} - F(\rho)]/j\lambda_1\rho \ . \qquad (5.12)$$

An integration of expression (5.12) between the limits $\rho = 0$ and $\rho = r$ gives the total vehicle distance travelled within the Central Area, and this result, divided by n, yields the average vehicle distance travelled. The average distance $\bar{d}$ can be written as

$$\bar{d} = \pi[r^2 + 2r\exp(-Kr)/K - 2/K^2 + 2\exp(-Kr)/K^2]/nj\lambda_1 - P/j\lambda_1 \ . \qquad (5.13)$$

5.4 The Residential Zone

Reversible Roads

There are two opposite flows of commuter traffic in every region of the Residential zone. In the case of reversible roads, the total width of road required at any place is obtained simply by multiplying the total commuter flow there with the parameter λ_2, where λ_2 is the width of road required for a journey in the Residential zone.

It is assumed that at a distance ρ from the town centre O $(r \leqslant \rho \leqslant R)$, a fraction $h(\rho)$ of the ground area is used for residential purposes, and a fraction

$[1 - h(\rho)]$ for commuter roads. With the same reasoning as in the previous section, the number of commuters living in the annulus, bounded by circles of radii r and ρ, centred at O, is $\int_r^\rho [2\pi\xi h(\xi)/L]d\xi$, where L is the area of ground required by each worker for purposes such as living, leisure and casual travel.

Since the fraction of workers commuting radially outwards to the Ring Road in the morning is y, the number of commuters crossing the annulus, bounded by circles of radii ρ and $\rho + d\rho$, and proceeding outwards is $y\int_r^\rho [2\pi\xi h(\xi)/L]d\xi$. Similarly the number of commuters passing radially inwards through the same annulus is $x\int_\rho^R [2\pi\xi h(\xi)/L]d\xi$, where x is the fraction of workers going inwards in the morning. The equation for the road space requirements in this annulus is therefore

$$\lambda_2 d\rho \{y\int_r^\rho [2\pi\xi h(\xi)/L]d\xi + x\int_\rho^R [2\pi\xi h(\xi)/L]d\xi\} = 2\pi\rho[1 - h(\rho)]d\rho \quad . \qquad (5.14)$$

The total number of workers in the Residential zone is

$$N = \int_r^R [2\pi\xi h(\xi)/L]d\xi \ . \qquad (5.15)$$

Equation (5.14) is then simplified to

$$Nx\lambda_2 + (y-x)\lambda_2\int_r^\rho [2\pi\xi h(\xi)/L]d\xi = 2\pi\rho[1 - h(\rho)] \ . \qquad (5.16)$$

Differentiation with respect to ρ gives

$$\rho\frac{dh(\rho)}{d\rho} + (1 + B\rho)h(\rho) = 1 \ , \qquad (5.17)$$

where $B = (y - x)\lambda_2/L$ and the initial condition is $h(r) = 1 - (Nx\lambda_2/2\pi r)$. The solution to equation (5.17) is

$$h(\rho) = (r - 1/B - Nx\lambda_2/2\pi)\exp[-B(\rho-r)]/\rho + 1/B\rho \quad (y \neq x) \tag{5.18}$$

$$= 1 - N\lambda_2/4\pi\rho \; . \quad (y = x)$$

The size R of the satellite town is found by substituting $\rho = R$ in equation (5.16), which then yields

$$Ny\lambda_2/2\pi = R - (r - 1/B - Nx\lambda_2/2\pi)\exp[-B(R-r)] - 1/B \quad (y \neq x) \tag{5.19}$$

$$N/2\pi = (R^2 - r^2)/[2L + \lambda_2(R - r)] \; . \quad (y = x)$$

This gives the required relation between $R, r,$ and N.

Since the function $h(\rho)$ represents a fraction, it is clear that the following condition must hold

$$0 \leqslant h(\rho) \leqslant 1 \quad \text{for} \quad r \leqslant \rho \leqslant R \; . \tag{5.20}$$

This is easily proved. Consider, for example, the case $B > 0$ or equivalently $y > x$. Then for $h(\rho) \geqslant 0$, we must have

$$1/B \geqslant [1/B - (r - Nx\lambda_2/2\pi)]\exp[-B(\rho-r)] \tag{5.21}$$

since $\rho \geqslant r > 0$. The inequality (5.21) is clearly true for $r \geqslant Nx\lambda_2/2\pi$. Similarly for $h(\rho) \leqslant 1$, the following inequality must hold

$$\rho \geqslant (r - 1/B - Nx\lambda_2/2\pi)\exp[-B(\rho-r)] + 1/B \; , \tag{5.22}$$

and this is again true for $r \geqslant Nx\lambda_2/2\pi$. The condition

$2\pi r \geqslant Nx\lambda_2$ simply means that the circumference of the Central Area cannot be smaller than the total width of road required for the commuter journeys into the Central Area. The rest of the proof follows on exactly the same lines. In the future, obvious conditions such as these will not be mentioned.

As before, the distance travelled per unit area is obtained by using equation (5.14), and is simply $[1 - h(\rho)]/\lambda_2$ at a distance ρ from the town centre O. The average distance $\bar{d}$ travelled within the Residential zone is given by

$$\bar{d} = \pi(R^2 - r^2)/N\lambda_2 - L/\lambda_2 \quad . \qquad (5.23)$$

Equation (5.23) can be rearranged to give

$$N\bar{d}\lambda_2 + NL = \pi(R^2 - r^2) \quad , \qquad (5.24)$$

which is simply a statement of the fact that the total space within the Residential zone consists of the space required for commuter roads plus the space required for living purposes.

Irreversible Roads

With irreversible roads, the road space required at a point in the Residential zone is governed by the larger of the two opposite flows of commuter traffic at that point. If the number of commuters crossing a line in the morning in one direction is M, and the number crossing the same line in the opposite direction is $N (M > N)$, then the width of road required at this line is $2M\lambda_2$, where λ_2 is the width of road required for a journey in the Residential zone. The N stream of commuter traffic in the morning simply utilizes a portion of that half of the road, which is set aside for the use of the M stream in the evening. The flows are, of course, reversed in direction during the evening peak period.

It is assumed that at a distance ρ from the town centre O, where $r \leqslant \rho \leqslant a$, a function $h_1(\rho)$ is defined so that a fraction $h_1(\rho)$ of the ground area is used for living purposes and a fraction $[1 - h_1(\rho)]$ for commuter roads. A second function $h_2(\rho)$ plays a similar role in the range $a \leqslant \rho \leqslant R$. The boundary $\rho = a$ thus divides the Residential zone into two regions. The predominant flow is inwards for $r \leqslant \rho < a$, and outwards for $a < \rho \leqslant R$. The value of a is fixed by the following equation,

$$\int_r^a [2\pi\xi h_1(\xi)/L]d\xi = Nx \tag{5.25}$$

or equivalently,

$$\int_a^R [2\pi\xi h_2(\xi)/L]d\xi = Ny, \tag{5.26}$$

where N is the total number of workers in the satellite town and the other symbols have their usual significance. It is seen that the number of commuters crossing a circle of radius a, centred at O, in the inwards direction is equal to the number crossing outwards, and is just Nxy in each case.

Consider an annulus, bounded by circles of radii ρ and $\rho + d\rho$ centred at O. If $r \leqslant \rho < a$, the number of commuters crossing this annulus in the inward direction is greater than the number crossing outwards. The road space equation is thus

$$2\lambda_2 x d\rho \left\{ \int_\rho^a [2\pi\xi h_1(\xi)/L] d\xi + \int_a^R [2\pi\xi h_2(\xi)/L] d\xi \right\} = 2\pi\rho d\rho [1 - h_1(\rho)]. \tag{5.27}$$

Simplifying the equation and differentiating yields

$$\rho \frac{dh_1(\rho)}{d\rho} + (1 - 2\lambda_2 x\rho/L) h_1(\rho) = 1, \tag{5.28}$$

with the boundary condition $h_1(r) = 1 - Nx\lambda_2/\pi r$.

Similar considerations for the case $a < \rho \leqslant R$ results in the equation,

$$2\lambda_2 y d\rho \left\{ \int_r^a [2\pi\xi h_1(\xi)/L] d\xi + \int_a^\rho [2\pi\xi h_2(\xi)/L] d\xi \right\} = 2\pi\rho d\rho [1 - h_2(\rho)], \tag{5.29}$$

or correspondingly

$$\rho \frac{dh_2(\rho)}{d\rho} + (1 + 2\lambda_2 y\rho/L) h_2(\rho) = 1 \tag{5.30}$$

with boundary condition $h_2(R) = 1 - Ny\lambda_2/\pi R$.

The solutions to equation (5.28) and (5.30) are

$$h_1(\rho) = C \exp(2x\rho\lambda_2/L)/\rho - L/2x\rho\lambda_2 \tag{5.31}$$

$$h_2(\rho) = D \exp(-2y\rho\lambda_2/L)/\rho + L/2y\rho\lambda_2 \tag{5.32}$$

where

$$C = (r + L/2x\lambda_2 - Nx\lambda_2/\pi)\exp(-2xr\lambda_2/L) \qquad (5.33)$$

$$D = (R - L/2y\lambda_2 - Ny\lambda_2/\pi)\exp(2yR\lambda_2/L). \qquad (5.34)$$

The size R of the satellite town is found by substituting $\rho = a$ in equations (5.27) and (5.29), when the following expressions for C and D are obtained:

$$C = (a + L/2x\lambda_2 - Nxy\lambda_2/\pi)\ \exp\ (-2xa\lambda_2/L) \qquad (5.35)$$

$$D = (a - L/2y\lambda_2 - Nxy\lambda_2/\pi)\ \exp\ (2ya\lambda_2/L). \qquad (5.36)$$

Equations (5.33) - (5.36) provide the required relations between R,r, and N.

In the range $r \leqslant \rho \leqslant a$, the total distance travelled in an annulus, bounded by circles of radii ρ and $\rho + d\rho$, is

$$xd\rho\{\int_\rho^a [2\pi\xi h_1(\xi)/L]d\xi + \int_a^R [2\pi\xi h_2(\xi)/L]d\xi\} + yd\rho\int_r^\rho [2\pi\xi h_1(\xi)/L]d\xi. \qquad (5.37)$$

The distance travelled per unit area is therefore

$$(x-y)[1 - h_1(\rho)]/2x\lambda_2 + Ny/2\pi\rho. \qquad (5.38)$$

Similarly for $a \leqslant \rho \leqslant R$, the distance travelled per unit area, at a distance ρ from the town centre O, is

$$(y-x)[1 - h_2(\rho)]/2y\lambda_2 + Nx/2\pi\rho. \qquad (5.39)$$

The average distance $\bar{d}$ travelled in the Residential zone of the satellite town is now easily found and can be expressed as

$$\bar{d} = \pi(R^2-r^2)/2N\lambda_2 + \pi(y^2r^2 - x^2R^2)/2Nxy\lambda_2 + \pi a^2(x-y)/2Nxy\lambda_2 + (xR-yr) + a(y-x). \qquad (5.40)$$

5.5 The Ring Road

The size and structure of the Ring Road will depend on the position of the satellite town, relative to the main city and to other near-by large towns. These neighbouring urban communities will determine the amount of through traffic that has to be routed around the satellite town, and also the number and position of the external motorways that are to be connected to the Ring Road. A detailed analysis will therefore not be attempted in this Chapter, and only a very simple example will be discussed.

The Ring Road is assumed to have only two exits (or entrances) at M and S (Fig.10), and is utilized only by some of the residents of the satellite town. In the morning these residents commute radially to the edge of the satellite town. A fraction m of these residents then proceed circumferentially to M and a fraction $s = 1 - m$ to S. Consider a radius OZ, at an angle θ to MOS (Fig.10). Assuming that the necessary radial travel across the Ring Road is carried out at a different level from that of the Ring Road itself and that the road is reversible, the width $w(\theta)$ of the Ring Road at Z is

$$w(\theta) = Ny\theta m\lambda_3/2\pi + Ny(\pi-\theta)s\lambda_3/2\pi, \qquad (5.41)$$

where λ_3 is the width of road required for a journey along the Ring Road. N is the number of workers in the satellite town, and y is the fraction of workers proceeding outwards to the Ring Road in the morning.

The width $w(\theta)$ of the Ring Road is thus

$$w(\theta) = Ny\theta(m-s)\lambda_3/2\pi + Nys\lambda_3/2 \qquad (0 \leqslant \theta \leqslant \pi) \qquad (5.42)$$

$$= Ny\lambda_3/4 \qquad \text{if} \quad m = s = \tfrac{1}{2} \,.$$

In the case of an irreversible Ring Road, it is necessary to consider two separate cases for θ. In the first case $0 \leqslant \theta < \pi s$, the number of commuters proceeding circumferentially to S is greater than the number proceeding to M at all points in this section of the Ring Road. Hence the width $w(\theta)$ is found to be

$$w(\theta) = Nys(\pi-\theta)\lambda_3/\pi \,, \qquad (0 \leqslant \theta < \pi s) \qquad (5.43)$$

and this is just twice the width of road required for the commuter journeys to S. The corresponding formula for the second case $\pi s \leqslant \theta \leqslant \pi$ is

$$w(\theta) = Nym\theta\lambda_3/\pi \qquad (\pi s \leqslant \theta \leqslant \pi) \,. \qquad (5.44)$$

Formulae similar to (5.42) - (5.44) hold in the range $(\pi \leqslant \theta \leqslant 2\pi)$.

5.6 Numerical Calculations

In order to obtain some numerical results, it is necessary to have information on the values of the parameters that occur in the various formulae. Some very useful data on the road space required by a person to go a journey of one mile with different modes of transport are given in Table I. The assumptions behind this table have been listed in Chapter II of this thesis.

Since the population of the satellite town is not very large, the peak travel period will be taken to last for only 30 minutes in our case. It will be assumed that travel is on roads of width 24 feet at a speed of 10 miles/hour in the Central Area, on roads of width 44 feet at a speed of 30 miles/hour within the Residential zone, and on 12 feet lanes at a speed of 40 miles/hour along the Ring Road. Using Table I, the widths of road required for a journey in these three regions are found to be respectively,

$$\begin{aligned} \lambda_1 &= 0.032 \text{ feet} \\ \lambda_2 &= 0.013 \text{ feet} \\ \lambda_3 &= 0.008 \text{ feet} . \end{aligned} \qquad (5.45)$$

The space P required by a worker for parking will be taken to be 133 sq. ft., which is the figure given by Smeed [17] for off-street parking in London. Finally the space L required by a worker for living purposes will vary from place to place, depending on whether low- or high-density residences is the local custom. A figure of 4350 sq. ft. or 10 workers to an acre will be taken for illustrative purposes. It is possible now to obtain considerable information on the characteristics of the satellite town, and some of the results are given below.

<u>The size of the Central Area and of the Satellite Town</u>

The radius r of the Central Area of a satellite Residential Town has been calculated for various values of n the number of commuters entering, and the results are shown in Figure 11. It is assumed that $g(r) = 0.5$, i.e. that the fraction of ground area, reserved for commuter roads and parking, varies from zero at the town centre O to a value of 0.5 at the edge of the Central Area. It is seen that the flow regulation on the roads has a significant effect on the size of the Central Area for a given value of n. The ratio of the value of r with irreversible roads to that with reversible roads increases from 1.04 ($n = 10,000$) to 1.13 ($n = 100,000$), thus showing that the flow regulation plays an increasingly important role as the number of commuters entering the Central Area becomes greater.

The overall radius R of the satellite Residential Town, excluding the Ring Road, has been plotted against N (the total number of workers living in the town) in Figure 12. It is assumed that $x = y = 0.5$, and that the roads are of the same type throughout the satellite town i.e. if they are reversible in the Residential zone, they are also reversible

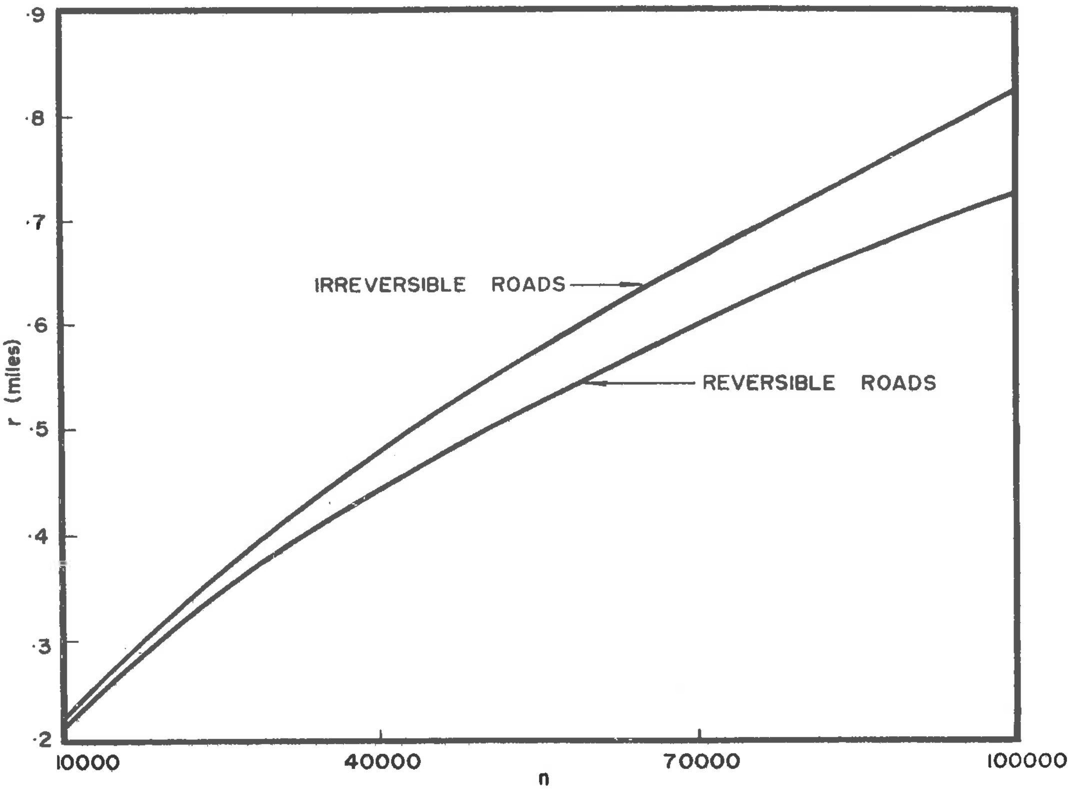

FIGURE 11. GRAPH OF r (MILES) = THE RADIUS OF THE CENTRAL AREA OF A SATELLITE RESIDENTIAL TOWN AGAINST n = THE NUMBER OF COMMUTERS ENTERING THE CENTRAL AREA IN THE PEAK TRAVEL PERIOD

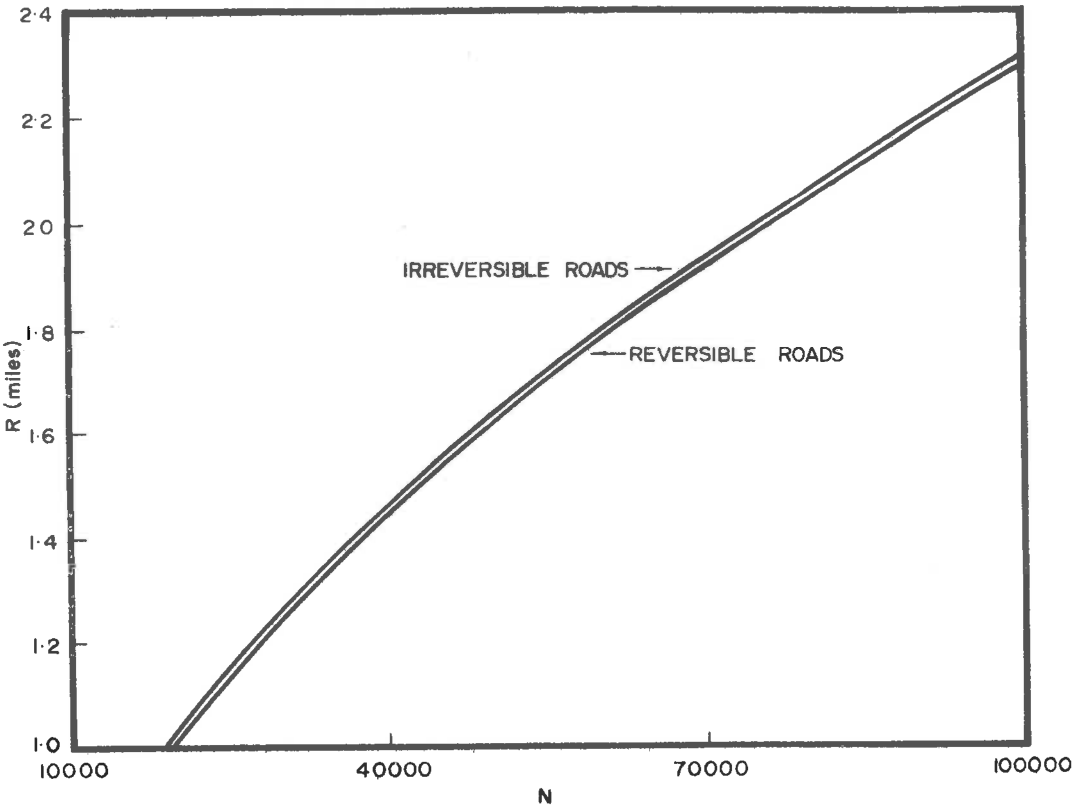

FIGURE 12. GRAPH OF R (MILES) = THE OVERALL RADIUS, EXCLUDING THE RING ROAD, OF THE SATELLITE RESIDENTIAL TOWN AGAINST N = THE TOTAL NUMBER OF WORKERS LIVING IN THE TOWN. THE FRACTION OF WORKERS GOING INWARDS IN THE MORNING IS ASSUMED TO BE EQUAL TO THE FRACTION GOING OUTWARDS

in the Central Area. Figure 12 shows that the flow regulation on roads has only a slight effect on R. This is due to the fact that each worker requires a relatively large area of ground for living purposes. The ratio of the value of R with irreversible roads to the value with reversible roads depends on N, and increases from 1.00 (N = 10,000) to 1.01 (N = 100,000).

The width of the Ring Road will, in general, vary from point to point. If, however, the Ring Road is assumed to be reversible and the number of commuters turning left on reaching the Ring Road is equal to the number turning right, then the width w is $Ny\lambda_3/4$ and is constant. For $y = 0.5$, w varies from 10 feet (N = 10,000) to 104 feet (N = 100,000).

The space required for commuter roads

A very important factor, from the point of view of costs or planning, is the ground space that will be required for the commuter roads. Within the Central Area the space, used for purposes other than commuter roads and parking, is

$$\int_0^r [2\pi\xi e^{-K\xi}]d\xi = 2\pi(1-e^{-Kr}-Kre^{-Kr})/K^2. \qquad (5.46)$$

Hence the space G_1 required by the commuter roads is

$$\pi r^2 - 2\pi(1 - e^{-Kr}-Kre^{-Kr})/K^2-nP. \qquad (5.47)$$

The fraction g_1 of the whole Central Area, that is required for commuter roads, is $G_1/\pi r^2$. The values of G_1 and g_1

for various values of n have been calculated and are shown in Table IV.

It is seen that the commuter roads occupy an appreciable amount of space in the Central Area. For $n = 100{,}000$, 13 per cent of the Central Area is used to provide road space with irreversible roads and 7 per cent with reversible roads. The ratio of the road space required with irreversible roads to that required with reversible roads rises from 2.10 ($n = 10{,}000$) to 2.36 ($n = 100{,}000$) as shown in Table V. This shows that the wastage of road space due to the use of irreversible roads becomes greater as the number of commuters entering the Central Area increases.

Within the Residential zone, the space G_2 that is required for commuter roads, is

$$\pi(R^2 - r^2) - NL, \tag{5.48}$$

if the Ring Road is excluded. The fraction g_2 of the Residential zone, that is used for commuter roads, is thus $G_2/\pi(R^2 - r^2)$. If the Ring Road is included, then the radius R is increased to $R + w$, where it is assumed that the necessary conditions hold so that the width w of the Ring Road is constant.

The values of G_2 and g_2 for various values of N are shown in Tables VI and VII, the Ring Road being excluded and included respectively. It is interesting to note that

TABLE IV

GROUND SPACE REQUIRED FOR COMMUTER ROADS IN CENTRAL AREA

Number of commuters entering Central Area	Space required for commuter roads (sq.miles)		Fraction of whole Central Area required for commuter roads	
	Reversible Roads	Irreversible Roads	Reversible Roads	Irreversible Roads
10,000	0.003	0.007	0.024	0.047
20,000	0.010	0.021	0.034	0.065
30,000	0.018	0.040	0.041	0.079
40,000	0.029	0.064	0.047	0.090
50,000	0.041	0.091	0.052	0.100
60,000	0.054	0.122	0.057	0.108
70,000	0.069	0.158	0.061	0.116
80,000	0.085	0.196	0.065	0.123
90,000	0.102	0.239	0.069	0.129
100,000	0.120	0.284	0.073	0.135

TABLE V

COMPARISON OF THE ROAD SPACES
REQUIRED FOR IRREVERSIBLE AND REVERSIBLE
COMMUTER ROADS IN THE CENTRAL AREA

Number of commuters entering Central Area	Ratio = Road Space Required with Irreversible Roads / Road Space Required with Reversible Roads
10,000	2.10
20,000	2.15
40,000	2.22
60,000	2.27
80,000	2.32
100,000	2.36

TABLE VI

GROUND SPACE REQUIRED FOR COMMUTER ROADS

IN THE RESIDENTIAL ZONE EXCLUDING THE RING ROAD

Total Number of workers living in the satellite town	Space required for commuter roads (sq.miles)		Fraction of whole Residential zone required for commuter roads	
	Reversible Roads	Irreversible Roads	Reversible Roads	Irreversible Roads
10,000	0.007	0.011	0.004	0.007
20,000	0.020	0.030	0.006	0.010
30,000	0.036	0.056	0.008	0.012
40,000	0.056	0.086	0.009	0.014
50,000	0.078	0.120	0.010	0.015
60,000	0.102	0.157	0.011	0.016
70,000	0.129	0.198	0.012	0.018
80,000	0.158	0.241	0.012	0.019
90,000	0.188	0.287	0.013	0.020
100,000	0.220	0.336	0.014	0.021

TABLE VII

GROUND SPACE REQUIRED FOR COMMUTER ROADS

IN THE RESIDENTIAL ZONE INCLUDING THE RING ROAD

Total Number of workers living in the satellite town	Space required for commuter roads (sq.miles)		Fraction of whole Residential zone required for commuter roads	
	Reversible Roads	Irreversible Roads	Reversible Roads	Irreversible Roads
10,000	0.016	0.020	0.010	0.013
20,000	0.045	0.056	0.014	0.018
30,000	0.083	0.103	0.017	0.021
40,000	0.128	0.158	0.020	0.025
50,000	0.179	0.221	0.022	0.028
60,000	0.235	0.240	0.024	0.030
70,000	0.296	0.366	0.026	0.032
80,000	0.362	0.447	0.028	0.034
90,000	0.432	0.533	0.030	0.037
100,000	0.506	0.624	0.031	0.038

detailed calculation shows that the ratio of the space required for commuter roads with irreversible roads to the space required with reversible roads decreases slightly as N increases, the Ring Road being excluded. The values for this ratio are given in Table VIII and range from 1.55 (N = 10,000) to 1.51 (N = 200,000). This indicates that as the number of workers living in the satellite town increases, there is a slight decrease in the wastage of road space with irreversible roads. This downward trend in the Residential zone is in contrast with the upward trend in the Central Area.

5.7 Conclusions

In this Chapter, a simple mathematical model of a satellite town has been presented. It is shown that the model proposed is flexible enough to cover both the cases of a satellite Residential Town as well as of a satellite New Town.

A detailed analysis of the Central Area of a satellite Residential Town is given. The size of the Central Area, the distance travelled per unit area, and the average distance travelled are evaluated. In the next section the Residential zone is similarly studied and separate calculations are made under the assumptions of reversible and irreversible roads being used. Some brief comments are made on the width of

TABLE VIII

COMPARISON OF THE ROAD SPACES REQUIRED
FOR IRREVERSIBLE AND REVERSIBLE COMMUTER
ROADS IN THE RESIDENTIAL ZONE
EXCLUDING THE RING ROAD

Total number of workers living in the satellite town	Ratio = $\frac{\text{Road Space Required with Irreversible Roads}}{\text{Road Space Required with Reversible Roads}}$
10,000	1.55
50,000	1.54
100,000	1.53
150,000	1.52
200,000	1.51

the Ring Road that surrounds the satellite town.

Three interesting conclusions emerge from the numerical results:

1. The flow regulation on the roads, i.e. whether the roads are reversible or irreversible, is a significant factor in determining the size of the Central Area, and this factor becomes more important as the number of commuters entering the Central Area increases.
2. Within the Central Area the ground space that is required for irreversible roads as compared with the space required for reversible roads increases as the number of commuters entering the Central Area becomes greater. This shows that the wastage of space due to the use of irreversible roads becomes progressively more severe.
3. The reverse effect, however, occurs in the Residential zone. Here the ratio of the ground spaces, required for irreversible and reversible roads, decreases slightly as the total number of workers living in the satellite town increases.

These conclusions are significant with regard to the planning of satellite towns, as they are an aid to the planner in determining the optimum size of these towns.

CHAPTER VI

SOME WORK TRIP DATA FOR ADELAIDE

6.1 Introduction

In order to construct a mathematical model for commuter traffic in cities, it is necessary to make four main assumptions concerning:

(a) The distribution of the homes of workers;

(b) The distribution of work places;

(c) The correlation between the positions of homes and work places;

(d) The routes by which workers go from their homes to their work places.

The usefulness of the mathematical model will, of course, be enhanced if these assumptions can be made so as to agree accurately with the actual data that is available.

The distribution of homes is closely linked with the distribution of population within cities. This has been studied by Clark [5,6], who observes that:

(a) In every large city, excluding the central business district which has few resident inhabitants, there are districts of dense population in the interior with density falling off as one proceeds to the outer suburbs.

(b) In most, but not all cities, as time goes on, density tends to fall in the most populus inner suburbs and to rise in the outer suburbs, and the whole city tends to spread itself out and expand.

Clark found that the density of population D (thousands per square mile) can be expressed in terms of the distance ρ (miles) from the centre of the city by the simple expression

$$D(\rho) = Ae^{-b\rho} \quad . \tag{6.1}$$

A and b are constants, which characterise each particular city.

It should be noted that Clark's formula does not give a good fit to empirical data when ρ is small i.e. $\rho < 2$ miles. However this is not a serious defect as the complications of zoning laws, high density flats, parks, and fountains make each city centre unique. It is thus unlikely that all the city centres will conform to some simple law. Clark's formula should be regarded as being applicable to the residential suburban regions, and he shows that a surprisingly good fit is obtained for a large number of cities at various periods of time. A slightly different expression

$$D(\rho) = Ae^{-b\rho^2} \tag{6.2}$$

has been proposed by Sherratt [16], and he shows that a somewhat better fit is obtained with this expression for some cases.

In the following two sections of this Chapter, the distribution of homes and of work places within metropolitan Adelaide will be analysed in some detail. The data for this analysis was collected by the Metropolitan Adelaide Transportation Study in 1965. The data included information on the origins and destinations of the journeys to work during a typical week day, and this is precisely what is required.

The survey area was divided into 564 regions, and it was estimated that about 1 in every 20 households within each region was contacted. The data was later processed and expanded to give the final figures for the total number of work trips per average week day in Adelaide. Regression-type equations were also used to forecast the traffic pattern in 1986, and the projected figures are also included in the analysis below.

Since the correlation between the positions of homes and work places is an integral part of the transportation planning process, this topic has been the subject of numerous thorough and extensive investigations. It will therefore not be discussed in detail in this thesis. This correlation is generally expressed in the following form

$$T_{ij} = KH_i W_j f(d_{ij}) \quad , \qquad (6.3)$$

where T_{ij} is the number of work trips from zone i to zone j, H_i is the number of homes in zone i, W_j is the number of work places in zone j, and K is a normalising constant. The function $f(d_{ij})$ is a deterrence function involving the quantity d_{ij} which is related in some way to the cost of travel between the zones. In the Metropolitan Adelaide Transportation Study a deterrence function of the form $\exp[f_1(d_{ij})]$, where $f_1(d_{ij})$ is a cubic polynomial in d_{ij}, was used.

Unfortunately the results, calculated from formulae such as (6.3), do not agree well with the empirical data and it is customary to include an arbitrary factor K_{ij} in the function $f(d_{ij})$. Since K_{ij} takes on a different value for each pair of zones, this makes such formulae virtually untestable. For the present at least, it appears that the correlation is too complicated to be represented by a simple expression.

Finally we note that almost no data is available concerning the routes by which commuters go from their homes to their work places. The cost of collecting and processing such data would probably be prohibitive. In any case one of the chief uses of a mathematical model for commuter traffic is to study the effect of various routeing systems, and for this purpose it is necessary to assume different forms for the routeing system.

6.2 The distribution of homes in Adelaide

In order to study the distribution of homes it is necessary to first define a reference point, which for convenience is usually the centre of the city. Adelaide has a well-defined Central City Area in the shape of a square. Since the General Post Office lies almost precisely at the centre of this square, it was taken to be the required centre of the city.

The origin-destination data from the Metropolitan Adelaide Transportation Study listed the total number of origins and of destinations for the journeys to work in each of the 564 traffic regions. These origins and destinations correspond to homes and work places respectively. For the sake of preciseness, we distinguish here between homes and households. It is obviously possible that two homes (or two origins) may belong to the same household. As the traffic regions are extremely irregular in shape, it was not possible to study the distribution of homes by plotting a simple scatter diagram showing the average density of each region as a function of its mean distance from the centre of the city.

Instead seven concentric circles were drawn about the centre of the city. The radii of these circles ranged from one to seven miles, increasing by a mile at each step. The total number of homes within each annular zone was then calculated, and this divided by the corresponding area gave the average density of homes for each annular zone. The first zone was, of course, simply a circle of radius one mile. Where the circles cut the traffic regions, apportionments of homes had to be made. This was done in proportion to the area of the traffic region lying in each annular zone. The two outmost annular zones extended into the sea, and thus the areas covered by the sea were excluded from the calculations.

The above method, which was first used by Clark [5,6], can be expected to give accurate results because

(a) The sizes of the traffic regions are very small when compared to the size of an annular zone;

(b) The areas of the traffic regions could be measured very accurately by means of a planimeter.

Since the annular zones are of constant width 1 mile, this means that their areas became progressively larger as one proceeds outwards from the centre of the city. Coupled with the falling-off in the density of homes, this ensured that the total number of homes in an annular zone does not change too violently from zone to zone. An alternative method is to use annular zones of constant area. This means, however, that the outmost annular zones are extremely 'thin', and hence the calculation of the total number of homes in an annular zone is now more susceptible to large errors.

The data for the total number and the average density of homes in the seven annular zones are summarized in Table IX for the years 1965 and 1986. It is seen that the density rises to a maximum between two and three miles, and then declines. The inner two zones can thus be regarded as constituting the Central City Area whereas the five outer ones are the suburban residential regions.

Figure 13 shows the logarithm to base 10 of the density of homes D_h plotted against the mean distance ρ for each of the five outer zones. Straight lines were fitted to the data by least squares. It is clear that surprising good fits are obtained for the two sets of data (1965 and 1986). This shows that the data is consistent with Clark's formula (6.1), and the A and b co-efficients can now be immediately obtained.

For further illustration the data has been replotted in Figures 14 and 15, which shows the density of homes D_h plotted against the mean distance ρ from the city centre for the years 1965 and 1986 respectively. It is seen that the five outer points lie fairly close in each case to the

TABLE IX

DISTRIBUTION OF HOMES WITHIN ADELAIDE

Annular Zone	Area of Zone (sq.miles)	Mean radius of zone (miles)	Total number of homes within zone		Average density of homes within zone (per square mile)	
			1965	1986	1965	1986
1	3.14	0.67	2,270	2,370	720	760
2	9.43	1.56	16,780	19,210	1,780	2,040
3	15.71	2.53	34,520	34,270	2,200	2,180
4	22.00	3.52	36,840	40,380	1,680	1,840
5	28.27	4.52	39,260	45,990	1,390	1,630
6	33.85	5.52	32,210	41,040	950	1,210
7	34.08	6.51	21,600	31,600	630	930

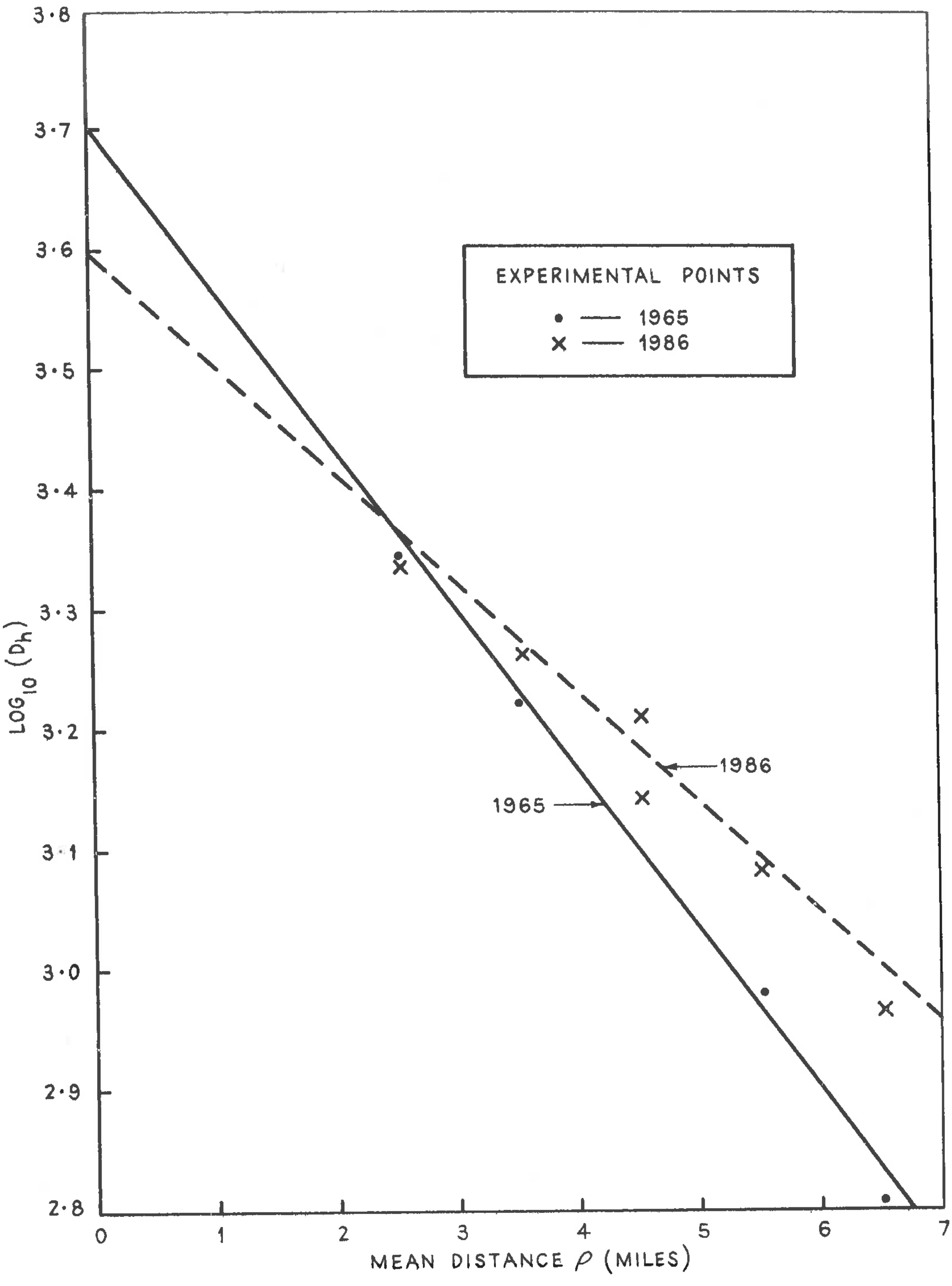

FIGURE 13. PLOT OF THE LOGARITHM TO BASE 10 OF THE DENSITY OF HOMES D_h (PER SQUARE MILE) AGAINST ρ, THE MEAN DISTANCE IN MILES FROM THE CENTRE OF THE CITY, FOR EACH OF THE FIVE OUTER ANNULAR ZONES. THE STRAIGHT LINES HAVE BEEN FITTED TO THE TWO SETS OF DATA BY LEAST SQUARES

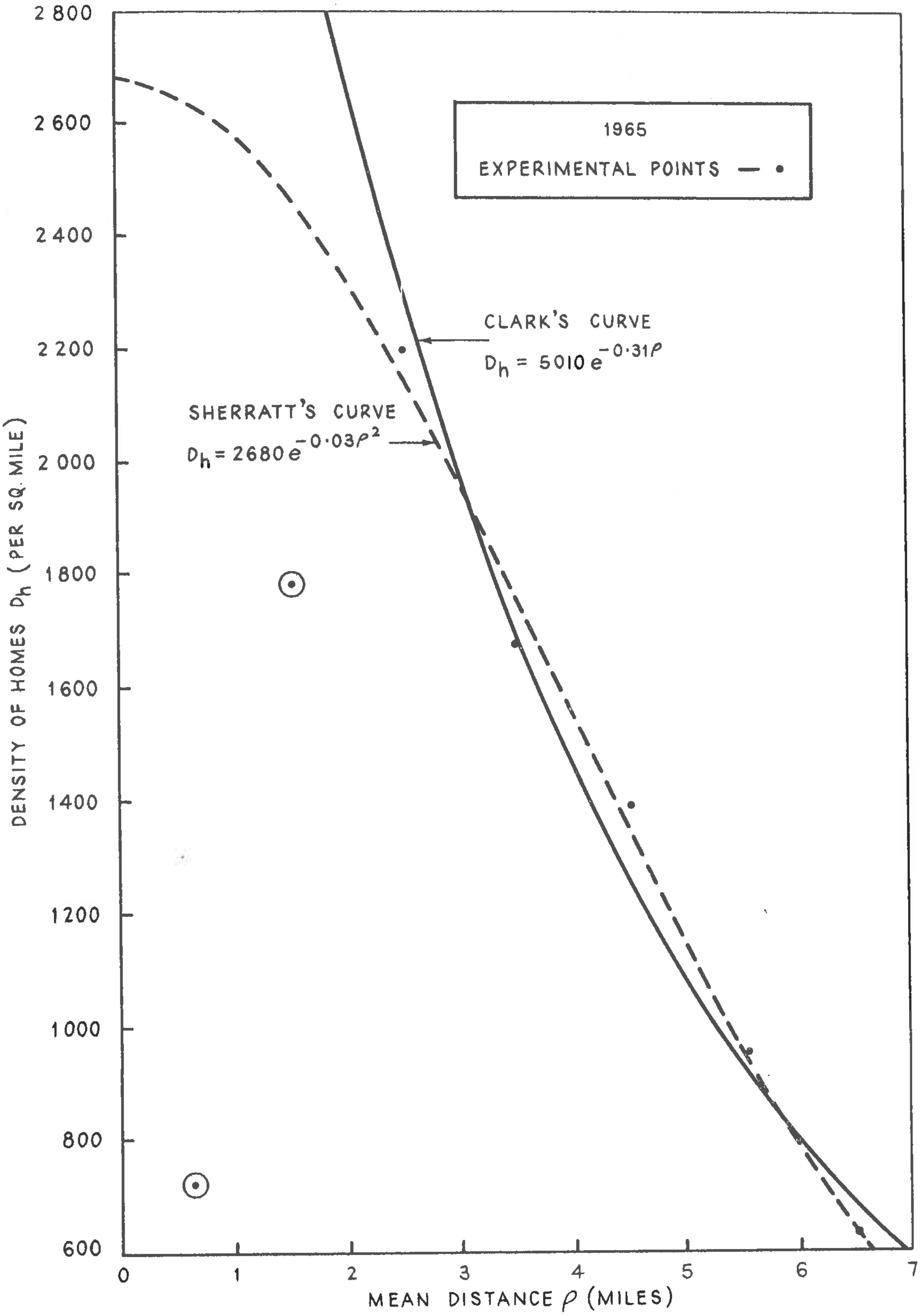

FIGURE 14. PLOT OF THE DENSITY OF HOMES D_h (PER SQUARE MILE) IN 1965 AGAINST ρ, THE MEAN DISTANCE IN MILES FROM THE CENTRE OF THE CITY, FOR EACH OF THE SEVEN ANNULAR ZONES. THE TWO CURVES HAVE BEEN FITTED BY LEAST SQUARES TO THE DATA, EXCLUDING THE ENCIRCLED POINTS

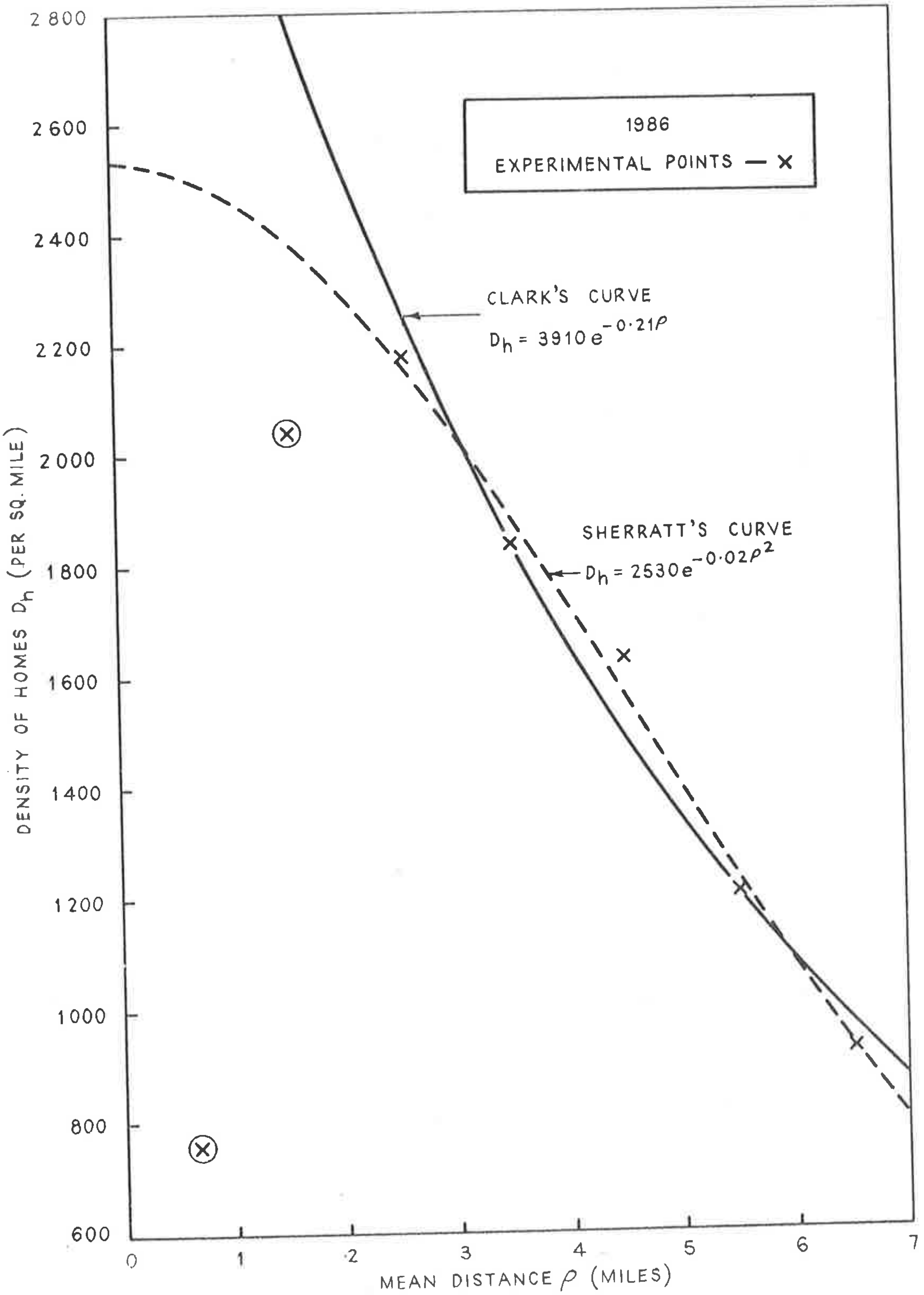

FIGURE 15. PLOT OF THE DENSITY OF HOMES D_h (PER SQUARE MILE) IN 1986 AGAINST ρ, THE MEAN DISTANCE IN MILES FROM THE CENTRE OF THE CITY, FOR EACH OF THE SEVEN ANNULAR ZONES. THE TWO CURVES HAVE BEEN FITTED BY LEAST SQUARES TO THE DATA, EXCLUDING THE ENCIRCLED POINTS

curve calculated by Clark's formula. This agreement is surprising in view of the numerous factors that might be expected to be involved in the distribution of homes.

Sherratt's formula (6.2) was also fitted to the data by least square, and the corresponding curves have been included in Figures 14 and 15. In the range for which data is available (approximately 2 to 7 miles), Sherratt's curves do not appear to give a very much better fit. Hence it may be concluded that the data indicates that Clark's simpler formula is sufficient to represent adequately the distribution of homes in Adelaide.

The A and b co-efficients in Clark's formula are as follows:

	Adelaide	
	1965	1986
A	5010	3910
b	0.31	0.21

Since the values for 1986 are lower than the corresponding values for 1965, it seems that (for Adelaide at least) town planners and transportation engineers agree with Clark's observation that cities tend to flatten out and expand as time goes on. The figures show that, between 1965 and 1986, the density of homes is expected to fall in the inner suburbs and to rise in the outer suburbs.

6.3 The distribution of work places in Adelaide.

The technique described above was also used to study the distribution of work places in Adelaide. The results are summarized in Table X and shown graphically in Figure 16.

The main feature to be observed here is the drastic drop in the density of work places as one proceeds outwards from the Central City Area to the surrounding zones. It does not seem possible to utilize a simple exponential-type formula to represent the density of work places as a function of the distance from the centre of the city. Of course, it is possible to fit a polynomial curve of sufficient complexity to the empirical results. This is, however, of little value because fitting polynomials requires the estimation of too many constants to enable any direct inter-city comparison to be made. The problem is further aggravated by the fact that such data is available only for a few cities.

It is doubtful in any case whether the distribution of work places in many cities can be represented by a simple formula. Zoning laws, for example, have a greater influence on the location of work places than on the location of homes. Furthermore only a comparatively few decisions, perhaps by company executives, may be sufficient to re-locate a factory from one place to another, and this can alter the distribution of work places appreciably. Except in

<u>TABLE X</u>

<u>DISTRIBUTION OF WORK PLACES WITHIN ADELAIDE</u>

Annular Zone	Area of Zone (sq.miles)	Mean radius of Zone (miles)	Total number of work places within zone		Average density of work places within zone (per square mile)	
			1965	1986	1965	1986
1	3.14	0.67	74,730	79,800	23,780	25,400
2	9.43	1.56	24,460	31,890	2,600	3,380
3	15.71	2.53	19,840	27,290	1,260	1,740
4	22.00	3.52	19,580	24,700	890	1,120
5	28.27	4.52	20,670	28,950	730	1,020
6	33.85	5.52	25,090	36,100	740	1,070
7	34.08	6.51	11,030	21,170	320	620

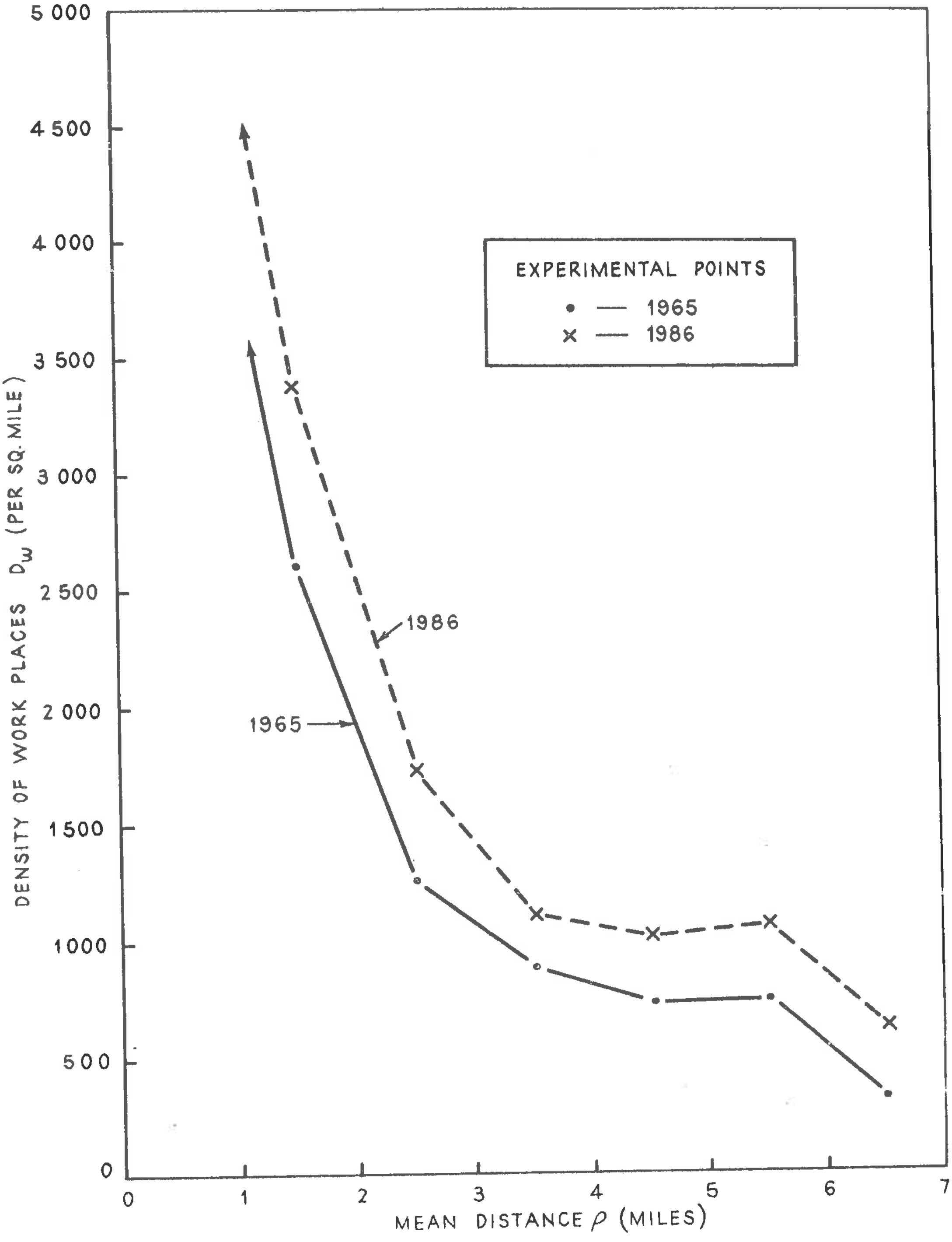

FIGURE 16. PLOT OF THE DENSITY OF WORK-PLACES D_W (PER SQUARE MILE) AGAINST ρ, THE MEAN DISTANCE IN MILES FROM THE CENTRE OF THE CITY, FOR EACH OF THE SEVEN ANNULAR ZONES. THE POINTS HAVE BEEN JOINED BY STRAIGHT LINES IN ORDER TO FACILITATE COMPARISON

countries where multi-storied flats are common, the distribution of homes is the result of a very large number of individual decisions, most of which are independent of one another. Statistical stability is more likely to occur under these circumstances.

6.4 The Central City Area

The Central City Area usually merits a special mention in any study of the commuter traffic pattern within a city. It is here that the worst bottlenecks and the most intractable traffic problems invariably occur. It is easy to see from the following data for Adelaide why such a difficult situation arises.

The Central City Area of Adelaide is roughly a square of area 1.5 square miles. In 1965, the total number of homes within this area was only 2400, the average density of homes being 1620 per square mile. On the other hand the total number of work places was 66,000, the average density thus being 44,600 per square mile. This huge disparity between homes and work places generates tidal waves of commuter traffic in the morning and evening. Consequently the road network within and around the Central City Area is overloaded at these peak periods.

The problem is expected to worsen in future years. In 1986 the number of homes is expected to fall to 2280 with an average density of 1540 per square mile, while the number of work places is expected to rise to 71,500 with an average density of 48,300 per square mile. The ratio of the number of work places to homes is thus expected to rise from 27.5 in 1965 to 31.4 in 1986. This slow rate of growth is actually very optimistic considering the experiences of other cities.

A final comment to be made here is that the above figures support very well the usual assumption of a Central City Area with a negligible number of homes and a high density of work places.

6.5 Conclusion and Discussion.

The most striking conclusion in the present chapter is the agreement between Clark's simple formula and the distribution of homes in Adelaide. It is possible to fit the empirical data more accurately by means of a more complex formula - perhaps one of the gamma family:

$$D_h(\rho) = A\rho^k \exp(-b\rho) \quad , \qquad (6.4)$$

where A, k, and b are parameters. This formula would take into account the drop in the density of homes as one approaches the centre of the city. However it is far more advantageous to have a simple formula which fits many cities approximately than to have a complicated formula which only fits one city well.

About four months of laborious work were required to obtain the figures quoted in this chapter. This is much too slow to permit the analysis of the data for a large number of cities. The principal need here is for some mechanical apparatus which will be able to 'read' a map and then calculate automatically the required distances, areas etc. Such an apparatus does not exist at present. It is also vitally necessary to persuade the highway authorities not to destroy the details of the data collected in traffic surveys. If this data can be recorded in some permanent form, they will form the bases for a quantitative study of urban growth and structure.

CHAPTER VII

GENERAL DISCUSSION

An attempt has been made in this thesis to develop useful models for the study of commuter traffic in cities. Following on a line of research initiated by Smeed, a simple model of an inhomogeneous city has been constructed. Basically the city is assumed to consist of an inner central business district, which contains only work places, surrounded by an annular residential zone, consisting only of homes. It is shown that a detailed analysis of this model yields significant information concerning a number of the characteristics of commuter travel in cities. Furthermore a variation of the model can be used to study the pattern of commuter traffic in Satellite Towns.

Various simplifying assumptions have been made in the thesis in order to render the calculations tractable. It is believed, however, that these assumptions are consistent with the most prominent feature of commuter traffic in many cities i.e. tidal flows of workers travelling from the suburbs to the central business district in the morning and in the reverse direction in the evening. Obviously the assumptions can be made more realistic, although it remains to be seen whether the corresponding increase in the complexity of the mathematics can be successfully handled. The primitive nature of the assumptions is, perhaps, best regarded as indicative of the rich variety of problems that are waiting to be solved.

In the light of the empirical data presented in Chapter VI, it is obvious that the next step in the theory is to modify the assumptions concerning the distribution of homes and work places. A shifted exponential or a gamma-type formula would be a more accurate representation of the distribution of homes in a city like Adelaide with a 'strong' central business district, and a widely dispersed suburban region consisting mainly of detached free-standing homes. Similarly the distribution of work places can be represented much better by a step function with a very high density in the central business district and a low density in an annular zone around it.

It is difficult to suggest a good assumption for the correlation between the positions of homes and work places as the evidence available is conflicting. A simple gravity-type formula may serve very well as a first approximation. Finally the ideal routeing systems to be considered will be made more realistic by assuming that special routes, such as those representing motorways, can be joined or left only at certain points which occur at regular intervals along the routes.

Until very recently, research in Town Planning has been conservative in its outlook. Most planners have assumed that the present pattern of settlement in a country or city will remain substantially the same in future years. Thus the solution to urban transportation difficulties lies in the provision of more highways or more buses. The suggestions for further work given above reflect the spirit of this philosophy.

This viewpoint, however, has lately been challenged and criticized by many authors, for example, the Editor of Nature [31]. These critics say that it is not sufficient to work on the basis that the physical fabric of the city will remain unchanged. A prominent aim in planning research should be to evolve a grand optimal strategy for the total urban development in a region, and thus provide guide lines for the framing of contemporary laws and regulations. Furthermore under the pressure of mounting urban problems it may well be that, in fifty years time, planning authorities will be given the necessary power to undertake radical and wide-spread changes in the landscape and townscape of a country. A disaster could then occur if these authorities were unprepared for such an eventuality.

It is clear that the techniques, developed in this thesis, constitute a natural and powerful tool for the analysis of ideal future city forms. Indeed it is even possible that these techniques may find their greatest potential for use in this area.

A favourite scheme for the best design of a growing city is to evade the problem of growth by creating a complex of cities of finite size. This solution is particularly attractive because of its flexibility. One proposal [31], for example, suggests an arrangement in which sub-cities with about a quarter of a million people each are spaced at distances of several miles in a country setting but are linked together by a fast and efficient transport system. If the urban region under consideration is centred on an existing major city, then a Doxiadis-type plan appears to be feasible. This plan generally consists of three stages:

(a) Determination of an axis of expansion centred on the existing city;

(b) Definition and widening of a corridor of reserved land along the axis;

(c) Promotion of urban growth along this corridor by the successive addition of subordinate cities, each of finite size, on either side of the original nucleus.

It is interesting to note that a very precise and concrete plan of the Doxiadis-type has been prepared for metropolitan Washington in the United States - a country which has no effective Federal Planning Authority. The Washington plan [32] is designed to cater for an expected expansion of the city population from two millions in 1963 to five millions by the year 2000. The recommended plan is termed the radial corridor plan, and consists of six roughly symmetrical fingers which would be pushed out from the existing city for distances of 25-30 miles. Each finger would comprise a succession of semi-independent suburban communities, strung along a radial communication corridor consisting of a first-class road and electric railway. Other plans that were considered included a ring of satellite New Towns, a series of new independent cities at a distance of about 70 miles, and a joined-up ring of towns on a circumferential line of communication.

All of the above plans can be studied quantitatively by an immediate extension of the model of a satellite town, discussed in Chapter V. Instead of a single town we now have a system of satellite New Towns and Residential Towns. The problem of routeing through traffic around the towns will now come into prominence. It is also possible to refine the model by introducing more complicated routeing systems in the Central Area of each town to cater for a variety of activities.

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As a final suggestion, we point out that no attempt has yet been made to take into account fully the interaction between the transportation system and the distribution of population in a region. A practically universal assumption in this field is that the land use pattern of settlement is determined first, and then a transportation system is designed to fit that pattern. This assumption, of course, ignores the fact that the provision of a major transport facility, such as a highway, has a considerable effect on the manner in which land is used around that facility. The inter-relationship between land use and transport is an extremely tangled one, and, as yet, even the glimmerings of a satisfactory theory to explain the process seem to be far away.

APPENDIX A

THE AVERAGE LENGTH OF A DIRECT ROUTE IN A SQUARE INHOMOGENEOUS CITY

The problem of calculating the average distance travelled with direct routeing in a square inhomogeneous city was considered in Chapter IV. The city is assumed to be in the form of a square with sides of length $2u$, within which is contained a square central business district with sides parallel to those of the first but of length $2a$ $(0 < a < u)$. Work places are assumed to be uniformly distributed over the inner square, while homes are assumed to be uniformly distributed over the surrounding annular zone. There is no correlation between the positions of homes and work places. The straight-line distance between a home $H(h,p)$ and a work place $G(\ell,m)$ is

$$d = \sqrt{[(h - \ell)^2 + (p - m)^2]} \,, \qquad (A1)$$

and thus the average distance travelled is

$$\bar{d} = \underset{ph\ell m}{EEEE} (d) \,. \qquad (A2)$$

The integrations with respect to the variables ℓ and m are facilitated by introducing two new variables f and g such that

$$f = h - \ell \qquad (A3)$$

and

$$g = p - m \,. \qquad (A4)$$

Then

$$4a^2 \underset{\ell m}{EE} (d) = \int_{-a}^{a} \int_{-a}^{a} \{\sqrt{[(h - \ell)^2 + (p - m)^2]}\} dm d\ell$$

$$= \int_{h_2}^{h_1} \int_{p_2}^{p_1} [\sqrt{(f^2 + g^2)}] dg df \quad , \qquad (A5)$$

where $h_1 = h + a \quad , \qquad h_2 = h - a \qquad (A6)$

and $p_1 = p + a \quad , \qquad p_2 = p - a \quad . \qquad (A7)$

Therefore

$$4a^2 \underset{\ell m}{EE} (d) = (\tfrac{1}{2}) \int_{h_2}^{h_1} \{p_1\sqrt{(p_1^2 + f^2)} + f^2 \ell n[p_1 + \sqrt{(p_1^2 + f^2)}]$$

$$- p_2\sqrt{(p_2^2 + f^2)} - f^2 \ell n[p_2 + \sqrt{(p_2^2 + f^2)}]\} df \quad . \qquad (A8)$$

By means of the relation:

$$\int \{f^2 \ell n[p + \sqrt{(p^2 + f^2)}]\} df = f^3 \ell n[p + \sqrt{(p^2 + f^2)}]/3 \qquad (A9)$$

$$+ pf\sqrt{(p^2 + f^2)}/6$$

$$- p^3 \ell n[f + \sqrt{(p^2 + f^2)}]/6$$

$$- f^3/9 \quad ,$$

equation (A8) can be reduced to

$$4a^2 \underset{\ell m}{EE}(d) = \frac{h_1^3}{6} . \ell n\left\{\frac{p_1 + \surd(p_1^2 + h_1^2)}{p_2 + \surd(p_2^2 + h_1^2)}\right\} - \frac{h_2^3}{6}\ell n\left\{\frac{p_1 + \surd(p_1^2 + h_2^2)}{p_2 + \surd(p_2^2 + h_2^2)}\right\}$$

$$+ \frac{p_1^3}{6}\ell n\left\{\frac{h_1 + \surd(h_1^2 + p_1^2)}{h_2 + \surd(h_2^2 + p_1^2)}\right\} - \frac{p_2^3}{6}\ell n\left\{\frac{h_1 + \surd(h_1^2 + p_2^2)}{h_2 + \surd(h_2^2 + p_2^2)}\right\}$$

$$+ \frac{h_1 p_1 \surd(h_1^2 + p_1^2)}{3} - \frac{h_2 p_1 \surd(h_2^2 + p_1^2)}{3}$$

$$+ \frac{h_2 p_2 \surd(h_2^2 + p_2^2)}{3} - \frac{h_1 p_2 \surd(h_1^2 + p_2^2)}{3} . \qquad \text{(A10)}$$

The next step is to consider the case when a home H lies somewhere within the narrow annulus bounded by squares with sides of lengths $2h$ and $2(h + dh)$. The probability of this event occurring is $8hdh/4(u^2 - a^2)$.

Since the figure is symmetrical, only the range $[0,h]$ need to be considered in calculating the expectation with respect to the variable p. Using some obvious substitutions and the relation:

$$\int\{p^3 \ell n[h + \surd(h^2 + p^2)]\}dp = p^4 \ell n[h + \surd(h^2 + p^2)]/4 - p^4/16$$
$$+ h(p^2 + h^2)^{3/2}/12$$
$$- h^3\surd(h^2 + p^2)/4 \quad , \qquad \text{(A11)}$$

the following result is obtained:

$$4a^2 h \underset{p\ell m}{EEE}(d) = \{h_1^3(5h - 3a)[\ell n(h_1) - \ell n[h_2 + \surd(h_1^2 + h_2^2)]]\}/24$$
$$+ \{h_2^3(5h + 3a)[\ell n(h_2) - \ell n[h_1 + \surd(h_1^2 + h_2^2)]]\}/24$$
$$+ \{[\ell n(1 + \surd 2)][5h^4 + 30a^2h^2 + 5a^4]\}/12$$
$$+ \surd 2(h^4 + 6a^2h^2 + a^4)/12$$
$$+ \surd(2h^2 + 2a^2)(9a^2h - h^3)/12 . \qquad \text{(A12)}$$

Since

$$\bar{d} = \frac{\int_a^u [8hEEE_{p\ell m}(d)]dh}{4(u^2 - a^2)}$$

$$= \left(\frac{1}{2a^2(u^2 - a^2)}\right) \int_a^u [4a^2 hEEE_{p\ell m}(d)]dh \quad , \qquad (A13)$$

an integration of equation (A12) between the limits a and u yields

$$\bar{d} = \frac{(u + a)^4(u - a)}{48(u^2 - a^2)a^2} \ell n \left(\frac{u + a}{u - a + \sqrt{(2u^2 + 2a^2)}}\right)$$

$$+ \frac{(u - a)^4(u + a)}{48(u^2 - a^2)a^2} \ell n \left(\frac{u - a}{u + a + \sqrt{(2u^2 + 2a^2)}}\right)$$

$$+ \left(\frac{5\ell n(1 + \sqrt{2}) + \sqrt{2}}{60}\right)\left(\frac{u^5 + 10u^3a^2 + 5ua^4 - 16a^5}{2(u^2 - a^2)a^2}\right)$$

$$- \frac{(u^4 - 18u^2a^2 + a^4)[\sqrt{(2u^2 + 2a^2)}]}{120(u^2 - a^2)a^2} - \frac{8a^5}{30(u^2 - a^2)a^2}$$

(A14)

In deriving equation (A14), relations of the following type have been used:

$$\int h^n \ell n[h - 2a + \sqrt{(2h^2 - 4ah + 4a^2)}]dh$$

$$= h^{n+1}\ell n[h - 2a + \sqrt{(2h^2 - 4ah + 4a^2)}]/(n + 1)$$

$$- \int \left\{\frac{h^{n+1}}{n+1} \left[\frac{2a}{h\sqrt{(2h^2 - 4ah + 4a^2)}} + \frac{1}{h} \right]\right\}dh \quad .$$

(A15)

If the area of the Central Business District is A and sA is the area of the whole city, then

$$(u/a) = \sqrt{s} \quad . \qquad (A16)$$

Equation (A14) can then be expressed in the dimensionless form

$$\frac{\bar{d}}{\sqrt{A}} = \frac{(\sqrt{s}+1)^3}{96}\,\ell n\left(\frac{\sqrt{s}+1}{\sqrt{s}-1+\sqrt{(2s+2)}}\right)$$

$$+\frac{(\sqrt{s}-1)^3}{96}\,\ell n\left(\frac{\sqrt{s}-1}{\sqrt{s}+1+\sqrt{(2s+2)}}\right)$$

$$+\frac{[5\ell n(1+\sqrt{2})+\sqrt{2}](s^2\sqrt{s}+10s\sqrt{s}+5\sqrt{s}-16)}{240(s-1)}$$

$$-\frac{[\sqrt{(2s+2)}](s^2-18s+1)+32}{240(s-1)}\quad, \qquad (A17)$$

and this is equation (4.24) of Chapter IV.

APPENDIX B

SUMMARY OF NOTATION

The more important symbols and definitions used in the thesis are given below, listed chapter-wise:

Chapter II

W = the width (e.g. in feet) of the roads used by commuters;

Q = the capacity (e.g. in vehicles per hour) of the roads used by commuters;

T = the duration (e.g. in hours) of the peak travel period in the morning or evening;

c = the number of commuters to a vehicle;

λ = the width (e.g. in feet) of road required for a person-journey.

These variables are related by the equation

$$\lambda = W/QcT \quad . \qquad \text{(B1)}$$

Chapter III

R = the overall radius (e.g. in miles) of the circular city;

r = the radius (e.g. in miles) of the circular central business district;

$H(h,0)$ = the position of a worker's home expressed in polar co-ordinates;

$G(\ell,\theta)$ = the position of a work place expressed in polar co-ordinates;

ψ = the ratio of the density of work places to homes;

d = the distance (e.g. in miles) travelled from home to work;

t = the time (e.g. in minutes) required to travel from home to work;

V_1, V_2, V_3 = the speed of travel (e.g. in miles per hour) in the outer residential zone, in the central business district, and along the Ring Road respectively.

The quantities R,r, and ψ are related by the equation

$$\psi = (R^2 - r^2)/r^2 \quad . \qquad \text{(B2)}$$

The expected value of a random variable δ is denoted by $\underset{\delta}{E}(\delta)$ or $\bar{\delta}$. If a function Φ depends on two random variables δ and θ, then the expected value of Φ is given by

$$\underset{\theta\,\delta}{EE}(\Phi) = \underset{\theta}{E}[\underset{\delta}{E}(\Phi/\theta)] \quad . \qquad \text{(B3)}$$

This means that the expected value of Φ is obtained by first calculating the conditional expected value of Φ for fixed θ, and then averaging over θ.

Chapter IV

The notation of Chapter IV follows that of Chapter III, but the following terms should be noted:

$2u, 2z$ = the overall width and length (e.g. in miles) of the rectangular city;

$2a, 2b$ = the width and length (e.g. in miles) of the inner rectangular central business district;

$H(h,p)$ = the position of a worker's home expressed in rectangular co-ordinates;

$G(\ell,m)$ = the position of a work place expressed in rectangular co-ordinates;

A = the area (e.g. in square miles) of the central business district;

sA = the area (e.g. in square miles) of the whole city;

α = the ratio of the average trip length in an inhomogeneous rectangular city to that in a homogeneous rectangular city;

V_1, V_2, V_3, V_4 = the speed of travel (e.g. in miles per hour) in the outer residential zone, along the radial or diagonal routes, at the edge of the central business district, and within the central business district respectively.

Chapter V

The following terms in this chapter should be noted:

R = the overall radius (e.g. in miles) of the circular satellite town excluding the Ring Road;

r = the radius (e.g. in miles) of the circular Central Area of the satellite town;

N = the total number of workers living in the satellite town;

n = the number of workers commuting into the Central Area in the morning peak period of travel;

x = the fraction of workers, living in a region, who commute inwards to the Central Area in the morning;

y = the fraction of workers, living in a region, who commute outwards to the Ring Road in the morning;

P = the ground area (e.g. in square feet) required by a worker for vehicle parking;

L = the ground area (e.g. in square feet) required by a worker for purposes such as living, leisure, and casual travel;

j = the parameter characterising the flow regulation on the commuter roads in the Central Area i.e. $j = 1$ if the roads are reversible and $j = 2$ if the roads are irreversible;

ρ or ξ = the distance from the town centre O;

$g(\rho)$ = the fraction of ground area, which is reserved for commuter roads and parking at a distance ρ within the Central Area;

$f(\rho)$ = the function specifying the division of the ground space reserved for commuter roads and parking. $f(\rho)$ is the fraction of the reserved space, at a distance ρ in the Central Area, which is to be used for parking purposes;

$h(\rho)$ = the fraction of ground area which is to be used for purposes other than commuter roads at a distance ρ in the Residential zone, for the case of Reversible roads;

$h_1(\rho), h_2(\rho)$ = the fraction of ground area which is to be used for purposes other than commuter roads at a distance ρ in the Residential zone, for the case of Irreversible roads. $h_1(\rho), h_2(\rho)$ hold in the ranges $(r \leqslant \rho \leqslant a)$ and $(a \leqslant \rho \leqslant R)$ respectively where

$$\int_r^a [2\pi\xi h_1(\xi)/L]d\xi = Nx;$$

$\lambda_1, \lambda_2, \lambda_3$ = the width (e.g. in feet) of road required for a commuter journey in the Central Area, Residential zone, and along the Ring Road respectively;

w = the width (e.g. in feet) of the Ring Road assuming that the road is reversible and that the number of commuters turning left on reaching the Ring Road is equal to the number turning right.

Chapter VI

ρ = the distance (e.g. in miles) from the town centre;

$D_h(\rho)$ = the density of homes (e.g. per sq. mile) at a distance ρ;

$D_w(\rho)$ = the density of work places (e.g. per sq. mile) at a distance ρ.

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