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Proper Efficiency and Proper Karush-Kuhn-Tucker Conditions for Smooth Multiobjective Optimization Problems *

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Abstract Proper Karush-Kuhn-Tucker (PKKT) conditions are said to hold when all the multipliers of the objective functions are positive. In 2012, Burachik and Rizvi introduced a new regularity condition under which PKKT conditions hold at every Geoffrion-properly efficient point. In general, the set of Borwein properly-efficient points is larger than the set of Geoffrion-properly efficient points. Our aim is to extend the PKKT conditions to the larger set of Borwein-properly efficient points.

Keywords Multiobjective optimization, Regularity conditions, Optimality conditions for efficient solutions, Geoffrion-properly efficient solutions, Borwein-properly efficient solutions, Proper Karush-Kuhn-Tucker conditions.

AMS subject classification. 90C29

1 Introduction

The process of optimizing a collection of objective functions simultaneously is called *multiobjective optimization* (or vector optimization) *problems* (MOP). Because of the conflicting nature of the objectives, it is hard to find a single solution that optimizes all the objectives simultaneously. Therefore, a set of alternative solutions is found and these solutions are called *Pareto optimal points* [1], *vector minimum points* [2], *non-inferior points* [3] or *efficient points* [4, 5]. We use the term *efficient point* for these solutions. More restricted definitions of efficiency, called proper

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efficiency, have been introduced by Kuhn-Tucker [6], Geoffrion [7] and Borwein [8]. We recall that every Geoffrion-properly efficient point is a Borwein-properly efficient point. The converse is not true in general. More details on the relationship between these notions is recalled in Section 2.1.

Optimality conditions are studied to find efficient points of multiobjective optimization problems. Many authors derive the first-order and second-order necessary and sufficient conditions for multiobjective optimization problems in the case of smooth [5, 9, 10] and non-smooth functions [11, 12, 13]. Initially, the optimality conditions of multiobjective problems used the same constraint qualifications as in the single-objective case [14, 15]. For these constraint qualifications we could obtain at least one positive multiplier corresponding to the objective function. This means that some of the multipliers may be equal to zero. As a consequence, some Lagrange multipliers have no role in the necessary conditions of optimality. To avoid this situation and to obtain positive Lagrange multipliers associated with each of the objective functions, we need a constraint qualification. In 1994, Maeda [5] introduced a generalized constraint qualification and ensured the existence of positive Lagrange multipliers. Later Bigi [2], and more recently, Burachik and Rizvi [4] introduced a generalized *regularity condition* for multiobjective optimization problems. They were able to ensure positivity of all Lagrange multipliers under these regularity conditions. The positivity of a multiplier corresponding to an objective function indicates an active role of that function in the identification of the solutions of the problem. When all the multipliers corresponding to the objective functions are positive, we say that *proper Karush-Kuhn-Tucker* (PKKT) conditions hold.

In [4], PKKT conditions are established for Geoffrion-properly efficient points. Our aim is to establish PKKT conditions for the larger set of Borwein-properly efficient points (see Theorem 3.2).

Note that generalized constraint qualification and regularity conditions introduced respectively in [5] and [4] do not guarantee KKT conditions for the non-smooth case. This is why, in this short note, we focus on the differentiable case.

The outline of the present paper is as follows: Section 2 contains notations, definitions and preliminary materials. Still in this section, we describe our theoretical setting, recall our regularity condition from [4], and prove a technical lemma. Finally, Section 3 contains the main results involving PKKT conditions under our regularity condition.

2 Preliminaries

In this section, we recall some basic tools of multiobjective optimization problems (MOP) which are used in the rest of the analysis. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For $x_i, y_i \in \mathbb{R}$, we say $x_i \geqslant y_i$ iff $(x_i - y_i)$ is non negative. For

$x, y \in \mathbb{R}^n$ we use from now on the following conventions [16].

$$x \geqq y, \text{ iff } \quad x_i \geqslant y_i, \quad i = 1, ..., n.$$

In other words, $\geqq$ represents in $\mathbb{R}^n$ the usual order coordinate by coordinate.

$$x \geq y, \text{ iff } \quad x \geqq y \text{ and } \quad x \neq y,$$

so $\geq$ means that each coordinate of x is greater than or equal each coordinate of y, and $x \neq y$.

$$x > y, \text{ iff } \quad x_i > y_i \text{ and } i = 1, ..., n.$$

Hence $>$ stands for the usual strict inequality, coordinate by coordinate. For $\ell, m \in \mathbb{N}$, we consider in this paper the following multiobjective optimization Problem (P):

$$(\mathrm{P}) \quad \begin{cases} \min & f(x) := [f_1(x), ..., f_\ell(x)], \\ \text{subject to} & g_j(x) \leqq 0, \quad j = 1, 2, ..., m. \end{cases} \tag{1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., \ell$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., m$ are continuously differentiable real-valued functions. We define $X = \{x \in \mathbb{R}^n \mid g_j(x) \leqq 0, \quad j = 1, 2, ..., m\}$ and $I(\bar{x}) := \{j | g_j(\bar{x}) = 0\}$. In our analysis, we do not make any convexity assumption on the functions involved in (P).

We next recall the definitions of efficient and local efficient points.

Definition 2.1 A point $\bar{x}$ is said to be *efficient* for Problem (P) if there is no $x \in X$ such that $f(x) \leq f(\bar{x})$. A point $\bar{x}$ is said to be *local efficient* to Problem (P) if there exists $\delta > 0$ such that $\bar{x}$ is an efficient point in $X \bigcap B(\bar{x}, \delta)$.

The analysis of optimality conditions depends on the concept of the contingent and linearizing cones, as well as their relationship. This relationship is used in many constraint qualifications. More studies on the relationship between contingent and linearizing cones can be found in [5, 13, 17, 18, 19, 20, 21]. We use the following two definitions in our analysis.

Definition 2.2 Let C be a subset of $\mathbb{R}^n$. The *contingent cone* to C at $\bar{x} \in C$ is the set defined by

$$T(C; \bar{x}) := \{d \in \mathbb{R}^n \mid \exists \{(x_n, t_n)\} \subseteq (C \times \mathbb{R}_{++}) \text{ such that } x_n \to \bar{x} \text{ and } t_n(x_n - \bar{x}) \to d\}.$$

The set $T(C; \bar{x})$ is a nonempty and closed cone. More properties of this cone can be seen in [22, 23, 24, 25, 26].

Definition 2.3 The *active linearizing cone* to X at $\bar{x} \in X$ is the set defined by

$$\tilde{L}(X; \bar{x}) := \left\{d \in \mathbb{R}^n \mid \nabla g_j(\bar{x})^T d \leqq 0, \quad j \in I(\bar{x})\right\}. \tag{2}$$

2.1 Proper Efficiency

In 1952, Kuhn and Tucker [6] pointed out that certain efficient points show unfavourable properties concerning the ratio of the rate of gain in one cost over the corresponding rate of loss in another cost. To avoid such pathological behaviour, they introduced the *properness* of the efficient solution of the multiobjective optimization problems. More general definitions of proper efficiency are given by Geoffrion [7] and Borwein [8]. For discussions about the relationship among different types of proper efficiency see [27]. In our analysis, we focus on the concepts of Borwein-proper efficiency and Geoffrion-proper efficiency. We use these concepts in Theorem 3.1 and Corollary 3.1 to ensure PKKT conditions under our proposed regularity condition. We recall below the relevant definitions of proper efficiency.

Definition 2.4 (See [6]) A point $\bar{x}$ is said to be *Kuhn-Tucker-properly efficient* (KTPE) of Problem (P) if it is efficient and the following system has no solution.

$$\nabla f_i(\bar{x})^T d \leqq 0,$$
$$\exists i_0 \in \{1, 2, \ldots, \ell\}, \text{ such that } \nabla f_{i_0}(\bar{x})^T d < 0,$$
$$\nabla g_j(\bar{x})^T d \leqq 0, \quad j \in I(\bar{x}).$$

Next we recall the definition of Geoffrion-properly efficient solution.

Definition 2.5 (See [7]) A point $\bar{x}$ is said to be *Geoffrion-properly efficient* (GPE) of Problem (P) if it is efficient and there exists a scalar $M > 0$ such that, for each i, we have

$$\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leqq M,$$

for some j such that $f_j(\bar{x}) < f_j(x)$ whenever $x \in X$ and $f_i(\bar{x}) > f_i(x)$.

An efficient point that is not properly efficient is said to be *improperly efficient*. Thus a point $\bar{x}$ is said to be *improperly efficient* of Problem (P) when for every scalar $M > 0$ (no matter how large) there is a point $x \in X$ and an index i such that $f_i(\bar{x}) > f_i(x)$ and

$$\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} > M,$$

for all j such that $f_j(\bar{x}) < f_j(x)$.
The concept of Borwein proper efficiency relies on the definition of contingent cone given in Definition 2.2.

Definition 2.6 (See [8]) A point $\bar{x}$ is said to be *Borwein-properly efficient* (BPE) of Problem (P) if it is efficient and

$$T(f(X) + \mathbb{R}^\ell_+, f(\bar{x})) \bigcap (-\mathbb{R}^\ell_+) = \{0\}.$$

Remark 2.1 In general, we have the following inclusions.

$$BPE \supseteq GPE, \tag{3}$$

(see [8, Proposition 1], where the inclusion " $\supseteq$ " stands for "less restrictive". However, if X is convex and f_i are convex functions then we have

$$BPE = GPE,$$

(see [28, Theorems 2.45(2) and 2.48]).

The following example (taken from [28, Exercise 2.13]), illustrates the case of a Borwein properly efficient point which is not properly efficient in the sense of Definitions 2.5 and 2.4.

Example 2.1 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x) := x$, and consider the problem

$$\min \ f(x),$$

$$\text{s.t. } x \in X = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geqq 0\} \bigcup \{x \in \mathbb{R}^2 \mid x_1 \geqq 1\} \bigcup \{x \in \mathbb{R}^2 \mid x_2 \geqq 1\},$$

and $X = f(X)$. Here, $\bar{x} = (0, 0)$ is an efficient point. It is easy to check that

$$T(f(X) + \mathbb{R}^2_+, f(\bar{x})) \bigcap (-\mathbb{R}^2_+) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geqq 0\} \bigcap (-\mathbb{R}^2_+) = \{0\}.$$

Hence, $\bar{x}$ is BPE. Note that

$$cl(cone(f(X) + \mathbb{R}^2_+ - f(\bar{x})) \bigcap (-\mathbb{R}^2_+) = L_1 \cup L_2 \neq \{0\},$$

where

$$L_1 := \{x \in \mathbb{R}^2 \mid x_2 = 0, x_1 \leqq 0\} \quad \text{and} \quad L_2 := \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \leqq 0\}.$$

We note that $\bar{x}$ is not GPE. Indeed, let $x = (-a, 1)$, $a > 0$, then we have that

$$\frac{f_1(x) - f_1(\bar{x})}{f_2(\bar{x}) - f_2(x)} = a. \tag{4}$$

The value of (4) can be made arbitrarily large by taking $a > 0$. Hence $\bar{x}$ is not a GPE. Next we prove that $\bar{x}$ is not KTPE. For this point, only the constraint $g_1(x) = x_1 + x_2 \geqq 0$ is active. If $d = (-1, -1)$, we have that

$$\nabla f_1(\bar{x})^T d < 0, \ \nabla f_2(\bar{x})^T d < 0, \nabla g_1(\bar{x})^T d < 0, \ \ 1 \in I(\bar{x}),$$

where $\nabla f_1(\bar{x}) = (1, 0)$, $\nabla f_2(\bar{x}) = (0, 1)$, and $\nabla g_1(\bar{x}) = (1, 1)$. Hence, $\bar{x}$ is not KTPE.

2.2 Regularity Conditions

In scalar optimization problems, constraint qualifications are used to ensure the positivity of the Lagrange multiplier associated with the objective function. However, constraint qualifications are not enough to guarantee the positivity of Lagrange multipliers in multiobjective optimization problems. To obtain PKKT conditions, i.e., positivity of Lagrange multipliers associated with each of the objective functions, some special assumptions on the objective functions and the constraints are required. These assumptions are called *regularity conditions* (RC). Fix $\bar{x} \in X$, and for f, g as in (1) define

$$Q := \{x \in \mathbb{R}^n \mid g(x) \leqq 0, \quad f(x) \leqq f(\bar{x})\}, \text{ and}$$

$$Q^i := \{x \in \mathbb{R}^n \mid g(x) \leqq 0, \quad f_k(x) \leqq f_k(\bar{x}), \quad k = 1, 2, \ldots, \ell \text{ and } k \neq i\}.$$

In our analysis we will need another concept of linearizing cone which is different from the one given by (2.3). This concept of cone uses the set Q defined above, it was introduced by Maeda in [5], and its definition is as follows.

Definition 2.7 The *linearizing cone* to Q at $\bar{x} \in Q$ is the set defined by

$$L(Q; \bar{x}) := \left\{d \in \mathbb{R}^n \mid \nabla f_i(\bar{x})^T d \leqq 0, \ \ i = 1, 2, \ldots, \ell, \nabla g_j(\bar{x})^T d \leqq 0, \ j \in I(\bar{x})\right\}.$$

Note that $L(Q; \bar{x}) \subseteq \tilde{L}(X; \bar{x})$, where $\tilde{L}$ as in Definition 2.3.

In 1994, Maeda [5] introduced the Guignard type regularity conditions.

Definition 2.8 (Generalized Guignard constraint qualification (GGCQ)) Let $\bar{x}$ be a feasible point of (P). Then (1) is said to satisfy the GGCQ if

$$L(Q; \bar{x}) \subseteq \bigcap_{i=1}^{\ell} clconvT(Q^i; \bar{x}). \tag{5}$$

The regularity condition we focus on has been introduced by Burachik and Rizvi in [4]. We recall now the definition of M^i from Rizvi ([29, 30]). For a fixed $\bar{x} \in X$,

$$M^i := \{x \in \mathbb{R}^n \mid g(x) \leqq 0, \quad f_i(x) \leqq f_i(\bar{x})\}, \quad i = 1, 2, \ldots, \ell.$$

Note that, by defining $M := \bigcap_{i=1}^{\ell} M^i$, we obtain $Q = M$. The relationship between these sets is obviously given by the equality

$$Q^i = \bigcap_{\substack{j=1 \\ j \neq i}}^{\ell} M^j; \ \ i = 1, ..., \ell.$$

Based on these sets, we will define the following linearizing cones.

Definition 2.9 The linearizing cone to M^i at $\bar{x} \in M^i$ is the set defined by

$$L(M^i;\bar{x}) := \left\{ d \in \mathbb{R}^n \mid \nabla f_i(\bar{x})^T d \leqq 0, \nabla g_j(\bar{x})^T d \leqq 0,\ j \in I(\bar{x}) \right\},\ \ i = 1, 2, \ldots, \ell,$$

and the linearizing cone to M at $\bar{x} \in M$ is the set defined by

$$L(M;\bar{x}) := \left\{ d \in \mathbb{R}^n \mid \nabla f_i(\bar{x})^T d \leqq 0,\ \ i = 1, 2, \ldots, \ell, \nabla g_j(\bar{x})^T d \leqq 0,\ j \in I(\bar{x}) \right\}.$$

In Theorem 3.1, we will use the following three simple lemmas.

Lemma 2.1 Let $h : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable function. Assume that
i) $z_n \to \bar{z}$,
ii) $h(z_n) \leqq h(\bar{z})$,
iii) $v = \lim_{n\to\infty} s_n(z_n - \bar{z})$ with $s_n > 0$ for all n then,

$$\nabla h(\bar{z})^T v \leqq 0.$$

Proof. See, [4, Lemma 3.1]. □

Lemma 2.2 Let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ be a continuously differentiable function. Assume that $\{t_n\} \subseteq \mathbb{R}_{++}$, $\{z_n\} \subseteq \mathbb{R}^n$ and $\bar{z} \in \mathbb{R}^n$, such that $z_n \to \bar{z}$ with $h = \lim_{n\to\infty} t_n(z_n - \bar{z})$. Then, we have that

$$\lim_{n\to\infty} t_n(f(z_n) - f(\bar{z})) = (\nabla f_1(\bar{z})^T h, \ldots, \nabla f_\ell(\bar{z})^T h)^T.$$

Proof. It is enough to show that $\lim_{n\to\infty} t_n(f_i(z_n) - f_i(\bar{z})) = \nabla f_i(\bar{z})^T h$, for all $i = 1, \ldots, \ell$. Fix $i = 1, \ldots, \ell$. From Taylor Theorem we have

$$t_n(f_i(z_n) - f_i(\bar{z})) = t_n \left(\nabla f_i(\bar{z})^T (z_n - \bar{z}) + o_i(\|z_n - \bar{z}\|) \right),$$

which can be rearranged as

$$t_n(f_i(z_n) - f_i(\bar{z})) = \nabla f_i(\bar{z})^T \left(t_n(z_n - \bar{z}) \right) + t_n o_i(\|z_n - \bar{z}\|).$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} t_n(f_i(z_n) - f_i(\bar{z})) = \nabla f_i(\bar{z})^T h + \lim_{n\to\infty} t_n o_i(\|z_n - \bar{z}\|).$$

Note that,

$$\lim_{n\to\infty} t_n o_i(\|z_n - \bar{z}\|) = \lim_{n\to\infty} t_n(z_n - \bar{z}) \left(\frac{o_i\|z_n - \bar{z}\|}{(z_n - \bar{z})} \right) = 0,$$

since $\frac{o_i\|z_n - \bar{z}\|}{(z_n - \bar{z})} \to 0$ as $\lim_{n\to\infty} t_n(z_n - \bar{z}) = h$. This completes the proof. □

The following lemma shows the relationship between the contingent cone $T(M^i;\bar{x})$ and the linearizing $L(M;\bar{x})$ cone.

Lemma 2.3 If $\bar{x}$ is a feasible solution to problem P, then

$$\bigcap_{i=1}^{\ell} cl\ conv\ T(M^i, \bar{x}) \subseteq L(M; \bar{x}) \quad \text{where} \quad M = \bigcap M^i. \tag{6}$$

Proof. See, [4, Lemma 3.2]. □

The reverse inclusion in (6) does not hold. Note that, when the sets M^i are closed and convex, then $cl\ conv\ T(M^i, \bar{x}) = T(M^i, \bar{x})$. Hence, we consider the assumption

$$L(M; \bar{x}) \subseteq \bigcap_{i=1}^{\ell} T(M^i, \bar{x}), \tag{7}$$

which is called *generalized Abadie regularity condition* (GARC), and was introduced in [4].

3 Main Results

3.1 Proper Karush-Kuhn-Tucker Conditions

A standard way to prove PKKT conditions is to show that KTPE holds for (P). In other words, we need to show the inconsistency of the system

$$\nabla f_i(\bar{x})^T d \leqq 0,$$

$$\exists i_0 \in \{1, 2, \ldots, \ell\}, \ \text{such that} \ \nabla f_{i_0}(\bar{x})^T d < 0,$$
$$\nabla g_j(\bar{x})^T d \leqq 0, \ \ j \in I(\bar{x}).$$

Remark 3.1 Theorem 4.2 in [13] establishes that PKKT conditions hold for any locally efficient solution (see Definition 2.1) of problem (P), under the following assumption.

$$L(Q; \bar{x}) \subseteq \bigcap_{i=1}^{l} clconvT(Q^i, \bar{x}). \tag{8}$$

Conditions (8) and (7) coincide when Q^i and M^i are all convex. However, condition (7) does not imply PKKT conditions for arbitrary efficient points (see, for instance, [4, Example 4.2]). On the other hand, it was recently established in [4, Theorem 5.3] that condition (7) implies PKKT conditions for Geoffrion-properly efficient points. Li (2000) presented an example in [25, Example 4.1] with a locally Lipschitz constraint, and shows that (8) does not guarantee KKT conditions for the non-smooth case. Regarding GARC, it was shown in [25, Example 4.1], that regularity condition (7) does not also guarantee KKT condition for the non-smooth case. This is why, as we mentioned in the introduction, we restrict our analysis to the differentiable case.

Next we prove that the result established in [4, Theorem 5.3] can be extended to the larger set of Borwein-properly efficient points.

Theorem 3.1 *Let $\bar{x} \in X$. Assume that GARC (condition (7)) holds at $\bar{x}$. Suppose that $\bar{x} \in X$ is a Borwein-properly efficient point of (P). Then the system*

$$\nabla f_i(\bar{x})^T d \leqq 0, \ \exists \ i_0 \in \{1, 2, \ldots, \ell\}, \quad \textit{such that } \nabla f_{i_0}(\bar{x})^T d < 0, \tag{9}$$

$$\nabla g_j(\bar{x})^T d \leqq 0, \ j \in I(\bar{x}), \tag{10}$$

has no solution $d \in \mathbb{R}^p$.

Proof. Assume that (9) and (10) have a solution $d \in \mathbb{R}^p$. Without loss of generality, one may assume

$$\nabla f_1(\bar{x})^T d < 0, \tag{11}$$

$$\nabla f_i(\bar{x})^T d \leqq 0, \quad i \in \{2, \ldots, \ell\}. \tag{12}$$

From (11) and (12), we can write $d \in L(M; \bar{x})$, since GARC holds. It implies that

$$d \in T(M^i; \bar{x}), \quad \forall i \in \{1, 2, ..., \ell\}.$$

Hence, for any $i \in \{2, 3, ..., \ell\}$, we have that $d \in T(M^i; \bar{x})$. This implies that there exist sequences $\{x^n\}_{n\in\mathbb{N}} \subseteq M^i$ and

$$\{s_n\} \subset \mathbb{R}_{++} \tag{13}$$

such that

$$\lim_{n\to\infty} x^n = \bar{x}, \quad \text{and } \lim_{n\to\infty} s_n(x^n - \bar{x}) = d. \tag{14}$$

Fix $i_0 \in \{2, 3, ..., \ell\}$. For this i_0, take the corresponding sequence $\{x^n\} \subseteq M^{i_0}$ as above. For this sequence we have

$$f_{i_0}(x^n) - f_{i_0}(\bar{x}) \leqq 0.$$

For every $n \in \mathbb{N}$, define the set

$$J_n := \{k \geqq 2 \mid f_k(x^n) > f_k(\bar{x})\} \subset \{2, 3, ..., \ell\}.$$

We claim that $J_n \neq \emptyset$ for all $n \in \mathbb{N}$. Indeed, if $J_n = \emptyset$ for some n, then this means that $\forall k \geqq 2$ (in particular, this includes $k = i_0$) we would have that

$$f_k(x^n) \leqq f_k(\bar{x}). \tag{15}$$

Using also the fact that $f_1(x^n) < f_1(\bar{x})$, we arrive at a contradiction with the efficiency of $\bar{x}$. Hence, $J_n \neq \emptyset$ for all $n \in \mathbb{N}$.

Moreover, $J_n \in \mathcal{P}(\{2, 3, ..., \ell\}) := \mathcal{P}_\ell$, the family of subsets of the finite set $\{2, 3, ..., \ell\}$. Consider the function $\varphi : N \to \mathcal{P}_\ell$ defined as $\varphi(n) = J_n$. Since the

set of natural numbers is infinite and $\mathcal{P}_\ell$ has a finite number of elements, there must exist a subset J in the range of φ such that $\varphi^{-1}(J) = \{n \in \mathbb{N} \mid \varphi(n) = J\}$ is infinite. Note that J is nonempty because $\varphi(n) = J_n \neq \emptyset, \ \forall\, n$. Hence, there exists $J \in \mathcal{P}_\ell$ such that $\varphi^{-1}(J) =: I_0 \subseteq \mathbb{N}$ is infinite. For the subsequence $\{x^n\}_{n\in I_0}$ we have $\varphi(n) = J$. In other words, φ is constant when restricted to $n \in I_0$. By restricting our analysis to the subsequence $\{x^n\}_{n\in I_0}$, we can assume without loss of generality that $\varphi(n) = J_n = J$ is constant for all n. Using Lemma 2.1 for $h = f_k$, (14) and (15), we have

$$\nabla f_k(\bar{x})^T d \geqq 0, \quad \forall k \in J_n = J, \ \ \forall n.$$

The above expression, together with (12) implies

$$\nabla f_k(\bar{x})^T d = 0, \ \text{ for all } \ k \in J. \tag{16}$$

Let

$$r_n = (-1/2s_n)(\nabla f_1(\bar{x})^T d, \ldots, \nabla f_\ell(\bar{x})^T d)^T \in \mathbb{R}^\ell_+, \tag{17}$$

where $\{s_n\}$ is as in (13) and define $\hat{h} \in \mathbb{R}^\ell$ as

$$\hat{h} = \lim_{n\to\infty} s_n(f(x_n) + r_n - f(\bar{x})). \tag{18}$$

From (11) and (17),

$$\lim_{n\to\infty} s_n r_n = (-1/2)(\nabla f_1(\bar{x})^T d, \ldots, \nabla f_\ell(\bar{x})^T d)^T \in \mathbb{R}^\ell_+ \setminus \{0\}. \tag{19}$$

By Lemma 2.2 applied to $z_n := x_n$, $\bar{z} := \bar{x}$ and $h = d = \lim_{n\to\infty} s_n(x^n - \bar{x})$, we have

$$\lim_{n\to\infty} s_n(f(x_n) - f(\bar{x})) = (\nabla f_1(\bar{x})^T d, \ldots, \nabla f_\ell(\bar{x})^T d)^T. \tag{20}$$

Thus, combining (18), (19) and (20) we obtain,

$$\hat{h} = (1/2)(\nabla f_1(\bar{x})^T d, \ldots, \nabla f_\ell(\bar{x})^T d)^T \in (-\mathbb{R}^\ell_+) \setminus \{0\}.$$

From (18) we have that $\hat{h} \in T(f(X) + \mathbb{R}^\ell_+, f(\bar{x}))$ and $\hat{h} \neq 0$, contradicting the Borwein proper efficiency of $\bar{x}$. Therefore, (9) and (10) have no solution. □

The following corollary is an immediate consequence of Theorem 3.1. It was obtained in [4, Theorem 5.3], using only the concept of Geoffrion-properly efficient points.

Corollary 3.1 *Let $\bar{x} \in X$. Assume that GARC (condition* (7)*) holds at $\bar{x}$. Suppose that $\bar{x} \in X$ is a Geoffrion-properly efficient point of (P). Then* (9) *and* (10) *have no solution.*

Proof. The conclusion follows from Theorem 3.1 and the fact that every Geoffrion-properly efficient point is Borwein-properly efficient (see [8, Proposition 1]). □

Using Theorem 3.1 we can establish PKKT conditions for multiobjective optimization problems.

Theorem 3.2 *Assume that the hypotheses of Theorem 3.1 hold. If $\bar{x} \in X$ is a Borwein properly efficient solution of Problem (P), then there exist vectors $u \in \mathbb{R}^\ell$, $v \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^{\ell} u_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{x}) = 0, \tag{21}$$

$$v_j g_j(\bar{x}) = 0, \ j = 1, 2, \ldots, m. \tag{22}$$

$$u > 0, \ v \geqq 0.$$

Proof. Let $\bar{x}$ be a Borwein properly efficient solution of Problem (P). Then, from Theorem 3.1 and Tucker's theorem [16], there exist $u \in \mathbb{R}^\ell$, $u_i > 0$ and $v_j \geqq 0$, $j \in I(\bar{x})$ such that

$$\sum_{i=1}^{\ell} u_i \nabla f_i(\bar{x}) + \sum_{j \in I(\bar{x})} v_j \nabla g_j(\bar{x}) = 0.$$

By setting $v_j = 0$ for every $j \notin I(\bar{x})$, we can write (21). Since $g_j(\bar{x}) = 0$ for $j \in I(\bar{x})$, we have (22), which completes the proof. □

Remark 3.2 Figure 1 demonstrates the relationship between proper efficiency and proper Karush-Kuhn-Tucker conditions, where A:= Convexity assumptions (i.e., X is convex and the functions f_i, g_j are convex) and B:= Generalized Abadie regularity condition (7). Here, $X \overset{\mathrm{U}}{\to} Y$ means that X implies Y under assumption U, and $X \to Y$ means that X implies Y without any additional assumption. The implications $BPE \overset{\mathrm{B}}{\to} KTPE$ and $GPE \overset{\mathrm{B}}{\to} KTPE$ are Theorem 3.1 and Corollary 3.1, respectively.

Corollary 14.3 of [31] proves that, when X is polyhedral and f_i are linear on $\mathbb{R}^n$, then any efficient point is Geoffrion properly efficient. In this setting, efficient points, Geoffrion properly efficient points and Borwein properly efficient points coincide (see [28, Theorem 2.45 and 2.48]). We therefore have the following corollary.

Corollary 3.2 *Let X be polyhedral and f_i be linear on $\mathbb{R}^n$. Let $\bar{x} \in X$ and assume that GARC (i.e., (7)) holds at $\bar{x}$. If $\bar{x} \in X$ is an efficient solution of Problem (P), then there exist vectors $u \in \mathbb{R}^\ell$, $v \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^{\ell} u_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{x}) = 0,$$

$$v_j g_j(\bar{x}) = 0, \ j = 1, 2, \ldots, m.$$

$$u > 0, \ v \geqq 0.$$

Proof. If X is polyhedral and f_i are linear on $\mathbb{R}^n$ then every efficient point is Borwein proper efficient point (see Remark 3.2). Hence by Theorem 3.2 we can conclude that PKKT conditions hold. □

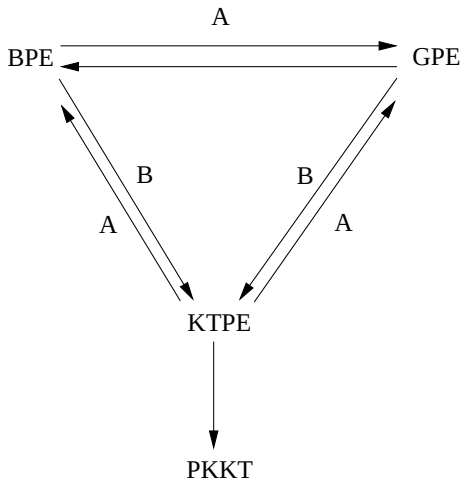


Figure 1: Relationship between various forms of proper efficiency and PKKT

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References

1. Maciel, S. A., Santos, S. A., Sottosanto, G. N.: Regularity conditions in differentiable vector optimization revisited. J. Optim. Theory Appl. 142, 385-398, (2009)

2. Bigi, G.: Optimality and Lagrangian regularity in vector optimization. PhD Thesis, Universita di Pisa (1999)

3. Chankong, V., Haimes, Y. Y.: Multiobjective Decision Making: Theory and Methodology. North-Holland, Amsterdam (1983)

4. Burachik, R. S., Rizvi, M. M.: On weak and strong Kuhn-Tucker conditions for smooth multiobjective optimization. J. Optim. Theory Appl., 155(2), 477-491 (2012)

5. Maeda, T.: Constraint qualification in multiobjective optimization problems: differentiable case. J. Optim. Theory Appl. 80, 483-500 (1994)

6. Kuhn, H. W., Tucker, A. W.: Nonlinear programming, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Berkeley, California, 481-492 (1952)

7. Geoffrion, A. M.: Proper efficiency and the theory of vector maximization, J. Math. Anal., 22, 618-630 (1968)

8. Borwein, J. M.: Proper efficient points for maximizations with respect to cones. SIAM J. Control Optim. 15, 57-63 (1977)

9. Chandra, S., Dutta, J., Lalitha, C. S.: Regularity conditions and optimality in vector optimization, Numerical Functional Anal. Optim. 25, 479-501 (2004)

10. Wang, S. Y.: Second-order necessary and sufficient conditions in multiobjective programming. Numerical Functional Anal. Optim. 12, 237-252 (1991)

11. Maeda, T.: Second-order conditions for efficiency in non-smooth multiobjective optimization problems. J. Optim. Theory Appl. 122(3), 521-538 (2004)

12. Li, X. F., Zhang, J. Z.: Stronger Kuhn-Tucker type conditions in nonsmooth multiobjective optimization: locally Lipschitz case. J. Optim. Theory Appl. 127, 367-388 (2005)

13. Giorgi, G., Jimenez, B., Novo, V.: Strong Kuhn-Tucker conditions and constraint qualifications in locally Lipschitz multiobjective optimization problems, TOP 17(2), 288-304 (2009)

14. Marusciac, I.: On Fritz John type optimality criterion in multiobjective optimization. Analyse Numerique et Theorie de I'Approximation, 11, 109-114 (1982)

15. Singh, C.: Optimality conditions in multiobjective differentiable programming. J. Optim. Theory Appl. 53, 115-123 (1987)

16. Mangasarian, O. L.: Nonlinear Programming. SIAM, Philadelphia (1994)

17. Bigi, G., Pappalardo, M.: Regularity conditions in vector optimization, J. Optim. Theory Appl. 102, 83-96 (1999)

18. Preda, V., Chitescu, I.: On constraint qualification in multiobjective optimization problems: semidifferentiable case. J. Optim. Theory Appl. 100, 417-433 (1999)

19. Bigi, G. and Pappalardo, M.: Regularity conditions for the linear separation of sets, J. Optim. Theory Appl. 99, 533-540 (1998)

20. Abadie, J.: On the Kuhn-Tucker Theorem, Nonlinear Programming. John Wiley, New York, 21-36 (1967)

21. Bazaraa, M., Shetty, C. M.: Nonlinear Programming: Theory and Algorithms. John Wiley and Sons, New York (1979)

22. Bazaraa, M. S., Goode, J. J., Nashed, M. Z.: On the cones of tangents with applications to mathematical programming. J. Optim. Theory Appl. 13, 389-426 (1974)

23. Giorgi, G., Guerraggio, A.: On the notion of tangent cone in mathematical programming. Optimization, 25, 11-23 (1992)

24. Giorgi, G., Guerraggio, A.: On a characterization of Clarke's tangent cone. J. Optim. Theory Appl., 74(2), 369-372 (1992)

25. Li, X. F.: Constraint qualifications in nonsmooth multiobjective optimization. J. Optim. Theory Appl., 106(2), 373-398 (2000)

26. Borwein, J. M.: The geometry of Pareto efficiency over cones. Math. Operationsforschung 11(2), 235-248 (1980)

27. Guerraggio, A., Molho, E., Zaffaroni, A.: On the notion of proper efficiency in vector optimization. J. Optim. Theory Appl. 82(1), 1-21 (1994)

28. Ehrgott, M.: Multicriteria Optimization. Springer Berlin Hiedelberg, New York (2005)

29. Rizvi, M. M., Hanif, M., Waliullah, G. M.: First-order optimality conditions in multiobjective optimization problems: - differentiable case. J. Bangladesh Math. Soc. (Ganit), 29, 99-105 (2009)

30. Rizvi, M. M., Nasser, M.: New second-order optimality conditions in multiobjective optimization problems: differentiable case, J. Indian Inst. Sci., 86, 279-286 (2006)

31. White D. J.: Concept of proper efficiency, European J. Oper. Res., 13, 180-188 (1983)